Born Geometry and Relative Locality

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Abstract

The theory of relative locality describes matter interacting in a curved phase space. We begin with a reconstruction of the main equations of relative locality. From simple geometric arguments, the same connection coefficients posited in the literature are recovered in a series expansion. Relative locality then motivates the study of string theory on curved phase space. One example of such a model is Born geometry, which is based around a symmetry called Born reciprocity present in the Hamiltonian formulation of classical mechanics as well as phase space quantum mechanics. We explicitly compute nontrivial connections on phase space which respect the required symmetries of Born geometry. It is found that primary Hopf surfaces naturally acquire a Born connection with nonzero curvature, while complex 2-tori are flat for any simple choice of metric. The existence of a nontrivial Born connection points to the possibility of relative locality effects in space-time arising as a result of fluxes in string theory.

Contents

1 Introduction

In the 1940s, Max Born investigated the duality between position and momentum space which is fundamental to many computations in quantum mechanics. During this time, Born conceptualized the principle of reciprocity - that the laws of physics should be symmetric under the exchange of space-time with momentum-energy [1]. This reciprocity is inherent in the Hamiltonian formulation of classical mechanics, where the equations of motion are invariant under transformations of the form $x_i \mapsto p_i$, $p_i \mapsto -x_i$. Indeed, the matrix mechanics formulation of quantum mechanics also follows this reciprocity as we can see from Heisenberg's equations of motion. However, the exact nature of this reciprocity is mysterious, and it is not obvious how to extend the principle of reciprocity to the theory of general relativity or modern quantum field theories such as string theory. Between 2014 and 2016, Laurent Freidel and collaborators constructed a geometric theory of phase space which naturally introduced Born reciprocity, and dubbed it Born Geometry [2, 3, 4, 5]. This geometry naturally generalizes the original setting of double field theory. Born Geometry takes inspiration from earlier attempts to endow phase space with a noncommutative structure, such as the principle of relative locality [6]. The area has since seen a lot of research activity. See [7] for a comprehensive survey on Born geometry and related models in double field theory.

Born Geometry is a theory of extended space-time, in which the typical n dimensional setting of physics is replaced with a general $2n$ dimensional manifold which we can think of as an analogue of phase space. This manifold carries with it a *Born structure*, which allows one to differentiate between the *space-time* sector of the manifold and the *energy-momentum* sector. The splitting of extended space-time is induced by a so-called para-hypercomplex structure, which is a set of tensor fields on the manifold whose eigenbundles determine natural splittings of extended space-time into dual submanifolds. Our goal will be to investigate connections on the tangent bundle of the Born manifold, in particular the Born connection first appearing in [8]. We will assume a general familiarity with differential geometry. For excellent references, see the books by Kobayashi and Nomizu, Tu, and Lee [9, 10, 11].

2 Connections and Curved Momentum Space

2.1 Parallel Transport

In Relative Locality, we model phase space as an arbitrary curved manifold. This manifold is equipped with a pseudo-Riemannian metric, a symplectic form, and a connection. For the first section we will not need to use the metric or the symplectic form, but they will be useful in our construction of Born manifolds in section 2. The connection on phase space generally has a non-vanishing torsion and curvature tensor, which is not the case in General Relativty. This produces strange effects on the Planck scale for any physical observables coupled to this connection [12]. The result is that observers with different momenta will disagree on the spatial coordinates of particles. In a scattering process the result is that momenta of incoming and outgoing particles must be added using a modified composition law [6]. This rule is given by the parallel transport induced by the Born connection, which we will recall as follows.

Definition 2.1 (Parallel Transport)**.** Let ∇ be a connection on a vector bundle $\pi : E \to M$ and let $\gamma : [0,1] \to M$ be a smooth curve. Then ∇ induces a map $\mathcal{P}(\gamma)_0^s : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(s))$ called the **parallel transport operator** which takes a vector $X_0 \in \pi^{-1}(\gamma(0))$ and transports it to a nearby fibre. Here we have parameterized the curve γ by a variable $s \in [0,1]$. Explicitly the map is given by solving the following differential equation:

$$\nabla_{\dot{\gamma}}\mathcal{P}(\gamma)_t^s X|_{\gamma(t)} = 0, \qquad t \in [0,1]. \tag{2.1.1}$$

2.2 Relative Locality

The typical phase space manifold studied in relative locality is $M = T^*P$, where P is the momentum space of some particle. Recall that P is equipped with a distinguished point x_0 which we call the origin. Given a particle which has some momentum $p(t)$ parametrized along its worldline, the position then given by a curve $x(t)$ in the cotangent space T_p^*P. The interpretation of this is that the observed worldline of a particle with momentum p as measured by some observer at the present position of the particle will depend on the momentum p. In this way, one could imagine two incoming photons from a faraway event with similar worldlines but wildly different momenta. The observer would then measure these photons to be coming from different directions [12]. This phenomenon can then be used to constrain the size $|\Gamma^i{}_{jk}|$ of the connection coefficients on momentum space by measuring the components of the torsion and curvature. If we wished to then write down conservation laws for different scattering processes, it would not be enough to just set the sum of the incoming momenta equal to the sum of the outgoing momenta. Addition of points on an arbitrary manifold is not in general well defined. Instead, we can define an addition law for momenta with a careful construction. We begin by requiring that for a composite system of particles the sum of the momenta $p \oplus q \in P$ is well defined without reference to a coordinate chart. A good way to do this would be to require that (at least, in a neighbourhood of the origin x_0) $p \oplus q$ induces an element $\ominus p$ in momentum space so that

$\ominus p \oplus (p \oplus q) = q$. This makes momentum space into a local Lie quasigroup, where the 'quasi' indicates that it is not necessarily associative. For a mathematical treatment of Local Lie groups which are not globally associative see [13].

Definition 2.2 (Local Lie Quasigroup)**.** Let M be a manifold with a distinguished point x_0, and let U, V be neighbourhoods of x_0. Suppose $\mu : M \times M \to M$ is a smooth map defined on $U \times \{x_0\} \cup \{x_0\} \times U$ and $\nu : V \to M$ is a smooth map such that $\nu(V) \times V \subseteq U$ and $V \times \nu(V) \subseteq U$. Then the data (M, μ, x_0) is called a **local Lie quasigroup** if the following conditions hold:

1. $\mu(x_0, p) = \mu(p, x_0) = p$ for all $p \in M$
2. $\mu(\nu(p), p) = \mu(p, \nu(p)) = x_0$ for all $p \in V$.

Definition 2.3 (Momentum Addition Rule)**.** Let P be the momentum space of some particle. In a neighbourhood U of x_0, for each $p \in U$ we can define a family of local diffeomorphisms $L_p|_U : U \to P$ whose components define the **addition operator**,

$$(L_p q)^i := (p \oplus q)^i \tag{2.2.1}$$

This family of diffeomorphisms must respect the following rules:

1. The data $(M, U, \oplus)$ satisfies the definition of a local Lie quasigroup.
2. The pushforward $(L_p)_*$ is a parallel transport operator.

