The optimal hours to tilt sun umbrellas: a geometry problem

1 The problem

Some sun umbrellas one can find in beaches can be tilted at their top. This makes possible both to move and to widen the projected shadow, like in the figure[1].

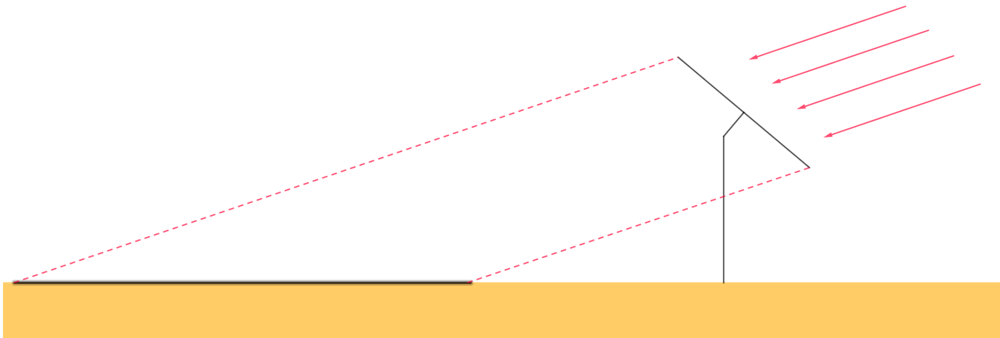

Figure 1: Tilted sun umbrella and its projected shadow.

The following obvious consideration hold: when the sun is at the *zenith* point (equivalently, when the Sun rays are perpendicular to the ground) tilting the umbrella is counterproductive, as the next figure shows: the area of the shadow of the tilted umbrella is smaller than the area of the shadow in the horizontal configuration.

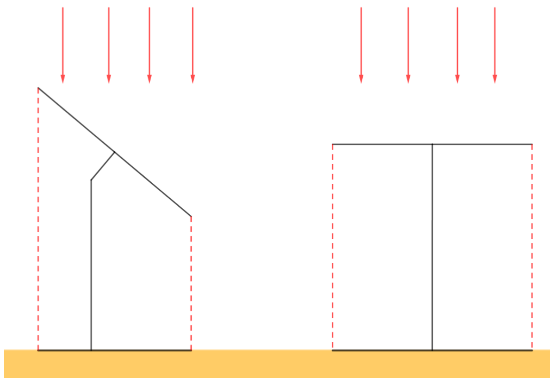

Figure 2: Shadow at the zenith for a tilted umbrella (left) and for an horizontal umbrella (right).

We would like to study in general when it is convenient to tilt the umbrella. We are now going to make other considerations in order to define the problem precisely. However, who wishes to figure out the problem entirely by themselves, it is advised to stop here.
We notice that the shadow projected by the horizontal umbrella is constant along the day, that is to say it is independent on the inclination of the Sun rays, which has only the effect to displace the shadow. Assuming the umbrella circular, the area of such shadow is $A = \pi r^2$.
Finally, when the sun is setting, the area of the shadow projected by the tilted umbrella is significantly elongated, such that it clearly becomes greater than A. For this reason, after the zenith, there must an angle of the Sun rays after which it is convenient to tilt the umbrella.
The **problem** we are here defining is obtaining such threshold value of the angle of Sun rays as a function of the tilting angle of the umbrella, usually fixed by how the umbrella is built.
Analytically, let ϕ be the angle of Sun rays with respect to the horizon ($\phi = 90°$ is the zenith) and θ be the tilting angle, defined as in the figure. We want to determine ϕ_{max} such that when $\phi < \phi_{max}$ we have $A_{shadow} > A$, where ϕ_{max} is a function of θ, so that we can write $\phi_{max}(\theta)$. Varying from the latitude and the day of the year, the information

[1] We approximate Sun rays to be parallel.

on Sun rays angle can be translated into hours of the day using tables that can be found easily online (we will see an example later).

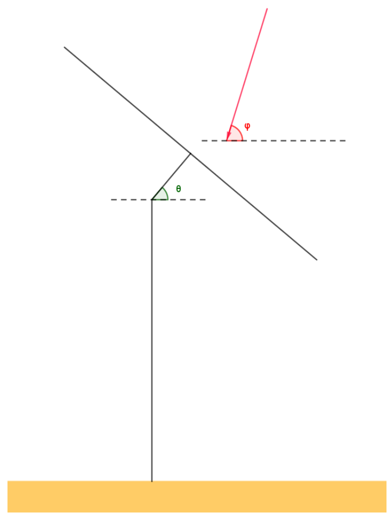

Figure 3: Definition of the angular variables θ e ϕ.

We show two different ways to get to the solution. The first makes use of basic trigonometry, that is sufficient to get $\phi_{max}(\theta)$, the second, using analytic geometry and linear algebra, makes possible to calculate easily also the exact analytical expression of the shadow cone generated by the Sun and by the umbrella, together with the equation of the shadow on the ground.

