Math 510 L15

Mike Snyder

TOTAL POINTS

9 / 10

QUESTION 1

1 15.1 **9 / 10**

- **- 0 pts** Correct
- **- 1 pts** Turned in after the late deadline but before the grader started.
- **- 1 pts** a) backwards stable
- **- 1 pts** b) backwards stable
- **- 1 pts** c) stable not backwards stable
- ✓ **- 1 pts d) backwards stable**
- ✓ **- 0 pts e) not stable, not backwards stable**
- ✓ **- 0 pts f) stable, not backwards stable**
- ✓ **- 0 pts g) stable not backwards stable**

Math 510 HW 15 Mike Snyder

15.1 Each of the following problems describes an algorithm implemented on a computer satisfying axioms (13.5) and (13.7). For each one, state whether the algorithm is backward stable, stable but not backward stable, or unstable and prove it or give a convincing argument. Be sure to follow the definitions, as given in the text.

(a) Data: $x \in \mathbb{C}$. Solution: $2x$, computed as $x \oplus x$.

Proof: Let $x \in \mathbb{C}$, then for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \leq \varepsilon_{machine}$

$$\tilde{f}(x) = fl(x) \oplus f(x)$$
$$= [x(1+\varepsilon_1) + x(1+\varepsilon_2)](1+\varepsilon_3)$$
$$= x(1+\varepsilon_1)(1+\varepsilon_3) + x(1+\varepsilon_2)(1+\varepsilon_3)$$
$$= x(1+\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3) + x(1+\varepsilon_2+\varepsilon_3+\varepsilon_2\varepsilon_3)$$
$$= x(1+\varepsilon_4) + x(1+\varepsilon_5)$$
$$= f(\tilde{x})$$

for some $|\varepsilon_4|, |\varepsilon_5| \leq 2\varepsilon_{machine} + O(\varepsilon_{machine}^2)$

Thus we have $\tilde{f}(x) = f(\tilde{x})$, now take $\varepsilon = \max\{|\varepsilon_4|, |\varepsilon_5|\}$

Then
$$\frac{|\tilde{x} - x|}{|x|} = \frac{|x(1+\varepsilon) - x|}{|x|}$$
$$= \frac{|x + x\varepsilon - x|}{x}$$
$$= \varepsilon$$
$$= O(\varepsilon_{machine}).$$

Therefore, $\tilde{f}(x) = x \oplus x$ is backward stable.

(b) Data $x \in \mathbb{C}$. Solution $2x$ computed as $x \otimes x$.

Proof: Let $x \in \mathbb{C}$ and $\tilde{f}(x) = fl(x) \otimes fl(x)$. Let $\varepsilon_m := \varepsilon_{machine}$.

Then for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \leq \varepsilon_{machine}$ we have

$$
\begin{aligned}
\tilde{f}(x) &= fl(x) \otimes fl(x) \\
&= x(1+\varepsilon_1) \times (1+\varepsilon_2)(1+\varepsilon_3) \\
&= x(1+\varepsilon_1) \times (1+\varepsilon_2+\varepsilon_3+\varepsilon_2\varepsilon_3) \\
&= x(1+\varepsilon_1) \times (1+\varepsilon_4) \\
&= \tilde{x}^2
\end{aligned}
$$

for some $|\varepsilon_4| \leq 2\varepsilon_m + O(\varepsilon_m^2)$. Let $\varepsilon = \max\{|\varepsilon_1|, |\varepsilon_4|\}$.

Then for some $\tilde{x}$,

$$
\begin{aligned}
\frac{|\tilde{x} - x|}{|x|} &= \frac{|x(1+\varepsilon) - x|}{|x|} \\
&= |\varepsilon| \\
&= O(\varepsilon_m).
\end{aligned}
$$

Therefore, $\tilde{f}(x)$ is backward stable.

(c) Data: $x \in \mathbb{C} \setminus \{0\}$. Solution: 1, computed as $x \oplus x$.

Proof: Let $x \in \mathbb{C}$ and $\tilde{f}(x) = fl(x) \oslash fl(x)$. Let $\varepsilon_m := \varepsilon_{machine}$.

