Complex Variables

Chapter-1: Complex Number

1.1

(g) $(4+2i)(2-3i)$

$= 8-12i+4i-6i^2$

$= 8-8i+6$

$= 14-8i$ Ans.

(k) $\dfrac{3-2i}{-1+i}$

$= \dfrac{(3-2i)\cdot(-1-i)}{(-1+i)(-1-i)}$

$= \dfrac{-3-3i+2i+2i^2}{(-1)^2-i^2}$

$= \dfrac{-3-i-2}{1+1}$

$= \dfrac{-5-i}{2}$

$= -\dfrac{5}{2}-\dfrac{1}{2}i$ A

(h) $(2-i)(-3+2i)(5-4i)$

$= \cancel{(2-i)(-15-12i+10i+8i^2)}$

$= \cancel{(2-i)(-7-}$

$= (2-i)(-15+12i+10i-8i^2)$

$= (2-i)(-7+22i)$

$= -14+44i+7i-22i^2$

$= 8+51i$ A

(m) $\dfrac{3i^{30}-i^{19}}{2i-1}$

$= \dfrac{3(i^2)^{15}-(i^2)^9 i}{2i-1}$

$= \dfrac{3(-1)^{15}-(-1)^9\cdot i}{2i-1}$

$= \dfrac{-3+i}{-1+2i}$

$= \dfrac{(-3+i)(-1-2i)}{(-1+2i)(-1-2i)}$

$= \dfrac{3+6i-i-2i^2}{1-4i^2}$

$= \dfrac{5+5i}{5} = 1+i$ Ans

(1) $\frac{5+5i}{3-4i}+\frac{20}{4+3i}$

$=\frac{(5+5i)(3+4i)}{(3-4i)(3+4i)}+\frac{20(4-3i)}{(4+3i)(4-3i)}$

$=\frac{15+20i+15i+20i^2}{(3)^2-(4i)^2}+\frac{80-60i}{(4)^2-(3i)^2}$

$=\frac{15+35i-20}{9-16i^2}+\frac{80-60i}{16-9i^2}$

$=\frac{-5+35i}{25}+\frac{80-60i}{25}$

$=\frac{-5+35i+80-60i}{25}$

$=\frac{75-25i}{25}=\frac{75}{25}-\frac{25}{25}i=3-i$ Ans

(1.2)

Given that,

$z_1=2+i$, $z_2=3-2i$

$z_3=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(P.T.O)

(a) $|3z_1 - 4z_2|$

$= |3(2+i) - 4(3-2i)|$

$= |6+3i-12+8i|$

$= |-6+11i|$

$= \sqrt{(-6)^2+(11)^2}$

$= \sqrt{36+121}$

$= \sqrt{157}$ Ans

~~(b) $z_1^3 - 3z_2^2 + 4z_1 - 8$~~

~~$= (2+i)^3 - 3(3-2i)^2 + 4(2+i) - 8$~~

~~$= (2)^3 + 3(2)^2 i + 3\cdot 2 i^2 + i^3 - 3(3^2 - 2\cdot 3\cdot 2i + 4i^2) + 8 + 4i - 8$~~

~~$= 8 + 12i + 6i^2 - i - 27 + 36i - 12i^2 + 4i$~~

~~$= -19 + 47i - 6i^2$~~

(b) $z_1^3 - 3z_1^2 + 4z_1 - 8$

$= (2+i)^3 - 3(2+i)^2 + 4(2+i) - 8$

$= 2^3 + 3(2)^2 i + 3\cdot 2i^2 + i^3 - 3(4+4i+i^2) + 4i$

$= 8 + \cancel{12i} + 6i^2 - i - 12 - \cancel{12i} - 3i^2 + 4i$

$= -4 \overset{+3i}{\cancel{+2i}} + 3i^2$

$= -4 + 3i + 3$

$= -\cancel{7} + 3i$

(c) $(\bar{z}_3)^4 = \left(\overline{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}\right)^4$

$= \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)^4$

$= \left\{\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)^2\right\}^2$

$= \left\{\left(-\frac{1}{2}\right)^2 - 2\cdot\left(-\frac{1}{2}\right)\cdot\frac{\sqrt{3}i}{2} + \left(\frac{\sqrt{3}i}{2}\right)^2\right\}^2$