Recall that in classical scattering theory, if two particles collide, merge, or otherwise interact through an attractive/repulsive potential, the momenta of the outgoing particles must equal the sum of the momenta of the outgoing particles. We want this type of formula to still hold in relative locality. We therefore must require that given incoming momenta p, q and outgoing momenta k, ℓ for a scattering process, the identity (which we write in coordinates) $x_0^i = ((p \oplus q) \ominus (k \oplus \ell))^i$ is invariant under infinitesimal diffeomorphisms. According to Amelino-Camelia et al, this provides a way to derive the coefficients of a connection on momentum space, and shows that L_p is a parallel transporter. We provide an alternative derivation of the connection coefficients by beginning with the assumption that $(L_p)_*$ is a parallel transport operator on momentum space. Therefore in our derivation we will make the assumptions in the opposite order.

To arrive at the same equations found in [6], we require that the pushforward $(L_p)_* : T_q P \to T_{p \oplus q} P$ of L_p must be covariantly constant along a geodesic connecting q and $p \oplus q$. This guarantees that the addition law is well defined, since the coordinates of an outgoing particle with momentum $p \oplus q$ should be computable from the coordinates of incoming particles with momentum p, q, given the conservation of momentum rule. So the addition operator which we use to define the conservation law must provide a way of comparing coordinates of particles with different momenta, i.e. it is a parallel transporter. Let $U^j{}_i = \frac{\partial L^i_p}{\partial p^j}$, which is the Jacobian matrix which represents $(L_p)_*$. Then in a neighbourhood U of x_0 we can write the following parallel transport equation:

$$\begin{aligned} 0 = \nabla_{\dot{\gamma}} U^j{}_i &= \dot{\gamma}^k \frac{\partial U^j{}_i}{\partial r^k} + \Gamma^j{}_{ak} U^k{}_i \dot{\gamma}^a \\ \implies \dot{\gamma}^k \frac{\partial U^j{}_i}{\partial r^k} &= -\Gamma^j{}_{ak} U^k{}_i \dot{\gamma}^a \\ \implies \frac{\partial U^j{}_i}{\partial t} &= - \left. \Gamma^j{}_{ak} U^k{}_i \dot{\gamma}^a \right|_{\gamma(t)}, \end{aligned} \tag{2.2.2}$$

which can be solved directly using the path ordered exponential:

$$U^j{}_i = P \exp\left(- \int_0^1 \Gamma^j_{ik}(\gamma(t)) \dot{\gamma}^k(t) \mathrm{d}t \right). \tag{2.2.3}$$

This equation can then be used to derive an expansion for the connection coefficients in terms of the addition operator as follows. If we expand (2.2.3) around the origin of momentum space in some path connected chart (q^i, U) containing p we get the following expression for $U^j{}_i = ((L_p)_*)^j_i$:

$$U^j{}_i = \delta^j_i + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^1 \ldots \int_0^1 \left(\Gamma^j{}_{i_1 k_1} \dot{\gamma}^{k_1}(t) \right) \ldots \left(\Gamma^{i_n}{}_{i k_n} \dot{\gamma}^{k_n}(t_n) \right) \mathrm{d}t \ldots \mathrm{d}t_n. \tag{2.2.4}$$

To first order, this can be written,

$$\begin{aligned} U^j{}_i &= \delta^j_i + \int_0^1 \Gamma^j{}_{ia} \dot{\gamma}^a(t) \mathrm{d}t \\ &\approx \delta^j_i + \Gamma^j{}_{ia}|_0 \int_0^1 \dot{\gamma}^a(t) \mathrm{d}t \\ &= \delta^j_i + \Gamma^j{}_{ia}|_0 p^a. \end{aligned}$$

In [14] the parallel propagator along a geodesic is expanded around the origin by iteratively computing $\gamma^{(n)}(t)$ for each n using the geodesic equation. The following coefficients given by this expansion are used to write a full expression for $U^j{}_i$:

$$\Gamma^j{}_{\alpha_1 \ldots \alpha_n i} = \partial_{\alpha_i} \Gamma^j{}_{\alpha_2 \ldots \alpha_n i} - \Gamma^\sigma{}_{\alpha_1 \alpha_k} \Gamma^j{}_{\alpha_1 \ldots \alpha_{k-1} \sigma \alpha_{k+1} \ldots \alpha_n i}.$$

This yields:

$$U^j{}_i = \delta^j_i - \sum_{n=1}^{\infty} \frac{1}{n!} \Gamma^j_{\alpha_1 \ldots \alpha_n i}|_0 p^{\alpha_1} \ldots p^{\alpha_n}. \tag{2.2.5}$$

Now observe that after Taylor expanding to first order,

$$L_p(q)^i \approx L_p(0)^i + [(L_p)_*]^i_j|_{p=0}\, q^j \tag{2.2.6}$$

This expression yields a simplified version of our addition rule:

$$(p \oplus q)^i \approx p^i + q^i + \Gamma^i{}_{jk} p^j q^k. \tag{2.2.7}$$

This also provides us with a way of computing the connection coefficients from an addition rule,

$$\Gamma^i{}_{jk} = \frac{\partial}{\partial p^j} \frac{\partial}{\partial q^k} \big((p \oplus q)^i \big)|_{p=q=0}. \tag{2.2.8}$$

This completes a justification for the use of parallel transport in [6, 12, 15], where equations (2.2.7) and (2.2.8) are given without proof. Observe that the non-commutativity of the addition rule yields the torsion:

$$T^i{}_{jk}(0) = \frac{\partial}{\partial p^j} \frac{\partial}{\partial q^k} \big((p \oplus q)^i - (q \oplus p)^i \big)|_{p=q=0}. \tag{2.2.9}$$

Additionally, the non-associativity gives the curvature tensor. Let $p, q, k \in P$, then we have,

$$R_{abc}{}^d(0) = 2 \frac{\partial}{\partial p^{[a}} \frac{\partial}{\partial q^{b]}} \frac{\partial}{\partial k^c} \big((p \oplus q) \oplus k - p \oplus (q \oplus k) \big)^d |_{q,p,k=0} \tag{2.2.10}$$

Using these expressions, one could in principle perform an experiment which scatters triplets of particles in a particular order, and use the discrepancy that appears due to non-associativity to constrain the curvature and torsion of the connection on momentum space. Due to the fact that the torsion is nonzero, it is not possible to reduce expression (2.2.7) to $p^i + q^i$, since even in metric normal coordinates we have $\Gamma^i{}_{jk} = \frac{1}{2} T^i{}_{jk}$. We see that this is therefore a physical phenomenon with definite consequences, although observers localized near an event will be unable to measure this effect to any reasonable degree of accuracy. Instead, we must perform experiments using distant events such as gamma ray bursts [16, 12]. A great example of a connection on phase space with non-vanishing torsion and curvature is the Born connection. This is a connection associated with equations of motion arising from flux backgrounds in string theory. We will move on to a brief introduction to the concepts of Born geometry, before computing some examples of nontrivial phase-space connections explicitly.

