Numerical Linear Algebra: Homework #7

Due on September 16, 2022 at 10:00PM

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Section 1

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Problem 7.3

Let A be an $m \times m$ matrix, and let a_j be its j^{th} column. Give an algebraic proof of *Hadmamard's inequality:*

$$|\det A| \leq \prod_{j=1}^{m} \left\|a_j\right\|_2 .$$

Also, give a geometric interpretation of this result, making use of the fact that the determinant equals the volume of a parallelepiped.

Proof. Let A be an $m \times m$ matrix, and let a_j be its j^{th} column. We will prove Hadamard's inequality using the following facts:

1. A has a QR factorization;
2. For matrices A and B, $\det AB = \det A \det B$;
3. By Theorem 5.6, $|\det A| = \prod \sigma_j$. Moreover, by Theorem 5.3, $\left\|A\right\|_2 = \sigma_1$. As a consequence of these two theorems, we have $|\det Q| \leq 1$;
4. $\det U = \prod u_{ii}$ for upper triangular U;
5. By algorithm 7.1, $r_{jj} = \left\|a_j\right\|_2$.

Using these facts, we have

$$\begin{array}{rcll} |\det A| & = & |\det QR| & \text{Fact 1} \\ & = & |\det Q \det R| & \text{Fact 2} \\ & \leq & |\det R| & \text{Fact 3} \\ & = & \prod_{j=1}^{n} r_{jj} & \text{Fact 4} \\ & = & \prod_{j=1}^{n} \left\|a_j\right\|_2 . & \text{Fact 5} \end{array}$$

□

Problem 7.4

Let $x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)}$ be nonzero vectors in $\mathbb{R}^3$ with the property that $x^{(1)}$ and $y^{(1)}$ are linearly independent and so are $x^{(2)}$ and $y^{(2)}$. Consider the two planes in $\mathbb{R}^3$,

$$P^{(1)} = \langle x^{(1)}, y^{(1)} \rangle, \qquad P^{(2)} = \langle x^{(2)}, y^{(2)} \rangle.$$

Suppose we wish to find a nonzero vector $v \in \mathbb{R}^3$ that lies in the intersection $P = P^{(1)} \cap P^{(2)}$. Devise a method for solving this problem by reducing it to the computation of QR factorizations of 3×2 matrices.

Solution: Let $A^{(1)} = \left[x^{(1)} \quad | \quad y^{(1)}\right]$, be the 3×2 matrix whose columns are the vectors $x^{(1)}$ and $y^{(1)}$. Similarly, form the matrix $A^{(2)} = \left[x^{(2)} \quad | \quad y^{(2)}.\right]$ Then

$$A^{(1)} = \left[x^{(1)} \quad | \quad y^{(1)}\right] = Q^{(1)}R^{(1)} = \left[q_1^{(1)} \quad | \quad q_2^{(1)} \quad | \quad q_3^{(1)}\right] R^{(1)},$$

where $Q^{(1)}R^{(1)}$ is the QR factorization of $A^{(1)}$ and hence by Algorithm 7.1, $q_1^{(1)}, q_2^{(1)}, q_3^{(1)}$ are the orthonormal columns of $Q^{(1)}$. But $range(A^{(1)})$ is spanned by the vectors $q_1^{(1)}, q_2^{(1)}$. Stated another way, $span(\{q_1^{(1)}, q_2^{(1)}\}) = P^{(1)}$. Since $q_3^{(1)}$ is orthogonal to $q_1^{(1)}$ and $q_2^{(1)}$, $q_3^{(1)}$ is also orthogonal to $P^{(1)}$. By similar analysis, we obtain, through QR factorization of the matrix $A^{(2)}$, the vector $q_3^{(2)}$ orthogonal to P^2.

Now consider the matrix $A^{(3)} = \left[q_3^{(1)} \quad | \quad q_3^{(2)}\right]$. By the same process as above we find the QR factorization of $A^{(3)}$, which yields the matrix $Q^{(3)} = \left[q_3^{(1)} \quad | \quad q_3^{(2)} \quad | \quad q_3^{(3)}\right]$. Again, $q_3^{(1)}$ and $q_3^{(2)}$ are orthogonal to $q_3^{(3)}$. But that puts $q_3^{(3)}$ in the intersection of the planes $P^{(1)}$ and $P^{(2)}$. Therefore, we have found the requested vector $v = q_3^{(3)}$ through QR factorization of 3×2 matrices.