Born Geometry and Relative Locality

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Abstract

Born reciprocity is a type of symmetry transformation which is compatible with the Hamiltonian formulation of classical mechanics as well as phase space quantum mechanics. In the double field theory literature, the target space for the field theory must include this Born reciprocity symmetry along with T-duality. The natural setting for such a model is called a Born manifold. We investigate connections on these manifolds, and explicitly compute nontrivial torsion-free connections which respect the required symmetries. It is found that primary Hopf surfaces naturally acquire a nontrivial Born connection, while complex 2-tori are flat for any simple choice of spacetime metric. In future work the possibility of nontrivial Born connections on the tangent bundle of spacetime will be considered along with their physical interpretation.

Contents

1 Introduction

In the 1940s, Max Born investigated the duality between position and momentum space which is fundamental to many computations in quantum mechanics. During this time, Born conceptualized the principle of reciprocity - that the laws of physics should be symmetric under the exchange of space-time with momentum-energy [1]. This reciprocity is inherent in the Hamiltonian formulation of classical mechanics, where the equations of motion are invariant under transformations of the form $x_i \mapsto p_i$, $p_i \mapsto -x_i$. Indeed, the matrix mechanics formulation of quantum mechanics also follows this reciprocity as we can see from Heisenberg's equations of motion. However, the exact nature of this reciprocity is mysterious, and it is not obvious how to extend the principle of reciprocity to the theory of general relativity or modern quantum field theories such as string theory. Between 2014 and 2016, Laurent Freidel and collaborators constructed a geometric theory of phase space which naturally introduced Born reciprocity, and dubbed it Born Geometry [2, 3, 4, 5]. This geometry naturally generalizes the original setting of double field theory. Born Geometry takes inspiration from earlier attempts to endow phase space with a noncommutative structure, such as the principle of relative locality [6].

Born Geometry is a theory of extended spacetime, in which the typical n dimensional setting of physics is replaced with a general $2n$ dimensional manifold which we can think of as an analogue of phase space. This manifold carries with it a *Born structure*, which allows one to differentiate between the *space-time* sector of the manifold and the *energy-momentum* sector. The splitting of extended spacetime is induced by a so-called para-hypercomplex structure, which is a set of tensor fields on the manifold who's eigenbundles determine natural splittings of extended spacetime into dual submanifolds. Our goal will be to investigate connections on the tangent bundle of the Born manifold, in particular the Born connection first appearing in [7].

1.1 Complex and Real Structures

The Born geometry model of extended space time will require us to recall some facts about smooth manifolds and vector fields. We will assume the reader is familiar with differential geometry at a level high enough to understand general relativity. This should include an understanding of smooth manifolds, tangent spaces, and connections. For some background, see the excellent textbooks by Lee, Kobayashi and Nomizu, and Tu [8, 9, 10]. On the other hand, complex geometry is not typically studied in an ordinary differential geometry course. As such, we will begin with complex structures. For a detailed overview of this material, see [11, 12].

Definition 1.1 (Almost-Complex Structure)**.** Let M be a smooth manifold and let TM be the tangent bundle on M, where π is the projection. Then an **almost-complex structure** on M is defined to be a tensor field $I \in \Gamma(\mathrm{End}(TM))$ such that $I_p^2 = -\mathbb{1}_{T_pM}$.

An almost-complex structure is a structure which allows one to identify the complexification of the tangent bundle TM with the vector bundle $TM \oplus ITM$. For a manifold to be considered complex in the most rigorous sense, we require that the transition maps are holomorphic. From this one can show that the transition maps are holomorphic by showing that the eigenbundles of I are integrable.

Definition 1.2 (Integrable Subbundle)**.** Let M be a smooth manifold and let TM be its tangent bundle. Then a subbundle E of TM is called **integrable** if it is closed with respect to the Lie Bracket. This means that for all $X, Y \in \Gamma(E)$ we have $[X, Y] = XY - YX \in \Gamma(E)$.

If we consider the special case of an almost complex structure J, then the question of integrability of the eigenbundles of J comes down to whether or not the Nijenhuis tensor vanishes.

Definition 1.3 (Nijenhuis Tensor)**.** Let M be a smooth manifold and TM its tangent bundle. Then the **Nijenhuis tensor** of a section $A \in \Gamma(\mathrm{End}(TM))$ is defined by the following formula.

$$N_A(X, Y) = -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY]. \tag{1.1.1}$$

Theorem 1.1 (Newlander-Nirenberg [13])**.** An almost-complex structure I has integrable eigenbundles if and only if $N_I = 0$.

