Math 510 **Instructor: Barker** **Exam 1** **Fall 2022**

Name:____

Instructions:

You may use the official course text, your own notes taken in class, and any resources Dr. Barker has made available on BYU Learning Suite. You may use your own homework assignments. You may use Python code you do from scratch to explore ideas. You may not talk with anyone about the exam other than Dr. Barker. You may not use the internet. Please do not talk to classmates about the exam until the exams have been returned in class. **This exam is due at 10:00pm on Thursday, September 29th, 2022.** Please turn it in on Gradescope as an uploaded PDF. Your code should be included in the PDF. Please comment your code well. Please also turn in your python code in Gradescope.

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I have not given or received any unauthorized assistance on this exam.

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1. Let $A \in \mathbb{R}^{2\times 2}$ be a matrix that satisfies

$$\sup_{\|x\|_2=1} \|Ax\|_2 = 3, \qquad \inf_{\|x\|_2=1} \|Ax\|_2 = 2.$$

   What are the singular values of A?

2. Suppose that $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition (5.1) of the $2m \times 2m$ hermitian matrix

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

3. We have studied the QR and SVD decompositions. This problem develops a variant of the QR decomposition, the QL decomposition. An $m \times m$ matrix of the form

$$K_m = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} = [k_{ij}],$$

   where $k_{i,m-i+1} = 1$, $1 \le i \le m$, and all other entries are zero is termed a *reversal matrix* and sometimes the *reverse identity matrix*.

   (a) Show that $K_m^2 = I$ (a very handy feature).

   (b) If A is an $m \times n$ matrix, $m \ge n$, what is the action of $K_m A$? What about AK_n?

   (c) If R is upper triangular $n \times n$ matrix, what is the form of the product $K_n R K_n$?

   (d) Let $AK_n = \hat{Q}\hat{R}$ be the reduced QR decomposition of AK_n, $m \ge n$. Show that $A = \left(\hat{Q}K_n\right)\left(K_n\hat{R}K_n\right)$, and from that deduce the decomposition

$$A = QL,$$

   where Q is an $m \times n$ matrix with orthogonal columns, and L is an $n \times n$ lower triangular matrix. This is a reduced QL decomposition.

4. Use the result of Problem 3 to develop a python function lq that takes as input a matrix A and returns Q and L where $A = QL$ is a QL decomposition, and test it with random matrices of sizes 10×7 and 75×50. Organize your code in a single Python notebook and include it in the PDF you turn in. By test your code, I mean that you compute the QL decomposition, that you verify Q has orthonormal columns, you verify that L is lower triangular, and that you compute $\|A - QL\|_2$, where A is the random matrix for which you compute the QL decomposition. I should be able to run your single file code and observe that you have done each of these verifications. I should also be able to see this by reading the PDF you turn in. To verify that Q is orthogonal, you might just compute the maximum absolute value of the off-diagonal entries of Q^*Q, for example.