Lecture 10 HW

Mike Snyder

10.2 (a) see attached code.

(b) see attached code

10.4 Consider the 2×2 orthogonal matrices

$$F = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}, \quad J = \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$

where $s = \sin\theta$ and $c = \cos\theta$ for some θ.

(a) Describe exactly what geometric effects left-multiplications by F and J have on the plane $\mathbb{R}^2$.

Solution: The matrix F takes $\binom{1}{0}$ accross the line $y = x$, by θ, the y-axis and $\binom{0}{1}$ across the line $y=x$, by θ. The matrix J is a counter-clockwise rotation of $\binom{1}{0}$ and $\binom{0}{1}$ by angle θ.

(b) Describe an algorithm for QR factorization that is analogous to Algorithm 10.1, but based on Givens rotations instead of Householder reflections.

Solution: We use a Givens rotation of angle θ, which we denote $J(\theta)$, to obtain the vector equivalent to $\|x\|_2 e_1$ in Algorithm 10.1. From Part (b), we know that J is a counterclockwise rotation, by angle θ. Thus, given

$$x = A_{k:m,k}$$

with angle

$$\theta = \cos^{-1}\left(\frac{x^* e_1}{\|x\|_2}\right),$$

from the x-axis,

we may rotate x by $-J(\theta)x$ to obtain $\|x\|_2 e_1$. Then we may form

$$v_k = \|x\|_2 e_1 - x$$
$$= -J(\theta)x - x.$$

The remainder of the algorithm to compute Q & R follow from the text, and we give them here.

$$\text{for } k = 1 \text{ to } n$$
$$x = A_{k:m,k}$$
$$\theta = \cos^{-1}\left(\frac{x^* e_1}{\|x\|_2}\right)$$
$$v_k = -J(\theta)x - x$$
$$v_k = v_k / \|v_k\|_2$$
$$A_{k:m,k:n} = A_{k:m,k:n} - 2v_k(v_k^* A_{k:m,k:n})$$

and using e_k for $k = n$ downto 1, we compute Q using

$$\text{for } k = n \text{ downto } 1$$
$$q_{k:m} = q_{k:m} - 2v_k(v_k^* q_{k:m}).$$

(c) Show that your algorithm involves six flops per entry operated on rather that 4, so that the asymptotic operation count is 50% greater than (10.9).

Solution: I am not sure where the explicit numbers 6 and 4 are coming from, but from my analysis,

$$v_k = -J(\theta)x - x$$

has n^2 multiplications and $n(n-1)$ additions for $-J(\theta)x$ and another n additions for $-J(\theta)x - x$. Thus, this computation has

$$n^2 + n(n-1) + n = 2n^2$$

flops.

Then

$$v_k = \text{sign}(x_1)\|x\|_2 e_1 + x$$

has $n+1$ multiplications for $\text{sign}(x)\|x\| e_1$, has $n-1$ additions for n multiplications and $n-1$ additions for $\|x\|_2$ and another n additions for $\text{sign}(x_1)\|x\|_2 e_1 + x$. Thus, we have

$$n+1 + n + n-1 + n = 4n$$

flops.

If you include the additional flops for computing θ, the Givens rotation method becomes even worse.