Notes on special functions

There are work-in-progress personal notes on past and ongoing studies on special functions for personal interest and research.

1 Gamma function

The **Gamma function** on positive real numbers is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

It may be defined as a continuation of the factorial function $n!$ from $\mathbb{N}$ to $\mathbb{R}$: notice that $\Gamma(1) = 1$ and the recursion relation

$$\Gamma(x) = (x-1)\Gamma(x-1),$$

therefore $\Gamma(n) = (n-1)!$. This gives an easy way to compute its integer value. The half-integer value can be computed easily using

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi}\frac{(2n)!}{4^n n!}$$

from a recursion relation.
The gamma function can be trivially analytically continued for positive complex numbers (Re$z > 0$)

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt.$$

However, this is not yet well defined for negative numbers, since for $t- > 0$ the power of t must be strictly greater than -1 for the integral to be finite. The analytical continuation to the whole complex plane can be performed by splitting the integration range in the following way

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \int_1^{+\infty} t^{z-1} e^{-t} dt = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{k+z-1} dt + \int_1^{+\infty} t^{z-1} e^{-t} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^{+\infty} t^{z-1} e^{-t} dt.$$

Now

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^{+\infty} t^{z-1} e^{-t} dt$$

is well defined in the complex plane except for the non-positive integers where there are simple poles. Equivalent to this one is another definition by Gauss

$$\Gamma(z) = \lim_n \frac{n^z n!}{z(z+1)\dots(z+n)}.$$

Using Stirling's approximation one can show with this the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z + \frac{1}{2}).$$

We report an interesting integral involving the Gamma function (for Rez¿1).

$$\int_0^{+\infty} dt \frac{t^{z-1}}{e^t - 1} = \Gamma(z)\zeta(z)$$

1.1 Beta function

We define the **Euler's Beta function**

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

which is linked to the Gamma function by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This relation can be shown by writing explicitly the RHS with the definition of the Gamma function as a double integral. The result can be derived by writing the double integral in the region $\mathbb{R}^+ \times \mathbb{R}^+$ from cartesian coordinates to polar coordinates.

1.2 Digamma function

The logarithmic derivative of the Gamma function defines the **Digamma function**

$$\psi(x) = \frac{d \log \Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Using the definition of the Gamma function we get the integral representation

$$\psi(x) = \frac{1}{\Gamma(x)} \int_0^{+\infty} t^{x-1} e^{-t} \log t \, dt.$$

The recursion relation of the Gamma function implies the similar recursion for the Digamma

$$\psi(x+1) = \psi(x) + \frac{1}{x},$$

hence for integer values

$$\psi(n+1) = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 + \psi(1) = H_n + \psi(1),$$

where H_n is the n-th harmonic number. For this reason, we could also define the Digamma function as a continuation of the harmonic function $H(n)$ to real numbers, translated by $\psi(1) = -\gamma_E$, called Euler's constant. Hence, for integer value the function is

$$\psi(n) = -\gamma_E + \sum_{k=1}^{n-1} \frac{1}{k},$$

while for real values, this generalizes to the series representation of the Digamma function

$$\psi(x) = -\gamma_E - \sum_{k=0}^{+\infty} \left(\frac{1}{x+k} - \frac{1}{1+k} \right).$$

The properties of the Digamma function give an expansion of the logarithm of the Gamma function around 1

$$\log \Gamma(1+x) = -\gamma_E x + \sum_{n=2}^{+\infty} \frac{(-1)^n x^n}{n} \zeta_n.$$

2 Polylogarithms

We define on real numbers a family of functions called **Polylogarithms** with the following series expansion

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

For $n = 1$ we have $\log x = -\log(1-x)$. This definition only holds for $|x| < 1$, that is the radius of convergence of the power series. These functions can be analytically continued to the whole complex plane by defining them in terms of the differential equations

$$\begin{cases} \frac{d}{dz} \log z = \frac{1}{z} \\ \log(1) = 0 \end{cases}$$

in the case of the logarithm, and, recursively, in the $n \geq 2$ case

$$\begin{cases} \frac{d}{dz} \mathrm{Li}_n(z) = \frac{1}{z} \mathrm{Li}_{n-1}(z) \\ \mathrm{Li}_n(0) = 0 \end{cases}.$$

As the logarithm, also polylogarithms have a branch cut that is from 1 to ∞. We can show this by computing the discontinuity along the real axis from the defining differential equation.

3 Proofs

3.1 Half-integer values of the Gamma function

We evaluate $\Gamma(n + \frac{1}{2})$ using the substitution $t = s^2$

$$\Gamma\left(n + \frac{1}{2}\right) = \int_0^{+\infty} t^{n-\frac{1}{2}} e^{-t} dt = 2 \int_0^{+\infty} s^{2n} e^{-s^2} ds.$$

This is a standard Gaussian integral that can be computed by recursion. Using integration by parts,

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{4}{2n+1}\int_0^{+\infty} s^{2n+2}e^{-s^2}ds = 2\frac{2}{2n+1}\Gamma(n+\frac{3}{2})$$

we get the already-known recursion relation

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{2n-1}{2}\Gamma\left(n-\frac{1}{2}\right).$$

Since

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^{+\infty} e^{-s^2}ds=\int_{-\infty}^{+\infty} e^{-s^2}ds=\sqrt{\pi},$$

we have

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)(2n-3)\dots 3\cdot 1}{2^n}\sqrt{\pi}=\frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Finally, notice that

$$\frac{(2n-1)!!}{2^n}=\frac{(2n)!}{4^n n!},$$

that can be shown by distributing the extra 2^n factor on the RHS on the even terms of the factorial at the numerator.

