Solutions to *Scattering Amplitudes in Quantum Field Theory*

Abstract

Personal solutions to *Scattering Amplitudes in Quantum Field Theory* by S. Badger, J. Henn, J. Plefka and S. Zoia. Work in progress.

1 Chapter 1

1.1 Manipulating spinor indices

(1) Expanding

$$(\sigma^\mu)_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}(\bar\sigma^\mu)^{\beta\dot\beta} = \epsilon_{\alpha 1}\epsilon_{\dot\alpha\dot 1}(\bar\sigma^\mu)^{\dot 1 1} + \epsilon_{\alpha 2}\epsilon_{\dot\alpha\dot 1}(\bar\sigma^\mu)^{\dot 1 2} + \epsilon_{\alpha 1}\epsilon_{\dot\alpha\dot 2}(\bar\sigma^\mu)^{\dot 2 1} + \epsilon_{\alpha 2}\epsilon_{\dot\alpha\dot 2}(\bar\sigma^\mu)^{\dot 2 2}$$

Using the definition of the ϵ tensor this is

$$\begin{cases} (\sigma^\mu)_{1\dot 1} = (\bar\sigma^\mu)^{\dot 2 2} \\ (\sigma^\mu)_{1\dot 2} = -(\bar\sigma^\mu)^{\dot 1 2} \\ (\sigma^\mu)_{2\dot 1} = -(\bar\sigma^\mu)^{\dot 2 1} \\ (\sigma^\mu)_{2\dot 2} = (\bar\sigma^\mu)^{\dot 1 1} \end{cases}$$

We see that clearly $\sigma^0 = \bar\sigma^0$ and $\sigma^i = -\bar\sigma^1$, keeping in mind that Pauli matrices are traceless, and for this reason $\bar\sigma^{\dot 1 1} = -\bar\sigma^{\dot 2 2}$.

(2) Obvious from the metric signature $\eta_{\mu\nu} = (+1, -1, -1, -1)$.

(3) Use the standard manipulation

$$\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] + \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] + \frac{1}{2}\mathrm{Tr}[\bar\sigma^\nu\sigma^\mu] = \frac{1}{2}\mathrm{Tr}[\{\sigma^\mu, \bar\sigma^\nu\}]$$

Now, when at least one of the two has Greek index 0, we clearly have e.g. $\{\sigma^0, \bar\sigma^\nu\} = 2\bar\sigma^\nu$, when both have Latin indices, then we can use the anticommutation relations between Pauli matrices $\{\sigma^i, \bar\sigma^j\} = -\{\sigma^i, \sigma^j\} = -2\delta^{ij}$. Combining together we get the desired identity, keeping in mind that the $1/2$ factor multiplying Tr cancels the extra 2 factors coming from the trace of the omitted spinorial identity matrix.

(4) Multiply and trace both sides by the Pauli matrix $\bar\sigma^\rho$

$$(\sigma^\mu)_{\alpha\dot\alpha}\mathrm{Tr}[\bar\sigma^\rho\sigma_\mu] = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}(\bar\sigma^\rho)^{\dot\beta\beta}.$$

Using the trace relation derived in the previous point on the LHS and contracting indices on the RHS we get an identity.

1.2 Massless Dirac equation and Weyl spinors

(a) Start from the helicity relations. Notice that

$$P_+\psi = \frac{1+\gamma_5}{2}\psi = \psi \to \gamma_5\psi = \psi$$

From the eq. (1.24) this implies that, defining $\psi = (\chi, \xi)$, $\xi = \chi$. In this way, all the helicity relations are readily verified. Using this property for u_+, we can simplify the Dirac equation $\gamma^\mu k_\mu \psi = 0$ focusing only on the first two components of the equation. These are

$$\begin{pmatrix} k^0 - k^3 & -(k_1 - ik_2) \\ -(k_1 + k_2) & k^0 + k^3 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

Notice that the determinant of the matrix is 0 (using $k^2 = 0$ since $m = 0$), therefore we can focus only on the first equation and fix a normalization. In particular

$$\chi_1 = \frac{k_1 - ik_2}{k^0 - k^3}\chi_2 = \frac{\sqrt{k^+}e^{-i\phi(k)}}{\sqrt{k^-}}\chi_2$$

Choosing $\chi_2 = \sqrt{k^-}e^{i\phi(k)}/\sqrt{2}$ we find u_+. Analogously we find u_-.

