Linear and Logistic Regression

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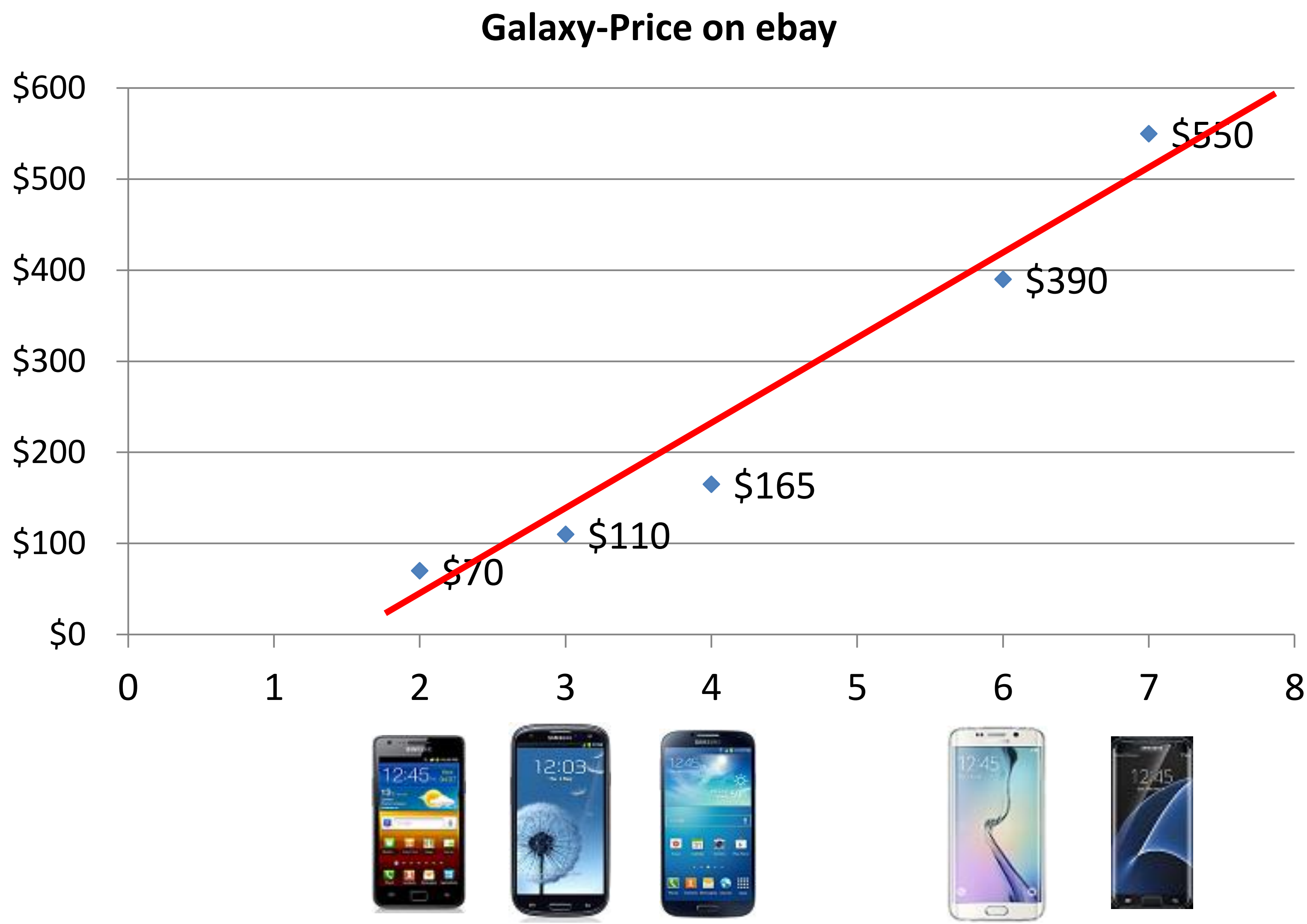
Estimating Galaxy-Phone Cost by Name



Formalizing Linear Regression

- y = wx + b
- What would we do if we had only 2 training examples?

We could solve 2 equations with 2 parameters

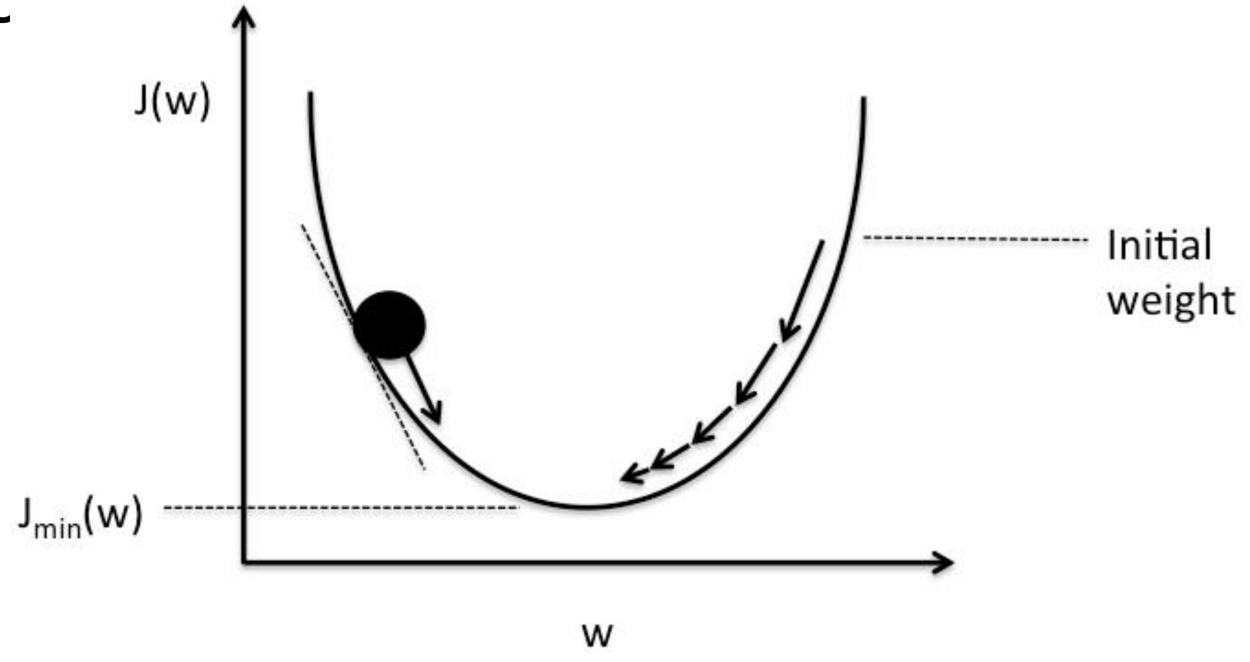
- Our prediction will be h(x) = wx + b
- $Y=\{y_1,y_2,y_3 \ldots y_m\}$, $X=\{x_1,x_2,x_3,\ldots x_m\}$
- Loss function: $J(w,b) = \frac{1}{m}\sum_{i=1}^{m} |wx_i + b - y_i|$
- Or: $J(w,b) = \frac{1}{2m}\sum_{i=1}^{m} (wx_i + b - y_i)^2$

