Parerga and paralipomena

1 p_T^2 integration

In the phase space decomposition employed for the rapidity distribution, the integration on p_T^2 is computed with a Dirac delta that leaves out a factor of $|\tilde{p}_T^2|^\epsilon$. This depends on the integration variables l^2 and k'^2. Explicitly, the Dirac delta in the frame of reference where l is at rest gives

$$p_H^0 + k'^0 - \sqrt{l^2} = 0. \tag{1}$$

Because initial gluon emissions are soft in the threshold limit, this frame of reference is also the frame of reference of the partonic CoM. The equation above can be rewritten using the usual parametrization of p_H. Therefore we have the equation

$$\sqrt{m_H^2 + p_T^2}\cosh y + \sqrt{k'^2 + p_T^2\cosh^2 y + m_H^2\sinh^2 y} - \sqrt{l^2} = 0. \tag{2}$$

Solving with Mathematica with respect to p_T^2 gives

$$\tilde{p}_T^2 = \frac{\left[(k'^2)^2 - 2k'^2 m_H^2 + m_H^4 - 2k'^2 l^2 + l^4 - 2m_H^2 l^2\cosh 2y\right]\operatorname{sech}^2 y}{4l^2} \tag{3}$$

Using $\cosh 2y = 2\cosh^2 y - 1$ this reduces to

$$\tilde{p}_T^2 = \frac{(k'^2 - m_H^2 - l^2)^2\operatorname{sech}^2 y}{4l^2} - m_H^2. \tag{4}$$

The expression is different than De Ros (eq. 4.68 pag. 46) because at that point he is still using as variable p_z instead of y.
In each passage of this calculation, one can verify that in the threshold limit, be it for fixed x_1 or x_2, $\tilde{p}_T^2 \to 0$ using $k'^2 \to 0, l^2 \to s$ and $s \to m_H^2/x_2$. Therefore $|\tilde{p}_T^2|^\epsilon$ carries a factor of $[(1-x_1)(1-x^2)]^\epsilon$ that **contributes to the collinear scale**. Possibly, as shown explicitly at NLO, it also carries a factor that regularizes the integrals in l^2 e k'^2.

2 Singling soft and collinear scales

We shortly revise and carry on the rewriting of the GS result that makes it possible to single out the soft and collinear scales $\ln t$ and $\ln Q_{max}^2$.
Equation 3.17 of GS is

$$\begin{aligned}
C_{sing}^{(2)} =& \delta(Q^2)\Bigg\{(11 + \delta + N_c U)g_{gg} \\
&+ (N_c - n_f)\frac{N_c}{3}\left[\frac{m_H^4}{s} + \frac{m_H^4}{t} + \frac{m_H^4}{u} + m_H^2\right]\Bigg\} \\
&+ \Bigg\{\left(\frac{1}{-t}\right)\left[-P_{gg}(z_t)\log\frac{\mu_F^2 z_t}{-t} + p_{gg}(z_t)\left(\frac{\log(1-z_t)}{1-z_t}\right)_+\right] g_{gg,t}(z_t) \\
&+ \left(\frac{1}{-t}\right)\left[-2n_f P_{qg}(z_t)\log\frac{\mu_F^2}{Q_{max}^2 q} + 2n_f z_t(1-z_t)\right] g_{qg,t}(z_t) \\
&+ \left(\frac{z_t}{-t}\right)\left[\left(\frac{\log(1-z_t)}{1-z_t}\right)_+ - \log\frac{Q_T^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right] \\
&\quad\cdot\frac{N_c^2}{2}\left[\frac{m_H^8 + s^4 + t^4 + u^4 + Q^8 + z_t z_u(m_H^8 + s^4 + Q^8 + (u/z_u)^4 + (t/z_t)^4)}{sut}\right] \\
&- \left(\frac{z_t}{-t}\right)\left(\frac{1}{1-z_t}\right)_+\frac{\beta_0}{2}N_c\left(\frac{m_H^8 + s^4 + z_t z_u((u/z_u)^4 + (t/z_t)^4)}{sut}\right) \\
&+ (t \leftrightarrow u)\Bigg\}
\end{aligned}$$

$$+ N_c^2\left[\frac{(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)(Q^2+Q_T^2)}{s^2Q^2Q_T^2}+\right.$$
$$\left.+\frac{2m_H^4((m_H^2-u)^4+(m_H^2-t)^4+t^4+u^4)}{sut(m_H^2-t)(m_H^2-u)}\right]\frac{1}{p_T^2}\log\frac{p_T^2}{Q_T^2}$$

