How to p-hack a robust result

Economists want to show that our results our robust. Consider Table 1 below: Column 1 contains the baseline model, with no covariates, and Column 2 controls for z. Because the coefficient on X is stable and significant across columns, we say that our result is robust.

Table 1: Robust results

	(1)	(2)
X	-0.375^{**}	-0.380^{**}
	(0.156)	(0.153)
z		1.076^{***}
		(0.159)
Observations	1,000	1,000
Adjusted R^2	0.005	0.048
Note:	*p<0.1; **p<0.05; ***p<0.01	

The twist: I p-hacked this result, using data where the true effect of X is zero.

In this post, I will show that, under certain conditions, it is easy to p-hack a robust result like this. The key is to use control variables that have a low partial-R^2. These variables have little effect on our main coefficient when excluded from the regression, and hence little effect when included. In contrast, controls with high partial-R^2 are more likely to kill a false positive, because they have more influence on the coefficient.

1 Setup

Let's see how this works. Consider data for $i = 1, ..., N$ observations generated according to

$$y_i = \sum_{k=1}^{K} \beta_k X_{k,i} + \gamma z_i + \varepsilon_i. \tag{1}$$

We have K potential treatment variables, $X_{1,i}$ to $X_{K,i}$, and a control variable z_i. I draw $X_{k,i} \sim N(0,1)$, $z_i \sim N(0,1)$, and $\varepsilon_i \sim N(0,1)$, so that $X_{k,i}$, z_i, and ε_i are all independent, but could be correlated in the sample. I set $\beta_k = 0$ for all k, so that X_k has no effect on y, and the true model is

$$y_i = \gamma z_i + \varepsilon_i. \tag{2}$$

I'm going to p-hack using the X_k's, running K regressions and selecting the k^* with the smallest p-value. I p-hack the baseline regression of y on X_k, by running K

regressions of the form

$$y_i = \alpha_{1,k} + \beta_{1,k} X_{k,i} + \nu_i. \tag{3}$$

I use the '1' subscript to indicate that this is the baseline model in Column 1. Out of these K regressions, I select the k^* with the smallest p-value on β_1. That is, I select the regression

$$y_i = \alpha_{1,k^*} + \beta_{1,k^*} X_{k^*,i} + \nu_i. \tag{4}$$

When $K \geq 20$, we expect $\hat{\beta}_{1,k^*}$ to have $p < 0.05$, since with a 5% significance level (i.e., false positive rate), the average number of significant results is $20 \times 0.05 = 1$. This is our p-hacked false positive.

To get a robust sequence of regressions, I need my full model including z to also have a significant coefficient on $X_{k^*,i}$. To test this, I run my Column 2 regression:

$$y_i = \alpha_{2,k^*} + \beta_{2,k^*} X_{k^*,i} + \gamma z_i + \varepsilon_i \tag{5}$$

Given that we p-hacked a significant $\hat{\beta}_{1,k^*}$, will $\hat{\beta}_{2,k^*}$ also be significant?

2 Homogeneous $\beta = 0$

First, I show a case where p-hacked results are not robust. I use the data-generating process from above with $\beta = 0$.

When regressing y on X_k in the p-hacking step, we have

$$y_i = \alpha_{1,k} + \beta_{1,k} X_{k,i} + \nu_i, \tag{6}$$

where

$$\begin{aligned} \nu_i &= \sum_{j \neq k}^{K} \beta_{1,j} X_{j,i} + \gamma z_i + \varepsilon_i \\ &= \gamma z_i + \varepsilon_i \end{aligned} \tag{7}$$

We estimate the slope coefficient as

$$\hat{\beta}_{1,k} = \frac{\widehat{Cov}(X_k, y)}{\widehat{Var}(X_k)} = \frac{\gamma \widehat{Cov}(X_k, z) + \widehat{Cov}(X_k, \varepsilon)}{\widehat{Var}(X_k)}. \tag{8}$$

Since $\beta = 0$, we should only find a significant $\hat{\beta}_{1,k}$ due to a correlation between X_k and the components of the error term ν_i:

1. $\gamma \widehat{Cov}(X_k, z)$

2. $\widehat{Cov}(X_k, \varepsilon)$

When $\gamma\widehat{Cov}(X_k, z)$ is the primary driver of $\hat{\beta}_{1,k}$, controlling for z in Column 2 will kill the false positive.