(b) First, using $\{\gamma_0, \gamma_5\} = 0$ we notice that $P_+\gamma_5 = \gamma_5 P_-$. Because, in addition, $P_+^\dagger = P_+$ and $P_-^\dagger = P_-$, the helicity relations for the conjugate expression are the same with $+$ and $-$ exchanged.

(c) Using $\gamma_0^2 = \mathbb{1}$ the transformation matrix becomes

$$U = \frac{1}{\sqrt{2}}(\mathbb{1} - i\gamma^1\gamma^2\gamma^3) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = \frac{1}{\sqrt{2}}(\mathbb{1} - \gamma^0\gamma^5) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}.$$

Applying the chiral transformation to the gamma matrices we see that

$$\gamma_C^0 = \gamma_D^5 \gamma_C^i = \gamma_D^i \qquad\qquad \gamma_C^5 = -\gamma_D^0.$$

Finally, applying U on $u_+ = (\chi, \chi)$ and on $u_- = (\chi, -\chi)$ we find $u_{+,C} = (0, \sqrt{2}\chi)$ and $u_{-,C} = (\sqrt{2}\chi, 0)$.

(d) In the chiral representation $(1 - \gamma_5) = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$. Consider the RHS

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\right],$$

since in the chiral representation, all matrices are off-diagonal, the last product is

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\begin{pmatrix} 0 & 0 \\ \bar{\sigma}^\tau & 0 \end{pmatrix}\right]$$

Rerunning the same argument we find

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\tau & 0 \\ 0 & 0 \end{pmatrix}\right] = \mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\tau].$$

1.3 SU(N_c) identities

(a) Trivial.

(b) Apply equation (1.46) twice on each term. On the first, for example, we get

$$[T^a, [T^b, T^c]] = [T^a, f^{bce}T^e] = f^{bce}[T^a, T^e] = f^{bce}f^{aeg}T^g,$$

where all coefficients have been omitted. Now, collecting T^g on all three terms, we find the desired expression using that T^a matrices are a basis of the algebra.

(c) Following the hint, we take a generic matrix M and expand it on a base of the corresponding vector space. Clearly, the dimension of the space is N_c^2, and since the dimension of the algebra is $N_c^2 - 1$, it takes one more linear independent matrix to get a basis. Since the identity matrix is not traceless, we can take this linear independent matrix to be $\mathbb{1}$. Therefore, in general

$$M = \sum_a c_a T^a + d\mathbb{1}.$$

where, tracing M with the different elements of the basis we find for the coefficients of the expansion

$$c_a = \mathrm{Tr}[T^a M] \qquad\qquad \mathrm{Tr}[M] = N_c d.$$

Substituting back in the expansion we get

$$M = \sum_a \mathrm{Tr}[T^a M]T^a + \frac{\mathrm{Tr}[M]}{N_c}\mathbb{1}.$$

Now we rewrite the expression in component form with rows i_1 and columns j_1

$$(M)_{i_1}^{j_1} = \sum_a (T^a)_k^l (M)_l^k (T^a)_{i_1}^{j_1} + \frac{1}{N_c}(M)_k^k \delta_{i_1}^{j_1}.$$

The expression is evaluated on the canonical basis of the vector space, made up of matrices with only a non-zero term in the $i_2 - j_2$ position, that in component form are $(E_{i_2 j_2})_{i_1}^{j_1} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1}$. Substituting

$$\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} = \sum_a (T^a)_{i_1}^{j_1}(T^a)_{i_2}^{j_2} + \frac{1}{N_c}(\delta)_{i_2}^{j_2}\delta_{i_1}^{j_1}.$$

This is the completeness relation.

1.4 Casimir operators (and another identity)

(a) To prove that it is a Casimir operator we calculate the commutator with a generic generator T^b.

$$\sum_a [T^a T^a, T^b] = \sum_a [T^a, T^b]T^a + T^a[T^a, T^b] = \sum_{a,c} i\sqrt{2} f^{abc}(T^c T^a + T^a T^c) = 0,$$

where in the last line we used that the product of an antisymmetric tensor with a symmetric tensor is 0.