$= \left(\frac{1}{4} + \frac{\sqrt{3}i}{2} + \frac{3i^2}{4}\right)^2$

$= \left(\frac{1}{4} - \frac{3}{4} + \frac{\sqrt{3}}{2}i\right)^2$

$= \left(\frac{-2}{4} + \frac{\sqrt{3}}{2}i\right)^2$

$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$

$= \left(-\frac{1}{2}\right)^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\cdot\frac{\sqrt{3}i}{2} + \left(\frac{\sqrt{3}}{2}i\right)^2$

(P.T.O)

$$= \frac{1}{4} - \frac{\sqrt{3}}{2}i + \frac{3i^2}{4}$$

$$= \frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}i$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad \text{Ans}$$

(d) $\left|\frac{2z_2 + z_1 - 5 - i}{2z_1 - z_2 + 3 - i}\right|^2$

$$= \left|\frac{2(3-2i) + (2+i) - 5 - i}{2(2+i) - (3-2i) + 3 - i}\right|^2$$

$$= \left|\frac{6 - 4i + 2 + \cancel{i} - 5 - \cancel{i}}{4 + 2i - \cancel{3} + 2i + \cancel{3} - i}\right|^2$$

$$= \left|\frac{3 - 4i}{4 + 3i}\right|^2$$

$$= \frac{|3-4i|^2}{|4+3i|^2} = \frac{\left(\sqrt{(3)^2 + (-4)^2}\right)^2}{\left(\sqrt{(4)^2 + (3)^2}\right)^2} = 1 \quad \text{Ans.}$$

1.13

(a) The centere can be represented by the complex number $-2+i$. If z is any points on the circle, the distance from z to $(-2+i)$ is – (P.T.O)

$$|z-(-2+i)| = 4$$

$$\Rightarrow |z+2-i| = 4$$

let, $z = x+iy$

$$\therefore |x+iy+2-i| = 4$$

$$\Rightarrow |x+2+i(y-1)| = 4$$

$$\Rightarrow \sqrt{(x+2)^2+(y-1)^2} = 4$$

$$\Rightarrow (x+2)^2+(y-1)^2 = 16$$

(b) The sum of the distance from any point of z on the ellipse, we have wrote

$$|z+3|+|z-3| = 10$$

let, $z = x+iy$

$$\therefore |x+iy+3|+|x+iy-3| = 10$$

$$\Rightarrow |x+3+iy|+|x-3+iy| = 10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2}+\sqrt{(x-3)^2+y^2} = 10$$

$$\Rightarrow \sqrt{(x+3)^2+y^2} = 10-\sqrt{(x-3)^2+y^2}$$

$$\Rightarrow (x+3)^2 + \cancel{y^2} = 100 - 20\sqrt{(x-3)^2+y^2} + (x-3)^2 + \cancel{y^2}$$

$$\Rightarrow \cancel{x^2} + 6x + \cancel{9} = 100 - 20\sqrt{(x-3)^2+y^2} + \cancel{x^2} - 6x + \cancel{9}$$

$$\Rightarrow 20\sqrt{(x-3)^2+y^2} = 100 - 12x$$

$$\Rightarrow 5\sqrt{(x-3)^2+y^2} = 25 - 3x$$

$$\Rightarrow 25(x^2 - 6x + 9 + y^2) = 625 - 150x + 9x^2$$

$$\Rightarrow 25x^2 - \cancel{150x} + 225 + 25y^2 = 625 - \cancel{150x} + 9x^2$$

$$\Rightarrow 16x^2 + 25y^2 = 400$$

$$\Rightarrow \frac{16x^2}{400} + \frac{25y^2}{400} = 1$$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\therefore \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

(1.16)