Turning to the full regression in Column 2, we get

$$\hat{\beta}_{2,k} = \frac{\widehat{Cov}(\hat{u}, y)}{\widehat{Var}(\hat{u})} = \frac{\widehat{Cov}((X_k - \hat{\lambda}_1 z), \varepsilon)}{\widehat{Var}(\hat{u})} = \frac{\widehat{Cov}(X_k, \varepsilon) - \hat{\lambda}_1 \widehat{Cov}(z, \varepsilon)}{\widehat{Var}(\hat{u})}. \tag{9}$$

This is from the two-step Frisch-Waugh-Lovell method, where we first regress X_k on z ($X_k = \lambda_0 + \lambda_1 z + u$) and take the residual $\hat{u} = X_k - \hat{\lambda}_0 - \hat{\lambda}_1 z$. Then we regress y on $\hat{u}$, using the variation in X_k that's not due to z, and the resulting slope coefficient is $\hat{\beta}_{2,k}$.[1] We can see that controlling for z literally removes the $\gamma\widehat{Cov}(X_k, z)$ term from our estimate.

Hence, to p-hack robust results, we want $\hat{\beta}_{1,k}$ to be driven by $\widehat{Cov}(X_k, \varepsilon)$, since that term is also in $\hat{\beta}_{2,k}$. If we have a significant result that's not driven by z, then controlling for z won't affect our significance.

2.1 Simulations

Setting $K = 20, N = 1000$, and $\gamma = 1$, I perform 1000 replications of the above procedure: I run 20 regressions, select the most significant X_{k^*} and record the p-value on $\hat{\beta}_{1,k^*}$, then add z to the regression and record the p-value on $\hat{\beta}_{2,k^*}$. As expected when using a 5% significance level, I find that out of the K regressions in the p-hacking step, the average number of significant results is 0.05. I find that $\hat{\beta}_{1,k^*}$ is significant in 663 simulations (=66%). But only 245 simulations (=25%) have both a significant $\hat{\beta}_{1,k^*}$ and a significant $\hat{\beta}_{2,k^*}$, meaning that only 37% (=245/663) of p-hacked Column 1 results have a significant Column 2. So in the $\beta = 0$ case, we infer that $\widehat{Cov}(X_k, \varepsilon)$ is small relative to $\gamma\widehat{Cov}(X_k, z)$. With these parameters, it's not easy to p-hack robust results.

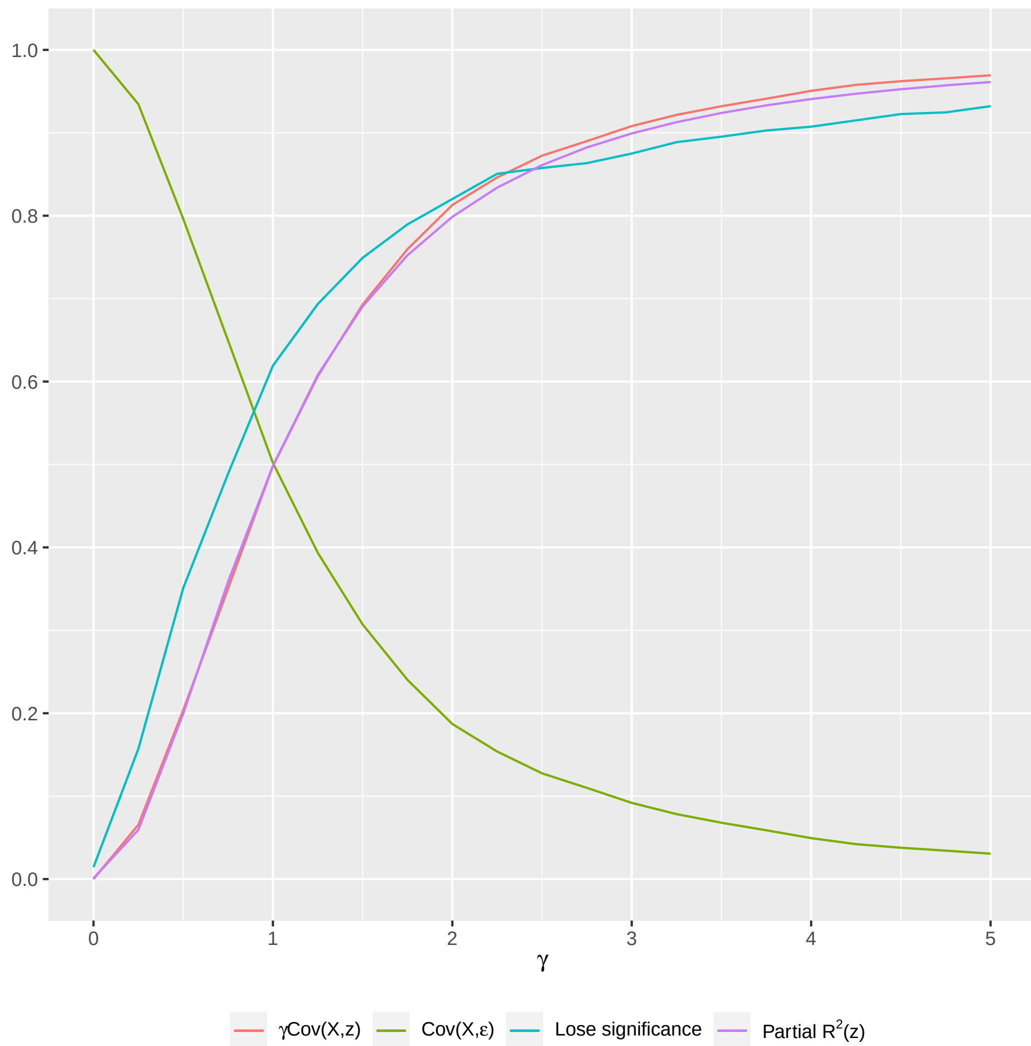
Figure 1 repeats this process for a range of γ's. I plot the shares of $\gamma\widehat{Cov}(X_k, z)$ and $\widehat{Cov}(X_k, \varepsilon)$ in $\hat{\beta}_{1,k}$.[2] We see that when $\gamma = 0$, $\gamma\widehat{Cov}(X_k, z)$ has 0 weight, but its share increases quickly. Closely correlated with this share is the fraction of significant results losing significance after controlling for z. Specifically, this is the fraction of simulations with a nonsignificant $\hat{\beta}_{2,k}$, out of the simulations with a significant $\hat{\beta}_{1,k}$. And even more tightly correlated with $\gamma\widehat{Cov}(X_k, z)$ is the partial R^2 of z.[3] Intuitively, as γ increases, the additional improvement in model fit from adding z also increases,