  $$J(w,b) = \frac{1}{2m}\sum_{i=1}^{m} (h(x_i) - y_i)^2$$

- Find w, b that will minimize J(w,b)

Gradient Descent

- Nc
  - w,b) has
  - erivative.



Schematic of gradient descent.

sets of f

nt update

What is the gradient (∇)?

- A vector which represents the derivation of a function, which has multiple parameters.
- Each entry is the function's derivative with respect to one of the parameters.

$$f_1(j_1, j_2) = 2j_1 \cdot j_2 + 7j_1$$

$$\nabla(f_1(j_1, j_2)) = (2j_2 + 7, 2j_1)$$

$$f_2(j_1, j_2, j_3) = 3j_1^2 j_2 j_3^3 + 5j_1 j_2$$

$$\nabla(f_2) = (6j_1 j_2 j_3^3 + 5j_2, 3j_1^2 j_3^3 + 5j_1, 9j_1^2 j_2 j_3^2)$$

The Gradient for Linear Regression

Our loss function: $J(w,b) = \frac{1}{2m}\sum_{i=1}^{m}(wx_i + b - y_i)^2$

$$\frac{\partial J}{\partial w} = \frac{1}{2m}\sum_{i=1}^{m} 2((wx_i + b - y_i)x_i)$$

$$= \frac{1}{m}\sum_{i=1}^{m}(wx_i + b - y_i)x_i$$

$$\frac{\partial J}{\partial b} = \frac{1}{m}\sum_{i=1}^{m}(wx_i + b - y_i)$$

$$\nabla(J) = (\tfrac{1}{m}\sum_{i=1}^{m}(wx_i + b - y_i)x_i, \tfrac{1}{m}\sum_{i=1}^{m}(wx_i + b - y_i))$$

Gradient Descent in Linear Regression

- Pick random w, b
- Select the learning rate, α, (hyper-parameter), e.g. 0.01
- Repeat until convergence:
  - Update w to w-α$\frac{1}{m}\sum_{i=0}^{n} x_i(h(x_i) - y_i)$

    $$w_{next} = w_{curr} - \alpha\frac{1}{m}\sum_{i=0}^{n} x_i(w_{curr}(x_i) + b_{curr} - y_i)$$

  - Update b to b-α$\frac{1}{m}\sum_{i=0}^{n} 1 \cdot (h(x_i) - y_i)$

    $$b_{next} = b_{curr} - \alpha\frac{1}{m}\sum_{i=0}^{n} 1 \cdot (w_{curr}(x_i) + b_{curr} - y_i)$$

Linear Regression with SGD in Python

(Using the Galaxy Data-set)

```
import numpy as np
galaxy_data = np.array([[2,70],[3,110],[4,165],[6,390],[7,550]])
w = 0
b = 0
alpha = 0.01
for iteration in range(10000):
   deriv_b  = np.mean(1*((w*galaxy_data[:,0]+b)-galaxy_data[:,1]))
   deriv_w = np.mean(galaxy_data[:,0] *((w*galaxy_data[:,0]+b)-galaxy_data[:,1]))
   b -= alpha*deriv_b
   w -= alpha*deriv_w
   if iteration % 200 == 0 :
      print("it:%d,  grad_w:%.3f, grad_b:%.3f, w:%.3f, b:%.3f" %(iteration, deriv_w, deriv_b, w, b))
print("Estimated price for Galaxy S5: ", w*5 + b)
```

$$b_{next} = b_{curr} - \alpha \frac{1}{m} \sum_{i=0}^{n} 1 \cdot (w_{curr}(x_i) + b_{curr} - y_i)$$

$$w_{next} = w_{curr} - \alpha \frac{1}{m} \sum_{i=0}^{n} x_i (w_{curr}(x_i) + b_{curr} - y_i)$$

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   w -= alpha*deriv_w
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print("Estimated price for Galaxy S5: ", w*5 + b)
```

$$b_{next} = b_{curr} - \alpha \frac{1}{m} \sum_{i=0}^{n} 1 \cdot (w_{curr}(x_i) + b_{curr} - y_i)$$

$$w_{next} = w_{curr} - \alpha \frac{1}{m} \sum_{i=0}^{n} x_i (w_{curr}(x_i) + b_{curr} - y_i)$$

Value of b in a certain iteration

galaxy_data = np.array([[2,70],[3,110],[4,165],[6,390],[7,550]])

deriv_b = np.mean(1*((w*galaxy_data[:,0]+b)-galaxy_data[:,1]))

w=11 ; b=5

$$mean\left(11 \times \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} + 5 - \begin{bmatrix} 70 \\ 110 \\ 165 \\ 390 \\ 550 \end{bmatrix}\right) = mean \begin{bmatrix} 22+5-70 \\ 33+5-110 \\ 44+5-165 \\ 66+5-390 \\ 77+5-550 \end{bmatrix} = mean \begin{bmatrix} -43 \\ -72 \\ -116 \\ -319 \\ -468 \end{bmatrix} = -203.6$$

b -= alpha*deriv_b

b =5- 0.1*(-203.6) = 25.36

Value of w in a certain iteration