(a) Here, $2+2\sqrt{3}i$

Modulus, $r = \sqrt{(2)^2+(2\sqrt{3})^2}$

$= \sqrt{4+12} = 4$

Argument, $\theta = \tan^{-1}\frac{2\sqrt{3}}{2} = \pi/3$

$\therefore 2+2\sqrt{3}i = r(\cos\theta + i\sin\theta)$

$= 4(\cos\pi/3 + i\sin\pi/3)$

$\therefore$ The Polar form $= \cancel{4e^{i\pi/3}} = 4e^{\pi i/3}$

(b) $-5+5i$

$\therefore$ Modulus, $r = \sqrt{25+25} = 5\sqrt{2}$

Argument, $\theta = \tan^{-1}\frac{5}{-5}$

$= \pi - \tan^{-1}1$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

$\therefore -5+5i = 5\sqrt{2}\left(\cos \frac{3\pi}{4} + i\sin \frac{3\pi}{4}\right)$

The Polar form $= 4e^{3\pi i/4}$

ⓒ Modulus, $r = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2}$

$= \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}$

Argument, $\theta = \tan^{-1} \frac{-\sqrt{2}}{-\sqrt{6}}$

$= \pi + \tan \frac{1}{\sqrt{3}}$

$= \pi + \frac{\pi}{6}$

$= \frac{7\pi}{6}$

$\therefore -\sqrt{6} - \sqrt{2}i = 2\sqrt{2}\left(\cos \frac{7\pi}{6} + i\sin \frac{7\pi}{6}\right)$

Polar form $= 4e^{7\pi i/6}$

(14.3)

(a) Given that, $2x+y=5$

Let, $z = x+iy$ ——— (i)

$\therefore \bar{z} = x-iy$ ——— (ii)

(i)+(ii)

$2x = z+\bar{z}$

$\therefore x = \frac{z+\bar{z}}{2}$

(i)−(ii)

$2iy = z-\bar{z}$

$\therefore y = \frac{z-\bar{z}}{2i}$

$\therefore 2x+y = 5$

$\Rightarrow \cancel{2}\left(\frac{z+\bar{z}}{\cancel{2}}\right) + \frac{z-\bar{z}}{2i} = 5$

$\Rightarrow z+\bar{z}+\frac{z-\bar{z}}{2i} = 5$

$\Rightarrow \frac{2zi+2\bar{z}i+z-\bar{z}}{2i} = 5$

$\Rightarrow (2i+1)z+(2i-1)\bar{z} = 10i$

(b) Let, $z = x + iy$ | $x = \frac{z+\bar{z}}{2}$

$\bar{z} = x - iy$ | $y = \frac{z-\bar{z}}{2i}$

$\therefore x^2 + y^2 = 36$

$\Rightarrow \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 = 36$

$\Rightarrow \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{z^2 - 2z\bar{z} + \bar{z}^2}{4i^2} = 36$

$\Rightarrow \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{-z^2 + 2z\bar{z} - \bar{z}^2}{4} = 36$

$\Rightarrow \frac{\cancel{z^2} + 2z\bar{z} + \cancel{\bar{z}^2} - \cancel{z^2} + 2z\bar{z} - \cancel{\bar{z}^2}}{4} = 36$

$\Rightarrow \frac{4z\bar{z}}{4} = 36$

$\therefore z\bar{z} = 36$

⇒ Cauchy Reimann Equations:

Necessary condition for $f(z)$ to be analytic:

Theorem: The necessary condition for a fun $f(z) = u(x,y) + iv(x,y)$ to be analytic in a Domain D is that in the Domain D, u and v satisfy Cauchy-Reimann equations.

In order for $f(z)$ to be analytic, the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{--- (1)}$$

Let, $z = x + iy$

$f(z) = u(x,y) + iv(x,y)$

$\Delta z = \Delta x + \Delta iy$

$\therefore z + \Delta z = x + iy + \Delta x + \Delta iy = (x+\Delta x) + i(y+\Delta y)$

$\therefore f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + \Delta iy}$$

(P.T.O)

Case-1: Along to X-axis

$\Delta y = 0$; $\Delta z = \Delta x$

$\Delta z \to 0$; $\Delta x \to 0$

$$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) + iv(x+\Delta x, y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + 0}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + iv(x+\Delta x, y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x}$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

Case-2: Along to Y-axis

$\Delta x = 0$; $\Delta z = \Delta y$

$\Delta z \to 0$; $\Delta y \to 0$

$$\therefore f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y+\Delta y) + iv(x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{0 + \Delta iy}$$

(P.T.O)

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta iy}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta iy} + \lim_{\Delta y \to 0} \frac{\cancel{i}\{v(x, y+\Delta y) - v(x,y)\}}{\cancel{i}\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{(ii)}$$

Compare (i) and (ii), we get,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}} \quad \text{and}$$

$\therefore$ The function is analytic.

