Numerical Linear Algebra: Homework #1

Due on August 31, 2022 at 10:00PM

Instructor: Professor Blake Barker
Section 1

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Problem 1.2

Suppose masses m_1, m_2, m_3, m_4 are located at positions x_1, x_2, x_3, x_4 in a line connected by springs with spring constants k_{12}, k_{23}, k_{34} whose natural lengths of extension are l_{12}, l_{23}, l_{34}. Let f_1, f_2, f_3, f_4 denote the rightward forces on the masses, e.g., $f_1 = k_{12}(x_2 - x_1 - l_{12})$.

(a) Write the 4x4 matrix equation relating the column vectors f and x. Let K denote the matrix in this equation.

Solution

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} -k_{12} & k_{12} & 0 & 0 \\ k_{12} & -(k_{12}+k_{23}) & k_{23} & 0 \\ 0 & k_{23} & -(k_{23}+k_{34}) & k_{34} \\ 0 & 0 & k_{34} & -k_{34} \end{pmatrix} + \begin{pmatrix} k_{12}l_{12} & 0 & 0 \\ k_{12}l_{12} & -k_{23}l_{23} & 0 \\ 0 & k_{23}l_{23} & -k_{34}l_{34} \\ 0 & 0 & k_{34}k_{34} \end{pmatrix}$$

(b) What are the dimensions of the entries of K in the physics sense.

Solution The entries of K are spring constants and have units of N/m or kg/s^2.

(c) What are the dimensions of $\det(K)$, again in the physics sense.

Solution The dimensions of $\det(K)$ are $(N/m)^4$ or $(kg/s^2)^4$.

(d) Suppose K is given numerical values based on the units meters, kilograms, and seconds. Now the system is rewritten with a matrix K' based on centimeters, grams, and seconds. What is the relationship of K' to K? What is the relationship of $\det(K')$ to $\det(K)$?

Solution Since $1kg = 1000g$, $K' = 1000K$ and $\det(K') = 1000^4 \det(K)$.

Problem 1.3

Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular.

Proof. Suppose R is a nonsingular $m \times m$ upper triangular matrix.

We list two useful relationships:

† The fact that R is nonsingular implies that $r_{ii} \neq 0$ for $1 \leq i \leq m$.

$\star$ The equation for (1.8) is $e_j = \sum_{i=1}^{m} z_{ij} r_i$, where z_{ij} is the ij-entry of $Z = R^{-1}$ and r_i is the i^{th} column of R.

We will use † and $\star$ to show by induction that $Z = R^{-1}$ is upper-diagonal. We begin with two base cases, R a 2×2 matrix and R a 3×3 matrix. In the 2×2 case, we have

$$RZ = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} \begin{pmatrix} z_{11} & r_{12} \\ z_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then using $\star$, we have

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = z_{11} \begin{pmatrix} r_{11} \\ 0 \end{pmatrix} + z_{21} \begin{pmatrix} r_{12} \\ r_{22} \end{pmatrix}$$

and

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = z_{12} \begin{pmatrix} r_{11} \\ 0 \end{pmatrix} + z_{22} \begin{pmatrix} r_{12} \\ r_{22} \end{pmatrix}$$

As has already been asserted, by †, $r_{22} \neq 0$. However, the second component of e_1, which we denote $(e_1)_2 = 0$. Thus, $z_{21} = 0$, and $Z_{2\times 2}$ is upper diagonal.

In the 3×3 case, we have

$$RZ = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Again, using $\star$, we have

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = z_{11} \begin{pmatrix} r_{11} \\ 0 \\ 0 \end{pmatrix} + z_{21} \begin{pmatrix} r_{12} \\ r_{22} \\ 0 \end{pmatrix} + z_{31} \begin{pmatrix} r_{13} \\ r_{23} \\ r_{33} \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = z_{12} \begin{pmatrix} r_{11} \\ 0 \\ 0 \end{pmatrix} + z_{22} \begin{pmatrix} r_{12} \\ r_{22} \\ 0 \end{pmatrix} + z_{32} \begin{pmatrix} r_{13} \\ r_{23} \\ r_{33} \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = z_{11} \begin{pmatrix} r_{11} \\ 0 \\ 0 \end{pmatrix} + z_{23} \begin{pmatrix} r_{12} \\ r_{22} \\ 0 \end{pmatrix} + z_{33} \begin{pmatrix} r_{13} \\ r_{23} \\ r_{33} \end{pmatrix}$$

From the 2×2 case, we already have $z_{21} = 0$. Using the same logic, we see that, since $(e_j)_i = 0$ for $i > j$ and $r_{ij} \neq 0$ for $j \geq i$, it is necessary that $z_{ij} = 0$ for $i > j$, $1 \leq i \leq 3$, $1 \leq j \leq 3$.

Applying induction, we now assume that $Z_{k\times k}$ is upper diagonal for $k > 3$, and that the result is verified for all matrices Z less than $k \times k$. Then since $r_{kk} \neq 0$ by †, we have $z_{kj} = 0$ for $k > j$. Otherwise, $(e_j)_k \neq 0$ for $k > j$. Therefore, the $m \times m$ matrix $Z = R^{-1}$ is an upper-diagonal matrix, as was to be shown.

□

Problem 1.4

Let $f_1, \ldots, f_2$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1, \ldots, 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.

Proof. Suppose to the contrary that $d_1, \ldots, d_8$ do not uniquely determine $c_1, \ldots, c_8$. Then $\exists\ a_1, \ldots, a_8$ such that

$$d_i = \sum_{j=1}^{8} a_j f_j(i) \quad i = 1, \ldots, 8.$$

But this means

$$0 = d_i - d_i = \sum_{i=1}^{8} (a_j - c_j) f_j(i), \quad i = 1, \ldots, 8.$$

But this implies $a_j = c_j$, contradicting our assumption that the d_i do not uniquely determine the c_j. Therefore, $d_1, \ldots, d_8$ uniquely determines $c_1, \ldots, c_8$. □

(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the ij-entry of A^{-1}?

Solution: The ij-entry of A^{-1} is $f_j(i)$, since if A maps data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$, then

$$\vec{c} = A\vec{d} \tag{1}$$

$$A^{-1}\vec{c} = A^{-1}A\vec{d} \tag{2}$$

$$A^{-1}\vec{c} = \vec{d} \tag{3}$$

which means A^{-1} is the matrix of functions $f_j(i)$.