Notes on special functions

The following are work-in-progress personal notes on past and ongoing studies on special functions for personal interest and research.

1 Riemann zeta function

The **Riemann zeta function** is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for Re$z > 1$. The sum on odd terms can be related to the original function with

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^z} = (1 - 2^{-z})\zeta(z),$$

as can be shown by splitting the series into odd and even terms. Similarly, one can relate the **Dirichlet eta function** to the zeta function with

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = (1 - 2^{1-z})\zeta(z).$$

Even values of the zeta function are proportional to increasing powers of π. In particular, $\zeta(2) = \frac{\pi^2}{6}$, as can be shown by using the summation of series via the residue theorem. For the other even terms

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!},$$

where B_{2n} denotes Bernoulli's numbers.

2 Gamma function

The **Gamma function** on positive real numbers is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

It may be defined as a continuation of the factorial function $n!$ from $\mathbb{N}$ to $\mathbb{R}$: notice that $\Gamma(1) = 1$ and the recursion relation

$$\Gamma(x) = (x-1)\Gamma(x-1),$$

therefore $\Gamma(n) = (n-1)!$. This gives an easy way to compute its integer value. The half-integer value can be computed easily using

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2n)!}{4^n n!}$$

from a recursion relation.
The gamma function can be trivially analytically continued for positive complex numbers (Re$z > 0$)

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt.$$

However, this is not yet well defined for negative numbers, since for $t- > 0$ the power of t must be strictly greater than -1 for the integral to be finite. The analytical continuation to the whole complex plane can be performed by splitting the integration range in the following way

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \int_1^{+\infty} t^{z-1} e^{-t} dt = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{k+z-1} dt + \int_1^{+\infty} t^{z-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^{+\infty} t^{z-1} e^{-t} dt.$$

Now

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^{+\infty} t^{z-1} e^{-t} dt$$

is well defined in the complex plane except for the non-positive integers where there are simple poles. Equivalent to this one is another definition by Gauss

$$\Gamma(z) = \lim_n \frac{n^z n!}{z(z+1)\dots(z+n)}.$$

Using Stirling's approximation one can show with this the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+\frac{1}{2}).$$

We report an interesting integral involving the Gamma function (for Rez¿1).

$$\int_0^{+\infty} dt \frac{t^{z-1}}{e^t - 1} = \Gamma(z)\zeta(z)$$

2.1 Beta function

We define the **Euler's Beta function**

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

which is linked to the Gamma function by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This relation can be shown by writing explicitly the RHS with the definition of the Gamma function as a double integral. The result can be derived by writing the double integral in the region $\mathbb{R}^+ \times \mathbb{R}^+$ from cartesian coordinates to polar coordinates.

2.2 Digamma function

The logarithmic derivative of the Gamma function defines the **Digamma function**

$$\psi(x) = \frac{d \log \Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Using the definition of the Gamma function we get the integral representation

$$\psi(x) = \frac{1}{\Gamma(x)} \int_0^{+\infty} t^{x-1} e^{-t} \log t \, dt.$$

The recursion relation of the Gamma function implies the similar recursion for the Digamma

$$\psi(x+1) = \psi(x) + \frac{1}{x},$$

hence for integer values

$$\psi(n+1) = \frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{2} + 1 + \psi(1) = H_n + \psi(1),$$

where H_n is the n-th harmonic number. For this reason, we could also define the Digamma function as a continuation of the harmonic function $H(n)$ to real numbers, translated by $\psi(1) = -\gamma_E$, called Euler's constant. Hence, for integer value the function is

$$\psi(n) = -\gamma_E + \sum_{k=1}^{n-1} \frac{1}{k},$$

while for real values, this generalizes to the series representation of the Digamma function

$$\psi(x) = -\gamma_E - \sum_{k=0}^{+\infty} \left(\frac{1}{x+k} - \frac{1}{1+k} \right).$$

The properties of the Digamma function give an expansion of the logarithm of the Gamma function around 1

$$\log \Gamma(1+x) = -\gamma_E x + \sum_{n=2}^{+\infty} \frac{(-1)^n x^n}{n} \zeta_n.$$

3 Polylogarithms

We define on real numbers a family of functions called **Polylogarithms** with the following series expansion

