CPS 5310 Extra Credit

I. (40 pts) Consider the model of a vibrating rectangular membrane with tension 12.5 lbs/ft and density 2.5 slugs/ft^2 and is fixed along its entire boundary in the xy-plane at all times $t \in [0, T]$. Suppose that the initial displacement is

$$g(x, y) = 0.1(4x - x^2)(2y - y^2)\ ft$$

and the initial velocity is zero. Furthermore, assume that this thin rectangular homogeneous elastic membrane is of dimension 4 ft $\times$ 2 ft and that the ground state of this membrane is described by a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ.

(a) Set up the partial differential equation (PDE) modeling the deflections $u(x, y, t)$ of the rectangular membrane over the time interval $[0, T]$. Make sure to specify all the boundary conditions and initial conditions.

Answer: The partial differential equation can be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

where $u = u(x, y, t)$ and $c^2 = \frac{T}{\rho}$.

Boundary Conditions:

$$\begin{aligned} u(0, y, t) = u(a, y, t) &= 0 \\ 0 \le y &\le b \\ t &\ge 0 \end{aligned} \tag{2}$$

$$\begin{aligned} u(x, 0, t) = u(x, b, t) &= 0 \\ 0 \le x &\le a \\ t &\ge 0 \end{aligned} \tag{3}$$

Initial Conditions:

$$u(x, y, t = 0) = g(x, y) = 0.1(4x - x^2)(2y - y^2)ft \tag{4}$$

$$\frac{\partial u(x, y, t)}{\partial t}\Big|_{t=0} = v(x, y) \tag{5}$$

where $(x, y) \subset R$

(b) Find the deflection $u(x,y,t)$ of the rectangular membrane at any point (x,y) of the membrane and at any time t.

Answer:

Using the separation of variables, we can write the generalized form of the solution as

$$u(x,y,t) = X(x)Y(x$$

$$u(x,y,t) = \sum_m \sum_n (Asin(\alpha x) + A'cos(\alpha x))(Bsin(\beta y) + B'cos(\beta y))(C_{mn}cos(\lambda_{mn}t) + C'_{mn}sin(\lambda$$

(6)

Using boundary conditions (2) and (3), we can reformulate the above equation as

$$u_{mn}(x,y,t) = sin(\frac{m\pi}{a}x)sin(\frac{n\pi}{b}y)(C_{mn}cos(\lambda_{mn}t) + C'_{mn}sin(\lambda_{mn}t)) \quad (7)$$

Using the initial condition (5), we find finally equation (7) as

$$u_{mn}(x,y,t) = sin(\frac{m\pi}{a}x)sin(\frac{n\pi}{b}y)(C_{mn}cos(\lambda_{mn}t) \quad (8)$$

At time, $t = 0$,

$$g(x,y) = 0.1(4x - x^2)(2y - y^2) = sin(\frac{m\pi}{a}x)sin(\frac{n\pi}{b}y)(C_{mn}cos(\lambda_{mn}t) \quad (9)$$

Integrating both sides, we find value of C_{mn},

$C_{mn} = \frac{4}{ab}\int_0^b \int_0^a 0.1(4x - x^2)(2y - y^2)sin(\frac{m\pi}{a}x)sin(\frac{n\pi}{b}y)dxdy$

Integrating by part and using the value $a = 4, andb = 2$, we find the C_{mn}

$$C_{mn} = \frac{0.426}{m^3n^3} \quad (10)$$

where m and n are both odd integer. The displacement, $u(x,y,t)$ becomes

$$u(x,y,t) = 0.426\sum_m \sum_n \frac{1}{m^3n^3}sin(\frac{m\pi}{a}x)sin(\frac{n\pi}{b}y)(cos(\lambda_{mn}t) \quad (11)$$

where $\lambda_{mn} = \sqrt{\frac{T}{\rho}}\pi\sqrt{((\frac{m}{4})^2) + (\frac{n}{2})^2)}$

(c) How does the underlying PDE obtained in (a) change if the vibrations are forced by an external force $f(x, y, t)$ acting perpendicular to the xy-plane ?

Answer: Equilibrium state of this elastic membrane = Total energy of system.