Definition 1.4 (Complex Manifold)**.** A manifold M is called almost-complex if there exists an almost-complex structure $I \in \Gamma(\mathrm{End}(TM))$. If the eigenbundles of I are integrable then I is called a **complex structure**, and M is called a **complex manifold**. This is equivalent to the existence of an atlas of charts $\{\phi_U | U \subseteq M\}$ where if $\phi_U : M \to \mathbb{R}^{2n} \cong \mathbb{C}^n$ and $\phi_V : M \to \mathbb{R}^{2n} \cong \mathbb{C}^n$ are coordinate charts for M then $\phi_V \circ \phi_U^{-1}$ is holomorphic.

Definition 1.5 (Almost-Real Structure)**.** An **almost-real structure** on M is defined to be a tensor field $K \in \Gamma(\mathrm{End}(TM))$ such that $K_p^2 = \mathbb{1}_{T_pM}$

On the other hand, an almost-real structure provides an involution on TM because it squares to the identity. It is mostly interesting to study real structures in Born Geometry because a real structure can induce integrable submanifolds of M, which correspond to natural splittings of extended spacetime. If K is an almost-real structure then the eigenvalues of K must be 1 and/or -1.

Definition 1.6 (Para-Complex Manifold)**.** Let M be a smooth manifold and let $K \in \Gamma(\mathrm{End}(TM))$ be an almost-real structure. The eigenbundles of K are:

$$T^+M = \coprod_{x\in M} \{v \in T_xM | K_x v = v\} \tag{1.1.2}$$

$$T^-M = \coprod_{x\in M} \{v \in T_xM | K_x v = -v\} \tag{1.1.3}$$

If T^+M and T^-M have the same rank and T^+M and T^-M are both integrable subbundles, then we call M a **para-complex manifold**.

1.2 Para-Quaternionic and Born Structures

Born geometry goes a step beyond a single complex structure - we can consider any possible combination of real and complex structures. This leads in naturally to the study of quaternionic structures and their relatives.

Definition 1.7 (Para-quaternions)**.** The **para-quaternions** are an associative algebra $\tilde{\mathbb{H}}$ over the real numbers with generators $\{1, J_1, J_2, J_3\}$. The generators define $\tilde{\mathbb{H}}$ according to the following multiplication rules.

$$J_1^2 = J_2^2 = -J_3^2 = 1 \qquad J_1J_2 = -J_2J_1 = J_3. \tag{1.2.1}$$

Definition 1.8 (Almost-Para-Hypercomplex Structure)**.** Let M be a smooth manifold, and let $I, J, K \in \Gamma(\mathrm{End}(TM))$ be defined as tensor fields with the property that $I^2 = J^2 = -K^2 = \mathbb{1}$. We call the quadruplet (M, I, J, K) an almost para-hypercomplex structure.

The natural setting of Born geometry is a para-hypercomplex manifold along with some extra requirements regarding the integrability of the eigenbundles of I, J, and K.

Definition 1.9 (Symplectic Form)**.** Let $\omega \in \Omega^2(M)$ be a totally antisymmetric two-form. That is, ω is a bilinear antisymmetric map from $(T^*M)^2$ to $\mathbb{R}$. We call ω **symplectic** if it is everywhere nonvanishing and non-degenerate.

A symplectic form on M can only exist if M is even dimensional (set $\dim M = 2n$). It is therefore interesting to consider symplectic forms on complex manifolds and tangent bundles. Next we need to define what it means for a manifold to be hyper-Hermitian, hyper-Kähler, and so on.

Definition 1.10 (Hermitian Manifold)**.** Let M be a complex manifold with tangent space $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. Then M is called **Hermitian** if there exists a Hermitian metric h on $T_{\mathbb{C}}M$. This essentially says that the components h_{ab} form a Hermitian matrix.

Definition 1.11 (Para-Hermitian Manifold)**.** Let M be a para-complex manifold with tangent space $TM = T^{+}M \oplus T^{-}M$. Suppose M has the data of a pseudo-Riemannian metric η. Then M is called **para-Hermitian** if $\eta(KX, Ku) = -\eta(X, Y)$.

It is in fact possible to define a symplectic form $\omega(X, Y) = \eta(KX, Y)$. Hence it just so happens that introducing a (possibly split) metric to a para-Hermitian manifold immediately produces for us a natural symplectic form. If we then add a Riemannian metric, one finally acquires a Born manifold.

