Mathematical Statistics Part B: Tutorial Sheet Week 5

Preamble *The solutions to this tutorial will be available on Moodle on Friday at 4pm UK time.*

To enhance your learning, you are invited to attempt the solutions to the 2nd and 3rd exercise **before** the tutorial class.

1. Correction of the assessed ST219 Quiz 2 [max 10 minutes].
   The solution to the ST219 Quiz 2 will be available on Moodle on Wednesday afternoon at 1pm UK time.

2. Exercise 2.2.6 from the Lecture notes. I restate it here for convenience.
   Let $X = (X_1, \dots, X_n)$ consist of iid random variables with marginal cdf F_θ given by

$$F_\theta(x_i) := \mathbb{P}_\theta(X_i \le x_i) = \begin{cases} 0 & x_i \le 0 \\ x_i^\theta & x_i \in (0,1) \\ 1 & x_i \ge 1 \end{cases},$$

   $i = 1, \dots, n$. Let $\theta > 0$ be the unknown parameter of interest.

   (a) Write down the statistical model $(S, \{f_\theta : \theta \in \Theta\})$.
   (b) Derive a sufficient statistic for θ.
   (c) Compute
      a) the moment estimator $\hat{\theta}_{\mathrm{ME}}(X)$ of θ.
      b) the maximum likelihood estimator $\hat{\theta}_{\mathrm{MLE}}(X)$ of θ;
      c) an estimator $\hat{\theta}_1(X)$ of θ applying the theorem 2.2.3 for the 1 param. natural exponential family. To do this,
         i. check that X can be factorized as

$$f_\eta(x) = 1_{\{x \in A^n\}} \exp\left(\eta T(x) + d_0(\eta) + M(x)\right);$$

         with statistic $T(X) = \sum_{i=1}^n T(X_i)$, support A^n and $\eta = \theta$.
         ii. check that the hypotheses of the theorem are fulfilled (see the footnote);
         iii. find the estimator by solving $\mathbb{E}_\theta[T(X)] = T(x)$ or $d_0'(\theta) = T(x)$. **Remark**"We write $\mathbb{E}_\theta$ and $d_0'(\theta)$ instead of $\mathbb{E}_\eta$ and $d_0'(\eta)$ because $\theta = \eta$ in this example.

3. Continuation of the Exercise from the Tutorial in Week 3. Is the derived ME $\hat{\theta}_{\mathrm{ME}}(X) = 2/\bar{X}$ an unbiased and efficient estimator of θ? Hint: Note that $X_i \sim \Gamma(2, \theta)$, with shape parameter 2 and rate parameter θ, and use the properties of the Gamma distribution to compute mean and variance of $\bar{X}$ and $\hat{\theta}_{\mathrm{ME}}(X)$.