2 First method

As we have discussed, it is necessary to find ϕ so that the shadow projected by the tilted umbrella is circular, that is to say equal to the shadow projected by the horizontal umbrella.

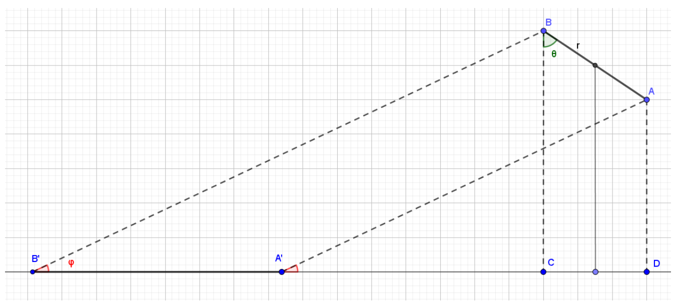

Figure 4: Scheme of the problem.

We are looking for ϕ such that $A'B' = 2r$. Let h be the height of the midpoint of the umbrella. Using trigonometric formulae, it is possible to calculate trivially all the sides of the angles CBB' e DAA' and, using finally,

$$A'B' = B'C' + CD - A'D$$

we can find the projection. We report here some intermediate passages. In the first place the height of the two triangles are

$$BC = h + r\cos\theta$$
$$AD = h - r\cos\theta.$$

The bases are

$$B'C = BC\text{cotg}\phi = (h + r\cos\phi)\text{cotg}\phi$$
$$A'D = AD\text{cotg}\phi = (h - r\cos\phi)\text{cotg}\phi.$$

Because $CD = 2r\sin\theta$, combining we have

$$AB = 2r(\cos\theta\text{cotg}\phi + \sin\theta).$$

By putting $A'B'(\phi) > 2r$, we find

$$\text{cotg}\phi > \frac{1-\sin\theta}{\cos\theta}.$$

Using $\text{cotg}\phi = \tan(90-\phi)$ we can write the solution in terms of the inverse tangent arctan,

$$90 - \phi > \arctan\frac{1-\sin\theta}{\cos\theta}.$$

The solution greatly simplifies by noting that $\arctan\frac{1-\sin\theta}{\cos\theta} = 45 - \frac{\theta}{2}$[2], whence

$$\boxed{\phi < 45^\circ + \frac{\theta}{2}}.$$

For example, if the umbrella has a tilting angle of 60°, it is convenient to tilt it when the Sun rays are lower than 75°. If the tilting angle is 45°, The Solar threshold angle is 67.5°. Notice that the angle is halfway between θ and 90°, since $45^\circ + \frac{\theta}{2} = \frac{90^\circ+\theta}{2}$.

3 Second method

We will now make use of analytic geometry to explicitly calculate the equations of the curves and surfaces involved in the problem. These can be drawn with computer programs such as Geogebra, obtaining results like in the figure.

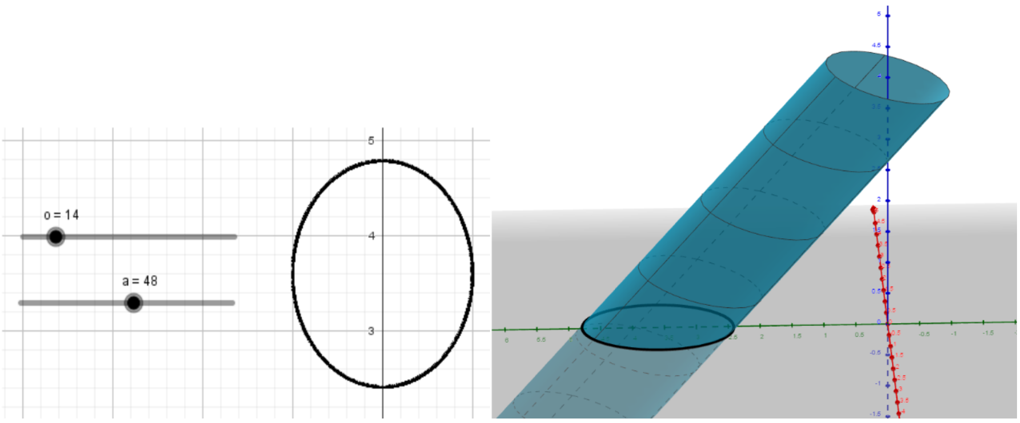

Figure 5: Drawing of the shape of the shadow on the ground (left) and of the shadow cone (right), for $\phi = 48^\circ$ e $\theta = 76^\circ$. Geogebra project in www.geogebra.org/m/f3vw5fpz.

