Solutions to *Scattering amplitudes in Quantum Field Theory*

Abstract

Personal solutions to *Scattering amplitudes in Quantum Field Theory* by S. Badger, J. Henn, J. Plefka and S. Zoia. Work in progress.

1 Chapter 1

1.1 Manipulating spinor indices

(1) Expanding

$$(\sigma^\mu)_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}(\bar\sigma^\mu)^{\beta\dot\beta} = \epsilon_{\alpha 1}\epsilon_{\dot\alpha\dot 1}(\bar\sigma^\mu)^{\dot 1 1} + \epsilon_{\alpha 2}\epsilon_{\dot\alpha\dot 1}(\bar\sigma^\mu)^{\dot 1 2} + \epsilon_{\alpha 1}\epsilon_{\dot\alpha\dot 2}(\bar\sigma^\mu)^{\dot 2 1} + \epsilon_{\alpha 2}\epsilon_{\dot\alpha\dot 2}(\bar\sigma^\mu)^{\dot 2 2}$$

Using the definition of the ϵ tensor this is

$$\begin{cases} (\sigma^\mu)_{1\dot 1} = (\bar\sigma^\mu)^{\dot 2 2} \\ (\sigma^\mu)_{1\dot 2} = -(\bar\sigma^\mu)^{\dot 1 2} \\ (\sigma^\mu)_{2\dot 1} = -(\bar\sigma^\mu)^{\dot 2 1} \\ (\sigma^\mu)_{2\dot 2} = (\bar\sigma^\mu)^{\dot 1 1} \end{cases}$$

We see that clearly $\sigma^0 = \bar\sigma^0$ and $\sigma^i = -\bar\sigma^1$, keeping in mind that Pauli matrices are traceless, and for this reason $\bar\sigma^{\dot 1 1} = -\bar\sigma^{\dot 2 2}$.

(2) Obvious from the metric signature $\eta_{\mu\nu} = (+1, -1, -1, -1)$.

(3) Use the standard manipulation

$$\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] + \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] + \frac{1}{2}\mathrm{Tr}[\bar\sigma^\nu\sigma^\mu] = \frac{1}{2}\mathrm{Tr}[\{\sigma^\mu, \bar\sigma^\nu\}]$$

Now, when at least one of the two has greek index 0, we clearly have e.g. $\{\sigma^0, \bar\sigma^\nu\} = 2\bar\sigma^\nu$, when both have Latin indices, then we can use the anticommutation relations between Pauli matrices $\{\sigma^i, \bar\sigma^j\} = -\{\sigma^i, \sigma^j\} = -2\delta^{ij}$. Combining together we get the desired identity, keeping in mind that the 1/2 factor multiplying Tr cancels the extra 2 factor coming from the trace of the omitted spinorial identity matrix.

(4) Multiply and trace both sides by the Pauli matrix $\bar\sigma^\rho$

$$(\sigma^\mu)_{\alpha\dot\alpha}\mathrm{Tr}[\bar\sigma^\rho\sigma_\mu] = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}(\bar\sigma^\rho)^{\dot\beta\beta}.$$

Using the trace relation derived in the previous point on the LHS and contracting indices on the RHS we get an identity.

1.2 Massless Dirac equation and Weyl spinors

(a) Start from the helicity relations. Notice that

$$P_+\psi = \frac{1+\gamma_5}{2}\psi = \psi \to \gamma_5\psi = \psi$$

From the eq. (1.24) this implies that, defining $\psi = (\chi, \xi)$, $\xi = \chi$. In this way all the helicity relations are readily verified. Using this property for u_+, we can simplify Dirac equation $\gamma^\mu k_\mu \psi = 0$ focusing only on the first two component of the equation. These are

$$\begin{pmatrix} k^0 - k^3 & -(k_1 - ik_2) \\ -(k_1 + k_2) & k^0 + k^3 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

Notice that the determinant of the matrix is 0 (using $k^2 = 0$ since $m = 0$), therefore we can focus only on the first equation and fix a normalization. In particular

$$\chi_1 = \frac{k_1 - ik_2}{k^0 - k^3}\chi_2 = \frac{\sqrt{k^+}e^{-i\phi(k)}}{\sqrt{k^-}}\chi_2 \tag{1}$$

Choosing $\chi_2 = \sqrt{k^-}e^{i\phi(k)}/\sqrt{2}$ we find u_+. Analogously we find u_-.