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

For $n = 1$ we have $\log x = -\log(1-x)$. This definition only holds for $|x| < 1$, that is the radius of convergence of the power series. These functions can be analytically continued to the whole complex plane by defining them in terms of the differential equations

$$\begin{cases} \frac{d}{dz} \log z = \frac{1}{z} \\ \log(1) = 0 \end{cases}$$

in the case of the logarithm, and, recursively, in the $n \geq 2$ case

$$\begin{cases} \frac{d}{dz} \mathrm{Li}_n(z) = \frac{1}{z} \mathrm{Li}_{n-1}(z) \\ \mathrm{Li}_n(0) = 0 \end{cases} .$$

As the logarithm, also polylogarithms have a branch cut that is from 1 to ∞. We can show this by computing the discontinuity along the real axis from the defining differential equation.

A Proofs

A.1 The Basel problem

We can show that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\pi^2}{6}$$

using that

$$\sum_{n \in \mathbb{Z}/\text{poles}} f(n) = -\sum_k \mathrm{Res}[f(z)\pi \cot(\pi z), z = z_k].$$

In this case, the formula reduces to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n \in \mathbb{Z}/0} f(n) = -\frac{1}{2} \mathrm{Res}\left[\frac{\pi \cot \pi z}{z^2}, z = 0\right] = \frac{\pi^2}{6},$$

where the residue can be most effectively calculated by computing the Laurent expansion of the function to the desired order. Notice that through this method one can calculate also the other even values of the zeta function.

A.2 Even values of Riemann zeta

The previous formula can be generalized to all even terms by expanding the cotangent up to the desired order. A closed formula can be obtained in terms of the Bernoulli numbers, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m^- x^m}{x!},$$

where $B_m^- = \left\{1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots\right\}$. Using series evaluation by residue theorem we have

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = -\frac{\pi}{2} \mathrm{Res}\left[\frac{\cot \pi z}{z^{2n}}, z = 0\right].$$

The cotangent can be expanded in terms of Bernoulli numbers. First, we rewrite it as

$$\cot z = i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = \frac{i}{2iz} \left[\frac{-2iz}{e^{-2iz} - 1} + \frac{2iz}{e^{2iz} - 1}\right],$$

then we use the series expansions and add together the two series by singling out the $m = 1$ terms which cancel between the two, thus leaving only the even terms. This amounts to

$$\cot z = \frac{1}{z} \sum_{m=0}^{\infty} (-1)^m \frac{B_{2m} (2z)^{2m}}{(2m)!}.$$

Then to get the residue we simply let $m = n$. Taking care of all the factors we find the desired formula.

A.3 Half-integer values of the Gamma function

We evaluate $\Gamma(n+\frac{1}{2})$ using the substitution $t=s^2$

$$\Gamma\left(n+\frac{1}{2}\right)=\int_0^{+\infty} t^{n-\frac{1}{2}}e^{-t}dt=2\int_0^{+\infty} s^{2n}e^{-s^2}ds.$$

This is a standard Gaussian integral that can be computed by recursion. Using integration by parts,

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{4}{2n+1}\int_0^{+\infty} s^{2n+2}e^{-s^2}ds=2\frac{2}{2n+1}\Gamma(n+\frac{3}{2})$$

we get the already-known recursion relation

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{2n-1}{2}\Gamma\left(n-\frac{1}{2}\right).$$

Since

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^{+\infty} e^{-s^2}ds=\int_{-\infty}^{+\infty} e^{-s^2}ds=\sqrt{\pi},$$

we have

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)(2n-3)\dots 3\cdot 1}{2^n}\sqrt{\pi}=\frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Finally, notice that

$$\frac{(2n-1)!!}{2^n}=\frac{(2n)!}{4^n n!},$$

that can be shown by distributing the extra 2^n factor on the RHS on the even terms of the factorial at the numerator.

A.4 Gauss's equivalent definition of the Gamma function

We write the first definition of the Gamma function as

$$\Gamma(z)=\int_0^{+\infty} t^{z-1}e^{-t}dt=\int_0^{+\infty} t^{z-1}\lim_n\left(1-\frac{t}{n}\right)^n=\lim_n\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n.$$

The last integral can be computed by applying integration by parts recursively. This gives Gauss's definition straightforwardly.

