Runge-Kutta 4th Order Method: Explanation and Worked Example

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Abstract

This note presents the classical 4th-order Runge-Kutta (RK4) method for solving ordinary differential equations (ODEs). It includes the derivation, a worked example, and discussion on accuracy and stability.

1 Introduction

The Runge-Kutta methods are a family of iterative techniques for solving initial value problems:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The 4th-order Runge-Kutta (RK4) method is one of the most commonly used due to its high accuracy and simplicity for single-step methods. It is fourth-order accurate, meaning the global truncation error scales with h^4.

2 Derivation of RK4 Method

The RK4 method computes intermediate slopes to approximate the solution:

$$\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\
k_4 &= f(t_n + h, y_n + hk_3)
\end{aligned}$$

The next step is then computed as a weighted average of these slopes:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Explanation: - k_1 is the slope at the beginning of the interval - k_2 and k_3 are slopes at the midpoint - k_4 is the slope at the end - The weighted average gives a fourth-order accurate approximation

3 Worked Example

Consider the differential equation:

$$\frac{dy}{dt} = t + y, \quad y(0) = 1$$

with exact solution:

$$y(t) = -t - 1 + 2e^t$$

Choose step size $h = 0.1$.

3.1 Step 1

$$\begin{aligned}
k_1 &= f(t_0, y_0) = 0 + 1 = 1 \\
k_2 &= f(t_0 + 0.05, y_0 + 0.05 \cdot 1) = f(0.05, 1.05) = 0.05 + 1.05 = 1.10 \\
k_3 &= f(t_0 + 0.05, y_0 + 0.05 \cdot 1.10) = f(0.05, 1.055) = 0.05 + 1.055 = 1.105 \\
k_4 &= f(t_0 + 0.1, y_0 + 0.1 \cdot 1.105) = f(0.1, 1.1105) = 0.1 + 1.1105 = 1.2105 \\
y_1 &= y_0 + \frac{0.1}{6}(1 + 2 \cdot 1.10 + 2 \cdot 1.105 + 1.2105) \\
&= 1 + 0.016667(1 + 2.2 + 2.21 + 1.2105) \\
&= 1 + 0.016667 \cdot 6.6205 \approx 1 + 0.1103 = 1.1103
\end{aligned}$$

3.2 Step 2 (Optional)

Repeat similarly to compute $y_2, y_3, \ldots$ for the interval of interest.

4 Discussion on Accuracy and Advantages

- RK4 is fourth-order accurate, much more precise than Euler (first-order) or Heun (second-order) for the same step size.
- It is a single-step method, requiring no previous history beyond the last step.
- More computationally intensive than Euler or Heun (4 slope evaluations per step), but highly stable and reliable for general ODEs.
- Provides a standard reference method for testing and benchmarking other numerical solvers.

5 Conclusion

The classical RK4 method is a robust and accurate numerical technique for solving ODEs. Its balance of accuracy, stability, and simplicity makes it widely used in scientific computing.