Now defining new coefficients this is rewritten as

$$\begin{aligned}
C^{(2)}_{sing,rap} = \int_0^1 dqJ(x_1,x_2,q)Q^2_{max} \quad & \Bigg\{\frac{\delta(q)}{Q^2_{max}}\mathcal{V}+ \\
& \frac{1}{-t}\mathcal{A}_t\left[-p_{gg}\left(-\left(\frac{\ln(1-z_t)}{1-z_t}\right)_+ + \ln\frac{\mu^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right)-\beta_0\ln\frac{\mu^2 z_t}{-t}\frac{\delta(q)}{Q^2_{max}}\right]+ \\
& \frac{1}{-t}\left[\mathcal{B}_{1,t}\ln\frac{1}{qQ^2_{max}}+\mathcal{B}_{2,t}\right]+ \\
& \frac{z_t}{-t}\mathcal{C}_t\left[\left(\frac{\ln(1-z_t)}{1-z_t}\right)_+ - \ln\frac{Q_T^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right]- \\
& \frac{z_t}{-t}\mathcal{D}_t\left(\frac{1}{1-z_t}\right)_+ + (t\leftrightarrow u) \\
& +\mathcal{E}\Bigg\}
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
&\mathcal{A}_t = g_{gg,t}(z_t)\\
&\mathcal{A}_u = g_{gg,u}(z_u)\\
&\mathcal{B}_{1,t} = -2n_f P_{qg}(z_t)g_{qg,t}(z_t)\\
&\mathcal{B}_{2,t} = 2n_f z_t(1-z_t)g_{qg,t}(z_t)\\
&\mathcal{B}_{1,u} = -2n_f P_{qg}(z_u)g_{gq,t}(z_t)\\
&\mathcal{B}_{2,t} = 2n_f z_u(1-z_u)g_{gq,t}(z_t),\\
&\mathcal{C}_t = \mathcal{C}_u = \frac{N_c^2}{2}\left[\frac{m_H^8+s^4+t^4+u^4+Q^8+z_t z_u(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)}{sut}\right]\\
&\mathcal{D}_t = \mathcal{D}_u = \frac{\beta_0}{2}N_c\left(\frac{m_H^8+s^4+z_t z_u((u/z_u)^4+(t/z_t)^4)}{sut}\right)\\
&\mathcal{E} = N_c^2\left[\frac{(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)(Q^2+Q_T^2)}{s^2Q^2Q_T^2}+\frac{2m_H^4((m_H^2-u)^4+(m_H^2-t)^4+t^4+u^4)}{sut(m_H^2-t)(m_H^2-u)}\right]\frac{1}{p_T^2}\log\frac{p_T^2}{Q_T^2}
\end{aligned}$$

notice that in the definition of the $\mathcal{B}$ coefficients, quarks and gluons functions are exchanged too. Rearranging

$$\begin{aligned}
C^{(2)}_{sing,rap} = \int_0^1 & dqJ(x_1,x_2,q)\Bigg\{ \\
& Q^2_{max}\left[\mathcal{E}+\frac{\mathcal{B}_{2,t}}{-t}+\frac{\mathcal{B}_{1,t}}{-t}\ln\frac{1}{qQ^2_{max}}\right] \\
& +\delta(q)\Bigg[\mathcal{A}_t\beta_0\ln t+N_c\mathcal{A}_t\ln^2 Q^2_{max}-N_c\mathcal{A}_t\ln^2 t-\mathcal{C}_t\ln Q^2_{max}\ln p^2_{T,max}+\frac{1}{2}\mathcal{C}_t\ln^2 Q^2_{max} \\
& -\frac{1}{2}\mathcal{C}_t\ln^2 t-\mathcal{D}_t\ln Q^2_{max}+\mathcal{D}_t\ln t+\mathcal{C}_t\ln Q_T^2\ln t+\mathcal{V}\Bigg] \\
& +\left(\frac{1}{q}\right)_+\left[\frac{\mathcal{A}_t p_{gg}}{z_t}\ln Q^2_{max}+\mathcal{C}_t\ln Q^2_{max}-\mathcal{C}_t\ln Q_T^2-\mathcal{D}_t\right] \\
& +\left(\frac{\ln q}{q}\right)_+\left[\frac{\mathcal{A}_t p_{gg}}{z_t}+\mathcal{C}_t\right]+(t\leftrightarrow u)\Bigg\},
\end{aligned} \tag{6}$$

where in terms proportional to $\delta(q)$ we have set $z_t = 1$ and therefore $p_{gg}(1) = 2N_c$. Here we see explicitly both scales. Their origin is discussed in the thesis