```
galaxy_data = np.array([[2,70],[3,110],[4,165],[6,390],[7,550]])
deriv_w = np.mean(galaxy_data[:,0] *((w*galaxy_data[:,0]+b)-galaxy_data[:,1]))
```

w=11 ; b=5

$$mean\left(\begin{bmatrix}2\\3\\4\\6\\7\end{bmatrix}\times\left(11\times\begin{bmatrix}2\\3\\4\\6\\7\end{bmatrix}+5-\begin{bmatrix}70\\110\\165\\390\\550\end{bmatrix}\right)\right)=mean\begin{bmatrix}2\\3\\4\\6\\7\end{bmatrix}\times\begin{bmatrix}11*2+5-70\\11*3+5-110\\11*4+5-165\\11*6+5-390\\11*7+5-550\end{bmatrix}=mean\left(\begin{bmatrix}2\\3\\4\\6\\7\end{bmatrix}\times\begin{bmatrix}-43\\-72\\-116\\-319\\-468\end{bmatrix}\right)=mean\begin{bmatrix}-86\\-219\\-464\\-357\\-3276\end{bmatrix}=-880.4$$

w -= alpha*deriv_w

w = 11 –0.1*(-880.4) = 99.04

Result:

it:0, grad_w:1464.000, grad_b:257.000, w:14.640, b:2.570

it:200, grad_w:3.825, grad_b:-19.693, w:70.598, b:-33.968

it:400, grad_w:2.859, grad_b:-14.720, w:77.230, b:-68.115

it:600, grad_w:2.137, grad_b:-11.003, w:82.188, b:-93.639

it:800, grad_w:1.597, grad_b:-8.224, w:85.893, b:-112.717

it:1000, grad_w:1.194, grad_b:-6.147, w:88.663, b:-126.977

it:1400, grad_w:0.667, grad_b:-3.434, w:92.280, b:-145.604

it:1800, grad_w:0.373, grad_b:-1.919, w:94.301, b:-156.010

it:2600, grad_w:0.116, grad_b:-0.599, w:96.062, b:-165.073

it:3600, grad_w:0.027, grad_b:-0.140, w:96.674, b:-168.226

it:4400, grad_w:0.008, grad_b:-0.044, w:96.802, b:-168.886

it:5400, grad_w:0.002, grad_b:-0.010, w:96.847, b:-169.116

it:6200, grad_w:0.001, grad_b:-0.003, w:96.856, b:-169.164

it:7000, grad_w:0.000, grad_b:-0.001, w:96.859, b:-169.179

it:7800, grad_w:0.000, grad_b:-0.000, w:96.860, b:-169.184

it:9800, grad_w:0.000, grad_b:-0.000, w:96.860, b:-169.186

Estimated price for Galaxy S5: 315.116281573

- Actual price was: **$250**
- Estimated price for Galaxy S1 is**: -72.22** ☹

- What would happen with alpha=0.5?

Adding Features

- Screen size
- Number of cores
- Core speed
- …

Adding Features (cont.)

- $x_1 = \{x_{11,}\ x_{12,}\ x_{13},\ldots,\ x_{1k}\}$
- $W = \{w_1, w_2, w_3,\ldots,w_k\}$
- Loss(W,b) = $\frac{1}{2m}\sum(y-(Wx+b))^2$

- Sometimes we set: $x_{i0} = 1$; w_0=b
  So that $x_i = \{x_{i0}\ ,\ x_{i1,}\ x_{i2,}\ x_{i3},\ldots,\ x_{ik}\}$
  => no need for "b"
- $W = \{w_0\text{=b}, w_1, w_2, \ldots\}$
- Loss = $\frac{1}{2m}\sum(y-Wx\ )^2$

Gradient Descent with Multiple Features (is actually the same…)

- ∇ is: $\sum_{i=0}^{n} \overleftarrow{x_i}\,(h(\overleftarrow{x_i}) - y_i)$
- Initialize W
- Repeat:
  - Calculate ∇: $\frac{1}{m}\sum_{i=0}^{n} \overleftarrow{x_i}\,(h(\overleftarrow{x_i}) - y_i)$

  Update: W = W - α ∇

Adding quadratic feature

- We can add a feature which is simply a square of the first feature.
- We will end up with 2 weights and one bias:
  - $w_1 x + w_2 x^2 + b$
- This way, instead of fitting a linear line, we are actually fitting a quadratic function (parabola).

Quadratic Feature

h(x) - y

```
import numpy as np
data_x = np.array([[2,4],[3,9],[4,16],[6,36],[7,49]])
data_y = np.array([70,110,165,390,550])
w1 = 0
w2 = 0
b = 0
alpha = 0.001
for iteration in range(1000000):
    deriv_b = np.mean(1*((w1*data_x[:,0]+w2*data_x[:,1]+b)-data_y))
    deriv_w1 = np.dot(((w1*data_x[:,0]+w2*data_x[:,1]+b)-data_y), data_x[:,0]) * 1.0/len(data_y)
    deriv_w2 = np.dot(((w1*data_x[:,0]+w2*data_x[:,1]+b)-data_y), data_x[:,1]) * 1.0/len(data_y)
    b -= alpha * deriv_b
    w1 -= alpha * deriv_w1
    w2 -= alpha * deriv_w2

print("Estimated price for Galaxy S5: ", np.dot(np.array([5,25]),np.array([w1, w2])) + b)
print("Estimated price for Galaxy S1: ", np.dot(np.array([1,1]),np.array([w1, w2])) + b)
```

Value of b in a certain iteration

```
data_x = np.array([[2,4],[3,9],[4,16],[6,36],[7,49]])
data_y = np.array([70,110,165,390,550])
deriv_b = np.mean(1*((w1*data_x[:,0]+w2*data_x[:,1]+b)-data_y))
```

w1=11; w2=7 ; b=5

$$mean\left(11 \times \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} + 7 \times \begin{bmatrix} 4 \\ 9 \\ 16 \\ 36 \\ 49 \end{bmatrix} + 5 - \begin{bmatrix} 70 \\ 110 \\ 165 \\ 390 \\ 550 \end{bmatrix}\right) = mean \begin{bmatrix} 22 + 28 + 5 - 70 \\ 33 + 63 + 5 - 110 \\ \ldots \\ \ldots \\ \ldots \end{bmatrix} = mean \begin{bmatrix} -15 \\ -9 \\ \ldots \\ \ldots \\ \ldots \end{bmatrix} = \ldots$$