Potential energy of the system is proportional to the change of surface of the membrane.

$$J(u) = TotalEnergy \tag{12}$$

$$J(u) = \int (\sqrt(1 + |\nabla|^2) - 1)dx - \int fudx \tag{13}$$

where the second term comes from external force, so the PDE in(a) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - fu \tag{14}$$

(d) Let the velocity be denoted by $v(x, y, t) = \frac{\partial u(x,y,t)}{\partial t}$.

i. Express the second order (in time t) PDE obtained in (c) as a system of first order (in time t) PDEs in terms of u and v.
This should be of the form

$$\frac{\partial \mathbf{X}}{\partial t} = \begin{bmatrix} F_1(x, y, t) \\ F_2(x, y, t) \end{bmatrix}, \quad \mathbf{X}|_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{15}$$

Please specify $\mathbf{X}$, $F_1(x, y, t)$, $F_2(x, y, t)$, u_0 and v_0.
Note: $\mathbf{X}$ should depend on (x, y) and t.
Answer: $\begin{bmatrix} \frac{\partial u(x,y,t)}{\partial t} \\ \frac{\partial v(x,y,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} v \\ \frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - fu \end{bmatrix}$

ii. Derive the variational formulation of the first order system (15) and show that it assumes the following form:
for each $0 < t < T$,
find $(u(t), v(t)) \in V \times V$

$$u(0) = u_0,\ v(0) = v_0, \tag{16}$$

$$(\partial_t u(t), \phi_1) - (v(t), \phi_1) = \ell_1(\phi_1), \tag{17}$$

$$(\partial_t v(t), \phi_2) - a(u(t), \phi_2) = \ell_2(\phi_2) \quad \forall(\phi_1, \phi_2) \in V \times V, \tag{18}$$

where $u(t) \equiv u(x, y, t)$, $v(t) \equiv v(x, y, t)$ and $\phi_i \equiv \phi_i(x, y)$, $i = 1, 2$.
Please specify the bilinear form $a(.,.)$, $\ell_1(.)$, $\ell_2(.)$ and the solution space V.

iii. Let $t_0 = 0 < t_1 \cdots t_N = T$ with $\tau = t_{i+1} - t_i$, $i = 0, \cdots, N - 1$. Please write down the backward Euler Method with time step τ for (16)-(18) and $N = 3$.
Answer:
using Backwark Euler Method, the above equation becomes
$\begin{bmatrix} \frac{u_{i+1} - u_i}{\tau} \\ \frac{v_{i+1} - v_i}{\tau} \end{bmatrix}$

$$= \begin{bmatrix} v_{i+1} \\ c^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u_{i+1} - fu_{i+1} \end{bmatrix}$$

So, organizing the above equations, we can write in this form as

$$u_{i+1} - \tau v_{i+1} = u_i \tag{19}$$

and

$$-\tau[c^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u_{i+1}] - fu_{i+1} + v_{i+1}v_i \tag{20}$$

So, the equations (13) and (14) can be written as more compact form as

$$\begin{bmatrix} 1 & -\tau \\ -\tau c^2\nabla^2 - f & -1 \end{bmatrix} \cdot \begin{bmatrix} u_{i+1} \\ v_{i_{+1}} \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix}$$

II. (20 pts) Consider a closed ecological system consisting of three species with populations $x(t)$, $y(t)$ and $z(t)$ at a given time t. The population of species $x(t)$ is preyed upon by the species with population $y(t)$ who, in turn, are the sole food source of species having population $z(t)$.

(a) Formulate the system of differential equations for the rate of change of the three populations $x(t)$, $y(t)$ and $z(t)$.

You may denote the parameters arising in the system by a notation of your choice.

(a) Formulate the system of differential equations for the rate of change of the three populations x(t), y(t) and z(t).

Answer:

$$\frac{dx}{dt} = a_a x - a_2 xy \tag{21}$$

$$\frac{dy}{dt} = -b_1 y + b_2 xy - b_3 yz \tag{22}$$

$$\frac{dz}{dt} = -c_1 z + c_2 yz \tag{23}$$

(b) What are the possible values of $x(t)$, $y(t)$ and $z(t)$ that lead to the equilibrium of this ecosystem? Please give your values in terms of the parameters.