Definition 1.12 (Born Manifold)**.** Let (M, K, η, ω) be a para-Hermitian manifold. Let $\mathcal{H}$ be a metric of signature $(2n, 0)$ on M. Then $(M, K, \eta, \mathcal{H})$ is called a Born manifold. The triplet $(M, K, \eta, \mathcal{H})$ is equivalent to the data of a para-hypercomplex manifold (M, I, J, K) where $I = \mathcal{H}^{-1}\omega$ and $J = \eta^{-1}\mathcal{H}$. It can also be shown that $K = \eta^{-1}\omega$, so it even holds that $(M, \eta, \mathcal{H}, \omega)$ is an equivalent formulation.

Since there are so many ways of characterizing a Born manifold, it is useful to split them up and interpret the different characterizations. The data $(M, \eta, \mathcal{H}, J)$ is called the **chiral representation** of M, and can be used to describe the spacetime inhabited by a closed string, which has a worldsheet defined by a mapping $X : S^1 \to M$. The dynamics of the string can then be determined completely by the constraints [2, 3, 4] $\mathcal{H}(\partial_\tau X, \partial_\tau X) = 0, \eta(\partial_\tau X, \partial_\tau X) = 0$, and $\partial_\tau X = J\partial_\sigma X$.

In order to actually do computations involving dynamics of objects in this spacetime, one needs a connection. This way dynamical fields can be compared between different points. It happens that if M is a Born manifold, it also admits a natural connection [7].

1.3 D-Brackets and the Born Connection

Much can be learned from the connections on Born manifolds. A Born manifold is also a para-Hermitian manifold, and it has been shown [12] that each para-Hermitian manifold has a canonically defined connection ∇^c called the **canonical connection**. We will construct this as follows. Suppose K is a real structure on

M. Then we define two projectors $\frac{1}{2}(\mathbb{1}+K) = P : TM \to TM^+$ and $\frac{1}{2}(\mathbb{1}-K) = \tilde{P} : TM \to TM^-$, where $TM^{\pm}$ are the positive and negative eigenbundles of K.

Definition 1.13 (Canonical Connection)**.** Let (M, η, K) be a para-Hermitian manifold. Let ∇^η be the Levi-Civita connection of η. Then the **canonical connection** of M is defined by

$$\nabla^c_X Y = P\nabla^\eta_X PY + \tilde{P}\nabla^\eta_X \tilde{P} \quad \forall X, Y \in \Gamma(TM). \tag{1.3.1}$$

From the expression we can glean that this connection somehow respects the two eigenbundles by separating all "interaction terms" between the two bundles by projecting onto them before and after differentiating. In [14] it was shown that the canonical connection ∇^c induces a bracket $[\![X, Y]\!]$ on TM. In fact, this bracket is shown to be unique and it is shown that there is a unique connection ∇ which is compatible in a certain way with the Born structure.

Definition 1.14 (Metric-Compatible Bracket)**.** A **metric-compatible bracket** $[\![X, Y]\!]$ on a manifold M is a bilinear operation on the algebra $\Gamma(TM)$ of vector fields which satisfies

$$X(\eta(Y, Z)) = \eta([\![X, Y]\!], Z) + \eta(Y, [\![X, Z]\!]), \tag{1.3.2}$$

$$[\![X, fY]\!] = f[\![X, Y]\!] + X(f)Y, \tag{1.3.3}$$

$$\eta(Y, [\![X, X]\!]) = \eta([\![Y, X]\!], X). \tag{1.3.4}$$

If M is a smooth manifold and η is a metric on M, the triple $(TM, \eta, [\![\cdot, \cdot]\!])$ defines a so-called metric algebroid. We will define a bracket on our extended spacetime using the Born metric η. On a Born manifold M there exists a unique bracket called the D-bracket which is defined as follows.

Definition 1.15 (D-Bracket)**.** Let $(M, \eta, \mathcal{H}, K)$ be a Born manifold. Then there exists a unique metric-compatible bracket called the **D-Bracket** on M determined by the following equation

$$\eta([\![X, Y]\!], Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y). \tag{1.3.5}$$

Finally, we get to the most important connection on the manifold, the Born connection.

Definition 1.16 (Born Connection)**.** Let $(M, \eta, K, \mathcal{H})$ be a Born manifold. Then there is a unique connection ∇^B such that $\nabla^B_X I = \nabla^B_X J = \nabla^B_X K = 0$ for all $X \in \Gamma(TM)$. The connection is given by the following formula for all $X, Y \in \Gamma(TM)$:

$$\nabla^B_X Y = [\![\tilde{P}X, PY]\!] + [\![PX, \tilde{P}Y]\!] + P(K[\![PX, K\tilde{P}Y]\!]) + \tilde{P}(K[\![\tilde{P}X, KPY]\!]). \tag{1.3.6}$$

If we would like to study of spacetime as an integral submanifold of a Born manifold, we want our generalized connection on the tangent space to reduce to the Levi-Civita connection so that the model reproduces ordinary physics. It can be verified [7] that given a manifold M the Born connection on M is the unique connection which has vanishing torsion (with respect to the D-bracket) and is also metric-compatible. Furthermore, it restricts to the Levi-Civita connection on the integral submanifolds of T^+M and T^-M.