(b) Consider first C_F. Using the same trick as exercise 1.3 we show that

$$C_F = \frac{\mathrm{Tr}[T_F^a T_F^a]}{N_c}.$$

The trace can be computed in two ways. First, employing the orthonormality relation (1.47), we can find straightforwardly $\mathrm{Tr}\,(T^a T^a) = N_c^2 - 1$. Secondly, using the completeness relation. The trace in component form is $\mathrm{Tr}[T_F^a T_F^a] = (T_F^a T_F^a)_i^i = (T_F^a)_i^j (T_F^a)_j^i$. This suggests naturally applying the completeness relation

$$\mathrm{Tr}[T_F^a T_F^a] = (T_F^a)_i^j (T_F^a)_j^i = \delta_i^i \delta_j^j - \frac{1}{N_c}\delta_i^j \delta_j^i = N_c^2 - \frac{1}{N_c} N_c = N_c^2 - 1.$$

Therefore

$$C_F = \frac{N_c^2 - 1}{N_c}.$$

Analogously, for the adjoint representation

$$C_A = \frac{\mathrm{Tr}\,(T_A^a T_A^a)}{N_c^2 - 1}.$$

Notice that there's no analogous completeness relation in the adjoint representation since T_A^a are only $N_C^2 - 1$ in a space with dimension $(N_c^2 - 1)(N_c^2 - 1)$. However, we can still calculate the trace using the definition of the adjoint representation

$$\mathrm{Tr}\,(T_A^a T_A^a) = -2\mathrm{Tr}\,(f^a f^a) = -2 f^{abc} f^{abc}$$

Using the definition of the structure constants we have

$$\mathrm{Tr}\,(T_A^a T_A^a) = \mathrm{Tr}\left(T^a[T^b, T^c]\right) \mathrm{Tr}\left(T^a[T^b, T^c]\right),$$

where the generators on the RHS are now in the fundamental representation. Now we can simplify the expression using the antisymmetry in abc, however, it is also instructive to proceed with the calculation from this point using only the completeness relation. This makes it possible to derive another identity among structure constants. First, by expanding commutators and renaming indices, we get

$$\mathrm{Tr}\,(T_A^a T_A^a) = 2\left\{\mathrm{Tr}\left(T^a T^b T^c\right) \mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^a T^b T^c\right) \mathrm{Tr}\left(T^a T^c T^b\right)\right\},$$

which, collecting and rearranging in component form, is

$$\mathrm{Tr}\,(T_A^a T_A^a) = 2(T^a)_{i_1}^{j_1} (T^a)_{i_2}^{j_2} \left[(T^b)_{j_1}^{k_1} (T^b)_{j_2}^{k_2} (T^c)_{k_1}^{i_1} (T^c)_{k_2}^{i_2} - (T^b)_{j_1}^{k_1} (T^b)_{k_2}^{i_2} (T^c)_{k_1}^{i_1} (T^c)_{j_2}^{k_2}\right].$$

Applying the completeness relation on the two factors inside the square brackets, expanding and collecting separately factors proportional to $\delta_{j_1}^{i_1}\delta_{j_2}^{i_2}$ and to $\delta_{j_1}^{i_2}\delta_{j_2}^{i_1}$, we are left with

$$\mathrm{Tr}\,(T_A^a T_A^a) = 2(T^a)_{i_1}^{j_1} (T^a)_{i_2}^{j_2} \left[-N_c \delta_{j_1}^{i_2} \delta_{j_2}^{i_1} + \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}\right].$$

Using the completeness relation also on the first product and expanding we get

$$\mathrm{Tr}\,(T_A^a T_A^a) = 2N_c(N_c^2 - 1),$$

and, hence,

$$C_A = 2N_c.$$

Notice that by collecting N_c in the last square bracket we have again a completeness identity. This gives the following identity

$$\mathrm{Tr}\left(T^a[T^b, T^c]\right) \mathrm{Tr}\left(T^a[T^b, T^c]\right) = -2N_c \mathrm{Tr}\left(T^a T^b\right) \mathrm{Tr}\left(T^a T^b\right).$$

Writing the LHS in terms of the structure constants we find

$$-2\mathrm{Tr}\left(T^a T^d\right) \mathrm{Tr}\left(T^a T^e\right) f^{bcd} f^{bce} = -2N_c \mathrm{Tr}\left(T^a T^b\right) \mathrm{Tr}\left(T^a T^b\right).$$

which gives the identity

$$f^{abc} f^{abd} = N_c \delta^{cd}.$$

1.5 Spinor identities

(a) From the definitions in the chiral representation we have immediately

$$[i|\gamma^\mu|j\rangle = \begin{pmatrix}0 & (\tilde\lambda_i)_{\dot\alpha}\end{pmatrix}\begin{pmatrix}0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\bar\sigma^\mu)^{\dot\alpha\beta} & 0\end{pmatrix}\begin{pmatrix}(\lambda_j)_\beta \\ 0\end{pmatrix} = (\tilde\lambda_i)_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}(\lambda_j)_\beta.$$

(b) Analogous.