```
b -= alpha * deriv_b
```

Value of w1 in a certain iteration

```
data_x = np.array([[2,4],[3,9],[4,16],[6,36],[7,49]])
data_y = np.array([70,110,165,390,550])
deriv_w1 = np.dot(((w1*data_x[:,0]+w2*data_x[:,1]+b)-data_y), data_x[:,0])* 1.0/len(data_y)
```

w1=11; w2=7 ; b=5

$$\frac{\left(11 \times \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} + 7 \times \begin{bmatrix} 4 \\ 9 \\ 16 \\ 36 \\ 49 \end{bmatrix} + 5 - \begin{bmatrix} 70 \\ 110 \\ 165 \\ 390 \\ 550 \end{bmatrix}\right) \times \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 22 + 28 + 5 - 70 \\ 33 + 63 + 5 - 110 \\ \dots \\ \dots \\ \dots \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{bmatrix}}{\text{len(data\_y)}} = \frac{-1350}{5}$$

w1 -= alpha * deriv_w1

w1 =0.001*(-270)= -0.27

Quadratic Feature (Weights as Vectors)

```
data_x = np.array([[2,4],[3,9],[4,16],[6,36],[7,49]])
data_y = np.array([70,110,165,390,550])

w = np.array([0.,0])
b = 0
alpha = 0.001
for iteration in range(1000000):
    deriv_b = np.mean(1*((np.dot(data_x,w)+b)-data_y))
    gradient_w = 1.0/len(data_y) * np.dot(((np.dot(data_x,w)+b)-data_y), data_x)
    b -= alpha*deriv_b
    w -= alpha*dradient_w

