Numerical Linear Algebra: Homework #2

Due on September 2, 2022 at 10:00PM

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Section 1

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Problem 2.2

The Pythagorean Theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 \tag{1}$$

(a) Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$.

Proof. We begin by noting that by orthogonality, $x_i^* x_j = 0$ for $i \neq j$. Thus,

$$\begin{aligned}
\left\|\sum_{i=1}^{n} x_i\right\|^2 &= \|x_1 + x_2\|^2 \\
&= (x_1 + x_2)^*(x_1 + x_2) \\
&= x_1^* x_1 + x_1^* x_2 + x_2^* x_1 + x_2^* x_2 \\
&= 1 + 0 + 0 + 1 \\
&= 1 + 1 \\
&= x_1^* x_1 + x_2^* x_2 \\
&= \|x_1\|^2 + \|x_2\|^2 \\
&= \sum_{i=1}^{n} \|x_1\|^2 .
\end{aligned}$$

□

(b) Show that this computation also establishes the general case, by induction.

Proof. The base case for induction has been shown in Part (a). Note orthogonality is used to anhilate terms $x_i^* x_j$ for $i \neq j$ in (3) below. Now, suppose Equation (1) equation holds for $k > 2$. We now show, by induction, that this implies Equation (1) also holds for $k + 1$:

$$\begin{aligned}
\left\|\textstyle\sum_{i=1}^{k+1} x_1\right\|^2 &= (x_1 + \cdots + x_k + x_{k+1})^*(x_1 + \cdots + x_k + x_{k+1}) && (2) \\
&= \sum_{i=1}^{k} \|x_i\|^2 + \sum_{i=1}^{k} x_{k+1}^* x_i + x_{k+1}^* x_{k+1} && (3) \\
&= \sum_{i=1}^{k+1} \|x_i\|^2 . && (4)
\end{aligned}$$

□

Problem 2.5

Let $S \in \mathbb{C}^{m \times m}$ be *skew-hermitian*, i.e., $S^* = -S$.

(a) Show by using Exercise 2.1 that the eigenvalues of S are pure imaginary.

Proof. We show this directly without using Exerciese 2.1. Suppose $S \in \mathbb{C}^{m \times m}$ be *skew-hermitian.* Then

$$S^* = -S. \tag{5}$$

Futher, suppose that λ is an eigenvalue of S with associated eigenvector x. Then, using Equation (5), we have

$$\begin{aligned} S^* &= -S \\ S^*x &= -Sx \\ S^*x &= -\lambda x \\ x^*Sx &= -\overline{\lambda}x^*x \\ x^*\lambda x &= -\overline{\lambda}x^*x \\ \lambda\left\|x\right\| &= -\overline{\lambda}\left\|x\right\| \\ \left\|x\right\|(\lambda+\overline{\lambda}) &= 0 \end{aligned}$$

Since $x \neq 0$, it must be that $\lambda = -\overline{\lambda}$. If $\lambda = a + bi$, then we have

$$\begin{aligned} a+bi &= -(a-bi) \\ a+bi &= -a+bi \end{aligned}$$

But this implies $a = 0$. Therefore, λ is pure imaginary, as was to be shown. $\square$

(b) Show that $I - S$ is non-singular.

Proof. Suppose to the contrary that $I - S$ is singular. Then there exists $x \in C^m$ such that $x \neq 0$, but $(I-S)x = 0$. Using this fact, we have

$$\begin{aligned} (I-S)x &= 0 \\ Ix - Sx &= 0 \\ x &= Sx \end{aligned}$$

But this means $\lambda = 1$ is an eigenvalue of S. By Part (a), this is a contradiction since all eigenvalues of S are pure imaginary. Therefore, $I - S$ is non-singular. $\square$

(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the *Cayley transform* of S, is unitary.

Proof. In this proof, we use the fact from Part (a) that $S^* = -S$. Note that by definition, a matrix A is unitary if $A^* = A^{-1}$. Thus, if Q is unitary, then $QQ^* = QQ^{-1} = I$. We prove this fact as follows.

$$\begin{aligned} QQ^* &= (I-S)^{-1}(I+S)[(I-S)^{-1}(I+S)]^* \\ &= (I-S)^{-1}(I+S)(I+S)^*(I-S)^{-*} \\ &= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1} \\ &= (I-S)^{-1}(I-S+S-S^2)(I+S)^{-1} \\ &= (I-S)^{-1}(-S+I)(S+I)(I+S)^{-1} \\ &= (I-S)^{-1}(I-S)(I+S)(I+S)^{-1} \\ &= II \\ &= I. \end{aligned}$$

$\square$

Problem 2.7

A *Hadamard matrix* is a matrix whose entries are all ± 1 and whose transpose is equal to its inverse times a constant factor. It is know that if A is a Hadamard matrix of dimension $m > 2$, then m is a multiple of 4. It is not know, however, whether there is a Hadamard matrix for every such m, though examples have been found for all cases $m \leq 424$.

Show that the following recursive description provides a Hadamard matrix of each dimension $m = 2^k, k = 0, 1, 2, \ldots$:

$$H_0 = \begin{bmatrix} 1 \end{bmatrix}, \quad H_{k+1} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix}.$$

Proof. We use repeatedly the property that, if A is a Hadamard matrix, then $A^T = aA^{-1}$ for some constant a. We prove the hypothesis by induction. As a base case, consider $k = 0$.

$$H_1 = \begin{bmatrix} H_0 & H_0 \\ H_0 & -H_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Here all entries are ± 1 as required. Moreover, we note that the columns of H_1 are linearly independent, which can be verified by taking the dot product of the columns to obtain $1 - 1 = 0$. Factoring out a constant $a = 2$ so that the columns are of unit length yields a constant times the unitary matrix $aQ = H_1$. By defnition, Q unitary means $Q^* = Q^{-1}$. Since Q contains only real values, we have $Q^T = Q^{-1}$. Hence

$$aQ^T = H_1^T \Rightarrow aQ^{-1} = H_1^T.$$

But this implies that $aQ^{-1} = aH_1^{-1}$.
Now assume that the recurrence has been verified to produce Hadamard matrices for all $m \times m$ matrices for $m = 2^{k-1}$ for $k > 1$. Then considering k we have by way of the recurrence,

$$H_{k+1} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix}$$

Since H_{k+1} is composed of linearly independent submatrices which are linearly independent, the columns of H_{k+1} are linearly independent. Thus we can apply the same procedure to show that $H_{k+1}^T = aH_{k+1}^{-1}$. That is, we can factor out a constant to obtain a unitary matrix whose transpose is the inverse to obtain $aQ^{-1} = aH_1^{-1} = H_1^T$. This fact combined with the fact that all entries of H_{k+1} are ± 1, means that H_{k+1} is a Hadamard matrix of size $m = 2^{k+1}$. Therefore, for dimensions $m = 2^k, k = 0, 1, 2, \ldots$, the recurrence produces Hadamard matrix.

$\square$