3 Overview of Born Geometry

3.1 Complex and Real Structures

The Born geometry model of extended space-time is most commonly used in the setting of string theory. Many types of string theories incorporate extra dimensions into their models of space-time which can be represented by complex coordinates. To review this material we will begin with complex structures. For a detailed overview see [17, 18].

Definition 3.1 (Almost-Complex Structure)**.** Let M be a smooth manifold and let TM be the tangent bundle on M, where π is the projection. Then an **almost-complex structure** on M is defined to be a tensor field $I \in \Gamma(\mathrm{End}(TM))$ such that $I_p^2 = -\mathbb{1}_{T_pM}$.

An almost-complex structure is a structure which allows one to identify the complexification of the tangent bundle TM with the vector bundle $TM \oplus ITM$. For a manifold to be considered complex in the most rigorous sense, we require that the transition maps are holomorphic. From this one can show that the transition maps are holomorphic by showing that the eigenbundles of I are integrable.

Definition 3.2 (Integrable Subbundle)**.** Let M be a smooth manifold and let TM be its tangent bundle. Then a subbundle E of TM is called **integrable** if it is closed with respect to the Lie Bracket. This means that for all $X, Y \in \Gamma(E)$ we have $[X, Y] = XY - YX \in \Gamma(E)$.

If we consider the special case of an almost complex structure J, then the question of integrability of the eigenbundles of J comes down to whether or not the Nijenhuis tensor vanishes.

Definition 3.3 (Nijenhuis Tensor)**.** Let M be a smooth manifold and TM its tangent bundle. Then the

Nijenhuis tensor of a section $A \in \Gamma(\mathrm{End}(TM))$ is defined by the following formula.

$$N_A(X,Y) = -A^2[X,Y] + A([AX,Y] + [X,AY]) - [AX,AY]. \tag{3.1.1}$$

Theorem 3.1 (Newlander-Nirenberg [19])**.** An almost-complex structure I has integrable eigenbundles if and only if $N_I = 0$.

Definition 3.4 (Complex Manifold)**.** A manifold M is called almost-complex if there exists an almost-complex structure $I \in \Gamma(\mathrm{End}(TM))$. If the eigenbundles of I are integrable then I is called a **complex structure**, and M is called a **complex manifold**. This is equivalent to the existence of an atlas of charts $\{\phi_U | U \subseteq M\}$ where if $\phi_U : M \to \mathbb{R}^{2n} \cong \mathbb{C}^n$ and $\phi_V : M \to \mathbb{R}^{2n} \cong \mathbb{C}^n$ are coordinate charts for M then $\phi_V \circ \phi_U^{-1}$ is holomorphic.

Definition 3.5 (Almost-Real Structure)**.** An **almost-real structure** on M is defined to be a tensor field $K \in \Gamma(\mathrm{End}(TM))$ such that $K_p^2 = \mathbb{1}_{T_pM}$

On the other hand, an almost-real structure provides an involution on TM because it squares to the identity. It is mostly interesting to study real structures in Born Geometry because a real structure can induce integrable submanifolds of M, which correspond to natural splittings of extended space-time. If K is an almost-real structure then the eigenvalues of K must be 1 and/or -1.

Definition 3.6 (Para-Complex Manifold)**.** Let M be a smooth manifold and let $K \in \Gamma(\mathrm{End}(TM))$ be an almost-real structure. The eigenbundles of K are:

$$T^+M = \coprod_{x \in M} \{v \in T_xM | K_x v = v\} \tag{3.1.2}$$

$$T^-M = \coprod_{x \in M} \{v \in T_xM | K_x v = -v\} \tag{3.1.3}$$

If T^+M and T^-M have the same rank and T^+M and T^-M are both integrable subbundles, then we call M a **para-complex manifold**.

3.2 Para-Quaternionic and Born Structures

Born geometry goes a step beyond a single complex structure - we can consider any possible combination of real and complex structures. This leads in naturally to the study of quaternionic structures and their relatives.

Definition 3.7 (Para-quaternions)**.** The **para-quaternions** are an associative algebra $\tilde{\mathbb{H}}$ over the real numbers with generators $\{1, J_1, J_2, J_3\}$. The generators define $\tilde{\mathbb{H}}$ according to the following multiplication rules.

$$J_1^2 = J_2^2 = -J_3^2 = 1 \qquad J_1 J_2 = -J_2 J_1 = J_3. \tag{3.2.1}$$

Definition 3.8 (Almost-Para-Hypercomplex Structure)**.** Let M be a smooth manifold, and let $I, J, K \in \Gamma(\mathrm{End}(TM))$ be defined as tensor fields with the property that $I^2 = J^2 = -K^2 = \mathbb{1}$. We call the quadruplet (M, I, J, K) an almost para-hypercomplex structure.

The natural setting of Born geometry is a para-hypercomplex manifold along with some extra requirements regarding the integrability of the eigenbundles of I, J, and K.

Definition 3.9 (Symplectic Form)**.** Let $\omega \in \Omega^2(M)$ be a totally antisymmetric two-form. That is, ω is a bilinear antisymmetric map from $(T^*M)^2$ to $\mathbb{R}$. We call ω **symplectic** if it is everywhere nonvanishing and non-degenerate.

A symplectic form on M can only exist if M is even dimensional (set $\dim M = 2n$). It is therefore interesting to consider symplectic forms on complex manifolds and tangent bundles. Next we need to define what it means for a manifold to be hyper-Hermitian, hyper-Kähler, and so on.

Definition 3.10 (Hermitian Manifold)**.** Let M be a complex manifold with tangent space $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. Then M is called **Hermitian** if there exists a Hermitian metric h on $T_{\mathbb{C}}M$. This essentially says that the components h_{ab} form a Hermitian matrix.

Definition 3.11 (Para-Hermitian Manifold)**.** Let M be a para-complex manifold with tangent space $TM = T^+M \oplus T^-M$. Suppose M has the data of a pseudo-Riemannian metric η. Then M is called **para-Hermitian** if $\eta(KX, KY) = -\eta(X, Y)$.

It is in fact possible to define a symplectic form $\omega(X, Y) = \eta(KX, Y)$. Hence it just so happens that introducing a (possibly split) metric to a para-Hermitian manifold immediately produces for us a natural symplectic form. If we then add a Riemannian metric, one finally acquires a Born manifold.

Definition 3.12 (Born Manifold)**.** Let (M, K, η, ω) be a para-Hermitian manifold. Let $\mathcal{H}$ be a metric of signature $(2n, 0)$ on M. Then $(M, K, \eta, \mathcal{H})$ is called a Born manifold. The triplet $(M, K, \eta, \mathcal{H})$ is equivalent to the data of a para-hypercomplex manifold (M, I, J, K) where $I = \mathcal{H}^{-1}\,\omega$ and $J = \eta^{-1}\,\mathcal{H}$. It can also be shown that $K = \eta^{-1}\omega$, so it even holds that $(M, \eta, \mathcal{H}, \omega)$ is an equivalent formulation.