[1] $\widehat{Cov}(\hat{u}, y) = \widehat{Cov}(\hat{u}, \gamma z + \varepsilon) = \gamma\widehat{Cov}(\hat{u}, z) + \widehat{Cov}(\hat{u}, \varepsilon) = 0 + \widehat{Cov}(\hat{u}, \varepsilon)$, since the residual $\hat{u}$ is orthogonal to z.

[2] Note that these terms can be negative, so this is not strictly a share in $[0, 1]$. When the terms in the denominator almost cancel out to 0, we get extreme values. Hence, for each γ, I take the median share across all simulations, which is well-behaved.

[3] $R^2(z) = \frac{\sum \hat{u}_i^2 - \sum \hat{v}_i^2}{\sum \hat{u}_i^2}$, where $\hat{u}_i^2$ is the residual from the baseline model, and $\hat{v}_i^2$ is the residual from the full regression (where we control for z). In other words, partial $R^2(z)$ is the proportional reduction in the sum of squared residuals from adding z to the model.

Figure 1: Shares of $\hat{\beta}_{1,k}$, varying with γ



which by definition increases $R^2(z)$. Hence, $R^2(z)$ turns out to be a useful proxy for the share of $\gamma\widehat{Cov}(X_{k,i}, z_i)$, which we can't calculate in practice. Lesson: when partial-$R^2(z)$ is large, controlling for z is an effective robustness check for false positives. This is because a large $\gamma\widehat{Cov}(X_k, z)$ implies both (1) a large $R^2(z)$; and (2) that z is more likely to be the source of the false positive, and hence controlling for z will kill it. So now we have a new justification for including control variables, apart from addressing confounders: to rule out false positives driven by coincidental sample correlations.

3 Heterogeneous $\beta_i \sim N(0, 1)$

However, you might think that $\beta = 0$ is not a realistic assumption. As Gelman says: "anything that plausibly could have an effect will not have an effect that is exactly zero." So let's consider the case of heterogeneous β_i, where each individual i has their own effect drawn from $N(0, 1)$. For large N, the average effect of X on y will be 0, but this effect will vary by individual. This is a more plausible assumption than β being uniformly 0 for everyone. And as we'll see, this also helps for p-hacking, by increasing the variance of the error term.