First we calculated the equation of the tilted umbrella. In the horizontal configuration the equation is simply

$$\begin{cases} x(\alpha) = r\cos\alpha \\ y(\alpha) = r\sin\alpha \\ z(\alpha) = h \end{cases}$$

We now rotate the circumference around the x axis (WLOG we assume the Sun rays to be perpendicular to this axis) of an angle θ employing the rotation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} x(\alpha) \\ y(\alpha) \\ z(\alpha) \end{pmatrix}.$$

Calculating the product we find

$$\begin{cases} x(\alpha) = r\cos\alpha \\ y(\alpha) = r\sin\alpha\sin\theta + h\cos\theta \\ z(\alpha) = -r\sin\alpha\cos\theta + h\sin\theta \end{cases}.$$

The shadow cone (properly a cilinder, since the Sun rays are parallel), can be easily obtained departing from each

[2]Proof in the appendix

point of the tilted circumference a line, parametrized by the variable t, with direction vector $(0, -\cos\phi, -\sin\phi)$[3],

$$\begin{cases} x(\alpha, t) = r\cos\alpha \\ y(\alpha, t) = r\sin\alpha\sin\theta - \cos\phi\, t \\ z(\alpha, t) = -r\sin\alpha\cos\theta + h\sin\theta - \sin\phi\, t \end{cases} .$$

The equation of the projection on the ground can be obtained by imposing $z(\alpha, t) = 0$. This makes possible to write t in terms of α (o vice versa), getting so the equation of the desired curve. First, we find

$$\tilde{t} = \frac{r\sin\alpha\cos\theta + h\sin\theta}{\sin\phi},$$

whence

$$\begin{cases} x(\alpha) = r\cos\alpha \\ y(\alpha) = h\cos\theta + r\mathrm{cotg}\phi\sin\theta + r\sin\alpha\left[\sin\theta + \mathrm{cotg}\phi\cos\theta\right] \\ z(\alpha) = 0 \end{cases} .$$

The curve is a displaced ellipse of equation

$$\frac{x^2}{r^2} + \frac{(y - h\cos\theta - r\mathrm{cotg}\phi\sin\theta)^2}{\left[r(\sin\theta + \mathrm{cotg}\phi\cos\theta)\right]^2} = 1$$

with area

$$A = \pi r^2(\sin\theta + \mathrm{cotg}\phi\cos\theta).$$

By imposing $A > \pi r^2$ we find the same result of the first method.

4 Case study

Consider a tiltable Sun umbrella with tilting angle $\theta = 45°$ at the latitude of Naples. The threshold angle is $\phi_{max} = 67.5°$. For different days of summer we have (data from www.sunearthtools.com)

- On June 21st the maximum Sun angle (technically called *elevation*) is approximately 72.5° at 1 pm. At midday the elevation is ca. ϕ_{max}, after which it increases up to 72.5° and decreases to reach again the threshold angle at ca. 2:15 pm. Therefore, from midday to 2:15 pm it is better to keep the umbrella horizontal to have a larger shadow while for the rest of the day the tilted configuration is better;
- On July 10th the horizontal configuration is optimal from 12:10 pm to 2:05 pm ca.;
- From July 31st the maximum elevation of the Sun is less than ϕ_{max}: from this day it is always convenient to keep the sun umbrella tilted.

A Proof of $\arctan\frac{1-\sin\theta}{\cos\theta} = 45° - \frac{\theta}{2}$

The proof can be carried in two ways, using trigonometry or mathematical analysis. Let us start from the first way. Define $\xi = 90° - \theta$. The formula becomes

$$\arctan\frac{1-\cos\xi}{\sin\xi} = \frac{\xi}{2},$$

so that the proof trivially reduces to show that $\tan\frac{\xi}{2} = \frac{1-\cos\xi}{\sin\xi}$:

$$\tan\frac{\xi}{2} = \frac{\sin\frac{\xi}{2}}{\cos\frac{\xi}{2}} = \frac{\sqrt{1-\cos\xi}}{\sqrt{1+\cos\xi}} = \frac{\sqrt{1-\cos\xi}}{\sqrt{1+\cos\xi}}\frac{\sqrt{1-\cos\xi}}{\sqrt{1-\cos\xi}} = \frac{1-\cos\xi}{\sin\xi},$$

and this concludes the proof.

Alternatively, using mathematical analysis, we notice that

$$\frac{d}{d\theta}\arctan\frac{1-\sin\theta}{\cos\theta} = -\frac{1}{2},$$

that is to say, the inverse tangent we are evaluating is simply a line with angular coefficient -1/2. With $\theta = 0$ we find that the intercept is $\arctan 1 = 45°$, thus concluding the proof.

[3] Recall that any line in the three-dimensional space can be parametrized as

$$\begin{cases} x(t) = x_0 + v_x t \\ y(t) = y_0 + v_y t \\ z(t) = z_0 + v_z y \end{cases} ,$$

where (x_0, y_0, z_0) is the origin of the line and (v_x, v_y, v_z) is the vector that defines its direction.