2 Connections on Compact Manifolds

Born Geometry is a relatively new topic of study, and as such very few concrete examples of the relevant structures have been computed in the literature. In the Master's project of Boulter [15], a number of compact manifolds are considered. In this thesis, the para-hypercomplex structure of the Born manifold is explicitly constructed along with the metrics and symplectic form which define the Born structure. The goal of this section will be to extend these calculations to include the Born connection.

2.1 Complex 2-Tori

In what follows we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ by the natural isomorphism. Consider the complex vector space $\mathbb{C}^2$ along with the underlying real vector space $\mathbb{R}^4$. A collection of four vectors $\Lambda = \{v_1, v_2, v_3, v_4\}$ which span $\mathbb{R}^4$ generate a group action on $\mathbb{C}^2$ given by $w \mapsto w + v_i$. If we quotient by this group action, the resulting manifold is periodic in each direction, and as such can be identified with a product of 4 circles - a torus! This construction is called the complex 2-torus, and is an important and simple example in complex geometry.

The beginning of this construction is the same as Boulter's, 2018 [15]. Since $M = \mathbb{C}^2/\Lambda$ is a complex manifold, there is a natural complex structure $I \in \mathrm{End}(\Gamma(TM))$ given by the map $I|_p\mathbf{v}(p) = i\mathbf{v}(p)$, $p \in M$ whenever $\mathbf{v} \in \Gamma(TM)$. Let $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ denote the complex coordinate vector fields on M. We can tell from the built-in complex structure that there is at least one real structure K on M given by $K\frac{\partial}{\partial z_i} = \frac{\partial}{\partial \bar{z}_i}$ and $K\frac{\partial}{\partial \bar{z}_i} = \frac{\partial}{\partial z_i}$. Finally, to induce the Born reciprocity it is natural to consider a real structure $J\frac{\partial}{\partial z_1} = \frac{\partial}{\partial \bar{z}_2}, J\frac{\partial}{\partial z_2} = \frac{\partial}{\partial \bar{z}_1}$. We can also write down a metric given by the form $\eta = \mathrm{d}z_1^2 + \mathrm{d}\bar{z}_1^2 + \mathrm{d}z_2^2 + \mathrm{d}\bar{z}_2^2$. It is clear that $KI = J$, so we see that (I, J, K) are an almost para-hypercomplex structure on M. Along with the metric η, we get a Born structure on M. In general, any metric of the form $\eta = p(z_1,z_2)(\mathrm{d}z_1^2+\mathrm{d}z_2^2)+q(z_1,z_2)(\mathrm{d}\bar{z}_1^2+\mathrm{d}\bar{z}_2^2)+r(z_1,z_2)(\mathrm{d}z_1\mathrm{d}z_2+\mathrm{d}\bar{z}_1\mathrm{d}\bar{z}_2)$ produces a Born structure whenever $4pq-r^2$ is nowhere vanishing.

In component form, we can write η locally:

$$\eta = \begin{bmatrix} p(z_1,z_2) & r(z_1,z_2) & 0 & 0 \\ r(z_1,z_2) & p(z_1,z_2) & 0 & 0 \\ 0 & 0 & q(z_1,z_2) & r(z_1,z_2) \\ 0 & 0 & r(z_1,z_2) & q(z_1,z_2) \end{bmatrix}. \tag{2.1.1}$$

The Levi-Civita connection is the first thing we must compute, since it gives us the canonical connection. After this, the Levi-Civita connection defines the D-Bracket, which defines the Born connection. So we can compute ∇^η_X by computing its connection coefficients Γ^k_{ij}, which determine the connection on any local coordinate patch. The connection coefficients are given by the Koszul formula [9]:

$$\Gamma^\ell_{jk} = \frac{1}{2}\eta^{\ell r}(\partial_k\eta_{rj}+\partial_j\eta_{rk}-\partial_r\eta_{jk}). \tag{2.1.2}$$

From this we can compute all of the connection coefficients. Splitting the components up into matrices $\Gamma^1_{jk},\Gamma^2_{jk},\Gamma^3_{jk},\Gamma^4_{jk}$ we can write them as follows. See the apppendix on page 12 for the Mathematica code used to generate these.

