Untitled: A Life Insurance forecasting tool

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Abstract

This will be an abstract.

Contents

Introduction

In response to the ever-increasing computing power and its accessibility, unsurprisingly, software is increasingly being deployed as tentative solution to simpify our lives, be it business or personal, to mixed results. Especially in the business side of things, it feels like the complexity has simply been moved to the software layer. The user is no longer required to understand the intricacies of the real-world process being modelled, but in turn is asked to be acquainted with the ever-changing intricacies of whatever software is being used, which might prove to be a questionable trade-off.

The actuarial world is no different, with companies trying to reap as many productivity gains from the transition to the ever-more-fashionable choice of third-party vendored software. These usually present an issue to the company making the transition: to accomodate as many users as possible the developers tend to pack them with features to the point of introducing needless complexity, forcing the company to invest serious capital (both time and money) to achieve the advertised productivity gains. Large operations can easily absorb such costs and even marginal productivity increases here and there can really add up in this scale. Moreover, large operations tend to pick up several quirks in their deals simply by virtue of the volume of business, which warrants the complexity in the software layer.

TODO: Last Paragraph

Chapter 1

Theoretical Framework

1.1 Survival Models

1.1.1 Mortality

(Pretty much this whole section is out of Dickson et al. (2020))
Let (x) denote a life aged x for $x \geq 0$.

Definition 1 *Let the random variable T_x denote the future lifetime of (x) such that $x + T_x$ is equal to its full lifespan. Define, also, the distribution function $F_x(t)$ which represents the probability that (x) will not survive more than t years, namely:*

$$F_x(t) = Pr(T_x \leq t) \tag{1.1}$$

In simpler terms, $F_x(t)$ is equal to the probability that (x) dies between ages x and t. Though this construct is incredibly important, it is often more useful for actuaries to think of it in terms of its counterpart, the **survival function**.

Definition 2 (Survival Function) *Define the survival function $S_x(t)$ to denote the probability that (x) survives t years:*

$$S_x(t) = Pr(T_x \geq t) = 1 - F_x(t) \tag{1.2}$$

Finally, to complete our model we make the following assumption (standard, more discussion on its validity in Dickson, Hardy and Waters (2020)):

$$Pr(T_x \geq t) = Pr(T_0 \geq x + t \mid T_0 > x) \tag{1.3}$$

Now equipped with this assumption and the following fact from basic probability:

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}.$$

we can assert a very convenient relationship:

$$Pr(T_x \geq t) = \frac{Pr(T_0 \geq x + t)}{Pr(T_0 > x)}.$$

Or, more conveniently expressed as:

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \quad \text{or} \quad S_0(x+t) = S_0(x)S_x(t) \tag{1.4}$$

This result is very important but it can be even further extended. Consider:

$$S_x(t+k) = \frac{S_0(x+t+k)}{S_0(x)} = \frac{S_0(x+t)}{S_0(x)}\frac{S_0(x+t+k)}{S_0(x+t)} = S_x(t)S_{x+t}(k) \tag{1.5}$$

In sum, equation (1.4) tells us that the probability of survival from birth to an age $x+t$ can be split up as the product of the probability of surviving from birth to x and the probability of surviving from x to $x+t$. Equation (1.5) expands on this result by lifting the requirement of the starting age being 0. We can, then, represent the probability of survival from age x to $x+k$ as the product of the probabilities of survival from x to $x+t$ and from $x+t$ to $x+k$ for some $t < k$.

1.1.2 International Actuarial Notation

The notation here onward used, unless otherwise noted, is in accordance with the so-called *International Actuarial Notation*. This set of conventions allows for a greatly increased expressivity in formulas and expressions while mostly maintaining their succintness, in exchange for a slight increase in complexity and recognizability. Appropriate notation will be introduced gradually in tandem with the associated concepts, but let us define the following as a foundation:

$$_tp_x = S_x(t) \tag{1.6}$$
$$_tq_x = 1 - S_x(t) = F_x(t) \tag{1.7}$$
$$_{k|t}q_x = {_kp_x}\ {_tq_{x+k}} = S_x(k) - S_x(k+t) \tag{1.8}$$