Since there are so many ways of characterizing a Born manifold, it is useful to split them up and interpret the different characterizations. The data $(M, \eta, \mathcal{H}, J)$ is called the **chiral representation** of M, and can be used to describe the space-time inhabited by a closed string, which has a worldsheet defined by a mapping $X : S^1 \to M$. The dynamics of the string can then be determined completely by the constraints [2, 3, 4] $\mathcal{H}(\partial_\tau X, \partial_\tau X) = 0, \eta(\partial_\tau X, \partial_\tau X) = 0$, and $\partial_\tau X = J\partial_\sigma X$.

In order to actually do computations involving dynamics of objects in this space-time, one needs a connection. This way dynamical fields can be compared between different points. It happens that if M is a Born manifold, it also admits a natural connection [8].

3.3 D-Brackets and the Born Connection

A Born manifold is also a para-Hermitian manifold, and it has been shown [18] that each para-Hermitian manifold has a canonically defined connection ∇^c called the **canonical connection**. We will construct this as follows. Suppose K is a real structure on M. Then we define two projectors $\frac{1}{2}(\mathbb{1} + K) = P : TM \to TM^+$ and $\frac{1}{2}(\mathbb{1} - K) = \tilde{P} : TM \to TM^-$, where $TM^\pm$ are the positive and negative eigenbundles of K.

Definition 3.13 (Canonical Connection)**.** Let (M, η, K) be a para-Hermitian manifold. Let ∇^η be the Levi-Civita connection of η. Then the **canonical connection** of M is defined by

$$\nabla^c_X Y = P\nabla^\eta_X PY + \tilde{P}\nabla^\eta_X \tilde{P} \quad \forall X, Y \in \Gamma(TM). \tag{3.3.1}$$

From the expression we can glean that this connection somehow respects the two eigenbundles by separating all "interaction terms" between the two bundles by projecting onto them before and after differentiating. In [20] it was shown that the canonical connection ∇^c induces a bracket $[\![X, Y]\!]$ on TM. This bracket is the key to acquiring the natural connection for Born geometry.

Definition 3.14 (Metric-Compatible Bracket)**.** A **metric-compatible bracket** $[\![X, Y]\!]$ on a manifold M

is a bilinear operation on the algebra $\Gamma(TM)$ of vector fields which satisfies

$$X(\eta(Y,Z)) = \eta([\![X,Y]\!],Z) + \eta(Y,[\![X,Z]\!]), \tag{3.3.2}$$

$$[\![X,fY]\!] = f[\![X,Y]\!] + X(f)Y, \tag{3.3.3}$$

$$\eta(Y,[\![X,X]\!]) = \eta([\![Y,X]\!],X). \tag{3.3.4}$$

If M is a smooth manifold and η is a metric on M, the triple $(TM,\eta,[\![\cdot,\cdot]\!])$ defines a so-called metric algebroid. We will define a bracket on our extended space-time using the Born metric η. On a Born manifold M, we might want to introduce (higher) gauge fields, such as an electromagnetic potential, the Kalb-Ramond 2-form [21], or something higher dimensional (see [22] for an introduction to higher gauge theory). The gauge transformations induced by such a field can then interrupt the integrability conditions we require for T^+M and T^-M, since it is not guaranteed that the eigenbundles transform invariantly under the transformation. In order to accommodate for this, a metric-compatible connection needs to be chosen so that the eigenbundles of K are still (weakly) integrable after a gauge transformation. The bracket we are looking for in this case is the D-bracket.

Definition 3.15 (D-Bracket). Let $(M,\eta,\mathcal{H},K)$ be a Born manifold. Then there exists a unique metric-compatible bracket called the **D-Bracket** on M determined by the following equation

$$\eta([\![X,Y]\!],Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y). \tag{3.3.5}$$

This bracket can be interpreted as a 'replacement' for the Lie bracket, so that under gauge transformations the eigenbundles of K remain integrable. However, integrability with respect to a general bracket does not guarantee the same properties as Frobenius integrability. Luckily, the D bracket comes equipped with a uniquely compatible connection: the Born connection.

Definition 3.16 (Born Connection). Let $(M,\eta,K,\mathcal{H})$ be a Born manifold. Then there is a unique connection ∇^B such that $\nabla^B_X I = \nabla^B_X J = \nabla^B_X K = 0$ for all $X \in \Gamma(TM)$. The connection is given by the following formula for all $X,Y \in \Gamma(TM)$:

$$\nabla^B_X Y = [\![\tilde{P}X,PY]\!] + [\![PX,\tilde{P}Y]\!] + P(K[\![PX,K\tilde{P}Y]\!]) + \tilde{P}(K[\![\tilde{P}X,KPY]\!]). \tag{3.3.6}$$

If we would like to study of space-time as an integral submanifold of a Born manifold, we want our generalized connection on the tangent space to reduce to the Levi-Civita connection so that the model reproduces ordinary physics. It can be verified [8] that given a manifold M the Born connection on M is the unique connection which has vanishing torsion (with respect to the D-bracket) and is also metric-compatible. Furthermore, it restricts to the Levi-Civita connection on the integral submanifolds of T^+M and T^-M.

4 Born Connections on Compact Manifolds

Born Geometry is a relatively new topic of study, and as such very few concrete examples of the relevant structures have been computed in the literature. In the Master's project of Boulter [23], a number of compact manifolds are considered and their Born structures are computed explicitly. The goal of this section will be to extend these calculations to include the Born connection.

4.1 Complex 2-Tori

In what follows we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ by the natural isomorphism. Consider the complex vector space $\mathbb{C}^2$ along with the underlying real vector space $\mathbb{R}^4$. A collection of four vectors $\Lambda = \{v_1, v_2, v_3, v_4\}$ which span $\mathbb{R}^4$ generate a group action on $\mathbb{C}^2$ given by $w \mapsto w + v_i$. If we quotient by this group action, the resulting manifold is periodic in each direction, and as such can be identified with a product of 4 circles - a torus! This construction is called the complex 2-torus, and is an important and simple example in complex geometry.