print("Estimated price for Galaxy S5: ", np.dot(np.array([5,25]),w) + b)
print("Estimated price for Galaxy S1: ", np.dot(np.array([1,1]),w) + b)
```

$$\nabla^{\vec{w}} = \left(\begin{bmatrix} 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 6 & 36 \\ 7 & 49 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \mathrm{b} - \begin{bmatrix} 70 \\ 110 \\ 165 \\ 390 \\ 550 \end{bmatrix} \right) \times \begin{bmatrix} 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 6 & 36 \\ 7 & 49 \end{bmatrix} = \begin{bmatrix} 2 * w_1 + 4 * w_2 + b - 70 \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 6 & 36 \\ 7 & 49 \end{bmatrix} = \begin{bmatrix} \nabla w_1 \\ \nabla w_2 \end{bmatrix}$$

Results

Prediction for Galaxy S5: $263.63

(actual: $250) 😊

Prediction for Galaxy S1: $71.81

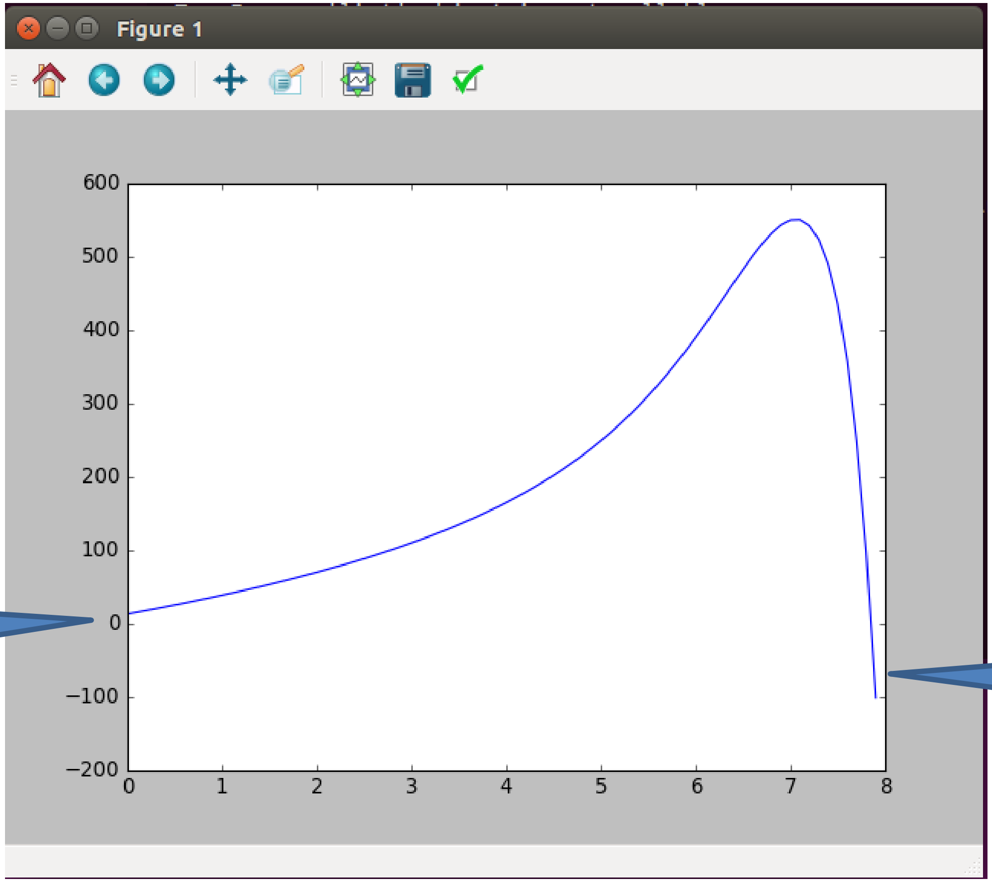
(actual: $30) 😔

Adding (too) Many Features

- We can add more and more features by adding additional polynomial terms (e.g. x^3, x^4, etc.)
- (We need to make sure not to overflow, so we should really normalize the values we get).
- Should we add 20 features? What can happen?

Over-Fitting

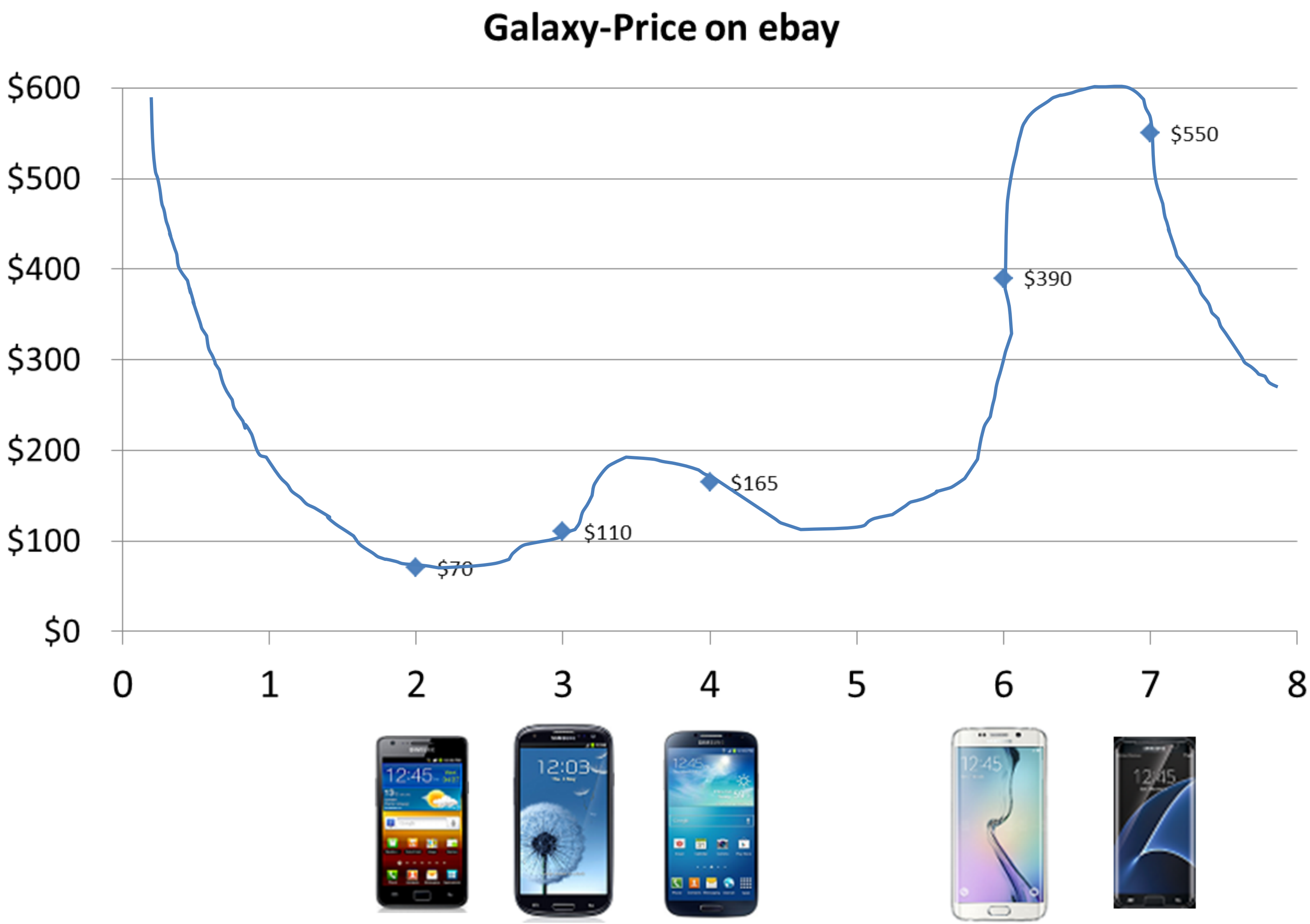
Loss = $5.3*10^{-7}$!!!



Bad prediction for 0

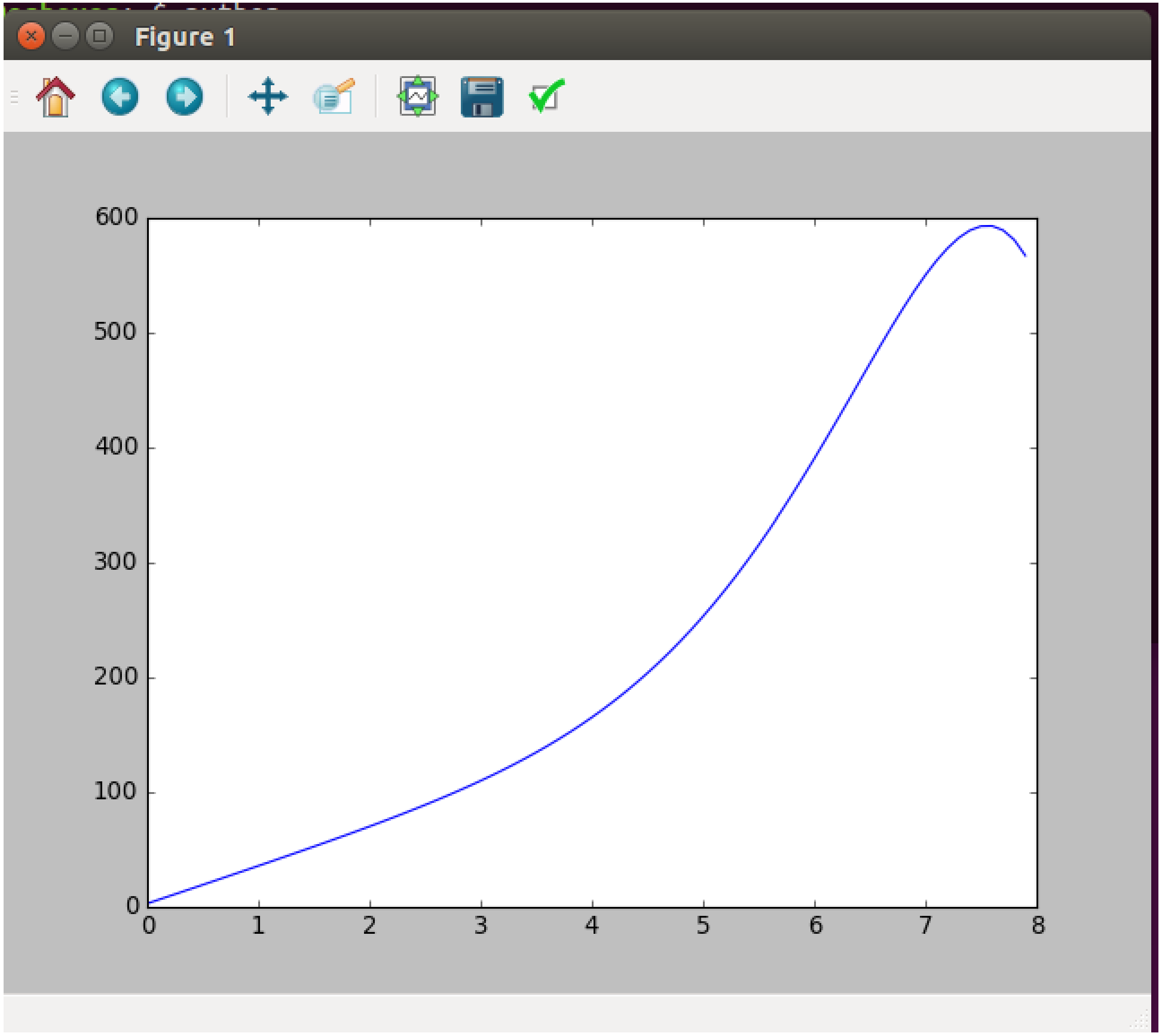
Horrible prediction for 8

Over-Fitting

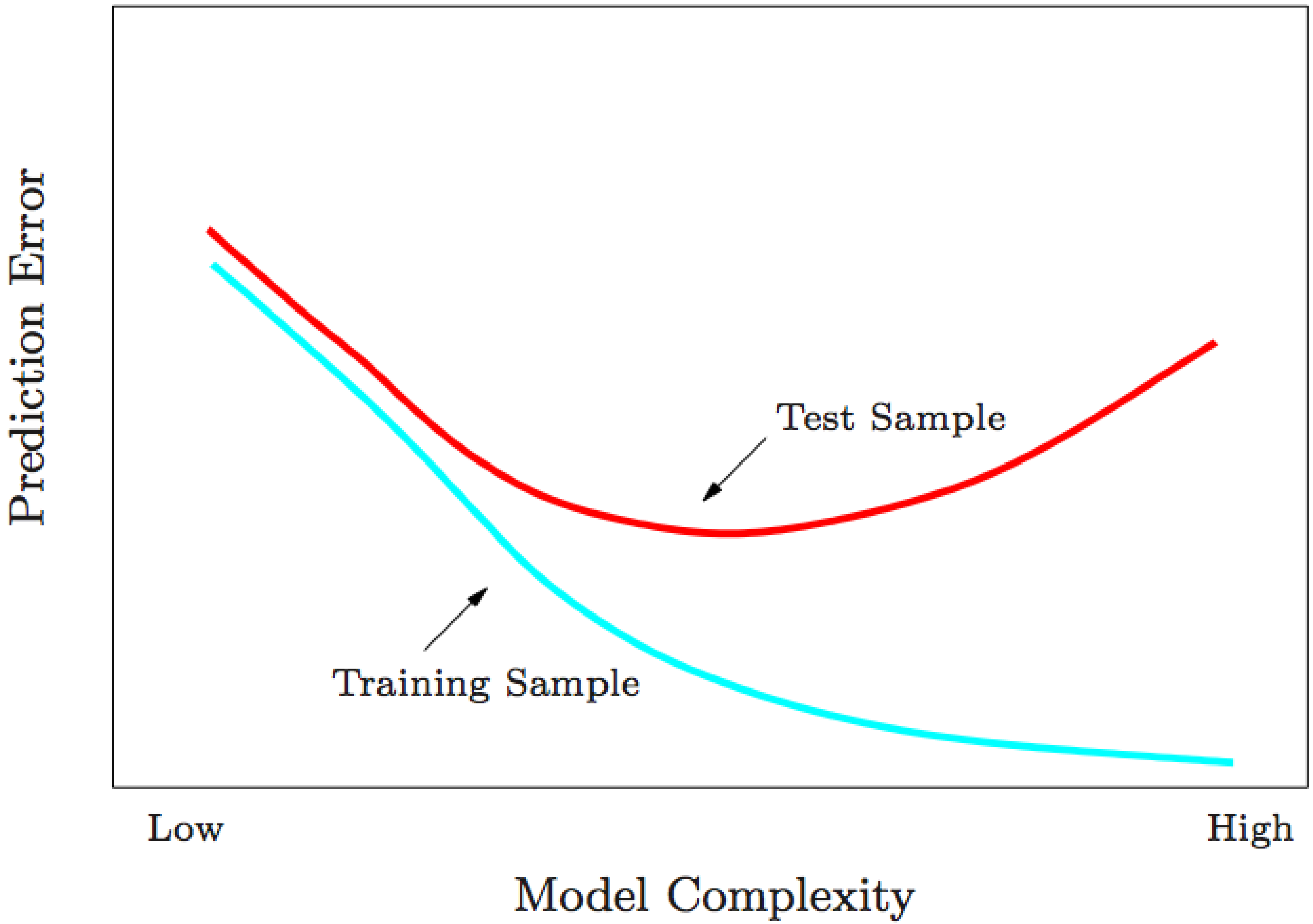


Results with "only" 10 features

Loss = 0.0049



Train/Test error



Credit: Collindcarroll.com

Train-Test-Validation

- If we want the test set to be a good prediction to what will happen with new data, it should be used only once.
- Therefore, we may want to have a validation set.
- The hyper-parameters / model complexity will be determined such that they maximize the accuracy of the validation set.

Closed form solution

- Linear regression has a closed-form solution:

$$W = (X^T X)^{-1} X^T Y$$

