Euler Method: Explanation and Worked Example

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Abstract

This note presents the Euler method for solving ordinary differential equations (ODEs). It includes the derivation of the method, a step-by-step worked example, and discussion on accuracy and limitations.

1 Introduction

The Euler method is one of the simplest numerical techniques for approximating solutions of ordinary differential equations of the form:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

It provides a first-order approximation, meaning that the error decreases linearly with the step size.

2 Derivation of Euler Method

Starting from the Taylor expansion of $y(t)$ around t_n:

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + h\left.\frac{dy}{dt}\right|_{t_n} + \frac{h^2}{2}\left.\frac{d^2y}{dt^2}\right|_{\xi_n}, \quad \xi_n \in [t_n, t_{n+1}]$$

Neglecting the second-order term gives:

$$y_{n+1} \approx y_n + hf(t_n, y_n)$$

which is the Euler update formula.

3 Worked Example

Consider the differential equation:

$$\frac{dy}{dt} = -2y + 2 - e^{-4t}, \quad y(0) = 1$$

with exact solution:

$$y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}$$

We choose a step size $h = 0.1$ and compute the first few iterations:

$$\begin{aligned}
y_1 &= y_0 + hf(t_0, y_0) = 1 + 0.1(-2 \cdot 1 + 2 - e^0) = 1 - 0.1 = 0.9 \\
y_2 &= y_1 + hf(t_1, y_1) = 0.9 + 0.1(-2 \cdot 0.9 + 2 - e^{-0.4}) \approx 0.9 + 0.1(-1.8 + 2 - 0.6703) \\
&\approx 0.9 + 0.1(-0.4703) \approx 0.85297
\end{aligned}$$

Continuing this process, one can approximate $y(t)$ for any t in the interval of interest.

4 Discussion on Accuracy and Limitations

- Euler method is first-order accurate, with global truncation error proportional to the step size h.
- For small step sizes, the method gives reasonable approximations; however, for stiff equations or large h, it may become unstable.
- It serves as a foundation for more advanced methods such as Heun's method, Runge-Kutta methods, and multistep methods.
- The method is easy to understand and implement, making it ideal for introductory study and teaching.

5 Conclusion

Euler method provides a simple way to numerically solve ODEs. While limited in accuracy, its conceptual simplicity makes it a crucial stepping stone toward more sophisticated numerical schemes.