Numerical Linear Algebra: Homework #3

Due on September 7, 2022 at 10:00PM

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Section 1

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Problem 3.3

Vector and matrix p-norms are related by various inequalities, often involving dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.
(a) $\|x\|_\infty \leq \|x\|_2$

Proof.

$$\begin{aligned} \|x\|_\infty &= \max_{1\leq i\leq m} |x_i| && (1) \\ &= \max_{1\leq i\leq m} \left(|x_i|^2\right)^{1/2} && (2) \\ &\leq \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} && (3) \\ &= \|x\|_2 && (4) \end{aligned}$$

Note that from (2) to (3) we use the fact that adding all elements smaller than the max x_i to x_i is larger, hence the inequality. □

As an example where the equality holds, consider e_1. In this case,

$$\|e_1\|_\infty = 1 = \|e_1\|_2$$

.
(b) $\|x\|_2 \leq \sqrt{m}\,\|x\|_2$

Proof.

$$\begin{aligned} \|x\|_2 &= \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} && (5) \\ &\leq \left(m \cdot \max_{1\leq i\leq m} |x_i|^2\right)^{1/2} && (6) \\ &= \sqrt{m} \max_{1\leq i\leq m} |x_i| && (7) \\ &= \sqrt{m}\,\|x\|_\infty && (8) \end{aligned}$$

Note: The expression in (6) takes m of the max x_i, hence is larger than the sum of the x_i. □

As an example, consider the vector containing all 1s. In this case,

$$\|x\|_2 = \sqrt{m} = \sqrt{m} \cdot 1 = \sqrt{m}\,\|x\|_\infty .$$

(c) $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$

Proof. We will use (3.6) from page 19 of the text to prove the claim; the definition for induced matrix norms, given here.

$$\|A\|_{(m,n)} = \sup_{x\neq 0} \frac{\|Ax\|_{(m)}}{\|x\|_{(n)}}$$

By Part (a)

$$\|Ax\|_\infty \leq \|Ax\|_2 .$$

By Part (b),

$$\|x\|_2 \leq \sqrt{n}\,\|x\|_\infty .$$

This implies

$$\frac{1}{\|x\|_\infty} \leq \frac{\sqrt{n}}{\|x\|_2}.$$

Putting the above relationships together, we have

$$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sup_{x\neq 0} \frac{\sqrt{n}\,\|Ax\|_2}{\|x\|_2} = \sqrt{n}\,\|A\|_2$$

Therefore, $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$

□

As an example, Consider $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Computing the ∞-norm for Ax and x yields:

$$\|x\|_\infty = \left\| \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|_\infty = \max\{1/\sqrt{2}, 1/\sqrt{2}\} = 1/\sqrt{2},$$

and

$$\|Ax\|_\infty = \left\| \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_\infty = 1$$

Thus,

$$\|A\|_\infty = \frac{\|Ax\|_\infty}{\|x\|_\infty} = \frac{1}{1/\sqrt{2}} = \sqrt{2}.$$

Computing the 2-norm for Ax and x yields:

$$\|x\|_2 = \left\| \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|_2 = \sqrt{\left(1/\sqrt{2}\right)^2 + \left(1/\sqrt{2}\right)^2} = 1,$$

and

$$\|Ax\|_2 = \left\| \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2 = 1$$

Thus,

$$\sqrt{2}\,\|A\|_2 = \sqrt{2}\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{2} \cdot \frac{1}{1} = \sqrt{2}.$$

Putting the computations together, we have

$$\|A\|_\infty = \sqrt{2} = \sqrt{2}\,\|A\|_2$$

(d) $\|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$

Proof. Again, (3.6) is used to prove the assertion. By Part (b),

$$\|Ax\|_2 \leq \sqrt{m}\,\|Ax\|_\infty .$$

Then by Part (a)

$$\frac{1}{\|x\|_2} \leq \frac{1}{\|x\|_\infty}.$$

Putting it all together yields,

$$\left\|A\right\|_2 = \sup_{x \neq 0} \frac{\left\|Ax\right\|_2}{\left\|x\right\|_2} \leq \sup_{x \neq 0} \frac{\sqrt{m}\left\|Ax\right\|_\infty}{\left\|x\right\|_\infty} = \sqrt{m}\left\|A\right\|_\infty .$$

Therefore,

$$\left\|A\right\|_2 \leq \sqrt{m}\left\|A\right\|_\infty .$$

$\square$

As an example, Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Computing the ∞-norm for Ax and x yields:

$$\left\|x\right\|_\infty = \left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|_\infty = \max\{1, 0\} = 1,$$

and

$$\left\|Ax\right\|_\infty = \left\|\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|_\infty = \left\|\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\|_\infty = 1$$

Thus,

$$\sqrt{2}\left\|A\right\|_\infty = \sqrt{2}\frac{\left\|Ax\right\|_\infty}{\left\|x\right\|_\infty} = \sqrt{2} \cdot \frac{1}{1} = \sqrt{2}.$$

Computing the 2-norm for Ax and x yields:

$$\left\|x\right\|_2 = \left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|_2 = \sqrt{(1)^2 + (0)^2} = 1,$$

and

$$\left\|Ax\right\|_2 = \left\|\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|_2 = \left\|\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\|_2 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus,

$$\left\|A\right\|_2 = \frac{\left\|Ax\right\|_2}{\left\|x\right\|_2} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

Putting the computations together, we have

$$\left\|A\right\|_2 = \sqrt{2} = \sqrt{2}\left\|A\right\|_\infty$$

Problem 3.4

Let A be an $m \times n$ matrix and let b be a submatrix of A, that is, a $\mu \times \nu$ matrix ($\mu \leq m, \nu \leq n$) obtained by selecting certain rows and columns of A.

(a) Explain how B can be obtained by multiplying A by certain row and column "deletion matrices" as in step 7 of Exercise 1.1.

Solution: The $\mu \times \nu$ matrix B can be obtained from the $m \times n$ matrix A by left multiplication by a $\mu \times m$ matrix L and a $n \times \nu$ matrix R. To remove row i from A, column i of L should have only zeros. Every column of L must contain exactly one 1. To remove column j from A, row j of R should have all zeros. Every row of R must contain exactly one 1.

(b) Using this product, show that $\left\|B\right\|_p \leq \left\|A\right\|_p$ for any p with $1 \leq p \leq \infty$.

Proof. Since L and R have columns and rows containing at most a single 1, the p-norms for L and R are at most 1, that is

$$\left\|L\right\|_p \leq 1, \qquad \left\|R\right\|_p \leq 1$$

Then using (3.14) from page 22 of the text, we have

$$\left\|B\right\|_p = \left\|LAR\right\|_p \leq \left\|L\right\|_p \left\|A\right\|_p \left\|R\right\|_p \leq 1 \cdot \left\|A\right\|_p \cdot 1 = \left\|A\right\|_p .$$

Therefore, $\left\|B\right\|_p \leq \left\|A\right\|_p$ □