⇒ Harmonic function: (3.6)

If $f(z)$ is analytic in R then the Cauchy Reimann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ ——— (i) and}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ ——— (ii)}$$

are satiesfied in R.

By differentiating both sides of (i) with respect x and (ii) with respect to y,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ ——— (iii) and}$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ ——— (iv)}$$

from (iii) and (iv), we get,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

(u is harmonic)

/ P.T.O)

$\therefore$ i.e. u is harmonic.

$\because$ Similarly, by differentiating ~~(i) and~~ both sides of (i) with respect y and (ii) with respect to x, we get

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$$

i.e. v is harmonic.

3.7

(a) Given that,

$$u = e^{-x}(x\sin y - y\cos y)$$

$$\Rightarrow \frac{\delta u}{\delta x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - e^{-x}(x\sin y - y\cos y)$$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} = -e^{-x}\sin y - e^{-x}(\sin y) + e^{-x}(x\sin y - y\cos y)$$

$$= -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \quad \text{--- (1)}$$

(P.T.O)

Again,

$$u = e^{-x}(x\sin y - y\cos y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + y\cos y + \sin y + \sin y)$$

$$= -xe^{-x}\sin y + ye^{-x}\cos y + 2e^{-x}\sin y \quad \text{(ii)}$$

(i) + (ii), and we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus u is harmonic (proved)

(b) from 'a' we get,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{(iii)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -xe^{-x}\cos y - ye^{-x}\sin y + e^{-x}\cos y$$

$$= e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y \quad \text{(iv)}$$

(P.T.O)

Now, intigrating (iii) with respect to y

$$v=\int\left(e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y\right)dy$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$

$$= \cancel{-e^{-x}\cos y} + xe^{-x}\cos y + ye^{-x}\sin y + \cancel{e^{-x}\cos y} + F(x)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + F(x) \quad \text{——— (v)}$$

Differentiating with respect to x,

$$\cancel{\frac{\partial v}{\partial x} = e^{-x}(-\sin}$$

$$\frac{\partial v}{\partial x} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y + F'(x) \quad \text{——— (vi)}$$

from (iv) and (vi), we get,

$$F'(x) = 0$$

$$\Rightarrow F(x) = c, \text{ where } c \text{ is an arbitrary constant.}$$

$$\therefore V = xe^{-x}\cos y + ye^{-x}\sin y + c$$

$$= e^{-x}(x\cos y + y\sin y) + c$$

A

[Another rule]

3.14

Given that,

$$v(x,y) = 6xy - 5x + 3$$

$$\Rightarrow \frac{\partial v}{\partial x} = 6y - 5 \text{ and } \frac{\partial v}{\partial y} = 6x$$

Here;

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

$$= 6x\,dx - (6y-5)dy$$

Now, integrating, we get

$$u = \int 6x\,dx - \int (6y-5)dy$$

$$= 3x^2 - 3y^2 + 5y + c$$

$$\therefore f(z) = u + iv$$

$$= 3x^2 - 3y^2 + 5y + c + i(6xy - 5x + 3)$$

$$= 3x^2 - 3y^2 + 5y + 6ixy - 5ix + 3i + c$$

$$= 3(x^2 - y^2 + 2ixy) + 5(-ix + y) + 3i + c$$

$$= 3\{x^2 + 2x \cdot iy + (iy)^2\} + 5(-i)(x + iy) + 3i + c$$

$$= 3(x+iy)^2 - 5i(x+iy) + 3i + c$$

$$= 3z^2 - 5iz + 3i + c$$

3.26 [Page - 3.24]

(a) We have,

$$w = e^{z} = e^{x+iy}$$
$$= e^{x}(\cos y + i\sin y)$$
$$= e^{x}\cos y + e^{x} i\sin y$$
$$= u + iv$$