Here we have data generated according to

$$y_i = \sum_{k=1}^{K} \beta_{k,i} X_{k,i} + \gamma z_i + \varepsilon_i, \tag{10}$$

where $\beta_{k,i} \sim N(0, 1)$.

Then, when regressing y on X_k, we have

$$y_i = \alpha_{1,k} + \delta_{1,k} X_{k,i} + v_i, \tag{11}$$

where

$$v_i = -\delta_{1,k} X_{k,i} + \beta_{k,i} X_{k,i} + \sum_{j \neq k}^{K} \beta_{j,i} X_{j,i} + \gamma z + \varepsilon_i. \tag{12}$$

When effects are heterogeneous (i.e., we have $\beta_{k,i}$ varying with i), a regression model with a constant slope $\delta_{1,k}$ is misspecified. To emphasize this, I include $-\delta_{1,k} X_{k,i}$ in the error term.[4]

The estimated slope coefficient is

$$\begin{aligned} \hat{\delta}_{1,k} &= \frac{\widehat{Cov}(X_{k,i}, y_i)}{\widehat{Var}(X_{k,i})} \\ &= \frac{\sum_{j=1}^{K} \widehat{Cov}(X_{k,i}, \beta_{j,i} X_{j,i}) + \gamma \widehat{Cov}(X_{k,i}, z_i) + \widehat{Cov}(X_{k,i}, \varepsilon)_i}{\widehat{Var}(X_{k,i})} \end{aligned} \tag{13}$$

[4]We could write $\beta_{k,i} = \bar{\beta}_{k,i} + (\beta_{k,i} - \bar{\beta}_{k,i}) := b_k + b_{k,i}$, and then have $y_i = \alpha_{1,k} + b_k X_{k,i} + v_i$, with $v_i = b_{k,i} X_{k,i} + \sum_{j \neq k}^{K} \beta_{j,i} X_{j,i} + \gamma z_i + \varepsilon_i$. However, $\hat{b}_k$ does not generally converge to $b_k = \bar{\beta}_{k,i}$, as I discuss below.

From Aronow and Samii (2015), we know that the slope coefficient converges to a weighted average of the $\beta_{k,i}$'s:

$$\hat{\delta}_{1,k} \rightarrow \frac{E[w_i \beta_{k,i}]}{E[w_i]}, \tag{14}$$

where w_i are the regression weights: the residuals from regressing X_k on the other controls. In this case, as we're using a univariate regression, the residuals are simply demeaned X_k (when regressing X on a constant, the fitted value is $\bar{X}$).

Because $\beta_{k,i} \sim N(0,1)$, we have $E[w_i \beta_{k,i}] = 0$ and hence $\hat{\delta}_{1,k}$ converges to 0. So any statistically significant $\hat{\delta}_{1,k}$ that we estimate will be a false positive.

There are three terms that make up $\hat{\delta}_{1,k}$ and could drive a false positive.

1. $\sum_{j=1}^{K} \widehat{Cov}(X_{k,i}, \beta_{j,i} X_{j,i})$
2. $\gamma \widehat{Cov}(X_k, z)$
3. $\widehat{Cov}(X_k, \varepsilon)$

Now we have a new source of false positives, case (1), due to heterogeneity in $\beta_{k,i}$. Note that controlling for z will only affect one out of three possible drivers, so now we should expect our false positives to be more robust to control variables, compared to when $\beta = 0$. To see this, note that when controlling for z in the full regression, we have

$$\begin{aligned}
\hat{\delta}_{2,k} &= \frac{\widehat{Cov}(\hat{u}_i, y_i)}{\widehat{Var}(\hat{u}_i)} \\
&= \frac{\sum_{j=1}^{K} \widehat{Cov}(X_{k,i} - \hat{\lambda}_1 z_i, \beta_{j,i} X_{j,i}) + \widehat{Cov}(X_{k,i} - \hat{\lambda}_1 z_i, \varepsilon_i)}{\widehat{Var}(\hat{u}_i)} \\
&= \frac{\sum_{j=1}^{K} \widehat{Cov}(X_{k,i}, \beta_{j,i} X_{j,i}) + \widehat{Cov}(X_{k,i}, \varepsilon_i)}{\widehat{Var}(\hat{u}_i)} - \hat{\lambda}_1 \frac{\left[\sum_{j=1}^{K} \widehat{Cov}(z_i, \beta_{j,i} X_{j,i}) + \widehat{Cov}(z_i, \varepsilon_i)\right]}{\widehat{Var}(\hat{u}_i)}
\end{aligned} \tag{15}$$

Here $\hat{u}$ is the residual from a regression of X_k on z: $X_k = \lambda_0 + \lambda_1 z + u$. We obtain $\hat{\delta}_{2,k}$ by regressing y on $\hat{u}$, via FWL, and using the variation in X_k that's not due to z.