3 Computation of the rapidity distribution at NNLO

To compare our result with Tackmann's it is necessary to have explicitly the rapidity distribution at NNLO in terms of the threshold variables x_1 and x_2. Since there is no published result for such an observable in the threshold limit, we compute it by integrating the fully differential distribution obtained by Glosser and Schmidt starting from the previous equation. In this section, we report the final result and discuss some details of the computation.

It should be noted what has been already observed in the thesis. First, Glosser and Schimit are only interested in computing the fully differential distribution and, for this reason, they do not include the two loop contributions from the $gg \to H$ process, since its kinematic is also trivial in the hadronic frame of reference. These terms would contribute to terms proportional to $\delta(p_T^2)\delta(1-x_1)\delta(1-x_2)$, which in turn are not trivial in rapidity distribution.

Secondly, they do not regularize the p_T^2 divergence, that is to say, there is no plus distribution in p_T^2. This is of course linked to the first point since this would add more poles and terms proportional to Dirac deltas.

We here ignore all these contributions, thus obtaining the rapidity distribution at NNLO in the threshold limit without all the terms proportional to Dirac deltas of x_1 and x_2.

We write here the result obtained. It is not yet in the most compact form since, to compute the Mellin-Mellin transform and compare it with Tackmann's expression it is important to keep track of the origin of the different terms.

$$
\begin{aligned}
C_{sing}(m^2, x_1, x_2) = J(1, x_2, 0)m^2 \Bigg\{ \\
&\frac{1}{(1-x_1)(1-x_2)}\left[N_c^2\left(-\ln x_2(1-x_2) f_I(x_2) + f_{II}(x_2) + f_{IV}(x_2) + f_V(x_2) + f_{VI}(x_2)\right) + N_c\frac{\beta_0}{2} f_{III}(x_2)\right] + \\
&+ \frac{\ln(1-x_1)(1-x_2)}{(1-x_1)}\left[N_c^2 f_I(x_2) + C_f n_f f_{VIII,2}(x_2)\right] + (x_1 \leftrightarrow x_2) \\
&+ \frac{f_{LO}(x_1,x_2)}{(1-x_1)(1-x_2)}\Bigg\{\Delta + \frac{67}{18} - \frac{5}{9}n_f \\
&- \beta_0 N_c \ln\frac{(1-x_1)(1-x_2)}{x_2} \\
&+ N_c^2\Bigg[\ln^2(1-x_1)(1-x_2) \\
&+ \ln(1-x_1)(1-x_2) f(x_2) \\
&- \ln^2(1-x_1) - 2\ln(1-x_1)\ln(1-x_2) - \ln^2(1-x_2) \\
&+ (\ln(1-x_1) + \ln(1-x_1)) g(x_2) \\
&+ \ln[(1-x_1)(1-x_2)](\ln(1-x_1) + \ln(1-x_2)) \\
&+ F.P(x_2)\Bigg]\Bigg\} \\
&+ (N_c - n_f)\frac{N_C}{3}\left[1 + \frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right]\Bigg\}.
\end{aligned}
$$

The explicit expression of the coefficients can be found in the Mathematica notebook.

3.1 Integration of the terms proportional to $\delta(q)$

Terms multiplied by $\delta(q)$ are straightforward to compute using $\delta(q)z_t = \delta(q)z_u = 1$. The relevant coefficients become

$$
\begin{aligned}
\mathcal{A}_t &= \mathcal{A}_u = g_{LO} \\
\mathcal{C}_t &= \mathcal{C}_u = N_c g_{LO} \\
\mathcal{D}_t &= \mathcal{D}_u = \frac{\beta_0}{2} g_{LO}
\end{aligned}
$$

We plug this into the equation above.