Here, $u = e^{x}\cos y$
$v = e^{x}\sin y$

Since, $\frac{\partial u}{\partial x} = e^{x}\cos y = \frac{\partial v}{\partial y}$ and

$$\frac{\partial v}{\partial x} = e^{x}\sin y = -\frac{\partial u}{\partial y}$$

the Caucy-Riemann equations are satisfied

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^{x}\cos y + ie^{x}\sin y$$
$$= e^{z}$$

(Proved)

3.27:

(a) We have,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\Rightarrow \frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right)$$

$$= \frac{1}{2i}\frac{d}{dz}e^{iz} - \frac{1}{2i}\frac{d}{dz}e^{-iz}$$

$$= \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz}$$

$$= \cos z \quad \text{(Proved)}$$

(b) We have,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\Rightarrow \frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right)$$

$$= \frac{1}{2}\frac{d}{dz}e^{iz} + \frac{1}{2}\frac{d}{dz}e^{-iz}$$

$$= \frac{i}{2}e^{iz} - \frac{i}{2}e^{-iz}$$

$$= -\frac{1}{2i}e^{iz} + \frac{1}{2i}e^{-iz}$$

$$= -\sin z \quad \text{(Proved)}$$

(c) $\frac{d}{dz}\tan z = \frac{d}{dz}\left(\frac{\sin z}{\cos z}\right)$

$$= \frac{\cos z \frac{d}{dz}\sin z - \sin z \frac{d}{dz}\cos z}{\cos^2 z}$$

$$= \frac{\cos^2 z + \sin^2 z}{\cos^2 z}$$

$$= 1 + \frac{\sin^2 z}{\cos^2 z}$$

$$= 1 + \tan^2 z$$

$$= \sec^2 z \quad \text{(Proved)}$$

3.29:

Let, $w = \ln z$

$\Rightarrow z = e^w$

$\Rightarrow \frac{dz}{dw} = e^w = z$

$\cancel{\Rightarrow z = \frac{dz}{dw}}$

Hence, $\frac{d}{dz}\ln z = \frac{dw}{dz} = \frac{1}{dz/dw} = \frac{1}{z}$

$\therefore \frac{d}{dz}\ln z = \frac{1}{z}$ (Proved)

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3.33:

Differentiating with respect to ~~x~~ z,

$$\frac{d}{dz}(w^3) - \frac{d}{dz}3z^2w + \cancel{4\ln} \frac{d}{dz}4\ln z = 0$$

$$\Rightarrow 3w^2\frac{dw}{dz} - 3z^2\frac{dw}{dz} - w.6z + \frac{4}{z} = 0$$

$$\Rightarrow \frac{dw}{dz}(3w^2 - 3z^2) = 6wz - \frac{4}{z}$$

$$\therefore \frac{dw}{dz} = \frac{6wz - 4/z}{3w^2 - 3z^2}$$ **Ans.**

3.35

We have,

$$\frac{dw}{dz} = \frac{6wz - 4/z}{3w^2 - 3z^2}$$

$$\Rightarrow \frac{d^2w}{dz^2} = \frac{(3w^2-3z^2)\frac{d}{dz}(6wz - 4/z) - (6wz - 4/z)\frac{d}{dz}(3w^2-3z^2)}{(3w^2-3z^2)^2}$$

$$= \frac{(3w^2-3z^2)\left(6z\frac{dw}{dz} + 6w + \frac{4}{z^2}\right) - (6wz - \frac{4}{z})\left(6w\frac{dw}{dz} - 6z\right)}{(3w^2-3z^2)^2}$$

3.80: [Page-3.40]