$$\Gamma^1_{jk} = \frac{1}{2(p^2-r^2)}\begin{bmatrix} pp_{z_1}+r(p_{z_2}-2r_{z_1}) & pp_{z_2}+rp_{z_1} \\ pp_{z_2}+rp_{z_1} & -(rp_{z_2}+p(p_{z_1}-2r_{z_2})) \end{bmatrix} \oplus \begin{bmatrix} rq_{z_2}-pq_{z_1} & rr_{z_2}-pr_{z_1} \\ rr_{z_2}-pr_{z_1} & rq_{z_2}-pq_{z_1} \end{bmatrix},$$

$$\Gamma^2_{jk} = \frac{1}{2(p^2-r^2)}\begin{bmatrix} -(rp_{z_1}+p(p_{z_2}-2r_{z_1})) & pp_{z_1}-rp_{z_2} \\ pp_{z_1}-rp_{z_2} & pp_{z_2}+r(p_{z_1}-2r_{z_2}) \end{bmatrix} \oplus \begin{bmatrix} rq_{z_1}-pq_{z_2} & rr_{z_1}-pr_{z_2} \\ rr_{z_1}-pr_{z_2} & rq_{z_1}-pq_{z_2} \end{bmatrix},$$

$$\Gamma^3_{jk} = \frac{1}{2(q^2-r^2)}\begin{bmatrix} 0 & 0 & qq_{z_1}-rr_{z_1} & qr_{z_1}-rq_{z_1} \\ 0 & 0 & qq_{z_2}-rr_{z_2} & qr_{z_2}-rq_{z_2} \\ qq_{z_1}-rr_{z_1} & qr_{z_2}-rq_{z_2} & 0 & 0 \\ qr_{z_1}-rq_{z_1} & qr_{z_2}-rq_{z_2} & 0 & 0 \end{bmatrix},$$

$$\Gamma^4_{jk} = \frac{1}{2(q^2 - r^2)} \begin{bmatrix} 0 & 0 & qr_{z_1} - rq_{z_1} & qq_{z_1} - rr_{z_1} \\ 0 & 0 & qr_{z_2} - rq_{z_2} & qq_{z_2} - rr_{z_2} \\ qr_{z_1} - rq_{z_1} & qr_{z_2} - rq_{z_2} & 0 & 0 \\ qq_{z_1} - rr_{z_1} & qq_{z_2} - rr_{z_2} & 0 & 0 \end{bmatrix}. \tag{2.1.3}$$

From here we must compute the coefficients of the canonical connection. The canonical connection obeys the rule $\nabla^c_X Y = P\nabla^\eta_X PY + \tilde{P}\nabla^\eta_X \tilde{P}Y$.

$$\begin{aligned} \nabla^c_X Y &= P\nabla^\eta_X PY + \tilde{P}\nabla^\eta_X \tilde{P}Y \\ &= \frac{1}{4}(\mathbb{1} + K)\nabla^\eta_X(\mathbb{1} + K)Y + \frac{1}{4}(\mathbb{1} - K)\nabla^\eta_X(\mathbb{1} - K)Y \\ &= \frac{1}{2}\left(\nabla^\eta_X Y + K\nabla^\eta_X KY\right). \end{aligned} \tag{2.1.4}$$

We can see this as a kind of "symmetrization" of the Levi-Civita connection with respect to conjugation by K. In coordinates, this is written as

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\Gamma^k_{ij} + \frac{1}{2}\Gamma^\ell_{ir}K^r_j K^k_\ell, \tag{2.1.5}$$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.1.6}$$

Using this, we can compute the canonical connection coefficients.

$$\tilde{\Gamma}^1_{jk} = \tfrac{1}{2(p^2 - r^2)} \begin{bmatrix} pp_{z_1} - rr_{z_1} & pr_{z_1} - rp_{z_1} & 0 & 0 \\ pp_{z_2} - rr_{z_2} & pr_{z_1} - rp_{z_1} & 0 & 0 \\ 0 & 0 & rq_{z_1} - pq_{z_1} & rr_{z_2} - pr_{z_1} \\ 0 & 0 & rr_{z_2} - pr_{z_1} & rq_{z_2} - pq_{z_1} \end{bmatrix},$$