$$
\begin{aligned}
C^{(2)}_{sing,rap} = &\Bigg[g_{LO}\beta_0 \ln t + N_c g_L O \ln^2 Q^2_{max} - N_c g_{LO} \ln^2 t - N_c g_{LO} \ln Q^2_{max} \ln p^2_{T,max} + \frac{1}{2} N_c g_{LO} \ln^2 Q^2_{max} \\
&- \frac{1}{2} N_c g_{LO} \ln^2 t - \frac{\beta_0}{2} g_{LO} \ln Q^2_{max} + \frac{\beta_0}{2} g_{LO} \ln t + N_c g_{LO} \ln Q^2_T \ln t + \mathcal{V}\Bigg] J(x_1, x_2, 0) \\
&+ \int_0^1 dq J(x_1, x_2, q) \Bigg\{ \\
&Q^2_{max}\left[\mathcal{E} + \frac{\mathcal{B}_{2,t}}{-t} + \frac{\mathcal{B}_{1,t}}{-t} \ln \frac{1}{q Q^2_{max}}\right] \\
&+ \left(\frac{1}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} \ln Q^2_{max} + \mathcal{C}_t \ln Q^2_{max} - \mathcal{C}_t \ln Q^2_T - \mathcal{D}_t\right] \\
&+ \left(\frac{\ln q}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} + \mathcal{C}_t\right] + (t \leftrightarrow u)\Bigg\},
\end{aligned}
\tag{7}
$$

Simplifying

$$
\begin{aligned}
C^{(2)}_{sing,rap} = &\Bigg[-\frac{3}{2} N_c g_{LO} \ln^2 t + \frac{3}{2} N_c g_{LO} \ln^2 Q^2_{max} - N_c g_{LO} \ln Q^2_{max} \ln Q^2_T + N_c g_{LO} \ln Q^2_T \ln t \\
&+ \frac{3}{2}\beta_0 g_{LO} \ln t - \frac{1}{2}\beta_0 g_{LO} \ln Q^2_{max} + \mathcal{V}\Bigg] J(x_1, x_2, 0) \\
&+ \int_0^1 dq J(x_1, x_2, q) \Bigg\{ \\
&Q^2_{max}\left[\mathcal{E} + \frac{\mathcal{B}_{2,t}}{-t} + \frac{\mathcal{B}_{1,t}}{-t} \ln \frac{1}{q Q^2_{max}}\right] \\
&+ \left(\frac{1}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} \ln Q^2_{max} + \mathcal{C}_t \ln Q^2_{max} - \mathcal{C}_t \ln Q^2_T - \mathcal{D}_t\right] \\
&+ \left(\frac{\ln q}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} + \mathcal{C}_t\right] + (t \leftrightarrow u)\Bigg\},
\end{aligned}
\tag{8}
$$

3.2 General properties of the integration in q

The integrand is singular both for $q \to 0$ and for $q \to 1$. The second is carried by terms proportional to $1/p_T^2$. Both extrema are regularized by plus distributions. The plus distributions centred around 0 come from the phase space factor $(Q^2)^{-\epsilon}$, while the plus distributions around 1 come from $|p_T^2|^{-\epsilon}$. The latter distributions are not explicitly written in the result since GS is only interested in the doubly differential distribution.
The generic term to integrate is

$$
\int_0^1 dq f(x_1, x_2, q) \left(\frac{1}{q}\right)_+ \left(\frac{1}{1-q}\right)_+ = \int_0^1 dq \frac{f(x_1, x_2, q) - q f(x_1, x_2, 1) - (1-q) f(x_1, x_2, 0)}{q(1-q)}.
$$

The integration can be simplified by expanding the integrand with respect to x_1 around 1 and retaining the leading terms in the singly soft limit $x_1 \to 1$. Since the integrand is rational this can be done by the dominated convergence theorem.

3.3 Coefficients in the singly soft limit

The gluon fusion LO in terms of x_1 and x_2 at $q = 0$ is

$$
g_{LO}(x_1, x_2, 0) = N_c \frac{(1 + x_1^4 x_2^4)(x_1 + x_2)^4 + (1 - x_1^2)^4 x_2^4 + (1 - x_2^2)^4 x_1^4}{x_1^2 x_2^2 (x_1 + x_2)^2 (1 - x_1)^2 (1 - x_2)^2}.
$$

In the limit $x_1 \to 1$, this is

$$
g_{LO}(1, x_2, 0) = N_c \frac{(1 + x_2)(1 - x_2 + x_2^2)^2}{x_2^2} \frac{1}{(1 - x_1)(1 - x_2)}.
$$

