Poiseuille Flow: Analytical and Numerical Study

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Outline

Flow Configuration

Poiseuille flow describes laminar flow of a viscous fluid driven by a pressure gradient in a confined geometry.

- Steady, incompressible, and fully developed
- Common geometries:
  - Flow between two parallel plates
  - Flow inside a circular pipe

Governing equations (steady-state, incompressible):

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u},$$
$$\nabla\cdot\mathbf{u} = 0$$

Analytical Solution

For 2D plane Poiseuille flow between two plates $y = 0$ and $y = H$:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}, \qquad u(0) = u(H) = 0$$

Integrating twice:

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - Hy\right)$$

- Parabolic velocity profile
- Maximum velocity at mid-plane: $u_{max} = -\frac{H^2}{8\mu}\frac{dp}{dx}$

Discretization

- Discretize in y using second-order central differences
- Solve linear system:

$$-u_{j-1} + 2u_j - u_{j+1} = \frac{(\Delta y)^2}{\mu} \frac{dp}{dx}$$

- Apply boundary conditions: $u_0 = u_N = 0$
- Solve using matrix inversion or iterative methods (Jacobi, Gauss-Seidel)

Comparison of Results

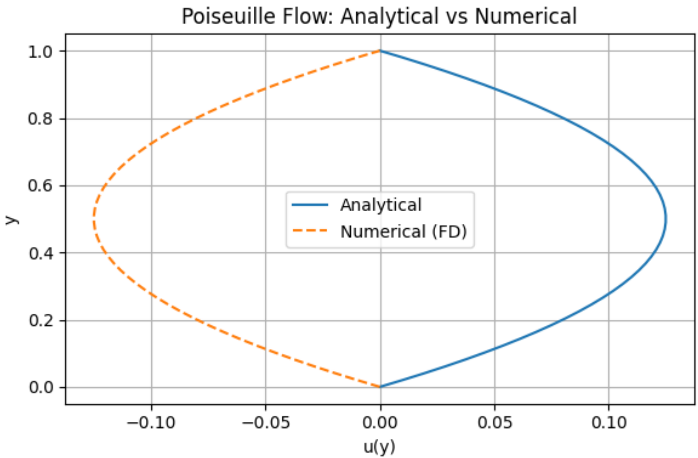

Figure: Velocity profile for plane Poiseuille flow: analytical vs. numerical

Observations

- Numerical solution matches analytical parabolic profile
- Higher grid resolution improves accuracy
- Benchmark case for testing viscous solvers and discretization accuracy

Conclusion

- Poiseuille flow demonstrates viscous, laminar motion under pressure gradient
- Analytical solution serves as a validation test for CFD solvers
- Numerical methods capture the parabolic velocity distribution efficiently