(a) $\frac{d}{dz}\sec z = \frac{d}{dz}\frac{1}{\cos z}$

$$= \frac{\cos z \cdot \frac{d}{dz}(1) - \frac{d}{dz}\cos z}{\cos^2 z}$$

$$= \frac{0 + \sin z}{\cos^2 z}$$

$$= \frac{1}{\cos z} \cdot \frac{\sin z}{\cos z}$$

$$= \sec z \cdot \tan z \text{ (Proved)}$$

(b) $\frac{d}{dz}\cot z = \frac{d}{dz}\left(\frac{\cos z}{\sin z}\right)$

$$= \frac{\sin z \cdot (-\sin z) - \cos z \cdot \cos z}{\sin^2 z}$$

$$= \frac{-\sin^2 z - \cos^2 z}{\sin^2 z}$$

$$= -1 - \frac{\cos^2 z}{\sin^2 z}$$

$$= -(1 + \cot^2 z)$$

$$= -\operatorname{cosec}^2 z \text{ (Proved)}$$

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3.89:

(a) Differentiatting with respect to z

$$\frac{d}{dz}w^2 - \frac{d}{dz}2w + \frac{d}{dz}\sin 2z = 0$$

$$\Rightarrow 2w\frac{dw}{dz} - 2\frac{dw}{dz} + 2\cos 2z = 0$$

$$\Rightarrow \frac{dw}{dz}(2w-2) = -2\cos 2z$$

(i) $$\Rightarrow \frac{dw}{dz} = \frac{-2\cos 2z}{2w-2} = \frac{-\cos 2z}{w-1}$$

$$= \frac{\cos 2z}{1-w}$$ Ans.

(b) We have

$$\frac{dw}{dz} = \frac{\cos 2z}{1-w}$$

$$\Rightarrow \frac{d^2w}{dz^2} = \frac{(1-w)\frac{d}{dz}\cos 2z - \cos 2z\frac{d}{dz}(1-w)}{(1-w)^2}$$

$$= \frac{(1-w)(-2\sin 2z) - \cos 2z\cdot\left(-\frac{dw}{dz}\right)}{(1-w)^2}$$

(i) $$= \frac{w\cdot 2\sin 2z - 2\sin 2z + \cos 2z\frac{dw}{dz}}{(1-w)^2}$$

3.20: [Page– 3.21]

Given that,

$$u = \cos x \cosh y$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\sin x \cosh y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$$

Again,

$$\frac{\partial u}{\partial y} = \cos x \sinh y \text{ ——— (ii)}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y$$

$$= 0$$

Thus, u is a harmonic function.

Let, v be its conjugate harmonic function then we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\sin x \cosh y \text{ ——— (i)}$$

and,

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\cos x \sinh y \longrightarrow (ii)$$

Now, integrating (i) with respect to y, we have

$$v = -\sin x \sinh y + F(x) \longrightarrow (iii)$$

Differentiating (iii) with respect to x,

$$\frac{\partial v}{\partial x} = -\cos x \sinh y + F'(x) \longrightarrow (iv)$$

From (ii) and (iv), we get,

$$F'(x) = 0$$

$$\Rightarrow F(x) = c$$

where c is an arbitrary constant.

$$\therefore v = -\sin x \sinh y + c$$

3.21:

Given that,

$$u = y^3 - 3x^2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy \quad \text{and} \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -6y \qquad \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, u is a harmonic function.

Let, v be its conjugate harmonic function

then, we have,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -6xy \text{ ------------- (i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 3x^2 - 3y^2 \text{ ------------- (ii)}$$

Integrating (i) with respect to y, we have

$$v = -3xy^2 + f(x) \text{ ------------- (iii)}$$

(P.T.O)

Differentiating (iii) with respect to x,

$$\frac{\partial v}{\partial x} = -3y^2 + F'(x) \quad \text{——— (iv)}$$

From (ii) and (iv), we have

$$-3y^2 + F'(x) = 3x^2 - 3y^2$$

$$\Rightarrow F'(x) = 3x^2$$

$$\Rightarrow F(x) = x^3 + c$$

where, c is an arbitrary constant

$\therefore v = -3xy^2 + x^3 + ic$ Ans:

3.22:

Given that,

$$u = x^3 - 3xy^2$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \text{ and } \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{——— (i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy \quad \text{——— (ii)}$$