By convention, also, the right subscript may be dropped in the case that it is 1. For instance:

$$_1q_x = q_x \quad \text{and} \quad {_1p_x} = p_x.$$

1.1.3 Curtate Lifetime

So far our modelling of mortality, based on the random variable defined on section 1.1.1, has been continuous. However, to better represent the reality of premium/claim payments this random variable needs to be adapted to a discrete paradigm. Let this new random variable K_x represent the integer part of T_x, like:

$$K_x = \lfloor T_x \rfloor,$$

where "$\lfloor x \rfloor$" denotes the floor function. From the above relation we can, then, assert the following:

$$Pr(K_x = k) = Pr(k \leq T_x < k+1). \tag{1.9}$$

Which we can rewrite it using our newly developed notation:

$$\begin{aligned} Pr(K_x = k) &= Pr(k \leq T_x) - Pr(T_x > k+1) \\ &= {_kp_x} - {_{k+1}p_x} \\ &= {_kp_x}\ q_{x+k} \end{aligned}$$

, which culminates in the following definition.

Definition 3 (Curtate Lifespan) *Let the Curtate Lifespan $K_x = \lfloor T_x \rfloor$ be a discrete random variable representing the lifespan of a life (x) in integer years (rounded down), defined by the following distribution:*

$$Pr(K_x = k) = {}_k p_x \, q_{x+k} = {}_{k|}q_x \tag{1.10}$$

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1.1.4 Life Tables

TODO.

1.2 Discounting

The discounting model used here onwards is a pretty standard discrete period discounting. As a refresher: we can find the value of a cashflow of C t periods in the future by the following relation:

$$FV = C(1 + i_0)(1 + i_1) \ldots (1 + i_{t-1}) \tag{1.11}$$

where i_t is the interest rate for period t. We call this the *future value* of the cashflow. Usually, in this sort of model interest is assumed to be constant and yields the perhaps more recognizable formula:

$$FV = C(1 + i)^t,$$

where i is this fixed rate. For the sake of customization, however, this assumption will not be made prematurely, so Equation 1.11 will be used. Similarly, we can have a notion of *present value* as follows:

$$PV = \frac{C}{(1 + i_0) \ldots (1 + i_{t-1})} = C(v_0) \ldots (v_{t-1}) \tag{1.12}$$

, where the $v_t = (1 + i_t)^{-1}$. Up until now, notation has been fairly standard, but for the sake of brevity in the coming chapters let us make the following, admittedly non-standard, definition:

Definition 4 *Let v_t stand for the discounting factor for period t. Now, define ${}_t\mathcal{V}_x$ to be the discounting factor from time $x + t$ back to time x, such that:*

$${}_t\mathcal{V}_x = (v_x) \ldots (v_{t-1}) = \prod_{i=x}^{t-1} v_i \tag{1.13}$$

For a more thorough discussion of the Theory of Interest one can consult Kellison (2008).

1.3 Insurance

TODO.

1.4 Annuities

These products function much like their certain counterparts except, of course, for being contingent on the benefitiary being alive at the time of the payment, requiring us to do mortality discounting on top of interest discounting. Moreover, just like insurance, there is no way to calculate a present value since they are contingent on a random variable, so we can only deal with an expected present value.

1.4.1 Whole Life

Definition 5 *Let a_x denote the expected present value of a whole-life annuity immediate to a life (x), meaning every year (x) survives they will get a payment, starting a year from inception. Then:*

$$a_x = a_{\overline{K_x}|} = \sum_{k=x}^{\infty} \left({}_k\mathcal{V}_x\right) {}_kp_x.$$

1.4.2 Term

Definition 6 *Let $a_{x:\overline{n}|}$ denote the expected present value of a term life annuity immediate to a life (x), meaning every year (x) survives up to and including $x+n$ they will get a payment, starting a year from inception. Then:*

$$a_{x:\overline{n}|} = \begin{cases} a_{\overline{K_x}|}, & K_x < n \\ a_{\overline{n}|}, & K_x \geq n \end{cases} = \sum_{k=x}^{x+n} \left({}_k\mathcal{V}_x\right) {}_kp_x$$

1.5 Premiums

TODO.

1.6 Reserves

TODO (If there is enough time.)

Chapter 2

Methodology and Algorithms

2.1 Insurance

TODO.

2.2 Annuities

TODO.

2.3 Premiums

TODO.

2.4 Reserves

TODO (Contingent on 1.6)

Conclusion

Bibliography

Dickson, D. C. M., Hardy, M., & Waters, H. R. (2020). *Actuarial mathematics for life contingent risks*. Cambridge University Press.

Kellison, S. (2008). *Theory of interest* (3rd ed.). McGraw-Hill Education.