Then for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \leq \varepsilon_m$ we have

$$
\begin{aligned}
\tilde{f}(x) &= fl(x) \oslash fl(x) \\
&= [x(1+\varepsilon_1) \div x(1+\varepsilon_2)](1+\varepsilon_3) \\
&= \frac{x(1+\varepsilon_1)(1+\varepsilon_3)}{x(1+\varepsilon_2)} \\
&= \frac{(1+\varepsilon_4)}{(1+\varepsilon_2)}
\end{aligned}
$$

for some $|\varepsilon_4| \leq 2\varepsilon_m + O(\varepsilon_m^2)$.

The solution for the problem f was defined to be 1. Thus, $\tilde{f}$ is not backward stable.

However, we find that

$$\frac{|\tilde{f}(x) - f(x)|}{|f(x)|} = \frac{\left|\frac{1+\varepsilon_4}{1+\varepsilon_2} - 1\right|}{|1|}$$

$$= \left|\frac{1+\varepsilon_4 - (1+\varepsilon_2)}{1+\varepsilon_2}\right|$$

$$= \frac{|\varepsilon_4 - \varepsilon_2|}{|1+\varepsilon_2|}$$

$$= O(\varepsilon_m)$$

since $1+\varepsilon_2 \approx 1$ and $\varepsilon_4 - \varepsilon_2 \leq \varepsilon_m$,

since we also have, for $\varepsilon = \max\{|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|, |\varepsilon_4|\}$,

$$\frac{|\tilde{x} - x|}{|x|} = \frac{|x(1+\varepsilon) - x|}{|x|} = \varepsilon \leq \varepsilon_m,$$

$\tilde{f}$ is stable, just not backward stable.

(d) Data $x \in \mathbb{C}$. Solution: 0, computed as $x \ominus x$.

$\tilde{f}(x) = f(x) \ominus fl(x)$. Let $\varepsilon_m := \varepsilon_{machine}$.

Proof: Let $x \in \mathbb{C}$ and

Then, for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \leq \varepsilon_m$,

$$\tilde{f}(x) = [x(1+\varepsilon_1) - x(1+\varepsilon_2)](1+\varepsilon_3)$$

$$= x(1+\varepsilon_4) - x(1+\varepsilon_5)$$

for some $|\varepsilon_4|, |\varepsilon_5| \leq 2\varepsilon_m + O(\varepsilon_m^2)$.

But

$$x(1+\varepsilon_4) - x(1+\varepsilon_5)$$

$$= x + x\varepsilon_4 - x - x\varepsilon_5$$

$$= x(\varepsilon_4 - \varepsilon_5)$$

$$\neq 0.$$

$\tilde{f}$ is not backward stable. Moreover if we try to show that $\tilde{f}$ is stable, we find

$$\frac{|\hat{f}(x) - f(x)|}{|f(x)|} = \frac{|x(\varepsilon_4 - \varepsilon_5)|}{|0|}$$

which is undefined.

Therefore, $\tilde{f}$ is unstable.

(e) Data: none. solution: e, computed by summing $\sum_{k=0}^{\infty} \frac{1}{k!}$ from left to right using $\otimes$ and $\oplus$, stopping when a summand is reached of magnitude $< \varepsilon_{machine}$.

<u>solution</u>: By definition for backward stable and stable, this statement is vacuously true since these definitions depend on the condition "for all $x \in X$". Here X is the empty set. Therefore the statement holds, and $\tilde{f}$ is both backward stable & stable,

(f) Data: none. Solution: e, computed by the same algorithm as above except with the series summed from right to left.

<u>solution</u>: The explanation is the same as Part (e), the algorithm is vacuously backward stable & stable since there is no data.

(g) Data: none. solution: π, computed by doing an exhaustive search to find the smallest floating point number x in the interval $[3,4]$ such that $s(x) \otimes s(x') \leq 0$. Here $s(x)$ is an algorithm that calculates $\sin(x)$ stably in the given interval, and x' denotes the next floating point number after x in the floating point system.

<u>solution</u>: Again, since there is no data, this seems to be vacuously true in the same way that (e) and (f) were.

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