$$\tilde{\Gamma}^2_{jk} = \tfrac{1}{2(p^2 - r^2)} \begin{bmatrix} pr_{z_1} - rr_{z_2} & pp_{z_1} - rp_{z_2} & 0 & 0 \\ pp_{z_1} - rp_{z_2} & pr_{z_1} - rr_{z_2} & 0 & 0 \\ 0 & 0 & rq_{z_1} - pq_{z_2} & rr_{z_1} - pr_{z_2} \\ 0 & 0 & rr_{z_1} - pr_{z_2} & rq_{z_1} - pq_{z_2} \end{bmatrix},$$

$$\tilde{\Gamma}^3_{jk} = \tfrac{1}{4(q^2-r^2)} \begin{bmatrix} 0 & 0 & q(r_{z_2}+qq_{z_1})-r(q_{z_2}+r_{z_1}) & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) \\ 0 & 0 & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) \\ q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & 0 & 0 \\ q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & 0 & 0 \end{bmatrix},$$

$$\tilde{\Gamma}^4_{jk} = \tfrac{1}{4(q^2-r^2)} \begin{bmatrix} 0 & 0 & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) \\ 0 & 0 & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) \\ q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & 0 & 0 \\ q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & 0 & 0 \end{bmatrix}.$$

Next, we can use the definition

$$\eta([\![X,Y]\!],Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y) \tag{2.1.7}$$

to compute the D-bracket. The D-bracket is a bilinear function, so it should have a coordinate representation given by $[\![\partial_i,\partial_j]\!] = D^k_{ij}\partial_k$ for some tensorial coefficients D^k_{ij}. If we insert $X=\partial_i$, $Y=\partial_j$, and $Z=\partial_k$ we find the coefficients:

$$\eta_{k\ell}D^\ell_{ij} = \eta_{k\ell}\tilde{\Gamma}^\ell_{ij} - \eta_{k\ell}\tilde{\Gamma}^\ell_{ji} + \eta_{j\ell}\tilde{\Gamma}^\ell_{ki},$$

$$\eta^{sk}\eta_{k\ell}D^\ell_{ij} = \delta^s_\ell D^\ell_{ij} = D^s_{ij} = \tilde{\Gamma}^s_{ij} - \tilde{\Gamma}^s_{ji} + \eta_{j\ell}\eta^{sk}\tilde{\Gamma}^\ell_{ki},$$

$$D^s_{ij} = \tilde{\Gamma}^s_{ij} - \tilde{\Gamma}^s_{ji} + \eta_{j\ell}\eta^{sk}\tilde{\Gamma}^\ell_{ki}. \tag{2.1.8}$$

The Born connection is then related to the D bracket by the following.

$$\begin{aligned} \nabla^B_X Y &= [\![\tilde{P}X, PY]\!] + [\![PX, \tilde{P}Y]\!] + PK[\![PX, K\tilde{P}Y]\!] + \tilde{P}K[\![\tilde{P}X, KPY]\!] \\ &= \frac{1}{4}[\![X-KX, Y+KY]\!] + \frac{1}{4}[\![X+KX, Y-KY]\!] \\ &\quad + \frac{1}{8}(\mathbb{1}+K)[\![X+KX, KY-Y]\!] + \frac{1}{8}(K-\mathbb{1})[\![X-KX, Y+KY]\!] \\ &= \frac{1}{4}[\![X,Y]\!] - \frac{1}{4}[\![KX,KY]\!] + \frac{1}{4}K[\![X,KY]\!] - \frac{1}{4}K[\![KX,Y]\!]. \end{aligned}$$

So the Born connection coefficients, which we will denote $\hat{\Gamma}^k_{ij}$, are related to D^k_{ij} by the following formula

$$\hat{\Gamma}^k_{ij} = \frac{1}{4}\left(D^k_{ij} - D^k_{ab}K^a_iK^b_j + D^c_{ib}K^k_cK^b_j - D^c_{aj}K^k_cK^a_i\right).$$

With some more mathematica, we can compute these coefficients for the torus. They are listed on the following page. Remarkably, $\Gamma^3_{ij} = \Gamma^4_{ij} = 0$ even though the D-bracket is highly nontrivial:

$$\hat{\Gamma}^1_{ij} = \tfrac{1}{4(p^2-r^2)} \begin{bmatrix} p(r_{z_2} - p_{z_1}) + r(p_{z_2} - r_{z_1}) & r(r_{z_2} - p_{z_1}) + p(p_{z_2} - r_{z_1}) & 0 & 0 \\ r(r_{z_2} - p_{z_1}) + p(p_{z_2} - r_{z_1}) & p(r_{z_2} - p_{z_1}) + r(p_{z_2} - r_{z_1}) & 0 & 0 \\ 0 & 0 & p(r_{z_2} - q_{z_1}) + r(q_{z_2} - r_{z_1}) & r(r_{z_2} - q_{z_1}) + p(q_{z_2} - r_{z_1}) \\ 0 & 0 & r(r_{z_2} - q_{z_1}) + p(q_{z_2} - r_{z_1}) & p(r_{z_2} - q_{z_1}) + r(q_{z_2} - r_{z_1}) \end{bmatrix},$$