A.5 An integral involving the Gamma function

We multiply and divide by e^{-t} and expand the geometric series

$$\begin{aligned}\int_0^{+\infty} dt\frac{t^{z-1}}{e^t-1}&=\int_0^{+\infty} dt\frac{t^{z-1}e^{-t}}{1-e^{-t}}=\int_0^{+\infty} dt\, t^{z-1}e^{-t}\sum_{k=0}^{\infty}e^{-tk}\\&=\sum_{k=0}^{\infty}\int_0^{+\infty} dt\, t^{z-1}e^{-t(k+1)}.\end{aligned}$$

Defining the variable $s=t(k+1)$

$$\int_0^{+\infty} dt\frac{t^{z-1}}{e^t-1}=\sum_{k=0}^{\infty}\frac{1}{(k+1)^z}\int_0^{+\infty} ds\, s^{z-1}e^{-s}=\Gamma(z)\zeta(z).$$

A.6 The Beta function identity

The product of two Gamma functions is

$$\Gamma(x)\Gamma(y)=\int_0^{+\infty} u^{x-1}e^{-x}du\int_0^{+\infty} v^{y-1}e^{-y}dv=4\int_0^{+\infty} da\int_0^{+\infty} db\, a^{2x-1}b^{2y-1}e^{-(a^2+b^2)},$$

where we have defined $a=u^2$ and $b=v^2$. Changing to polar coordinates ($dadb=rdrd\theta$)

$$\Gamma(x)\Gamma(y)=4\int_0^{+\infty} r^{2x+2y-1}e^{-r^2}dr\int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta=\Gamma(x+y)2\int_0^{\pi/2}(\cos^2\theta)^{x-\frac{1}{2}}(1-\cos^2\theta)^{y-\frac{1}{2}}d\theta.$$

Defining $t=\cos^2\theta$ we finally find $\Gamma(x)\Gamma(y)=\Gamma(x+y)\beta(x,y)$.

A.7 Recursion of Digamma function

From the recursion relation of the Gamma function we have

$$\log \Gamma(1+x) = \log \Gamma(x) + \log x.$$

Differentiation gives

$$\psi(1+x) = \psi(x) + \frac{1}{x},$$

and therefore

$$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}.$$

A.8 Series representation of the Digamma function

Applying the recursion relation we notice that

$$\psi(x+n) - \psi(1+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k} - \psi(1) - \sum_{k=0}^{n-1} \frac{1}{1+k} = \psi(x) + \gamma_E + \sum_{k=0}^{n-1} \left(\frac{1}{x+k} - \frac{1}{1+k} \right).$$

We take the limit for $n \to \infty$. The LHS can be evaluated using Stirling's approximation $\Gamma(1+x) = \sqrt{2\pi x} x^x e^{-x} \left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right]$. Taking the logarithm and then the derivative we find

$$\psi(1+x) = \left(\frac{1}{2x} + \log x \right) \left[1 + \mathcal{O}\left(\frac{1}{x} \right) \right].$$

Therefore,

$$\psi(x+n) - \psi(1+n) = \frac{1}{2(x+n-1)} + \log(x+n-1) - \frac{1}{n} = \log n + \mathcal{O}\left(\frac{1}{n}\right) = \log\left(1 + \frac{x-1}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right) = 0,$$

and the desired representation follows.

A.9 Logarithmic expansion of the Gamma function

Consider the Taylor series

$$\log \Gamma(1+x) = \sum_{k=0}^{\infty} \frac{x^n}{n!} \frac{d^n}{dx^n} \log \Gamma(1+x) \Big|_{x=0}.$$

The first coefficient is $\log \Gamma(1) = 0$. The second is $\psi(1) = -\gamma_E$. The others can be calculated using the series representation of the Digamma function. For example

$$\frac{d^2}{dx^2} \log \Gamma(1+x) = \frac{d}{dx} \psi(1+x) = -\frac{d}{dx} \left(\frac{1}{1+x+k} - \frac{1}{1+k} \right) = \sum_{k=0}^{+\infty} \frac{1}{(x+k)^2}.$$

At $x=0$ this is ζ_2. Clearly, further derivatives gives $(-1)^n (n-1)! \zeta_n$, and therefore

$$\log \Gamma(1+x) = -\gamma_E x + \sum_{n=2}^{+\infty} \frac{(-1)^n x^n}{n} \zeta_n.$$

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