Comparing $\hat{\delta}_{1,k}$ to $\hat{\delta}_{2,k}$, we see that $\sum_{j=1}^{K} \widehat{Cov}(X_{k,i}, \beta_{j,i} X_{j,i}) + \widehat{Cov}(X_{k_i}, \varepsilon_i)$ shows up in both estimates. Hence, if our p-hacking selects for a $\hat{\delta}_{1,k}$ with a large value of these terms, we're also selecting for the majority of the components of $\hat{\delta}_{2,k}$. In contrast to the $\beta = 0$ case, now we should expect $\gamma \widehat{Cov}(X_{k,i}, z_i)$ to be dominated, and significance in Column 1 should carry over to Column 2.

3.1 Simulations

I repeat the same procedure as before, running $K = 20$ regressions of y on X_k and z, taking the X_k with the smallest p-value, X_{k^*}, and then running another regression while excluding z. Again, I use $\gamma = 1$ and perform 1000 replications. Here I use robust standard errors to address heteroskedasticity.

I find that $\hat{\delta}_{1,k^*}$ is significant in 650 simulations (=65%). But this time, 569 simulations (=57%) have both a significant $\hat{\delta}_{1,k^*}$ and a significant $\hat{\delta}_{2,k^*}$. So 88% (=569/650) of p-hacked Column 1 estimates also have a significant Column 2. Compare this to 37% in the $\beta = 0$ case. That's what I call p-hacking a robust result! We infer that $\gamma\widehat{Cov}(X_{k,i}, z_i)$ is too small relative to the other components for its presence or absence to affect our estimates very much.

To illustrate how $\hat{\delta}_{1,k}$ is determined, I plot the shares of its three constituent terms while varying γ.[5] As shown in Figure 2, when γ is small, most of the weight in $\hat{\delta}_{1,k}$ is from $\sum_{j=1}^{K} \widehat{Cov}(X_{k,i}, \beta_{j,i}X_{j,i})$, indicating that its K terms provide ample opportunity for correlations with $X_{k^*,i}$. But as γ increases, this share falls, while the share of $\gamma\widehat{Cov}(X_{k,i}, z_i)$ rises linearly. The share of $\widehat{Cov}(X_{k,i}, \varepsilon_i)$ is small and decreases slightly. Looking at robustness, we see that the fraction of significant results losing significance rises much more slowly than in the $\beta = 0$ case. And we again see a tight link between partial-$R^2(z)$ and the share of z in $\hat{\delta}_{1,k}$.[6]

Overall, we can see why controlling for z is less effective with heterogeneous effects: $\hat{\delta}_{1,k}$ is mostly *not* determined by $\gamma\widehat{Cov}(X_{k,i}, z_i)$, so removing it (by controlling for z) has little effect. In other words, when variables have low partial-R^2, controlling for them won't affect false positives.

4 Conclusion

In general, economists think about robustness in terms of addressing potential confounders. I haven't seen any discussion of robustness to false positives based on coincidental sample correlations. This is possibly because it seems hopeless: we always have a 5% false positive rate, after all. But as I've shown, adding high partial-R^2 controls is an effective robustness check against p-hacked false positives.[7] So we have a new weapon to combat false positives: checking whether a result remains significant as high partial-R^2 controls are added to the model.

[5] Similar results hold when varying $Var(\beta_i)$ or $Var(\varepsilon)$.

[6] Note that the overall R^2 in Column 1 is irrelevant. For $\alpha = 0.05$, we will always have a false positive rate of 5% when the null hypothesis is true. Controlling for z is effective when $\gamma\widehat{Cov}(X_{k,i}, z_i)$ has a large share in $\hat{\delta}_{1,k}$. And a large share also means that $R^2(z)$ is large. This is true whether the overall R^2 is 0.01 or 0.99, since partial R^2 is defined in relative terms, as the decrease in the sum of squared residuals relative to a baseline model.

[7] Note that this holds regardless of which regression is p-hacked. Here, I've p-hacked the baseline regression. But the results are actually identical when you work backwards, p-hacking the full regression and then excluding z.

Figure 2: Shares of $\hat{\delta}_{1,k}$ and robustness