(c) Using $(\sigma^\mu)_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon\dot\alpha\dot\beta(\bar\sigma^\mu)^{\dot\beta}\beta$ the proof is straightforward.

(d)

$$\langle i|\gamma^\mu|i] = (\tilde\lambda_i)^{\dot\alpha}(\sigma^\mu)_{\dot\alpha\alpha}(\lambda_i)^\alpha = (\sigma^\mu)_{\alpha\dot\alpha}p_i^{\dot\alpha\alpha} = (\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\alpha}p_\nu = 2\eta^{\mu\nu}p_\nu = 2p_i^\mu,$$

where we have used the definition of the helicity spinor and the trace relation from exercise 1.1.

(e) Expanding

$$\langle\lambda_1\lambda_2\rangle\lambda_3^\alpha + \langle\lambda_3\lambda_1\rangle\lambda_2^\alpha + \langle\lambda_2\lambda_3\rangle\lambda_1^\alpha = \epsilon_{\beta\gamma}(\lambda_1^\alpha\lambda_2^\beta\lambda_3^\gamma + \lambda_1^\gamma\lambda_2^\alpha\lambda_3^\beta + \lambda_1^\beta\lambda_2^\gamma\lambda_3^\alpha).$$

Using the antisymmetry of $\epsilon_{\beta\gamma}$, this is explicitly

$$\begin{aligned}\langle\lambda_1\lambda_2\rangle\lambda_3^\alpha + \langle\lambda_3\lambda_1\rangle\lambda_2^\alpha + \langle\lambda_2\lambda_3\rangle\lambda_1^\alpha &= \lambda_1^\alpha\lambda_2^1\lambda_3^2 + \lambda_1^2\lambda_2^\alpha\lambda_3^1 + \lambda_1^1\lambda_2^2\lambda_3^\alpha \\ &\quad - \lambda_1^\alpha\lambda_2^2\lambda_3^1 + \lambda_1^1\lambda_2^\alpha\lambda_3^2 + \lambda_1^2\lambda_2^1\lambda_3^\alpha\end{aligned}$$

Assigning $\alpha = 1$ and $\alpha = 2$ one can explicitly verify that all terms cancel out.

(f) Employing the first point we have

$$[i|\gamma^\mu|j\rangle[k|\gamma_\mu|l\rangle = (\tilde\lambda_i)_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}(\lambda_j)_\alpha(\tilde\lambda_k)_{\dot\beta}(\bar\sigma^\mu)^{\dot\beta\beta}(\lambda_l)_\beta.$$

Rearranging and using the usual completeness relation we are left with

$$[i|\gamma^\mu|j\rangle[k|\gamma_\mu|l\rangle = (\tilde\lambda_i)_{\dot\alpha}(\tilde\lambda_k)_{\dot\beta}(\lambda_j)_\alpha(\lambda_l)_\beta 2\epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}.$$

Contracting we get the desired identity.

1.6 Lorentz generators in the spinor-helicity formalism

(a) Consider the Lorentz generators in the real space acting on a test function $f(x)$ of the Schwartz space

$$M^{\mu\nu}f(x) = i\left(x^\mu\frac{\partial}{\partial x^\nu} - x^\nu\frac{\partial}{\partial x^\mu}\right)f(x).$$

Now let us compute the Fourier transform

$$\int[M^{\mu\nu}f(x)]e^{-ipx}d^4xi\int\left[\left(x^\mu\frac{\partial}{\partial x_\nu} - x^\nu\frac{\partial}{\partial x_\mu}\right)f(x)\right]e^{-ipx}d^4x,$$

using integration by parts (recall that since f is rapidly decreasing the border term is zero) we have

