Subject: Date:

problem 1: Ans:

Step 1: A set G is with a binary operation $*$ is a group if it satisfies-

(i) closure: For all $a, b \in G$, $a*b \in G$

(ii) Associativity: $(a*b)*c = a*(b*c)$

(iii) Identity element: any $e \in G$ such, $a*e = e*a = a$ for all $a \in G$

(iv) Inverse element: For each $a \in G$, there exist $a^{-1} \in G$ such $a * a^{-1} = e$

if $a*b = b*a$ for all $a, b \in G$, the group is abelian

step 2: Take set of odd integer

Let,

$$O = \{\cdots -3, -1, 1, 3 \cdots\}$$

with binary operation $(+)$ (usual addition)

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Subject: Date:

Step 3: verify group axioms -

(i) closure odd + odd = Even

Thus closure fails

(ii) Associativity -

$(a+b)+c = a+(b+c)$ ~~but irre~~

but irrelevent since closure failed.

(iii) Identity element

The additive identity in $(\mathbb{Z},+)$ is 0

0 is not odd so fails.

(iv) inverse element

For an odd integer a, its inverse under addition is $-a$.

Example: if $3 \in O$, inverse is -3, which is also odd.

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Subject: Date:

step 4: since closure and identity fails the set of odd integer with + is not a group. Therefore it cannot be an abelian group either.

Final ans:

The set of odd integers under addition is not an abelian group because-

→ it's not closed (odd + even $\notin O$)

→ It does not contain the identity element (since 0 is not odd)

Subject: Date:

problems 2 Ans:

① if $|G| = pq$ with distinct prime p, q then G is abelian. Ans: false.

why: By ~~stly~~ sylow theory the sylow-q subgroup normal, so G so G is a semidirect product $P \times Q$. if $p \nmid (q-1)$ the semidirect product force to be direct but $p \mid (q-1)$ nontrival.

~~problem 2~~

② if $G = p^2$ (p = prime) the G is abelian iff it has $(p+1)$ subgroup of order p.

Ans: True.

why: Only group of order p^2 are

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Subject: Date:

C_{p^2} and $C_p \times C_p$. Thus having $p+1$ subgroup of order p characterized abelian $C_p \times C_p$ case

(3) For finite G and proper $H \le G$ union of all conjugate H cannot equal G

Ans: True

why: Let conj $H_1, H_2 \ldots H_k$. Each intersect another in a proper subset and shows

$$|\cup_i H_i| \le k(|H|-1)+1 \le [G:H](|H|-1)+1 < |G|$$

so union is smaller than G.

(4) If $N \trianglelefteq G$, N cyclic and G/N cyclic then G is abelian.

Ans: False.

Subject: Date:

why: In genearal nonabelian group tension elements need to be closed under multiplication. Example: finite dihedral group D_8. it rotation subgroup $N \cong C_4$ is cyclic and normal. $D_8/N \cong C_2$ is cyclic yet D_8 nonabelian

(5) Any group G set of finite order forms a sub group.

Ans: False.

$\Rightarrow$ nonabelian group tension element need not to be closed under multiplication

Example: the finite dihedral D_8 has many reflection of order 2.

CEFALO

Subject: ________ Date:

(7) if $a^4 = b^2$ and $ab = ba$ then $(ab)^6 = e$

Ans: False.

why: if a, b commute then $(ab)^6 = a^6 b^6$.

From $b^2 = a^4$ we get $b^6 = (b^3)^2 = a^{12}$.

so $(ab)^6 = a^{18}$ there is no reason $a^{18} = e$ in

general. in finite cycle group take

$a = g$, $b = g^2$ then $a^4 = g^4 = b^2$

they commute $(ab)^6 = g^{18} \neq e$.

(8) $[G:H] = n$, for any $x \in G$, $x^n \in H$.

Ans: False.

why: General true statement is $x^{n!} \in H$

for all $x \in G$. Reason the perm action

of G of n cost Gives. $\varphi: G \to S_n$

The order of $\varphi(x)$ divides $n!$ so $x^{n!} \in \ker \varphi =$

$\bigcap_{g \in G} gHg^{-1} \subseteq H$. exponent n is not sufficient

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Subject: Date:

(9) If G has exactly one subgroup of order p^k for each $k \leq n$ (and $p^n \,||\, |G|$) the G has normal sylow p-subgroup.

Ans: True.

why! Let P be the subgroup of order p^n any conjugate of P has the same order of p^n the ~~th~~ hence must P equal by uniqueness. Thus P is normal and is the sylow p subgroup.

(10) If $|G| = p^n m$ with p prime and $p \nmid m$ and if $H \leq G$ with $|H| = p^n$ then H is normal in G.

Ans: True.

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Subject: Date:

why: A subgroup of order p^n is a sylow p-subgroup. By sylow theorem the number n_p of such subgroup divides m and satisfies $n_p \equiv 1 \pmod{p}$.

since $p \nmid m$, the only divisor of m congruent to $1 \pmod{p}$ is 1, so $n_p = 1$

uniqueness implies ~~normalty~~ normality.

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