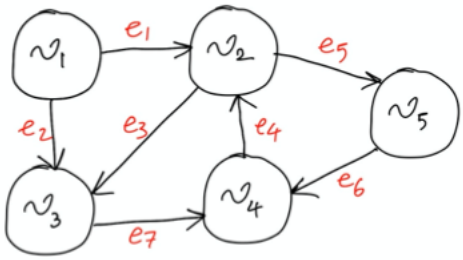
Graph Theory. Consider the following directed Graph $G(V, E)$.



(i) Find the adjacency matrix (call it A) and the incidence matrix (call it B) of this graph.

(ii) Remove the direction of the arcs in the graph, and find the adjacency matrix of the undirected version of this graph. Call the matrix C.

(iii) Find rank(B).

(iv) Show that $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, where $\vec{\mathbf{1}}$ is the (column) vector whose components are all equal to 1, and $\vec{\mathbf{0}}$ is the (column) vector whose components are all equal to 0. Based on $B^T\vec{\mathbf{1}} = \vec{\mathbf{0}}$, what can you conclude about eigenvalues and eigenvectors of BB^T?

(v) Set $L = BB^T$. Show that $L = D - C$, where $D = \text{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_5))$, namely, the degrees of vertices are on the main diagonal of D.

(vi) Find $\det(L)$ without direct calculation.

(vii) Without direct calculation, show that all the eigenvalues of L are nonnegative.

(viii) Without direct calculation, show that one of the eigenvalues of L is zero and the corresponding eigenvector is $\vec{\mathbf{1}}$.