Uppsala universitet
1TD184 Optimisation

Assignment 2 – basic option
October 27, 2023

Assignment 2: Portfolio optimisation

The Markowitz framework for portfolio composition

The return of investment instruments such as publicly-traded stocks are unknown at the time when the investment is made. Typically, there is a tradeoff between risk and return: a higher return is associated with a higher risk. A quantitative study of such tradeoffs is made particularly simple using an analysis from the 1950's due to Markowitz assuming that the returns of the investments are random variables with a known multivariate normal distribution.

An *asset* is an investment instrument that can be bought or sold. The *rate of return* of the asset is the number r satisfying $X_1 = (1+r)X_0$, where X_0 and X_1 are the prices of the asset at purchase and selling, respectively. As an example, the rate of return from deposits in a bank account is the interest rate. Assume that an investor wants to select a portfolio of n possible assets. The investor puts the fraction w_i of available funds into asset i with rate of return r_i, $i = 1, \dots, n$. We assume that all available funds are invested, so $\sum_{i=1}^n w_i = 1$. Thus, rate of return for the portfolio will be $r = \sum_{i=1}^n r_i w_i$.

The situation when a weight w_i is negative corresponds to a short selling of the asset, that is, the investor borrows/loans (but does not buy) the asset from someone and sells it to someone else. In practice there are terms attached to the loan (see 'Securities Lending' in Wikipedia) and therefore not everyone has the ability to short sell assets.

Short selling can significantly increase the total return of the portfolio at the price of a substantially increased risk. When short selling is not allowed, we require that $w_i \geq 0$, whereas there are no constraints on w_i when unlimited short selling of the asset is allowed.

The rates of returns are often not known in advance. We assume in the Markowitz framework that we possess estimates of their expected values $\bar{r}_i = E(r_i)$ and their covariances $\sigma_{ij} = E\big((r_i - \bar{r}_i)(r_j - \bar{r}_j)\big)$. These estimates can for instance be obtained by analysis of previous performance of the assets. The properties of the expected value and variance yields that the expected rate of return and the variance of the portfolio will be $\bar{r} = \sum_{i=1}^n \bar{r}_i w_i$ and $\sigma = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}$, respectively.

We will consider portfolios of five assets with yearly expected rates of returns and covariances according to Table 1. Note that any matrix of covariances is symmetric and positive semidefinite.

Tasks

1. Formulate the linear program of maximising the expected return of the portfolio when short selling is not allowed. Solve the linear program by "inspection" (that

is, do not use software or hand calculation). What is the variance of the portfolio?

2. Formulate the quadratic program of minimising the variance of the portfolio (short selling allowed). Set up the KKT system and solve it using MATLAB. Report the computed weights and the variance of corresponding portfolio and compare with 1.

   Note that MATLAB can handle block matrices. Assume that A is an n-by-n matrix and b a column vector of dimension n, and that these can be blocked as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

   where

   - matrix A_{11} has as many rows as matrix A_{12}, and matrix A_{21} has as many rows as matrix A_{22},
   - matrix A_{11} has as many columns as matrix A_{21}, and matrix A_{12} has as many columns as matrix A_{22};
   - both b_1 and b_2 are column vectors.

   Then MATLAB matrices `A` and `b` can be specified as

```
A = [A_11, A_12; A_21, A_22];
b = [b_1; b_2];
```

   where `A_11`, `A_12`, `A_21`, `A_22`, `b_1`, and `b_2` are previously defined matrices with dimensions that match as above. Note that elements on the same row are separated with "`,`" (a separating space is also OK), whereas a new row is indicated by "`;`" (inserting a new row by pressing the return key is also OK).

3. An extremely *risk averse* investor would choose the strategy in 2 whereas an extremely *risk preferring* investor would choose the strategy in 1. Most investors choose a strategy in between these extremes. For instance, if an investor requires a particular expected rate of return $\bar{r}$, it makes sense to calculate the particular portfolio that meets that goal at the minimum variance.

   (a) Formulate the quadratic program that minimises the portfolio variance, subject to a precise value $\bar{r} = \rho$ of the expected rate of return for the portfolio. Formulate the problem both for the case when short selling is and is not allowed.

   (b) Solve the problem for $\rho = 0.1$ with the MATLAB algorithm `quadprog` using a medium-scale active set algorithm. Check the MATLAB documentation for the options you need to set. Report weights and variance of the portfolio.

   (c) What is the result for $\rho = 0.2$? Interpret! (Hint: Look at the values of $\bar{r}$.)

4. Let σ and $\bar{r}$ be the portfolio variance and expected rate of return calculated as in 3 above. The *efficient frontier* is the set of all possible pairs $(\sigma, \bar{r})$. A good way of computing the efficient frontier is to solve, for a parameter $0 \le \alpha \le 1$, the

quadratic program

$$
\begin{aligned}
\min \quad & \left(\alpha \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} - (1-\alpha) \sum_{i=1}^{n} \bar{r}_i w_i \right) \\
\text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1, \ \text{and, when short selling is not allowed,} \\
& w_i \geq 0, \ \text{for } i = 1, \dots, n.
\end{aligned}
\tag{1}
$$

(a) Which cases do the extreme values $\alpha = 0$ and $\alpha = 1$ correspond to?

(b) Solve (1) for $\alpha = 0.05, 0.1, 0.15, \dots, 1.0$ using `quadprog`, both for the case when short selling is and is not allowed. Plot and compare the efficient frontiers (i.e., plot $\bar{r}$ over σ).

Table 1: Covariances and rates of returns for five assets. Note that the numbers in the table should be multiplied by 10^{-2}. For instance, $\bar{r}_1 = 0.130$ and $\sigma_{11} = 0.0401$. These values are borrowed from, and slightly modified compared to the values in, Example 6.11 of Chapter 6 in Luenberger: *Investment Science*, Oxford Univ. Press (1997).

Asset	Covariances ($\times\ 10^{-2}$)					$\bar{r}_i$ (%)
1	4.01	-1.19	0.60	0.74	-0.21	13.0
2	-	1.12	0.21	-0.54	0.55	4.4
3	-	-	3.31	0.77	0.29	12.1
4	-	-	-	3.74	-1.04	7.1
5	-	-	-	-	2.6	11.7

Acknowledgements: Original assignment idea created by Daniel Noreland, Cheng Gong, Markus Kowalewski
Javier Cano, Martin Berggren, Katharina Kormann,
Martin Kronbichler, Per Lötstedt, Eddie Wadbro