The beginning of this construction is the same as Boulter's, 2018 [23]. Since $M = \mathbb{C}^2/\Lambda$ is a complex manifold, there is a natural complex structure $I \in \mathrm{End}(\Gamma(TM))$ given by the map $I|_p\mathbf{v}(p) = i\mathbf{v}(p)$, $p \in M$ whenever $\mathbf{v} \in \Gamma(TM)$. Let $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ denote the complex coordinate vector fields on M. We can tell from the built-in complex structure that there is at least one real structure K on M given by $K\frac{\partial}{\partial z_i} = \frac{\partial}{\partial \bar{z}_i}$ and $K\frac{\partial}{\partial \bar{z}_i} = \frac{\partial}{\partial z_i}$. Finally, to induce the Born reciprocity it is natural to consider a real structure $J\frac{\partial}{\partial z_1} = \frac{\partial}{\partial \bar{z}_2}, J\frac{\partial}{\partial z_2} = \frac{\partial}{\partial \bar{z}_1}$. We can also write down a metric given by the form $\eta = \mathrm{d}z_1^2 + \mathrm{d}\bar{z}_1^2 + \mathrm{d}z_2^2 + \mathrm{d}\bar{z}_2^2$. It is clear that $KI = J$, so we see that (I, J, K) are an almost para-hypercomplex structure on M. Along with the metric η, we get a Born structure on M. In general, any metric of the form $\eta = p(z_1, z_2)(\mathrm{d}z_1^2 + \mathrm{d}z_2^2) + q(z_1, z_2)(\mathrm{d}\bar{z}_1^2 + \mathrm{d}\bar{z}_2^2) + r(z_1, z_2)(\mathrm{d}z_1\mathrm{d}z_2 + \mathrm{d}\bar{z}_1\mathrm{d}\bar{z}_2)$ produces a Born structure whenever $4pq - r^2$ is nowhere vanishing.

In component form, we can write η locally:

$$\eta = \begin{bmatrix} p(z_1,z_2) & r(z_1,z_2) & 0 & 0 \\ r(z_1,z_2) & p(z_1,z_2) & 0 & 0 \\ 0 & 0 & q(z_1,z_2) & r(z_1,z_2) \\ 0 & 0 & r(z_1,z_2) & q(z_1,z_2) \end{bmatrix}. \tag{4.1.1}$$

The Levi-Civita connection is the first thing we must compute, since it gives us the canonical connection. After this, the Levi-Civita connection defines the D-Bracket, which defines the Born connection. So we can compute ∇^{η}_X by computing its connection coefficients ${\Gamma^k}_{ij}$, which determine the connection on any local coordinate patch. The connection coefficients are given by the Koszul formula [9]:

$${\Gamma^\ell}_{jk} = \frac{1}{2}\eta^{\ell r}(\partial_k \eta_{rj} + \partial_j \eta_{rk} - \partial_r \eta_{jk}). \tag{4.1.2}$$

From this we can compute all of the connection coefficients. Splitting the components up into matrices ${\Gamma^1}_{jk}, {\Gamma^2}_{jk}, {\Gamma^3}_{jk}, {\Gamma^4}_{jk}$ we can write them as follows. See the apppendix on page 17 for the Mathematica code used to generate these.

$${\Gamma^1}_{jk} = \frac{1}{2(p^2-r^2)} \begin{bmatrix} pp_{z_1} + r(p_{z_2} - 2r_{z_1}) & pp_{z_2} + rp_{z_1} \\ pp_{z_2} + rp_{z_1} & -(rp_{z_2} + p(p_{z_1} - 2r_{z_2})) \end{bmatrix} \oplus \begin{bmatrix} rq_{z_2} - pq_{z_1} & rr_{z_2} - pr_{z_1} \\ rr_{z_2} - pr_{z_1} & rq_{z_2} - pq_{z_1} \end{bmatrix},$$

$${\Gamma^2}_{jk} = \frac{1}{2(p^2-r^2)} \begin{bmatrix} -(rp_{z_1} + p(p_{z_2} - 2r_{z_1})) & pp_{z_1} - rp_{z_2} \\ pp_{z_1} - rp_{z_2} & pp_{z_2} + r(p_{z_1} - 2r_{z_2}) \end{bmatrix} \oplus \begin{bmatrix} rq_{z_1} - pq_{z_2} & rr_{z_1} - pr_{z_2} \\ rr_{z_1} - pr_{z_2} & rq_{z_1} - pq_{z_2} \end{bmatrix},$$

$${\Gamma^3}_{jk} = \frac{1}{2(q^2-r^2)} \begin{bmatrix} 0 & 0 & qq_{z_1} - rr_{z_1} & qr_{z_1} - rq_{z_1} \\ 0 & 0 & qq_{z_2} - rr_{z_2} & qr_{z_2} - rq_{z_2} \\ qq_{z_1} - rr_{z_1} & qr_{z_2} - rq_{z_2} & 0 & 0 \\ qr_{z_1} - rq_{z_1} & qr_{z_2} - rq_{z_2} & 0 & 0 \end{bmatrix},$$

$${\Gamma^4}_{jk} = \frac{1}{2(q^2-r^2)} \begin{bmatrix} 0 & 0 & qr_{z_1} - rq_{z_1} & qq_{z_1} - rr_{z_1} \\ 0 & 0 & qr_{z_2} - rq_{z_2} & qq_{z_2} - rr_{z_2} \\ qr_{z_1} - rq_{z_1} & qr_{z_2} - rq_{z_2} & 0 & 0 \\ qq_{z_1} - rr_{z_1} & qq_{z_2} - rr_{z_2} & 0 & 0 \end{bmatrix}. \tag{4.1.3}$$

From here we must compute the coefficients of the canonical connection. The canonical connection obeys the rule $\nabla^c_X Y = P\nabla^\eta_X PY + \tilde{P}\nabla^\eta_X \tilde{P}Y$.

$$\begin{aligned} \nabla^c_X Y &= P\nabla^\eta_X PY + \tilde{P}\nabla^\eta_X \tilde{P}Y \\ &= \frac{1}{4}(\mathbb{1}+K)\nabla^\eta_X(\mathbb{1}+K)Y + \frac{1}{4}(\mathbb{1}-K)\nabla^\eta_X(\mathbb{1}-K)Y \\ &= \frac{1}{2}\left(\nabla^\eta_X Y + K\nabla^\eta_X KY\right). \end{aligned} \tag{4.1.4}$$

We can see this as a kind of "symmetrization" of the Levi-Civita connection with respect to conjugation by K. In coordinates, this is written as

$$\tilde{\Gamma}^k{}_{ij} = \frac{1}{2}\Gamma^k{}_{ij} + \frac{1}{2}\Gamma^\ell_{ir}K^r_j K^k_\ell, \tag{4.1.5}$$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.1.6}$$

Using this, we can compute the canonical connection coefficients.

