Math 510 HW 19 Michael Snyder

19.1 Given $A \in \mathbb{C}^{m \times n}$ of rank n and $b \in \mathbb{C}^m$, considering the block 2×2 system of equations

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

where I is the $m \times m$ identity. Show that this system has a unique solution $(r, x)^T$, and that the vectors r and x are the residual and the solution of the least squares problem (18.1).

<u>Proof</u>: Considering the given system of equations, we see that

$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} r + Ax = b \\ A^* r + 0 = 0 \end{matrix}$$

$$\Rightarrow \begin{matrix} r = b - Ax \\ A^* r = 0 \end{matrix}$$

r is the residual of the least squares problem.

$$\Rightarrow A^*(b - Ax) = 0$$

$$\Rightarrow A^* b - A^* A x = 0$$

$$\Rightarrow A^* b = A^* A x$$

$$\Rightarrow b = Ax.$$

$\Rightarrow x$ is a solution of $Ax = b$.

To show uniqueness we consider another vector $\hat{x} \in \text{range}(A)$ such that $A\hat{x} = b$.

Since both $Ax, A\hat{x} \in \text{range}(A)$, $Ax - A\hat{x} \in \text{range}(A)$.
But this means $Ax - A\hat{x} \perp b - Ax$.
Thus, by the Pythagorean Theorem,

$$\|b - A\hat{x}\|_2^2 = \|b - Ax\|_2^2 + \|Ax - A\hat{x}\|_2^2 > \|b - Ax\|_2^2.$$

Thus, $\hat{x} \neq x$ as previously assumed, since $\hat{x}$ was arbitrary, x is the unique solution to $Ax = b$.