(b) First, using $\{\gamma_0, \gamma_5\} = 0$ we notice that $P_+\gamma_5 = \gamma_5 P_-$. Because, in addition, $P_+^\dagger = P_+$ and $P_-^\dagger = P_-$, the helicity relations for the conjugate expression are the same with $+$ and $-$ exchanged.

(c) Using $\gamma_0^2 = \mathbb{1}$ the transformation matrix becomes

$$U = \frac{1}{\sqrt{2}}(\mathbb{1} - i\gamma^1\gamma^2\gamma^3) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = \frac{1}{\sqrt{2}}(\mathbb{1} - \gamma^0\gamma^5) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}.$$

Applying the chiral transformation on the gamma matrices we see that

$$\gamma_C^0 = \gamma_D^5 \gamma_C^i = \gamma_D^i \qquad\qquad \gamma_C^5 = -\gamma_D^0.$$

Finally, applying U on $u_+ = (\chi, \chi)$ and on $u_- = (\chi, -\chi)$ we find $u_{+,C} = (0, \sqrt{2}\chi)$ and $u_{-,C} = (\sqrt{2}\chi, 0)$.

(d) In the chiral representation $(1 - \gamma_5) = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$. Consider the RHS

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\right],$$

since in the chiral representation all matrices are off-diagonal, the last product is

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho \begin{pmatrix} 0 & 0 \\ \bar{\sigma}^\tau & 0 \end{pmatrix}\right]$$

Rerunning the same argument we find

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\tau & 0 \\ 0 & 0 \end{pmatrix}\right] = \mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\tau].$$

1.3 SU(N_c) identities

(a) Trivial.

(b) Apply equation (1.46) twice on each term. On the first, for example, we get

$$[T^a, [T^b, T^c]] = [T^a, f^{bce}T^e] = f^{bce}[T^a, T^e] = f^{bce}f^{aeg}T^g,$$

where all coefficients have been omitted. Now, collecting T^g on all the three terms, we find the desired expression using that T^a matrices are a basis of the algebra.

(c) Following the hint, we take a generic matrix M and expand it on a base of the corresponding vector space. Clearly, the dimension of the space is N_c^2, and since the dimension of the algebra is $N_c^2 - 1$, it takes one more linear independent matrix to get a basis. Since the identity matrix is not traceless, we can take this linear independent matrix to be $\mathbb{1}$. Therefore, in general

$$M = \sum_a c_a T^a + d\mathbb{1}.$$

where, tracing M with the different elements of the basis we find for the coefficients of the expansion

$$c_a = \mathrm{Tr}[T^a M] \qquad\qquad \mathrm{Tr}[M] = N_c d.$$

Substituting back in the expansion we get

$$M = \sum_a \mathrm{Tr}[T^a M]T^a + \frac{\mathrm{Tr}[M]}{N_c}\mathbb{1}.$$

Now we rewrite the expression in component form with rows i_1 and columns j_1

$$(M)_{i_1}^{j_1} = \sum_a (T^a)_k^l (M)_l^k \, (T^a)_{i_1}^{j_1} + \frac{1}{N_c}(M)_k^k \, \delta_{i_1}^{j_1}. \tag{2}$$

The expression is evaluated on the canonical basis of the vector space, made up of matrices with only a non-zero term in the $i_2 - j_2$ position, that in component form are $(E_{i_2 j_2})_{i_1}^{j_1} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1}$. Substituting

$$\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} = \sum_a (T^a)_{i_1}^{j_1}(T^a)_{i_2}^{j_2} + \frac{1}{N_c}(\delta)_{i_2}^{j_2}\,\delta_{i_1}^{j_1}.$$

This is the completeness relation.