$$\tilde{\Gamma}^1{}_{jk} = \tfrac{1}{2(p^2-r^2)} \begin{bmatrix} pp_{z_1} - rr_{z_1} & pr_{z_1} - rp_{z_1} & 0 & 0 \\ pp_{z_2} - rr_{z_2} & pr_{z_1} - rp_{z_1} & 0 & 0 \\ 0 & 0 & rq_{z_1} - pq_{z_1} & rr_{z_2} - pr_{z_1} \\ 0 & 0 & rr_{z_2} - pr_{z_1} & rq_{z_2} - pq_{z_1} \end{bmatrix},$$

$$\tilde{\Gamma}^2{}_{jk} = \tfrac{1}{2(p^2-r^2)} \begin{bmatrix} pr_{z_1} - rr_{z_2} & pp_{z_1} - rp_{z_2} & 0 & 0 \\ pp_{z_1} - rp_{z_2} & pr_{z_1} - rr_{z_2} & 0 & 0 \\ 0 & 0 & rq_{z_1} - pq_{z_2} & rr_{z_1} - pr_{z_2} \\ 0 & 0 & rr_{z_1} - pr_{z_2} & rq_{z_1} - pq_{z_2} \end{bmatrix},$$

$$\tilde{\Gamma}^3{}_{jk} = \tfrac{1}{4(q^2-r^2)} \begin{bmatrix} 0 & 0 & q(r_{z_2} + qq_{z_1}) - r(q_{z_2} + r_{z_1}) & q(q_{z_2} + r_{z_1}) - r(r_{z_2} + q_{z_1}) \\ 0 & 0 & q(q_{z_2} + r_{z_1}) - r(r_{z_2} + q_{z_1}) & q(r_{z_2} + q_{z_1}) - r(q_{z_2} + r_{z_1}) \\ q(r_{z_2} + q_{z_1}) - r(q_{z_2} + r_{z_1}) & q(q_{z_2} + r_{z_1}) - r(r_{z_2} + q_{z_1}) & 0 & 0 \\ q(q_{z_2} + r_{z_1}) - r(r_{z_2} + q_{z_1}) & q(r_{z_2} + q_{z_1}) - r(q_{z_2} + r_{z_1}) & 0 & 0 \end{bmatrix},$$

$$\tilde{\Gamma}^4{}_{jk} = \frac{1}{4(q^2-r^2)} \begin{bmatrix} 0 & 0 & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) \\ 0 & 0 & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) \\ q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & 0 & 0 \\ q(r_{z_2}+q_{z_1})-r(q_{z_2}+r_{z_1}) & q(q_{z_2}+r_{z_1})-r(r_{z_2}+q_{z_1}) & 0 & 0 \end{bmatrix}.$$

Next, we can use the definition

$$\eta([\![X,Y]\!],Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y) \tag{4.1.7}$$

to compute the D-bracket. The D-bracket is a bilinear function, so it should have a coordinate representation given by $[\![\partial_i,\partial_j]\!] = D^k{}_{ij}\partial_k$ for some tensorial coefficients $D^k{}_{ij}$. If we insert $X=\partial_i$, $Y=\partial_j$, and $Z=\partial_k$ we find the coefficients:

$$\eta_{k\ell}D^\ell{}_{ij} = \eta_{k\ell}\tilde{\Gamma}^\ell{}_{ij} - \eta_{k\ell}\tilde{\Gamma}^\ell{}_{ji} + \eta_{j\ell}\tilde{\Gamma}^\ell{}_{ki},$$

$$\eta^{sk}\eta_{k\ell}D^\ell{}_{ij} = \delta^s_\ell D^\ell{}_{ij} = D^s{}_{ij} = \tilde{\Gamma}^s{}_{ij} - \tilde{\Gamma}^s{}_{ji} + \eta_{j\ell}\eta^{sk}\tilde{\Gamma}^\ell{}_{ki},$$

$$D^s{}_{ij} = \tilde{\Gamma}^s{}_{ij} - \tilde{\Gamma}^s{}_{ji} + \eta_{j\ell}\eta^{sk}\tilde{\Gamma}^\ell{}_{ki}. \tag{4.1.8}$$

The Born connection is then related to the D bracket by the following.

$$\begin{aligned} \nabla^B_X Y &= [\![\tilde{P}X, PY]\!] + [\![PX, \tilde{P}Y]\!] + PK[\![PX, K\tilde{P}Y]\!] + \tilde{P}K[\![\tilde{P}X, KPY]\!] \\ &= \frac{1}{4}[\![X-KX, Y+KY]\!] + \frac{1}{4}[\![X+KX, Y-KY]\!] \\ &+ \frac{1}{8}(\mathbb{1}+K)[\![X+KX, KY-Y]\!] + \frac{1}{8}(K-\mathbb{1})[\![X-KX, Y+KY]\!] \\ &= \frac{1}{4}[\![X,Y]\!] - \frac{1}{4}[\![KX,KY]\!] + \frac{1}{4}K[\![X,KY]\!] - \frac{1}{4}K[\![KX,Y]\!]. \end{aligned}$$

So the Born connection coefficients, which we will denote $\hat{\Gamma}^k{}_{ij}$, are related to $D^k{}_{ij}$ by the following formula

$$\hat{\Gamma}^k{}_{ij} = \frac{1}{4}\left(D^k{}_{ij} - D^k{}_{ab}K^a_iK^b_j + D^c{}_{ib}K^k_cK^b_j - D^c{}_{aj}K^k_cK^a_i\right).$$

With some more mathematica, we can compute these coefficients for the torus. They are listed on the following page. Remarkably, $\Gamma^3_{ij} = \Gamma^4_{ij} = 0$ even though the D-bracket is highly nontrivial:

$$\hat{\Gamma}^1{}_{ij} = \tfrac{1}{4(p^2-r^2)} \begin{bmatrix} p(r_{z_2}-p_{z_1})+r(p_{z_2}-r_{z_1}) & r(r_{z_2}-p_{z_1})+p(p_{z_2}-r_{z_1}) & 0 & 0 \\ r(r_{z_2}-p_{z_1})+p(p_{z_2}-r_{z_1}) & p(r_{z_2}-p_{z_1})+r(p_{z_2}-r_{z_1}) & 0 & 0 \\ 0 & 0 & p(r_{z_2}-q_{z_1})+r(q_{z_2}-r_{z_1}) & r(r_{z_2}-q_{z_1})+p(q_{z_2}-r_{z_1}) \\ 0 & 0 & r(r_{z_2}-q_{z_1})+p(q_{z_2}-r_{z_1}) & p(r_{z_2}-q_{z_1})+r(q_{z_2}-r_{z_1}) \end{bmatrix},$$

$$\hat{\Gamma}^2{}_{ij} = \tfrac{1}{4(p^2-r^2)} \begin{bmatrix} r(p_{z_1}-r_{z_2})+p(r_{z_1}-p_{z_2}) & p(p_{z_1}-r_{z_2})+r(r_{z_1}-p_{z_2}) & 0 & 0 \\ p(p_{z_1}-r_{z_2})+r(r_{z_1}-p_{z_2}) & r(p_{z_1}-r_{z_2})+p(r_{z_1}-p_{z_2}) & 0 & 0 \\ 0 & 0 & r(q_{z_1}-r_{z_2})+p(r_{z_1}-q_{z_2}) & p(q_{z_1}-r_{z_2})+r(r_{z_1}-q_{z_2}) \\ 0 & 0 & p(q_{z_1}-r_{z_2})+r(r_{z_1}-q_{z_2}) & r(q_{z_1}-r_{z_2})+r(r_{z_1}-q_{z_2}) \end{bmatrix}.$$

From the above we can see that if all of the components of the metric are constant along the torus, the connection coefficients vanish. Therefore in the case of the most simple metrics on the torus we find that the Born connection is flat.

