Numerical Linear Algebra: Homework #6

Due on September 14, 2022 at 10:00PM

Instructor: Professor Blake Barker
Section 1

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Problem 6.1

If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Proof. Suppose P is an orthogonal projector. A matrix Q is unitary if $Q = Q^{-1}$, or $Q^*Q = QQ^* = I$. Using this relationship, we find

$$\begin{aligned}(I-2P)^*(I-2P) &= I^*I - 2I^*P - 2P^*I + 2P^*(2P) && (1)\\ &= I - 4P + 4P && (2)\\ &= I. && (3)\end{aligned}$$

This verifies that $I - 2P$ is unitary. □

Geometric Interpretation Suppose $P \in C^{m\times m}$ is a projector. Then P partitions $C^{m\times m}$ into subspaces range(P) = null($I - P$) and null(P) = range($I - P$). Thus we consider vectors in these complementary spaces. Let the vector $v \in$ range(P). Then

$$(I - 2P)v = [(I - P) - P]v = (I - P)v - Pv = -v.$$

That is, $I - 2P$ reflects vectors in range(P). If we instead consider $w \in$ null(P) = range($I - P$), then we have

$$(I - 2P)w = [(I - P) - P]w = (I - P)w - Pw = w.$$

Thus, $I - 2P$ is the identity for vectors in range($I - P$).

Problem 6.3

Given $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Proof. ($\Rightarrow$) Suppose first that $A \in \mathbb{C}^{m \times n}$ such that A^*A is nonsingular. This means that $A^*Ax = 0$ if and only if $x = 0$. That is, $\text{null}(A^*A) = \{0\}$. Suppose to the contrary that A is not full rank. Then there exists $x \neq 0$ such that $y = Ax = 0$. But this means

$$(A^*A)x = A^*(Ax) = A^*y = 0.$$

Since $x \neq 0$, this contradicts the fact that A^*A is nonsingular. Therefore, x must equal zero and A must have full rank.

($\Leftarrow$) Now suppose that $A \in \mathbb{C}^{m \times n}$ has full rank. Then by Theorem 1.3, 0 is not a singular value of A. By Theorem 5.3 this also means that $\lambda = 0$ is not an eigenvalue of $A^*A = V\Sigma^2V^*$. Therefore, again by Theorem 1.3, A^*A is nonsingular.

□