

基礎方程式:

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla P - \frac{1}{\text{Re}_0}(\mathbf{u} - U_y \hat{\mathbf{y}}) + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \sin(ny) \hat{\mathbf{x}}, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

$$\mathbf{U} = \frac{a}{4\pi^2} \int dx dy \mathbf{u} = (0, U_y) \quad : \text{const.}$$

新しい変数:

$$\mathbf{v} = \mathbf{u} - \mathbf{U} = \nabla \times (\psi \hat{\mathbf{z}}) = (\partial_y \psi) \hat{\mathbf{x}} + (-\partial_x \psi) \hat{\mathbf{y}}$$

変換した基礎方程式:

$$\begin{aligned}\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} &= -\nabla P - \frac{1}{\text{Re}_0} \mathbf{v} + \frac{1}{\text{Re}} \nabla^2 \mathbf{v} + \sin(ny) \hat{\mathbf{x}}, \\ \nabla \cdot \mathbf{v} &= 0, \\ \frac{a}{4\pi^2} \int dx dy \mathbf{v} &= 0\end{aligned}$$

$\nabla \times$  を掛けたい:

$$\begin{aligned}\nabla \times \mathbf{v} &= \nabla \times \nabla \times (\psi \hat{\mathbf{z}}) = \nabla(\nabla \cdot \psi \hat{\mathbf{z}}) - \nabla^2(\psi \hat{\mathbf{z}}) = (-\nabla^2 \psi) \hat{\mathbf{z}} \\ \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) &= J(\psi, \nabla^2 \psi) \hat{\mathbf{z}}, \\ \nabla \times (\mathbf{U} \cdot \nabla \mathbf{v}) &= \mathbf{U} \cdot \nabla(\nabla \times \mathbf{v}) = (-U_y \partial_y \nabla^2 \psi) \hat{\mathbf{z}}\end{aligned}$$

ただし

$$J(a, b) = (\partial_x a)(\partial_y b) - (\partial_y a)(\partial_x b)$$

基礎方程式に  $\nabla \times$  を掛ける:

$$-\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) - U_y \partial_y \nabla^2 \psi = \frac{1}{\text{Re}_0} \nabla^2 \psi - \frac{1}{\text{Re}} \nabla^2 \nabla^2 \psi - n \cos(ny)$$

整理:

$$\begin{aligned}\partial_t \nabla^2 \psi &= n \cos(ny) - \frac{1}{\text{Re}_0} \nabla^2 \psi - U_y \partial_y \nabla^2 \psi + \frac{1}{\text{Re}} \nabla^2 \nabla^2 \psi \\ &\quad + J(\psi, \nabla^2 \psi)\end{aligned}$$

$\omega = (-\nabla^2 \psi) \cdot \hat{\mathbf{z}} = (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{z}}$  において整理:

$$\begin{aligned}\partial_t \omega &= -n \cos(ny) - \frac{1}{\text{Re}_0} \omega - U_y \partial_y \omega + \frac{1}{\text{Re}} \nabla^2 \omega \\ &\quad + J(\psi, \omega)\end{aligned}$$

線形部分 + cos 部分:

$$\partial_t \omega = -n \cos(ny) - \frac{1}{\text{Re}_0} \omega - U_y \partial_y \omega + \frac{1}{\text{Re}} \nabla^2 \omega$$

Fourier 变换  $F[\omega](k, \ell)$ :

$$\partial_t F[\omega] = -F[n \cos(ny)] - \frac{1}{\text{Re}_0} F[\omega] - i\ell U_y F[\omega] - \frac{1}{\text{Re}}(k^2 + \ell^2) F[\omega]$$

整理:

$$\begin{aligned}\partial_t \{F[\omega]e(t)\} &= -F[n \cos(ny)]e(t), \\ e(t) &= \exp(t\alpha), \\ \alpha &= \frac{1}{\text{Re}_0} + i\ell U_y + \frac{1}{\text{Re}}(k^2 + \ell^2)\end{aligned}$$

积分:

$$\begin{aligned}F[\omega](t) &= F[\omega](t_0)e(-t + t_0) - e(-t) \int_{t_0}^t F[n \cos(ny)]e(s)ds \\ &= F[\omega](t_0)e(-t + t_0) - F[n \cos(ny)] \cdot \frac{1 - e(-t + t_0)}{\alpha} \\ &= \left( F[\omega](t_0) + \frac{F[n \cos(ny)]}{\alpha} \right) e(-t + t_0) - \frac{F[n \cos(ny)]}{\alpha}\end{aligned}$$