$$\hat{\Gamma}^2_{ij} = \tfrac{1}{4(p^2-r^2)} \begin{bmatrix} r(p_{z_1} - r_{z_2}) + p(r_{z_1} - p_{z_2}) & p(p_{z_1} - r_{z_2}) + r(r_{z_1} - p_{z_2}) & 0 & 0 \\ p(p_{z_1} - r_{z_2}) + r(r_{z_1} - p_{z_2}) & r(p_{z_1} - r_{z_2}) + p(r_{z_1} - p_{z_2}) & 0 & 0 \\ 0 & 0 & r(q_{z_1} - r_{z_2}) + p(r_{z_1} - q_{z_2}) & p(q_{z_1} - r_{z_2}) + r(r_{z_1} - q_{z_2}) \\ 0 & 0 & p(q_{z_1} - r_{z_2}) + r(r_{z_1} - q_{z_2}) & r(q_{z_1} - r_{z_2}) + r(r_{z_1} - q_{z_2}) \end{bmatrix}.$$

From the above we can see that if all of the components of the metric are constant along the torus, the connection coefficients vanish. Therefore in the case of the most simple metrics on the torus we find that the Born connection is flat.

2.2 Hopf Surfaces

A Hopf surface is a complex manifold obtained by quotienting $\mathbb{C}^2 \setminus \{0, 0\} = A$ by a so-called dilation. A dilation is a transformation of the form $(z_1, z_2) \mapsto (az_1 + \lambda z_2^m, bz_2)$ where $a, b \in \mathbb{C}$ are such that $0 < |a| \leq |b| < 1$, $m \in \mathbb{Z}^+$, and $\lambda(a - b^m) = 0$. We then get a Hopf surface M by taking A/Γ, where Γ represents the action of the dilation on A. It happens that all Hopf surfaces are diffeomorphic, and so we can restrict our attention to particular cases to see if any of them are nontrivial. A particularly simple case is when $\lambda = 0$ and $b = \bar{a}$. For now we will just consider this case.

We can give this manifold a Born structure by introducing two linear maps on $\mathbb{C}^2$ and then descending to the quotient. The map $J_2 : \partial_{z_i} \mapsto \partial_{\overline{z}_i},\ \partial_{\overline{z}_i} \mapsto -\partial_{z_i}$ is a real structure on $\mathbb{C}^2$, and commutes with the differential of the group action $(z_1, z_2) \mapsto (az_1, \overline{a}z_2)$ thereby telling us that this is a real structure on X. We can also use the complex structure $J_1 : \partial_u \mapsto i\partial_u$ which descends from $\mathbb{C}^2$. From here we just need a metric and a second complex structure (which is given by $J_1 J_2$). A natural split metric on X is given by applying the pushforward of the quotient map to the following metric on $\mathbb{C}^2$

$$g = \frac{\mathrm{d}z_1 \mathrm{d}z_2 + \mathrm{d}\overline{z}_1 \mathrm{d}\overline{z}_2}{z_1 \overline{z}_1 + z_2 \overline{z}_2}.$$

We can see that this is compatible with the quotient structure, since if we take $z_i \mapsto az_i$, $\overline{z}_i \mapsto \overline{a}\,\overline{z}_i$ we get a factor of $|a|^2$ on both the numerator and denominator. It can also be checked that the complex and real structures both pull back the metric so that $J_1^* g = -g$ and $J_2^* g = g$ as expected. As a result, the quadruplet (X, J_1, J_2, J_1J_2, g) satisfies the definition of a Born manifold. Given this data, we would like to compute the Born connection on the Hopf surface. Despite the metric being the most simple metric we could write down for a Hopf surface, it turns out that the Born connection is not flat, unlike the torus. The connection coefficients are given by the following matrices:

$$\hat{\Gamma}^1_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -\overline{z}_1 & \overline{z}_2 & 0 & 0 \\ \overline{z}_2 & -\overline{z}_1 & 0 & 0 \\ 0 & 0 & -\overline{z}_1 & \overline{z}_2 \\ 0 & 0 & \overline{z}_2 & -\overline{z}_1 \end{bmatrix}, \quad \hat{\Gamma}^2_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -\overline{z}_2 & \overline{z}_1 & 0 & 0 \\ \overline{z}_1 & -\overline{z}_2 & 0 & 0 \\ 0 & 0 & -\overline{z}_2 & \overline{z}_1 \\ 0 & 0 & \overline{z}_1 & -\overline{z}_2 \end{bmatrix},$$