The Jacobian at $q = 0$ is

$$
J(x_1, x_2, 0) = \frac{2(1 + x_1 x_2) x_1 x_2}{(x_1 + x_2)^2}
$$

In the limit $x_1 \to 1$

$$J(1, x_2, 0) = \frac{2x_2}{1+x_2}$$

Hence,

$$J(x_1, x_2, q) g_{LO}(x_1, x_2, q) \xrightarrow[q=0]{x_1 \to 1} = N_c \frac{2(1-x_2+x_2^2)^2}{x_2} \frac{1}{(1-x_1)(1-x_2)}.$$

Now consider q-dependent quantities in the singly soft limit, the first leading terms for some variables are.

$$\begin{aligned}
t &= -m^2 \frac{(1-x_1)[2+q(1-x_2)]}{(1+x_2)} + \mathcal{O}(1-x_1) \\
p_T^2 &= m^2 \frac{2(1-q)(1-x_1)(1-x_2)}{1+x_2} + \mathcal{O}((1-x_1)^0) \\
\frac{p_T^2}{t} &= \frac{2(1-q)(1-x_2)}{2+q(1-x_2)} + \mathcal{O}((1-x_1)^0) \\
z_t &= \frac{[2+q(1-x_2)]x_2}{2x_2+q(1-x_2)(1+2x_2)} + \mathcal{O}((1-x_1)^0) \\
J &= \frac{2x_2}{(1+x_1)} + \mathcal{O}((1-x_1)^0)
\end{aligned}$$

3.4 Integration of terms proportional to $\left(\frac{1}{q}\right)_+$

3.4.1 Integration of $\frac{\mathcal{A}_u p_{gg}}{z_t}$

Consider first

$$I(x_1, x_2) = \int_0^1 J(x_1, x_2, q) \frac{\mathcal{A}_t(x_1, x_2, q) p_{gg}(z_t(x_1, x_2, q))}{z_t(x_1, x_2, q)} \left(\frac{1}{q}\right)_+ \left(\frac{1}{1-q}\right)_+ \ln Q_{max}^2.$$

Expanding with respect to x_1 and integrating gives

$$I(x_1, x_2) = m^2 N_c^2 \frac{f_I(x_2)}{1-x_1} [\ln(1-x_1)(1-x_2) - \ln x_2].$$

Notice that even if $\mathcal{A}_t$ is proportional to $[(1-x_1)(1-x_2)]^{-1}$, the integral is singular only for $x_1 \to 1$.

3.4.2 Integration of $\mathcal{C}_t$

$$I(x_1, x_2) = \int_0^1 J(x_1, x_2, q) \mathcal{C}_t(x_1, x_2, q) \left(\frac{1}{q}\right)_+ \ln \frac{Q_{max}^2(x_1, x_2, q)}{Q_T^2(x_1, x_2, q)}.$$

The log term is

$$\ln \frac{Q_{max}^2(x_1, x_2, q)}{Q_T^2(x_1, x_2, q)} = \ln \frac{1+x_2}{q - x_2 q + 2x_2} + \wr((1-x)^0).$$

Integration gives

$$I(x_1, x_2) = m^2 N_c \frac{f_{II}(x_2)}{(1-x_1)(1-x_2)}$$

The function g contains dilogarithms.

3.4.3 Integration of $\mathcal{D}_t$

$$I(x_1, x_2) = \int_0^1 J(x_1, x_2, q) \mathcal{D}_t(x_1, x_2, q) \left(\frac{1}{q}\right)_+$$

gives

$$I(x_1, x_2) = m^2 N_c \frac{\beta_0}{2} \frac{f_{III}(x_2)}{(1-x_1)(1-x_2)}$$

3.5 Integration of terms proportional to $\left(\frac{\ln q}{q}\right)_+$

3.5.1 Integration of the fourth term

$$I(x_1, x_2) = \int_0^1 dq J(x_1, x_2, q) \frac{\mathcal{A}_t(x_1, x_2, q) p_{gg}(z_t(x_1, x_2, q))}{z_t(x_1, x_2, q)} \left(\frac{\ln q}{q}\right)_+$$

gives

$$I(x_1, x_2) = m^2 N_c \frac{f_{IV}(x_2)}{(1-x_1)(1-x_2)}$$

3.5.2 Integration of the fifth term

$$I(x_1, x_2) = \int_0^1 dqJ(x_1, x_2, q)\mathcal{C}_t(x_1, x_2, q)\left(\frac{\ln q}{q}\right)_+$$