$$\begin{aligned}\int[M^{\mu\nu}f(x)]e^{-ipx}d^4x &= -i\int\left[\frac{\partial}{\partial x_\nu}(x^\mu e^{-ipx}) - \frac{\partial}{\partial x_\mu}(x^\nu e^{-ipx})\right]f(x)d^4x \\ &= -i\int\left[\delta^\mu_\nu e^{-ipx} - \delta^\nu_\mu e^{-ipx} + (-ip^\nu)x^\mu e^{-ipx} - (ip^\mu)x^\nu e^{-ipx}\right]f(x)d^4x \\ &= -i\int\left[p^\nu(-ix^\mu)e^{-ipx} - p^\mu(-ix^\nu)e^{-ipx}\right]f(x)d^4x \\ &= -i\int\left[p^\nu\frac{\partial}{\partial p_\mu} - p^\mu\frac{\partial}{\partial p_\nu}\right]e^{-ipx}f(x)d^4x \\ &= i\left[p^\mu\frac{\partial}{\partial p_\nu} - p^\nu\frac{\partial}{\partial p_\mu}\right]\tilde f(p),\end{aligned}$$

where we finally see explicitly $\tilde M^{\mu\nu}$.

(b) Consider the first case,

$$m_{\alpha\beta} = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_{\alpha\beta}i\left(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}\right) = -\frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_{\alpha\beta}\left(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}\right).$$

Focus on the first products

$$(\sigma^\mu\bar\sigma^\nu)_{\alpha\beta}p_\mu\frac{\partial}{\partial p^\nu} = (\sigma^\mu)_\alpha^{\dot\alpha}(\bar\sigma^\nu)_{\dot\alpha\beta}p_\mu\frac{\partial}{\partial p^\nu} = \lambda_\alpha\tilde\lambda^{\dot\alpha}(\bar\sigma^\nu)_{\dot\alpha\beta}\frac{\partial}{\partial p^\nu},$$
$$(\sigma^\mu\bar\sigma^\nu)_{\alpha\beta}p_\mu\frac{\partial}{\partial p^\nu} = (\sigma^\mu)_\alpha^{\dot\alpha}(\bar\sigma^\nu)_{\dot\alpha\beta}p_\nu\frac{\partial}{\partial p^\mu} = \lambda_\beta\tilde\lambda_{\dot\alpha}(\sigma^\mu)_\alpha^{\dot\alpha}\frac{\partial}{\partial p^\mu}$$

To write the last derivative in terms of the helicity spinor we can use the chain rule. We are still acting implicitly on a test function $f(p)$, but now as $f(\lambda(p))$. Notice that in principle $f(\lambda(p),\tilde\lambda(p))$ is not correct since the two helicity spinors are not independent. We can therefore choose to write the test function in terms of one of the two helicity spinors. We choose the first so that $\frac{\partial}{\partial p^\nu} = \frac{\partial\lambda^\gamma}{\partial p^\nu}\frac{\partial}{\partial\lambda^\gamma}$. To calculate the coefficient of the derivative we need to know how to write the helicity spinor in terms of the four-momentum or vice versa. Using the definition of the helicity spinors the second turns out to be the easiest to achieve. Start from the definition

$$p^{\dot\alpha\alpha} = (\bar\sigma_\mu)^{\dot\alpha\alpha}p^\mu,$$

multiply both sides by $(\sigma_\nu)_{\alpha\dot\alpha}$ and use the trace relation

$$(\sigma_\nu)_{\alpha\dot\alpha}p^{\dot\alpha\alpha} = (\sigma_\nu)_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\alpha\alpha}p^\mu = 2\eta_{\mu\nu}p^\mu = 2p_\nu \;\;\rightarrow\;\; p_\nu = \frac{1}{2}(\sigma_\nu)_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha}\lambda^\alpha.$$

This leads to

$$\frac{\partial p_\nu}{\partial\lambda^\gamma} = \frac{1}{2}(\sigma_\nu)_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha}\epsilon^{\alpha\gamma}.$$

Now, to get the desired derivative we should invert this relation. This is not obvious since the norm of the spinor is 0. To get to the final result, however, we could set it to a finite value and then take the limit. Even if this is technically possible is more practical to change strategy and start from the derivative of the spinor

$$\frac{\partial}{\partial\lambda^\gamma} = \frac{\partial p^\nu}{\partial\lambda^\gamma}\frac{\partial}{\partial p^\nu} = \frac{1}{2}(\sigma_\nu)_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha}\epsilon^{\alpha\gamma}\frac{\partial}{\partial p^\nu}.$$

But this exactly mimics the structure of the second product we have found above. Combining together we have $-2\lambda_\beta\frac{\partial}{\partial\lambda^\alpha}$. To take care of the first we can use the same steps exchanging σ with $\bar\sigma$ employing the definition of σ (exercise 1.1). Combining gives the desired expression. Analogous steps hold for the other generator.