Integrating (i) with respet to y,

$$V = 3x^2y - y^3 + F(x) \quad \text{———} \quad (iii)$$

Differentiating (iii) with respect to x

$$\frac{\partial v}{\partial x} = 6xy + F'(x) \quad \text{———} \quad (iv)$$

From (ii) and (iv), we get

$$F'(x) = 0$$

$$\Rightarrow F(x) = c$$

$$\therefore v = 3x^2y - y^3 + c$$

where c is an arbitrary constant.

$$\therefore f(z) = u + iv$$

$$= x^3 - 3xy^2 + i(3x^2y - y^3 + c)$$

$$= x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3 + ic$$

$$= (x + iy)^3 + ic$$

$$= z^3 + ic$$

$$\Rightarrow f'(z) = 3z^2$$

which exist for all finite values of z

Hence, $f(z)$ is analytic in any finite region.

3.65: [Page - 3.39]

Given that,

$$u = 2x(1-y) = 2x - 2xy$$

$$\Rightarrow \frac{\delta u}{\delta x} = 2 - 2y \quad \text{and} \quad \frac{\delta u}{\delta y} = -2x$$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} = 0 \qquad \frac{\delta^2 u}{\delta y^2} = 0$$

$$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

Hence, u is a harmonic function. (Proved)

Let, v be its conjugate harmonic function,

Then we have,

$$\frac{\delta v}{\delta y} = \frac{\delta u}{\delta x} = 2 - 2y \text{ ——— (i)}$$

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y} = 2x \text{ ——— (ii)}$$

Integrating (i) with respect to y, we get,

$$v = 2y - y^2 + F(x) \text{ ——— (iii)}$$

$$\Rightarrow \frac{\delta v}{\delta x} = 0 + F'(x) \text{ ——— (iv)}$$

$\therefore F'(x) = 2x$

$\Rightarrow F(x) = x^2 + C$

$\therefore v = 2y - y^2 + x^2 + c$ [c = constant]

$f(z) = u + iv$

$= 2x - 2xy + i(2y - y^2 + x^2 + c)$

$= 2x - 2xy - 2iy - iy^2 + ix^2 + ic$

$\cancel{= i(x^2 - y^2) - 2xy - 2(x+iy) + ic}$

$\cancel{= i(x^2}$

$= 2x + 2xyi^2 - 2iy - iy^2 + ix^2 + ic$

$= i(x^2 + 2xiy - y^2) + 2(x+iy) + ic$

$= i(x^2 + 2xiy + i^2y^2) + 2(x+iy) + ic$

$= i(x+iy)^2 + 2z + ic$

$\Rightarrow f(z) = iz^2 + 2z + ic$

$\Rightarrow f'(z) = 2iz + 2$

which is analytic for any values of z

3.66:

Given that,

$$u = x^2 - y^2 - 2xy - 2x + 3y$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x - 2y - 2 \text{ and } \frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, u is a harmonic function.

Let, v be its conjugate harmonic function, then we have,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x - 2y - 2 \quad \text{——— (i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 2x - 3 \quad \text{——— (ii)}$$

$$v = 2xy - y^2 - 2y + F(x) \quad \text{——— (iii)}$$

$$\Rightarrow \frac{\partial v}{\partial x} = 2y + F'(x) \quad \text{——— (iv)}$$

$\therefore 2y + F'(x) = 2y + 2x - 3$

$\Rightarrow F'(x) = 2x - 3$

$\Rightarrow F(x) = x^2 - 3x + c$

$\therefore v = 2xy - y^2 - 2y + x^2 - 3x + c$

Now,

$f(z) = u + iv$

$= x^2 - y^2 - 2xy - 2x + 3y + i(2xy - y^2 - 2y + x^2 - 3x$

$= x^2 - y^2 - 2xy - 2x + 3y + 2ixy - iy^2 - 2iy + ix^2$
$- 3ix + ic$

$= x^2 + 2xiy + i^2y^2 + ix^2 + 2xi^2y + i^3y^2 - 2x + 3y$
$- 3ix - 2iy + ic$

$= (x+iy)^2 + i(x^2 + 2xiy + i^2y^2) - 2(x+iy)$
$- 3i(x+iy) + i$

$= z^2 + i(x+iy)^2 - 2z - 3iz + ic$

$f(z) = z^2 + iz^2 - 2z - 3iz + ic$