gives

$$I(x_1, x_2) = m^2 N_c \frac{f_V(x_2)}{(1-x_1)(1-x_2)}$$

3.6 Integration of terms proportional to Q^2_{max}

3.6.1 Integration of the sixth term

First, we write $\mathcal{E}(x_1, x_2, q)$ as

$$\mathcal{E}(x_1, x_2, q) = \mathcal{E}_0(x_1, x_2, q)\left[\ln(1-q) + \ln\frac{2x_2}{q+2x_2-qx_2} + \wr((1-x_1)^0)\right],$$

this gives the two integrals that we compute separately

$$I(x_1, x_2) = \int_0^1 dqJ(x_1, x_2, q)\left[\mathcal{E}_{0,reg}(x_1, x_2, q)\left(\frac{\ln(1-q)}{1-q}\right)_+ + \mathcal{E}_{0,reg}(x_1, x_2, q)\ln\frac{2x_2}{q+2x_2-qx_2}\left(\frac{1}{1-q}\right)_+\right]$$

Which together give

$$I(x_1, x_2) = m^2 N_c^2 \frac{f_{VI}(x_2)}{(1-x_1)(1-x_2)}$$

3.6.2 Integration of the seventh term

$$I(x_1, x_2) = \int_0^1 dqJ(x_1, x_2, q)Q^2_{max}\frac{\mathcal{B}_{2,t}}{-t}$$

is

$$I(x_1, x_2) = m^2 C_f n_f \frac{f_{VII}(x_2)}{1-x_1}$$

3.6.3 Integration of the eighth term

$$I(x_1, x_2) = \int_0^1 dqJ(x_1, x_2)Q^2_{max}\frac{\mathcal{B}_{1,t}}{-t}\ln\frac{1}{qQ^2_{max}} = \int_0^1 dqJ(x_1, x_2)Q^2_{max}n_f[z_t^2 + (1-z_t)^2]\frac{g_{qg,t(z_t)}}{-t}\ln qQ^2_{max}$$

The logarithm is divided into two terms two single out the enhanced log

$$\ln qQ^2_{max} = \ln\frac{q}{x_2} + \ln(1-x_1)(1-x_2).$$

Finally, this gives

$$I(x_1, x_2) = m^2 C_f n_f\left[\frac{f_{VIII,1}}{1-x_1} + \frac{\ln(1-x_1)(1-x_2)}{1-x_1}f_{VIII,2}\right] \tag{9}$$

3.7 Virtual terms

Combining virtual terms together with terms coming from the $\delta(q)$ terms in the integration we have (we absorbed all the possible m^2 inside the Mandelstam invariants)

$$\begin{aligned} J(x_1, x_2, 0)\frac{f_{LO}(x_1, x_2)}{(1-x_1)(1-x_2)}&\left\{\Delta + \frac{67}{18} - \frac{5}{9}n_f\right.\\ &- \beta_0 N_c \ln Q^2_{max}\\ &+ N_c^2\left[\frac{\pi^2}{3} - 2\ln p^2_{T,max}\ln Q^2_{max} + 3\ln Q^2_{max} + \ln p^2_{T,max}\ln t - \ln s\ln t - \ln^2 t + (t\to u) - 2\ln t\ln u + f_{fin,virt}(t,u)\right]\left.\vphantom{\frac{1}{1}}\right\}\\ &+ (N_c - n_f)\frac{N_C}{3}\left[1 + \frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right] \end{aligned}$$

where $f_{fin,virt}(t,u)$ are finite (not zero) terms in the threshold limit. Writing the third line in terms of x_1 and x_2 in the $x_1 \to 1$ limit gives

$$J(x_1, x_2, 0)\frac{f_{LO}(x_1, x_2)}{(1-x_1)(1-x_2)}\left\{\Delta + \frac{67}{18} - \frac{5}{9}n_f\right.$$

$$
\begin{aligned}
&- \beta_0 N_c \ln \frac{(1-x_1)(1-x_2)}{x_2} \\
&+ N_c^2 \Bigg[\ln^2 (1-x_1)(1-x_2) \\
&+ \ln(1-x_1)(1-x_2) f(x_2) \\
&- \ln^2(1-x_1) - 2 \ln (1-x_1) \ln (1-x_2) - \ln^2(1-x_2) \\
&+ (\ln(1-x_1) + \ln(1-x_1)) g(x_2) \\
&+ \ln[(1-x_1)(1-x_2)](\ln(1-x_1) + \ln(1-x_2)) \\
&+ F.P(x_2) \Bigg] \Bigg\} \\
&+ (N_c - n_f) \frac{N_C}{3} \left[1 + \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right] .
\end{aligned}
$$