Martingale Argument for Concentration

The martingale argument is due to Lovett and Meka (2012). Consider $X_1(t), \ldots, X_n(t)$, n simple random walks (SRWs) on $\mathbb{Z}$. There are s stationary points and m moving points at any instant; $m + s = n$. The top s are frozen. We're interested in the maximum process $W(t)$:

$$W(t) := \sup_{k \leq t} \sup_{i \leq n} \{|X_i(k)|\} \tag{1}$$

Let the filtration $\mathcal{F}_t$ be "information about all the points up to time t. Then $X_i(t)$ is a martingale with respect to the filtration for each i because,

$$X_i(t) = X_i(t-1) + \begin{cases} 0 & \text{if frozen} \\ \pm 1 & \text{with equal probability if moving} \end{cases} \tag{2}$$

Let T be the final time in the martingale, and Doob's inequality gives us,

$$\mathbb{P}(\sup_{k \leq T} |X_i(k)| \geq a) \leq \int_{\sup_{k \leq T} |X_i(k)| \geq a} X_T d\mathbb{P} \tag{3}$$

However, the X_i are bounded by 1 in absolute value, and Azuma's inequality gives us

$$\mathbb{P}(\sup_{k \leq T} |X_i(k)| \geq u\sqrt{T}) \leq 2e^{-u^2/2}. \tag{4}$$

which is great. Now, define the random variable Y as follows:

$$Y := \#\{i | \sup_{k \leq T} |X_i(k)| \geq u\sqrt{T}\} \tag{5}$$

and its expectation is

$$E[Y] = n2e^{-u^2/2} \tag{6}$$

using (4) and linearity over indicators. The following set inclusion is easy

$$\{W(T) \geq u\sqrt{T} + 1\} \subset \{Y \geq s + 1\} \tag{7}$$

since one needs at least one point to be able to move when $s + 1$ are at $u\sqrt{T}$ to be able to get to $u\sqrt{T} + 1$. Then,

$$\mathbb{P}(W(T) \geq u\sqrt{T} + 1) \leq \frac{n}{s+1} \exp(-u^2/2) \tag{8}$$

gives us concentration for fixed n, s. This tells us that we ought to keep n/s constant if we're to send $n \to \infty$.

References

S. Lovett and R. Meka. Constructive discrepancy minimization by walking on the edges. *Arxiv preprint arXiv:1203.5747*, 2012.