Answer: In equilibrim condition

$$\frac{dx}{dt} = 0 = a_a x - a_2 xy$$

$$x = oory = \frac{a_1}{a_2}$$

Similary

$$\frac{dy}{dt} = 0 = -b_1 y + b_2 xy - b_3 yz$$

$$y(-b_1 + b_2 x - b_3 z) = 0$$

$$y = 0$$

or

$$-b_1 + b_2 x - b_3 z = 0 \tag{24}$$

if

$$x = 0$$
$$z = -\frac{b_1}{b_3}$$

if

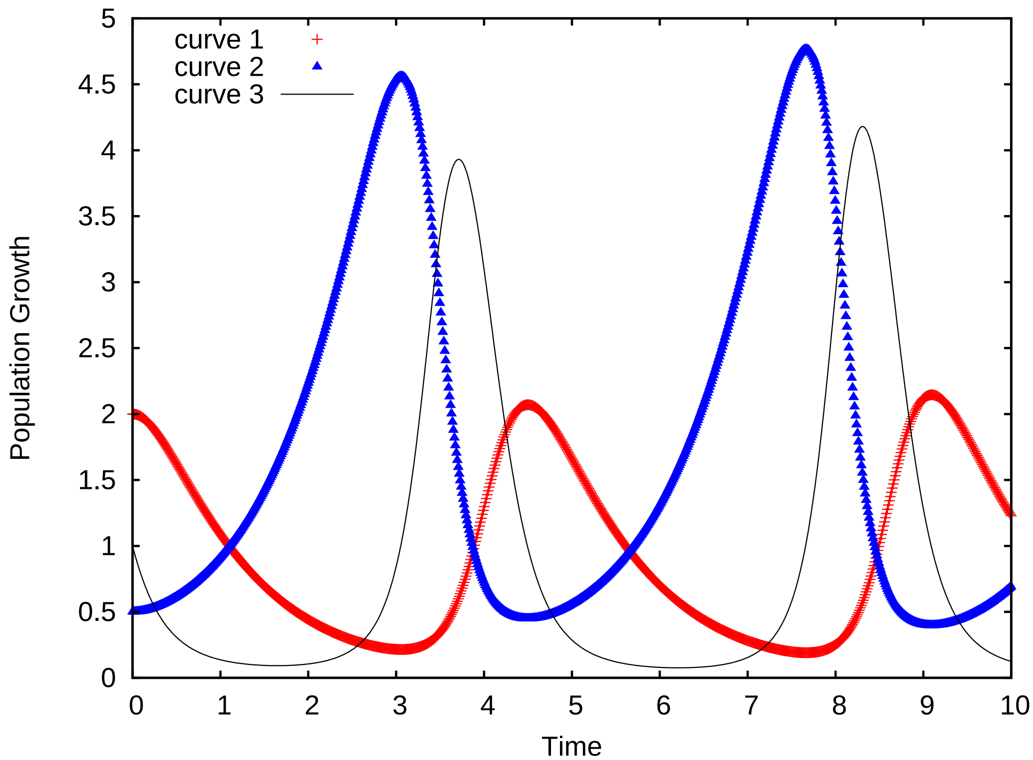
$$z = 0$$
$$x = \frac{b_1}{b_2}$$

(c) The figure below depicts the plots of particular periodic populations with the choices of all the parameters (appearing in the system) set to 1. Which of the curves in the figure below represent $x(t), y(t)$ and $z(t)$?

(d) Now assume that the population $x(t)$ in isolation follows the logistic equation. Keeping this assumption in mind, please modify the differential equations obtained in (a) and write the modified system of equations you obtain.

Answer:

$$\frac{dx}{dt} = a_1 x(1 - \frac{x}{k}) \tag{25}$$
$$\frac{dy}{dt} = -b_1 y - b_3 yz \tag{26}$$
$$\frac{dz}{dt} = -c_1 z + c_2 yz \tag{27}$$
$$\tag{28}$$



Answer:
Curve 2 corresponds x(t), which is prey
Curve 3 corresponds, y(t), which is predator but prey of z(t)
Curve 2 corresponds, z(t).