(c) Because we consider two momenta we consider the sum of two generators. We call for simplicity the two helicity spinor λ and μ, the Lorentz generators are

$$m_{\alpha\beta} = \lambda_\alpha\frac{\partial}{\partial\lambda^\beta} + \lambda_\beta\frac{\partial}{\partial\lambda^\alpha} + \mu_\alpha\frac{\partial}{\partial\mu^\beta} + \mu_\beta\frac{\partial}{\partial\mu^\alpha}$$
$$\bar m_{\dot\alpha\dot\beta} = \tilde\lambda_{\dot\alpha}\frac{\partial}{\partial\tilde\lambda^{\dot\beta}} + \tilde\lambda_{\dot\beta}\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}} + \tilde\mu_{\dot\alpha}\frac{\partial}{\partial\tilde\mu^{\dot\beta}} + \tilde\mu_{\dot\beta}\frac{\partial}{\partial\tilde\mu^{\dot\alpha}}.$$

Now, acting on $\langle ij\rangle = \lambda^\rho\mu\rho$ with the first generator we have

$$\left(\lambda_\alpha\frac{\partial}{\partial\lambda^\beta} + \lambda_\beta\frac{\partial}{\partial\lambda^\alpha} + \mu_\alpha\frac{\partial}{\partial\mu^\beta} + \mu_\beta\frac{\partial}{\partial\mu^\alpha}\right)\lambda^\rho\mu_\rho = \lambda_\alpha\epsilon^\rho_\beta\mu\rho + \lambda_\beta\epsilon^\rho_\alpha\mu_\rho + \mu_\alpha\epsilon_{\rho\beta}\lambda^\rho + \mu_\beta\epsilon_{\rho\alpha}\lambda^\rho =$$
$$= \lambda_\alpha\mu_\beta + \lambda_\beta\mu_\alpha - \mu_\alpha\lambda_\beta - \mu_\beta\lambda_\alpha = 0,$$

where we have used the contraction rule (eq. 1.99) and the antisymmetry of ϵ. The action of the second generator gives trivially 0. The same steps can be applied to $[ij]$ while the Lorentz invariance of s_{ij} follows trivially from eq 1.101 (alternatively the Lorentz generator in the scalar representation can be used too).

1.7 Gluon polarisations

(a) Following the hint we expand $\epsilon_\pm^{\alpha\dot\alpha}(p)$ on a basis. Since helicity spinors live in a two-dimensional phase space, the tensor with two indices must live in a four-dimensional space with the obvious base built on the four helicity spinors we have

$$\epsilon_\pm^{\alpha\dot\alpha} = A_\pm\lambda^\alpha\tilde\mu^{\dot\alpha} + B_\pm\lambda^\alpha\tilde\lambda^{\dot\alpha} + C_\pm\mu^\alpha\tilde\lambda^{\dot\alpha} + D_\pm\mu^\alpha\tilde\mu^{\dot\alpha}.$$

We now apply the different conditions. First, notice that the scalar product between four-vectors can be translated into the helicity spinor formalism using the relation in equation 1.101. In this way, the gauge choice becomes (for cleanness we suppress $\pm$)

$$0 = \epsilon^{\dot\alpha\alpha}r_{\dot\alpha\alpha} = A\langle\lambda\mu\rangle[\mu\mu] + B\langle\lambda\mu\rangle[\lambda\mu] + C\langle\mu\mu\rangle[\lambda\mu] + D\langle\mu\mu\rangle[\mu\mu].$$

Since $r^2=0$, $\langle\mu\mu\rangle = [\mu\mu] = 0$ we have $B=0$. Now consider the transverse condition

$$0 = \epsilon^{\dot\alpha\alpha}p_{\dot\alpha\alpha} = A\langle\lambda\lambda\rangle[\mu\lambda] + C\langle\mu\lambda\rangle[\lambda\lambda] + D\langle\mu\lambda\rangle[\mu\lambda],$$

Since $p^2 = 0$, $\langle\lambda\lambda\rangle = [\lambda\lambda] = 0$ we have $D = 0$. We now study the conjugation relation. From the last exercise, we have the identities

$$\epsilon_+^{\dot{\alpha}\alpha}(p) = A_+\lambda^\alpha\tilde{\mu}^{\dot{\alpha}} + C_+\mu^\alpha\tilde{\lambda}^{\dot{\alpha}},$$
$$\epsilon_-^{\dot{\alpha}\alpha}(p) = A_-\lambda^\alpha\tilde{\mu}^{\dot{\alpha}} + C_-\mu^\alpha\tilde{\lambda}^{\dot{\alpha}}.$$