- We won't be focusing on it since we will be moving beyond linear regression to problems that do not have closed-form solution.
- Furthermore, the closed form solution may be less practical for big data.
- We will be using gradient descent (and its variants) until the end of the course.

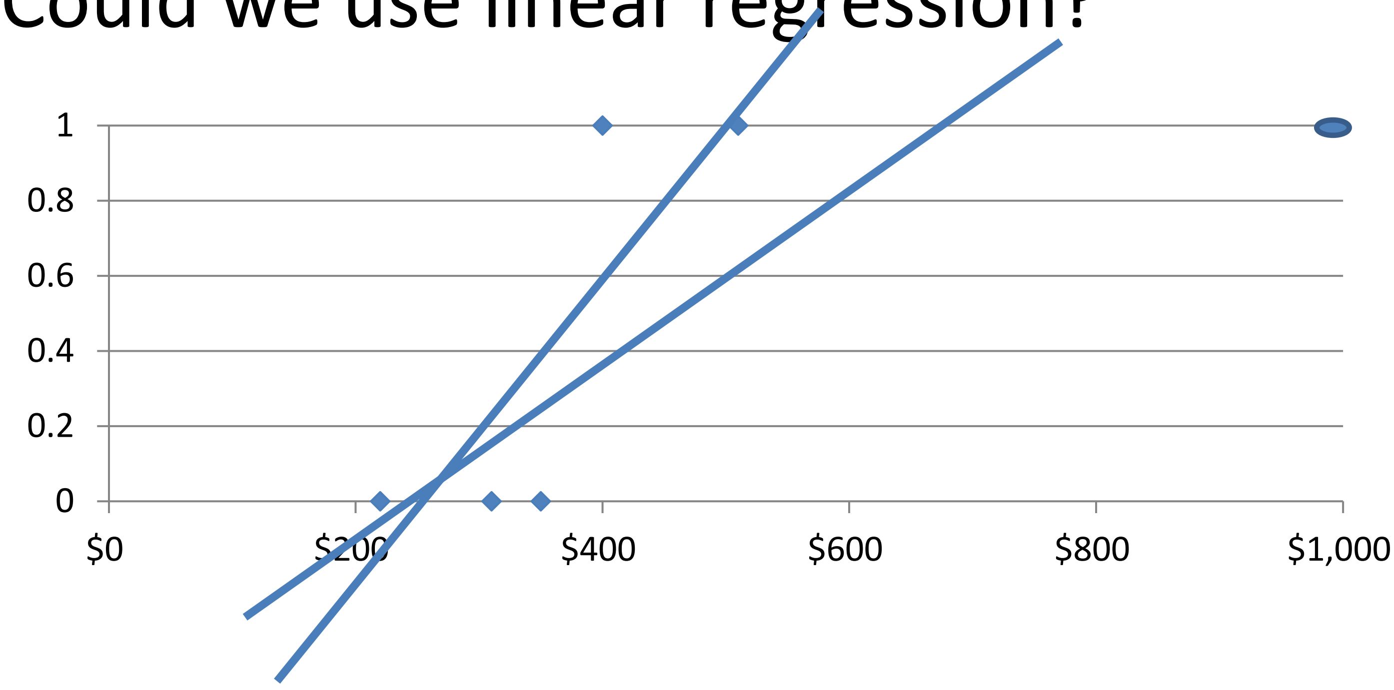
BGD, SGD, MB-GD

- Batch Gradient Descent (BGD): uses all Data-set to compute gradient (this is what we learnt).
  - May be too large, or take too long:
- Stochastic Gradient Descent (SGD): uses only a single example at a time (shuffle data first):
  - More iterations, since each iteration is less accurate
- Mini-Batch Gradient Descent (MB-GD): uses only a subset of the data-set, (e.g. 50) at a time:
  - A compromise which takes advantage of vectorization.

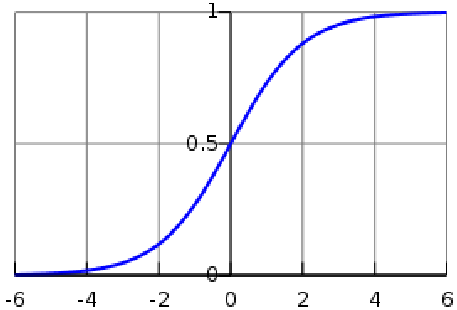
CLASSIFICATION

Classification

- Suppose we wanted to classify phones into new (1) or old/used (0), using the price as a single feature.
- Could we use linear regression?



Logistic Regression



- The Logistic function is:
  - $h(x) = \frac{1}{1+ e^{-(Wx+b)}}$
- A prediction of 1 will mean that we are **certain** that the value is 1.
- Instead of using least squares, we will use the following loss function:

This is actually the loss function we get when we try to maximize the likelihood of the data

$$\boxed{J(w,b) = -\frac{1}{m}\sum_{i=1}^{m}\left(y_i(log(h(x_i))) + (1-y_i)log(1-h(x_i))\right)}$$

- It turns out that the Gradient of the loss is:
  - $\frac{1}{m}\sum_{i=0}^{n} x_i(h(x_i) - y_i)$

Logistic Regression

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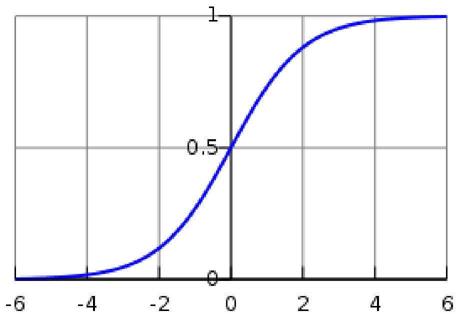

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$$J(w,b) = -\frac{1}{m}\sum_{i=1}^{m}\left(y_i(log(h(x_i))) + (1-y_i)log(1-h(x_i))\right)$$