3.2 Gauss's equivalent definition of the Gamma function

We write the first definition of the Gamma function as

$$\Gamma(z)=\int_0^{+\infty} t^{z-1}e^{-t}dt=\int_0^{+\infty} t^{z-1}\lim_n\left(1-\frac{t}{n}\right)^n=\lim_n\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n.$$

The last integral can be computed by applying integration by parts recursively. This gives Gauss's definition straightforwardly.

3.3 An integral involving the Gamma function

We multiply and divide by e^{-t} and expand the geometric series

$$\int_0^{+\infty} dt\frac{t^{z-1}}{e^t-1}=\int_0^{+\infty} dt\frac{t^{z-1}e^{-t}}{1-e^{-t}}=\int_0^{+\infty} dt\, t^{z-1}e^{-t}\sum_{k=0}^{\infty}e^{-tk}$$
$$=\sum_{k=0}^{\infty}\int_0^{+\infty} dt\, t^{z-1}e^{-t(k+1)}.$$

Defining the variable $s=t(k+1)$

$$\int_0^{+\infty} dt\frac{t^{z-1}}{e^t-1}=\sum_{k=0}^{\infty}\frac{1}{(k+1)^z}\int_0^{+\infty} ds\, s^{z-1}e^{-s}=\Gamma(z)\zeta(z).$$

3.4 The Beta function identity

The product of two Gamma functions is

$$\Gamma(x)\Gamma(y)=\int_0^{+\infty} u^{x-1}e^{-x}du\int_0^{+\infty} v^{y-1}e^{-y}dv=4\int_0^{+\infty} da\int_0^{+\infty} db\, a^{2x-1}b^{2y-1}e^{-(a^2+b^2)},$$

where we have defined $a=u^2$ and $b=v^2$. Changing to polar coordinates ($dadb=rdrd\theta$)

$$\Gamma(x)\Gamma(y)=4\int_0^{+\infty} r^{2x+2y-1}e^{-r^2}dr\int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta=\Gamma(x+y)2\int_0^{\pi/2}(\cos^2\theta)^{x-\frac{1}{2}}(1-\cos^2\theta)^{y-\frac{1}{2}}d\theta.$$

Defining $t=\cos^2\theta$ we finally find $\Gamma(x)\Gamma(y)=\Gamma(x+y)\beta(x,y)$.

3.5 Recursion of Digamma function

From the recursion relation of the Gamma function we have

$$\log\Gamma(1+x)=\log\Gamma(x)+\log x.$$

Differentiation gives

$$\psi(1+x)=\psi(x)+\frac{1}{x},$$

and therefore

$$\psi(x+n)=\psi(x)+\sum_{k=0}^{n-1}\frac{1}{x+k}.$$

3.6 Series representation of the Digamma function

Applying the recursion relation we notice that

$$\psi(x+n)-\psi(1+n)=\psi(x)+\sum_{k=0}^{n-1}\frac{1}{x+k}-\psi(1)-\sum_{k=0}^{n-1}\frac{1}{1+k}=\psi(x)+\gamma_E+\sum_{k=0}^{n-1}\left(\frac{1}{x+k}-\frac{1}{1+k}\right).$$

We take the limit for $n\to\infty$. The LHS can be evaluated using Stirling's approximation $\Gamma(1+x)=\sqrt{2\pi x}x^x e^{-x}\left[1+\mathcal{O}\left(\frac{1}{x}\right)\right]$. Taking the logarithm and then the derivative we find

$$\psi(1+x)=\left(\frac{1}{2x}+\log x\right)\left[1+\mathcal{O}\left(\frac{1}{x}\right)\right].$$

Therefore,

$$\psi(x+n)-\psi(1+n)=\frac{1}{2(x+n-1)}+\log\left(x+n-1\right)-\frac{1}{n}=\log n+\mathcal{O}\left(\frac{1}{n}\right)=\log\left(1+\frac{x-1}{n}\right)+\mathcal{O}\left(\frac{1}{n}\right)=0,$$

and the desired representation follows.

3.7 Logarithmic expansion of the Gamma function

Consider the Taylor series

$$\log\Gamma(1+x)=\sum_{k=0}^{\infty}\frac{x^n}{n!}\frac{d^n}{dx^n}\log\Gamma(1+x)\Big|_{x=0}.$$

The first coefficient is $\log\Gamma(1)=0$. The second is $\psi(1)=-\gamma_E$. The others can be calculated using the series representation of the Digamma function. For example

$$\frac{d^2}{dx^2}\log\Gamma(1+x)=\frac{d}{dx}\psi(1+x)=-\frac{d}{dx}\left(\frac{1}{1+x+k}-\frac{1}{1+k}\right)=\sum_{k=0}^{+\infty}\frac{1}{(x+k)^2}.$$

At $x=0$ this is ζ_2. Clearly, further derivatives gives $(-1)^n(n-1)!\zeta_n$, and therefore

$$\log\Gamma(1+x)=-\gamma_E x+\sum_{n=2}^{+\infty}\frac{(-1)^n x^n}{n}\zeta_n.$$

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