4.2 Hopf Surfaces

A Hopf surface is a complex manifold obtained by quotienting $\mathbb{C}^2 \setminus \{0, 0\} = A$ by a so-called dilation. A dilation is a transformation of the form $(z_1, z_2) \mapsto (az_1 + \lambda z_2^m, bz_2)$ where $a, b \in \mathbb{C}$ are such that $0 < |a| \leq |b| < 1$, $m \in \mathbb{Z}^+$, and $\lambda(a - b^m) = 0$. We then get a Hopf surface M by taking A/Γ, where Γ represents the action of the dilation on A. It happens that all Hopf surfaces are diffeomorphic, and so we can restrict our attention to particular cases to see if any of them are nontrivial. A particularly simple case is when $\lambda = 0$ and $b = \bar{a}$. For now we will just consider this case.

We can give this manifold a Born structure by introducing two linear maps on $\mathbb{C}^2$ and then descending to the quotient. The map $J_2 : \partial_{z_i} \mapsto \partial_{\overline{z}_i}$, $\partial_{\overline{z}_i} \mapsto -\partial_{z_i}$ is a real structure on $\mathbb{C}^2$, and commutes with the differential of the group action $(z_1, z_2) \mapsto (az_1, \overline{a}z_2)$ thereby telling us that this is a real structure on X. We can also use the complex structure $J_1 : \partial_u \mapsto i\partial_u$ which descends from $\mathbb{C}^2$. From here we just need a metric and a second complex structure (which is given by $J_1 J_2$). A natural split metric on X is given by applying the pushforward of the quotient map to the following metric on $\mathbb{C}^2$

$$g = \frac{\mathrm{d}z_1 \mathrm{d}z_2 + \mathrm{d}\overline{z}_1 \mathrm{d}\overline{z}_2}{z_1\overline{z}_1 + z_2\overline{z}_2}.$$

We can see that this is compatible with the quotient structure, since if we take $z_i \mapsto az_i$, $\overline{z}_i \mapsto \overline{a}\,\overline{z}_i$ we get a factor of $|a|^2$ on both the numerator and denominator. It can also be checked that the complex and real structures both pull back the metric so that $J_1^* g = -g$ and $J_2^* g = g$ as expected. As a result, the quadruplet (X, J_1, J_2, J_1J_2, g) satisfies the definition of a Born manifold. Given this data, we would like to compute the Born connection on the Hopf surface. Despite the metric being the most simple metric we could write down for a Hopf surface, it turns out that the Born connection is not flat, unlike the torus. The connection coefficients are given by the following matrices:

$$\hat{\Gamma}^1_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -\overline{z}_1 & \overline{z}_2 & 0 & 0 \\ \overline{z}_2 & -\overline{z}_1 & 0 & 0 \\ 0 & 0 & -\overline{z}_1 & \overline{z}_2 \\ 0 & 0 & \overline{z}_2 & -\overline{z}_1 \end{bmatrix}, \quad \hat{\Gamma}^2_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -\overline{z}_2 & \overline{z}_1 & 0 & 0 \\ \overline{z}_1 & -\overline{z}_2 & 0 & 0 \\ 0 & 0 & -\overline{z}_2 & \overline{z}_1 \\ 0 & 0 & \overline{z}_1 & -\overline{z}_2 \end{bmatrix},$$

$$\hat{\Gamma}^3_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -z_1 & z_2 & 0 & 0 \\ z_2 & -z_1 & 0 & 0 \\ 0 & 0 & -z_1 & z_2 \\ 0 & 0 & z_2 & -z_1 \end{bmatrix}, \quad \hat{\Gamma}^4_{ij} = \frac{1}{4(z_1\overline{z}_1 + z_2\overline{z}_2)} \begin{bmatrix} -z_2 & z_1 & 0 & 0 \\ z_1 & -z_2 & 0 & 0 \\ 0 & 0 & -z_2 & z_1 \\ 0 & 0 & z_1 & -z_2 \end{bmatrix}.$$

As we can see, these are nonzero for any choice of $z_1, z_2 \in A$. In the future, it would be interesting to explore space-times compactified on Hopf surfaces, which we have shown acquire a non-trivial Born structure on top of the underlying symmetries already present.

5 Conclusions and Physical Interpretation

In the first section we reconstruct some of the equations of relative locality. We showed that, given some simple geometric assumptions about the addition rule on momentum space, we can find an alternative way to recover the connection posited in [6]. This is the main contribution to PHYS437B. We then demonstrate the existence of a nontrivial Born connection on two different compact complex manifolds, which was done in PHYS437A. Such a connection can then be extended to a larger target space $TM \times C$ by writing it as a Whitney sum of a flat connection on the phase space TM and the Born connection on the compactification C. The gauge fields present in the compactification could then cause the torsion in ordinary momentum space to be nonzero. This would lead to relative locality arising from quantum gravity effects, as mentioned in [2].

It is also interesting to wonder whether there exist nontrivial Born connections on TM. In the future it will therefore be interesting to study Born connections on the double tangent bundle of a manifold. The goal would be to look for metrics on phase space which admit non-flat Born connections and examine the Levi-Civita connection induced by restricting domain of the Born connection to the integral submanifolds of K. As we saw in Section 2, this would open up the possibility for relative locality without string theory, as Born geometry is a general enough framework to apply to other field theories.

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Appendix: Mathematica Code

The following is the Mathematica code used to generate the Born connection coefficients for a particular metric and real structure. The functions `InverseMetric` and `ChristoffelSymbol` were written by the user Artes on Mathematica stack exchange [24].

```
InverseMetric[g_] := Simplify[Inverse[g]]


ChristoffelSymbol[g_, xx_] :=
 Block[{n, ig, res}, n = 4; ig = InverseMetric[g];
  res = Table[(1/2)*
     Sum[ig[[i,
         s]]*(-D[g[[j, k]], xx[[s]]] + D[g[[j, s]], xx[[k]]] +
          D[g[[s, k]], xx[[j]]]), {s, 1, n}], {i, 1, n}, {j, 1, n}, {k,
       1, n}];
  Simplify[res]]


CanonicalConnectionCoefficients[g_, xx_, K_] :=
 Block[{n, ig, res, Chr}, n = 4; ig = InverseMetric[g];
  Chr = ChristoffelSymbol[g, xx];
  res = Table[(1/2) Chr[[i, j, k]] + (1/2)*
      Sum[Chr[[i, a, b]] K[[a, j]] K[[b, k]], {a, 1, n}, {b, 1,
         n}], {i, 1, n}, {j, 1, n}, {k, 1, n}];
```

```
  Simplify[res]]

DBracketCoefficients[g_, xx_, K_] :=
 Block[{n, ig, res, canon}, n = 4; ig = InverseMetric[g];
  canon = CanonicalConnectionCoefficients[g, xx, K];
  res = Table[
    canon[[i, j, k]] - canon[[j, i, k]] +
     Sum[canon[[s, i, l]] g[[j, l]] ig[[k, s]], {l, 1, n}, {s, 1,
       n}], {i, 1, n}, {j, 1, n}, {k, 1, n}];
  Simplify[res]]

BornConnCoeffs[g_, xx_, Ka_] :=
 Block[{n, res, dbrak}, n = 4; dbrak = DBracketCoefficients[g, xx, Ka];
  res = Table[(1/4) (dbrak[[i, j, k]] -
       Sum[dbrak[[a, b, k]] Ka[[a, i]] Ka[[b, j]], {a, 1, n}, {b, 1,
          n}] + Sum[
         dbrak[[i, b, c]] Ka[[b, j]] Ka[[c, k]], {c, 1, n}, {b, 1,
          n}] - Sum[
         dbrak[[a, j, c]] Ka[[a, i]] Ka[[c, k]], {a, 1, n}, {c, 1,
          n}]), {i, 1, n}, {j, 1, n}, {k, 1, n}];
  Simplify[res]
  ]
```