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Gradient Descent in Logistic Regression

- Pick random w, b
- Select the learning rate, α, (hyper-parameter), e.g. 0.01
- Repeat until convergence:
  - Update w to w-α$\frac{1}{m}\sum_{i=0}^{n} x_i(h(x_i) - y_i)$
  - Update b to b-α$\frac{1}{m}\sum_{i=0}^{n} 1 \cdot (h(x_i) - y_i)$

How-come?

- Does the previous slide look familiar?
- The previous slide is identical to the slide we have seen in linear regression (I actually did a copy-paste and only changed the title).
- So how is logistic regression actually different than linear regression? If it is exactly the same algorithm, why not use linear regression?

Obviously, the algorithms are different because the hypothesis (h(x)) is totally different…

Employed or not?

- We have a data-base with all our users and we want to send out job-offers.
- For that, we need to know all unemployed users, though we only know this information on a fraction of the data.
- We would like to build a classifier that determines whether a user is employed or not, based on the user's age, gender and years of experience.

Our dataset

This is fake data, sorry about any gender/age biases introduced intentionally…

- Employed users:
  - Female, 28 years old, 4 years of experience
  - Female, 60 years old, 34 years of experience
  - Female, 25 years old, 3 year of experience
  - Male, 54 years old, 20 years of experience
  - Male, 24 years old, 2 years of experience
  - Male, 39 years old, 12 years of experience
  - Male, 30 years old, 4 years of experience
- Unemployed users:
  - Female, 36 years old 10 years of experience
  - Female, 26 years old 1 year of experience
  - Male, 44 years old, 9 years of experience

Do you think that a female, 49 year old with 8 years of experience employed?

What about a male, 29 years old with 3 years of experience?

And if it were a female?

