Numerical Linear Algebra: Exam #1

Due on October 1, 2022 at 5:00PM

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Section 1

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Problem 1

Let $A \in \mathbb{R}^{2\times 2}$ be a matrix that satisfies

$$\sup_{\|x\|_2=1} \|Ax\|_2 = 3, \qquad \inf_{\|x\|_2=1} \|Ax\|_2 = 2.$$

What are the singular values of A?

Solution:

Proof. By Theorem 4.1, A has an SVD. Let it be represented in the standard way, $A = U\Sigma V^*$. By Theorem 3.1, for any $A \in \mathbb{C}^{m\times n}$ and unitary $Q \in \mathbb{C}^{m\times m}$, $\|QA\|_2 = \|A\|_2$. Thus, $\|A\|_2 = \|U\Sigma V^*\|_2 = \|\Sigma\|_2$.
Using these relationships we find that

$$\dagger \quad 3 = \sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{\|x\|_2=1} \|\Sigma x\|_2 = \sup_{\|x\|_2=1} \sqrt{(\sigma_1 x_1)^2 + (\sigma_2 x_2)^2}.$$

By convention, singular values are arranged such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Hence the quantity $\sqrt{(\sigma_1 x_1)^2 + (\sigma_2 x_2)^2}$ is maximized for $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. But this means, completing the equalities in †, we have

$$\sup_{\|x\|_2=1} \sqrt{(\sigma_1 x_1)^2 + (\sigma_2 x_2)^2} = \sqrt{(\sigma_1 \cdot 1)^2 + (\sigma_2 \cdot 0)^2} = \sigma_1,$$

and thus, $\sigma_1 = 3$.

By similar logic, we have that $\sqrt{(\sigma_1 x_1)^2 + (\sigma_2 x_2)^2}$ is minimized for $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and thus,

$$2 = \inf_{\|x\|_2=1} \|Ax\|_2 = \inf_{\|x\|_2=1} \|\Sigma x\|_2 = \inf_{\|x\|_2=1} \sqrt{(\sigma_1 x_1)^2 + (\sigma_2 x_2)^2} = \sqrt{(\sigma_1 \cdot 0)^2 + (\sigma_2 \cdot 1)^2} = \sigma_2.$$

Thus, the singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$. □

Problem 2

Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition (5.1) of the $2m \times 2m$ hermitian matrix

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}.$$

Solution:

Proof. I can't seem to figure this one out... I've tried a few things. The thing that seemed the most promising was observing that if $B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$, then

$$B^*B = BB^* = BB = \begin{bmatrix} V\Sigma^2V^* & 0 \\ 0 & U\Sigma^2U^* \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} \Sigma^2 & 0 \\ 0 & \Sigma^2 \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix} = Q\Sigma^2Q^*$$

In a typical SVD $Q = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} = U_B$ would form the right singular vectors and $V_B = Q^*$ the left singular vectors. However, I cannot verify this, unless it is the case that $V = U$ in the SVD of A. If $V = U$, then

$$B = U_B\Sigma V_B^* = Q\Sigma Q^*$$

is an SVD of B. This would also be an eigenvalue decomposition of B.

□

Problem 3

We have studied the QR and SVD decompositions. This problem develops a variant of the QR decomposition, the QL decomposition. An $m \times m$ matrix of the form

$$K_m = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} = \left[k_{ij}\right],$$

where $k_{i,m-i+1} = 1$, $1 \leq i \leq m$, and all other entries are zero is termed a *reversal matrix* and sometimes the *reverse identity matrix.*

(a) Show that $K_m^2 = I$ (a very handy feature).

Proof. To avoid confusion in notation, for Part (a) we assume K is $m \times m$ and refer to the matrix K_m as K. Singular subscripts on K will refer to a columns of K, e.g., K_j is the j^{th} column of K. Double subscripts on K will refer to entries in K, i.e., K_{ij} is the ij-entry of K. With that out of the way, let $B = K^2$. Then by (1.6) from the text we have that the j^{th} column of B is

$$\star \quad b_j = \sum_{k=1}^{m} K_{kj} K_k,$$

that is, b_j is a linear combination of the columns of K whose coefficients are the j^{th} columns of K. But $K_{kj} = 0$ unless $k = m - j + 1$, in which case it equals 1. Thus, the formula $\star$ selects the $(m - j + 1)^{st}$ column of K_m for b_j. This selection effectively flips K_m across it's 'y-axis.' That is, $b_1 = K_{m-1+1} = K_m, b_2 = K_{m-2+1} = K_{m-2}, b_3 = K_{m-3+1} = K_{m-2}, \ldots, b_m = K_{m-m+1} = K_1$. The result is that

$$K_m^2 = I.$$

□

(b) If A is $m \times n$ matrix, $m \geq n$, what is the action of $K_m A$? What about AK_n?

Solution: $K_m A$ reflects A across it's 'x-axis.' This can be seen by considering the formula $\star$. The j^{th} column of $K_m A$ is given by $\sum_{k=1}^{m} a_{kj} K_j$, where again, K_j refers to the j^{th} column of K_m. Since the K_j column only has a one in it's $(m - j + 1)^{st}$ entry for $j = 1, \ldots, m$, this selects $a_{(m-k+1)j}$, that is, $b_j = a_{(m-k+1)j}$ for $k = 1, \ldots m$. As already stated, this is a reflection across A's ''x-axis.'

(c) If R is an upper triangular $n \times n$ matrix, what is the form of the product $K_n R K_n$?

Solution: As shown in Part (a), right multiplication by K_n results in a reflection of $K_n R$ across it's 'y-axis.' Thus, in $K_n R K_n$, we have a reflection first across R's x-axis, then that result is refelcted across it's y-axis.

(d) Let $AK_n = \hat{Q}\hat{R}$ be the reduced QR decomposition of AK_n, $m \geq n$. Show that $A = (\hat{Q}K_n)(K_n \hat{R} K_n)$, and from that deduce the decomposition

$$A = QL,$$

where Q is an $m \times n$ matrix with orthogonal columns, and L is an $n \times n$ lower triangular matrix. This is a reduced QL decomposition.

Solution:

Proof. We are given $AK_n = \hat{Q}\hat{R}$. By Part (a), $K_n^2 = I$. Moreover, since $K_n^* = K_n$, K_n is unitary. Using these facts, we have

$$\begin{aligned}
AK_n &= \hat{Q}\hat{R} \\
AK_n^2 &= \hat{Q}I\hat{R}K_n \\
A &= \hat{Q}K_nK_n\hat{R}K_n \\
A &= (\hat{Q}K_n)(K_n\hat{R}K_n)
\end{aligned}$$

By Part (c), $K_n\hat{R}K_n = L$ and by Part (a), $\hat{Q}K_n = Q$. Therefore,

$$A = (\hat{Q}K_n)(K_n\hat{R}K_n) = QL.$$

□

Problem 4

See attached code.