References

[1] Max Born. "Reciprocity theory of elementary particles". In: *Reviews of Modern Physics* 21.3 (1949), pp. 463–473. ISSN: 00346861. DOI: 10.1103/RevModPhys.21.463.

[2] Laurent Freidel, Robert G. Leigh, and Djordje Minic. "Born reciprocity in string theory and the nature of spacetime". In: *Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics* 730 (2014), pp. 302–306. ISSN: 03702693. DOI: 10.1016/j.physletb.2014.01.067. arXiv: 1307.7080. URL: http://dx.doi.org/10.1016/j.physletb.2014.01.067.

[3] Laurent Freidel, Robert G. Leigh, and Djorje Minic. “Quantum Gravity, Dynamical Phase Space and String Theory”. In: *International Journal of Modern Physics D* 23.12 (2014).

[4] Laurent Freidel, Robert G Leigh, and Djordje Minic. “Metastring Theory and Modular Space-time”. In: *Journal of High Energy Physics* 6 (2015). arXiv: `arXiv:1502.08005v1`.

[5] Laurent Freidel et al. “Theory of metaparticles”. In: *Physical Review D* 99.6 (2019), p. 66011. ISSN: 24700029. DOI: `10.1103/PhysRevD.99.066011`. URL: `https://doi.org/10.1103/PhysRevD.99.066011`.

[6] Giovanni Amelino-Camelia et al. “The principle of relative locality”. In: *Physical Review D - Particles, Fields, Gravitation and Cosmology* 84.8 (2011), pp. 1–12. ISSN: 15507998. DOI: `10.1103/PhysRevD.84.084010`. arXiv: `arXiv:1101.0931v2`.

[7] Vincenzo E Marotta and Richard J Szabo. “Para-Hermitian Geometry, Dualities, and Generalized Flux Backgrounds”. In: *Fortschritte der Physik* 67.3 (2019). DOI: `10.1002/prop.201800093`. arXiv: `arXiv:1810.03953v2`.

[8] Laurent Freidel, Felix J. Rudolph, and David Svoboda. “A Unique Connection for Born Geometry”. In: *Communications in Mathematical Physics* 372.1 (2019), pp. 119–150. ISSN: 14320916. DOI: `10.1007/s00220-019-03379-7`. arXiv: `1806.05992`.

[9] Shoshichi Kobiyashi and Katsumi Nomizu. *Foundations of Differential Geometry, Volume I.* Wiley & Sons Inc, 1963.

[10] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes.* Ed. by Sheldon Axler and Kenneth Ribet. 1st. Springer Nature, 2017, pp. 385–413. ISBN: 9783319550824. DOI: `10.1007/978-3-540-30721-1_12`.

[11] John M Lee. *Introduction to Smooth Manifolds.* 2nd Editio. Springer, 2000. ISBN: 978-1-4419-9981-8. DOI: `10.1007/978-1-4419-9982-5`.

[12] Laurent Freidel and Lee Smolin. “Gamma ray burst delay times probe the geometry of momentum space”. In: (2011), pp. 1–21. arXiv: `1103.5626`. URL: `http://arxiv.org/abs/1103.5626`.

[13] Peter J. Olver. “Non-associative local Lie groups”. In: *Journal of Lie Theory* 6.1 (1996), pp. 23–51. ISSN: 09495932.

[14] Roland Haas. “Self-force on point particles in orbit around a Schwarzschild black hole”. PhD thesis. 2009. URL: `https://atrium.lib.uoguelph.ca/xmlui/handle/10214/21812`.

[15] Jerzy Kowalski-Glikman. “Living in Curved Momentum Space”. In: *International Journal of Modern Physics A* 28.12 (2013). ISSN: 0217751X. DOI: 10.1142/S0217751X13300147. arXiv: 1303.0195.

[16] A. McCoy. “Gamma Ray Bursts, The Principle of Relative Locality and Connection Normal Coordinates”. In: *arXiv preprint arXiv:1201.1255* (2012). arXiv: 1201.1255.

[17] David Svoboda. “Born Geometry”. Doctor of Philosophy in Physics. University of Waterloo, 2020. URL: http://hdl.handle.net/10012/15772.

[18] Stefan Ivanov and Simeon Zamkovoy. “Para-Hermitian and Para-Quaternionic manifolds”. In: *Differential Geometry and its Applications* 23.2 (2005), pp. 204–234. DOI: https://doi.org/10.1016/j.difgeo.2005.06.002. arXiv: 0310415 [math]. URL: http://arxiv.org/abs/math/0310415.

[19] A Newlander and L Nirenberg. “Complex Analytic Coordinates in Almost Complex Manifolds”. In: *Annals of Mathematics* 65.3 (1957), pp. 391–404. DOI: 10.2307/1970051. URL: https://www.jstor.org/stable/1970051%7B%5C%%7D0AJSTOR.

[20] Laurent Freidel, Felix J. Rudolph, and David Svoboda. “Generalised kinematics for double field theory”. In: *Journal of High Energy Physics* 2017.11 (2017). ISSN: 10298479. DOI: 10.1007/JHEP11(2017)175. arXiv: 1706.07089.

[21] Michael Kalb and P Ramond. “Classical direct interstring action”. In: *Physical Review D* 9.8 (1974), pp. 2273–2284. DOI: doi:10.1103/physrevd.9.2273.

[22] John C Baez and John Huerta. “An Invitation to Higher Gauge Theory”. In: (2010), pp. 1–60. arXiv: arXiv:1003.4485v2.

[23] Eric Boulter and Ruxandra Moraru. *Born Metrics on Compact Complex Surfaces*. 2018.

[24] Artes and Sjoerd J de Vries. *How to calculate scalar curvature Ricci tensor and Christoffel symbols in Mathematica*. Dec. 2012. URL: https://mathematica.stackexchange.com/a/8908.