```python
import numpy as np
data_x = np.array([[1,28,4],[1,60,34],[1,25,3],[0,54,20],[0,24,2],[0,39,12],[0,30,4],[1,3
data_y = np.array([1,1,1,1,1,1,1,0,0,0])

def h(x,w,b):
    return 1 / (1+np.exp(-(np.dot(x,w) + b)))

w = np.array([0.,0,0])
b = 0
alpha = 0.001
for iteration in range(100000):
    gradient_b = np.mean(1*(data_y-(h(data_x,w,b))))
    gradient_w = np.dot((data_y-h(data_x,w,b)), data_x)*1/len(data_y)
    b += alpha*gradient_b
    w += alpha*gradient_w

print("User [1, 49, 8] prob of working: ", h(np.array([[1, 49, 8]]),w,b))
print("User [0, 29, 3] prob of working: ", h(np.array([[0, 29, 3]]),w,b))
print("User [1, 29, 3] prob of working: ", h(np.array([[1, 29, 3]]),w,b))
```

$$h(x) = \frac{1}{1 + e^{-(Wx+b)}}$$

Results

User [1, 49, 8] prob of working: [0.21107079]

User [0, 29, 3] prob of working: [0.66430518]

User [1, 29, 3] prob of working: [0.43735087]

Meaning of Logistic Regression Result

- The result given by logistic regression is suppose to relate to the probability of the instance belonging to the class p(y=1 | X).
- Logistic regression is a discriminative model, that is, it tries to model p(y | X) directly.
- Remember that in the naïve Bayes classifier (which is a generative model), we used the Bayes rule, and therefore had to model:

  p(y) and p(X | y) (and p(X))

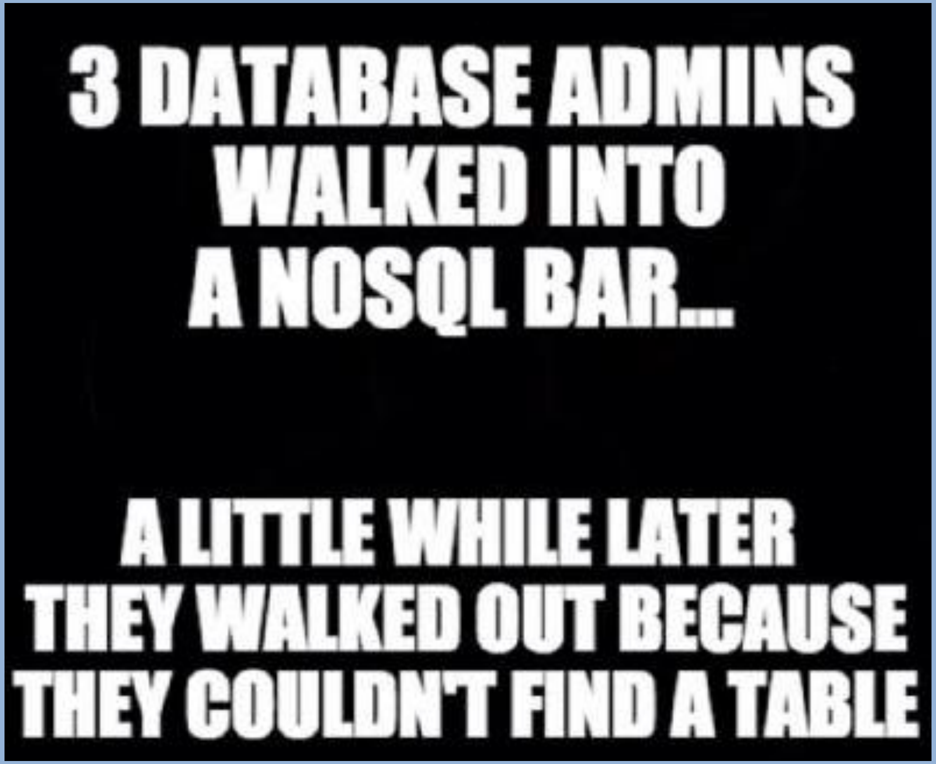
$$p(Y \mid X) = \frac{p(Y)p(X|Y)}{p(X)}$$

Multiple Classes

- Many times classification is into several labels. E.g. Classify an image to an object: cat/dog/airplane/sea/house
- We could build 5 different classifiers…
- More often we use one-hot vector representations for the labels. E.g.: cat = [1, 0, 0, 0, 0], sea = [0, 0, 0, 1, 0]
- **SoftMax**: as if we do logistic regression for each output alone and then scale:
    - $h(y = i \mid x) = \frac{e^{(W_i x + bi)}}{\sum_j^k e^{(W_j x + bj)}}$
  - Note that: $\sum_{i \in \{0,1,\ldots,k-1\}} h(y = i|x)$=1


THE
END
IS
NEAR

Closing notes



Closing notes

- Data and Database
- Sql databases
- Normalization
- XML & Json
- NoSql databases
- Java Streams & Spark
- Introduction to Machine Learning


THANK YOU FOR
LISTENING