$$\hat{\Gamma}^3_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -z_1 & z_2 & 0 & 0 \\ z_2 & -z_1 & 0 & 0 \\ 0 & 0 & -z_1 & z_2 \\ 0 & 0 & z_2 & -z_1 \end{bmatrix}, \quad \hat{\Gamma}^4_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -z_2 & z_1 & 0 & 0 \\ z_1 & -z_2 & 0 & 0 \\ 0 & 0 & -z_2 & z_1 \\ 0 & 0 & z_1 & -z_2 \end{bmatrix}.$$

As we can see, these are nonzero for any choice of $z_1, z_2 \in A$. Next we will investigate nontrivial Born connections on non-compact manifolds.

3 Conclusions and Physical Interpretation

We have so far shown the existence of a nontrivial Born connection on two different compact complex manifolds. Such a connection can then be extended to a larger target space $TM \times C$ by writing it as a Whitney sum of a flat connection on the phase space TM and the Born connection on the compactification C. It is also interesting to wonder whether there exist nontrivial Born connections on just the restriction to TM. In the future it will therefore be interesting to study Born connections on the double tangent bundle of a manifold. The goal would be to look for metrics on phase space which admit non-flat Born connections and examine the Levi-Civita connection induced by restricting domain of the Born connection to the integral submanifolds of K. The result would be that two observers witnessing an event from some distance will not necessarily measure the event to be at the same location. This could happen if, for example, the light signal that one observer measures travels on a closed trajectory in TM and picks up a 'phase' from the deformation of momentum space. This is the concept of relative locality, and is explored in [16, 6, 17]. Experimental tests of this have been considered in [18]. This idea will be explored more in PHYS437B.

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Appendix: Mathematica Code

The following is the Mathematica code used to generate the Born connection coefficients for a particular metric and real structure. The functions `InverseMetric` and `ChristoffelSymbol` were written by the user Artes on Mathematica stack exchange [19].

```
InverseMetric[g_] := Simplify[Inverse[g]]


ChristoffelSymbol[g_, xx_] :=
 Block[{n, ig, res}, n = 4; ig = InverseMetric[g];
  res = Table[(1/2)*
     Sum[ig[[i,
         s]]*(-D[g[[j, k]], xx[[s]]] + D[g[[j, s]], xx[[k]]] +
          D[g[[s, k]], xx[[j]]]), {s, 1, n}], {i, 1, n}, {j, 1, n}, {k,
       1, n}];
  Simplify[res]]


CanonicalConnectionCoefficients[g_, xx_, K_] :=
 Block[{n, ig, res, Chr}, n = 4; ig = InverseMetric[g];
  Chr = ChristoffelSymbol[g, xx];
  res = Table[(1/2) Chr[[i, j, k]] + (1/2)*
      Sum[Chr[[i, a, b]] K[[a, j]] K[[b, k]], {a, 1, n}, {b, 1,
         n}], {i, 1, n}, {j, 1, n}, {k, 1, n}];
  Simplify[res]]


DBracketCoefficients[g_, xx_, K_] :=
 Block[{n, ig, res, canon}, n = 4; ig = InverseMetric[g];
  canon = CanonicalConnectionCoefficients[g, xx, K];
  res = Table[
```

```
  canon[[i, j, k]] - canon[[j, i, k]] +
   Sum[canon[[s, i, l]] g[[j, l]] ig[[k, s]], {l, 1, n}, {s, 1,
     n}], {i, 1, n}, {j, 1, n}, {k, 1, n}];
 Simplify[res]]


BornConnCoeffs[g_, xx_, Ka_] :=
Block[{n, res, dbrak}, n = 4; dbrak = DBracketCoefficients[g, xx, Ka];
 res = Table[(1/4) (dbrak[[i, j, k]] -
      Sum[dbrak[[a, b, k]] Ka[[a, i]] Ka[[b, j]], {a, 1, n}, {b, 1,
        n}] + Sum[
       dbrak[[i, b, c]] Ka[[b, j]] Ka[[c, k]], {c, 1, n}, {b, 1,
        n}] - Sum[
       dbrak[[a, j, c]] Ka[[a, i]] Ka[[c, k]], {a, 1, n}, {c, 1,
        n}]), {i, 1, n}, {j, 1, n}, {k, 1, n}];
 Simplify[res]
 ]
```

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