Burgers' Equation: Numerical Solution

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Outline

Burgers' Equation

The one-dimensional Burgers' equation is a fundamental PDE given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},$$

where:

- $u(x, t)$ is the velocity field,
- ν is the kinematic viscosity,
- The first term represents nonlinear convection, and the second represents diffusion.

It is widely used as a simplified model for turbulence and shock formation.

Finite Difference Discretization

Consider a uniform grid:

$$x_0, x_1, \ldots, x_N \quad \text{with spacing } \Delta x, \quad t_0, t_1, \ldots, t_M \quad \text{with step } \Delta t$$

The explicit finite difference scheme is:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n) + \nu \frac{\Delta t}{(\Delta x)^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

- First term: nonlinear convection
- Second term: diffusion
- Simple explicit time-stepping

Initial and Boundary Conditions

We solve on the domain $x \in [0, 2]$ with:

- Initial condition: $u(x, 0) = -\sin(\pi x)$
- Boundary conditions: $u(0, t) = 0$, $u(2, t) = 0$

This setup allows us to observe shock formation and smoothing effects due to viscosity.

Illustrative Results

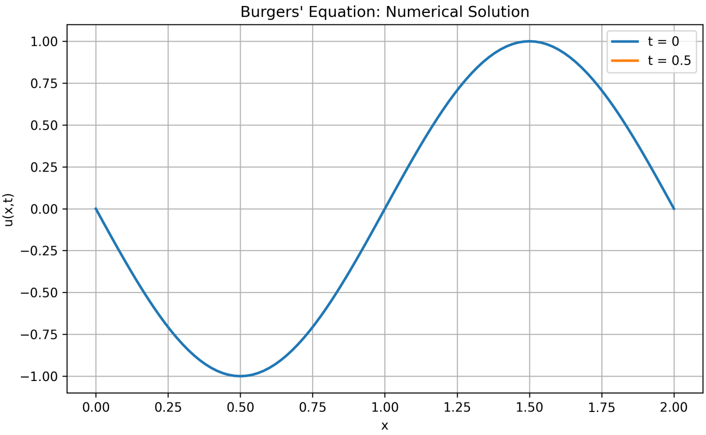

Figure: Numerical solution of Burgers' equation over time.

Observations

- Nonlinear convection leads to steep gradients (shock formation)
- Diffusion term smooths these gradients over time
- Stability requires careful choice of Δt relative to Δx (CFL condition)
- Burgers' equation is ideal for testing and understanding CFD solvers

Conclusion

- Burgers' equation demonstrates the interplay between convection and diffusion
- Numerical solutions illustrate shock formation and smoothing effects
- Serves as a foundational example for learning CFD methods