 Michael Snyder

16.1

(a) Let unitary matrices $Q_1, \dots, Q_k \in \mathbb{C}^{m\times m}$ be fixed and consider the problem of computing, for $A \in \mathbb{C}^{m\times n}$, the product $B = Q_k \cdots Q_1 A$. Let the computation be carried out from left to right by straightforward floating point operations on a computer satisfying (13.5) and (13.7). Show that this algorithm is backward stable.

<u>Proof</u>: Given the hypothesis, we have

$$f(A) = Q_k \cdots Q_1 A$$

and

$$\tilde{f}(A) = Q_k \cdots Q_1 A + Q_k \cdots Q_1 \delta A,$$

$$\cancel{\frac{\|Q_k \cdots Q_1 \delta A\|}{\|A\|} = \frac{\|\delta A\|}{\|A\|} = O(\varepsilon_m).}$$

for some $\delta A \in \mathbb{C}^{m\times n}$.

To show backward stability, we show that

$$\tilde{f}(A) = f(\tilde{A})$$

for some $\tilde{A}$ with

$$\frac{\|\tilde{A} - A\|}{\|A\|} = O(\varepsilon_m).$$

To this end, consider

$$\begin{aligned}\tilde{f}(A) &= Q_k \cdots Q_1 A + Q_k \cdots Q_1 \delta A \\ &= Q_k \cdots Q_1 (A + \delta A) \\ &= f(\tilde{A}).\end{aligned}$$

Moreover,

$$\frac{\|\tilde{A} - A\|}{\|A\|} = \frac{\|A + \delta A - A\|}{\|A\|} = \frac{\|\delta A\|}{\|A\|} = O(\varepsilon_m).$$

Therefore, $B = Q_k \cdots Q_1 A$ is backward stable.

(b) Give an example to show that this result no longer holds if the unitary matrices Q_j are replaced by arbitrary matrices $X_j \in \mathbb{C}^{m\times m}$.

Example Let $X_k \cdots X_1 = n(\delta A)^{-1}$, $n \in \mathbb{Z}^+$, Then

$$\tilde{f}(A) = X_k \cdots X_1 A + X_k \cdots X_1 \delta A$$

$$= X_k \cdots X_1 A + n(\delta A)^{-1}\delta A$$

$$= X_k \cdots X_1 A + n$$

$$\neq f(\tilde{A}),$$

where we have assumed $(\delta A)^{-1}$ exists, for this example.