Math 510 HW 14 Michael Snyder

14.1 True or False?

(a) $\sin x = O(1)$ as $x \to \infty$.

Soln: By definition, we seek a constant C such that

$$|\sin x| \le C \cdot 1,$$

as $x \to \infty$. Since $-1 \le \sin x \le 1 \quad \forall x \in \mathbb{R}$, we may set $C = 1$. Then $|\sin x| \le 1$ holds, and we conclude that $\sin x = O(1)$ as $x \to \infty$. The statement is True.

(b) $\sin x = O(1)$ as $x \to 0$. Again choose $C = 1$, then for all x,

$$-1 \le \sin x \le 1 \Rightarrow |\sin x| \le 1,$$

Thus it is also true for $x \to 0$. The statement is True.

(c) $\log x = O(x^{1/100})$ as $x \to \infty$.

Consider $x > x^{99/100} \Rightarrow \frac{1}{x} < \frac{1}{x^{99/100}}$, for $x \to \infty$.

Integrating both sides of this inequality yields

$$|\log x| < 100\, x^{1/100} \quad \text{as } x \to \infty.$$

Let $C = 100$, then we have

$$|\log x| < C x^{1/100} \quad \text{as } x \to \infty.$$

Therefore, $\log x = O(x^{1/100})$ as $x \to \infty$. The statement is true.

(d) $n! = O((n/e)^n)$ as $n \to \infty$.

This statement is false. Consider Stirling's approximation to $n!$, which has

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Also one of the coolest approximations in math, in my opinion. Anyway, This implies That $\nexists$ a single constant c with $n! \leq c\left(\frac{n}{e}\right)^n$, since c would have to be dependent on n. The statement is False.

(e) ~~$A = O((n/e)^n)$~~

$A = O(V^{2/3})$ as $V \to \infty$, when A & V are surface area and volume of a sphere measured in square miles and cubic microns, respectively.

First note that $1 \text{ mi}^3 = 4.168 \times 10^{27} \text{ microns}^3$.

Thus, $V_{miles} = \frac{1 \text{ mi}}{4.168 \times 10^{27} \text{ microns}^3} V_{microns}$. Let $c = 4\pi a^{2/3}$ with $a = \frac{1}{4.168 \times 10^{27}}$. Then since $r > 0$, we have

$$\begin{aligned}
A_{miles} = 4\pi r^2 &\leq 4\pi\left(\frac{\pi}{3}\right)^{2/3} r^2 \\
&= 4\pi\left(\frac{\pi}{3} r^3\right)^{2/3} \\
&\leq 4\pi (a V_{microns})^{2/3} \\
&= 4\pi a^{2/3} V_{microns}^{2/3} \\
&= V_{microns}^{2/3}.
\end{aligned}$$

Thus $A = O(V^{2/3})$ as $V \to \infty$. The statement is True.

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(f) $\underline{fl(\pi) - \pi = O(\varepsilon_{machine})}$.

By definition of $fl(\pi)$, $\exists\ \varepsilon$ with $|\varepsilon| < \varepsilon_{machine}$ such that

$$fl(\pi) = \pi(1+\varepsilon).$$

Thus

$$|\varepsilon| < \varepsilon_{machine}$$

$$\Rightarrow \pi|\varepsilon| < \pi\varepsilon_{machine}$$

$$\Rightarrow |\pi\varepsilon + \pi - \pi| < \pi\varepsilon_{machine}$$

$$\Rightarrow |\pi(1+\varepsilon) - \pi| < \pi\varepsilon_{machine}$$

$$\Rightarrow |fl(\pi) - \pi| < \pi\varepsilon_{machine}.$$

That is, we have found a constant $C = \pi$ such that $|fl(\pi) - \pi| \leq C\,\varepsilon_{machine}$. Thus, $fl(\pi) - \pi = O(\varepsilon_{machine})$. The statement is <u>True</u>.

(g) $\underline{fl(n\pi) - n\pi = O(\varepsilon_{machine})}$ uniformly for all integers n.

<u>Solution</u>: By definition of uniformly $O(\varepsilon_{machine})$, we need a constant C such that

$$|fl(n\pi) - n\pi| \leq C\,\varepsilon_{machine}$$

$\forall\ n \in \mathbb{Z}$.

But $fl(n\pi) \Rightarrow \exists\ \varepsilon$ with $|\varepsilon| \leq \varepsilon_{machine}$ such that $fl(n\pi) = n\pi(1+\varepsilon)$. But this means,

$|\varepsilon| \leq \varepsilon_{machine}$

$\Rightarrow |n\pi\varepsilon| < n\pi\,\varepsilon_{machine}$ for $n \in \mathbb{Z}$, $n \geq 0$.

$\Rightarrow |n\pi\varepsilon + n\pi - n\pi| \leq n\pi\,\varepsilon_{machine}$

$\Rightarrow |n\pi(1+\varepsilon) - n\pi| \leq n\pi\,\varepsilon_{machine}$

$\Rightarrow |fl(n\pi) - n\pi| \leq n\pi\,\varepsilon_{machine}$.

Here $C = n\pi$ is dependent on n. Since uniformly $O(\varepsilon_{machine})$ is $\forall\, n \in \mathbb{Z}$, $\nexists\, C$ a single constant such that $|fl(n\pi) - n\pi| \leq C\,\varepsilon_{machine}$

The statement is <u>False.</u>