From the identity derived in the previous exercise

$$\epsilon_+^\mu(p) = \frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}[A_+\lambda^\alpha\tilde{\mu}^{\dot{\alpha}} + C_+\mu^\alpha\tilde{\lambda}^{\dot{\alpha}}],$$
$$\epsilon_-^\mu(p) = \frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}[A_-\lambda^\alpha\tilde{\mu}^{\dot{\alpha}} + C_-\mu^\alpha\tilde{\lambda}^{\dot{\alpha}}].$$

Then, from the conjugation relation we have

$$(\sigma^\mu)^*_{\alpha\dot{\alpha}}[A_+^*\mu^\alpha\tilde{\lambda}^{\dot{\alpha}} + C_+^*\lambda^\alpha\tilde{\mu}^{\dot{\alpha}}] = (\sigma^\mu)_{\alpha\dot{\alpha}}[A_-\lambda^\alpha\tilde{\mu}^{\dot{\alpha}} + C_-\mu^\alpha\tilde{\lambda}^{\dot{\alpha}}],$$

Restricting to $\mu = 0$, we have $\sigma^0 = \mathbb{1}$ that gives

$$A_+^* = C_-$$
$$A_- = C_+^*.$$

Finally, the light-like condition gives

$$0 = \epsilon^{\dot{\alpha}\alpha}\epsilon_{\dot{\alpha}\alpha} = 2AC\langle\lambda\mu\rangle[\lambda\mu].$$

Since we are working with a generic r, we cannot conclude that $p^\mu r_\mu \propto \langle\lambda\mu\rangle[\mu\lambda] = 00$, therefore either A or C must be 0. We have the freedom to take any choice, in particular, we impose $A_+ = 0$ and therefore, consistently, $C_- = 0$. We have therefore

$$\epsilon_+^{\alpha\dot{\alpha}}(p) = C_+\mu^\alpha\tilde{\lambda}^{\dot{\alpha}},$$
$$\epsilon_-^{\alpha\dot{\alpha}}(p) = A_-\lambda^\alpha\tilde{\mu}^{\dot{\alpha}} = C_+^*\lambda^\alpha\tilde{\mu}^{\dot{\alpha}}.$$

The last condition gives

$$-1 = \frac{1}{2}\epsilon_+^{\alpha\dot{\alpha}}(\epsilon_-)_{\alpha\dot{\alpha}} = \frac{1}{2}|C_+|^2\langle\mu\lambda\rangle[\lambda\mu] = -\frac{1}{2}|C_+|^2|\langle\mu\lambda\rangle|^2,$$

using eq. (1.119). Therefore

$$|C_+|^2 = \frac{2}{|\langle\mu\lambda\rangle|^2}.$$

In extracting the coefficient we have the freedom to choose the phase, that we take to be 1. Therefore

$$\epsilon_+^{\alpha\dot{\alpha}}(p) = \sqrt{2}\frac{\mu^\alpha\tilde{\lambda}^{\dot{\alpha}}}{\langle\mu\lambda\rangle} = -\sqrt{2}\frac{\mu^\alpha\tilde{\lambda}^{\dot{\alpha}}}{\langle\lambda\mu\rangle},$$
$$\epsilon_-^{\alpha\dot{\alpha}}(p) = \sqrt{2}\frac{\lambda^\alpha\tilde{\mu}^{\dot{\alpha}}}{\langle\mu\lambda\rangle^*} = \sqrt{2}\frac{\lambda^\alpha\tilde{\mu}^{\dot{\alpha}}}{[\lambda\mu]}.$$

(b) Using the conjugation property we have

$$\epsilon_+^\mu\epsilon_+^{*\nu} + \epsilon_-^\mu\epsilon_-^{*\nu} = \epsilon_+^\mu\epsilon_-^\nu + \epsilon_-^\mu\epsilon_+^\nu,$$

so can just evaluate the first product and then symmetrize. We write the polarization vector in terms of the helicity spinors using the usual inversion identity

$$\epsilon_+^\mu\epsilon_-^\nu = -\frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}\frac{\tilde{\lambda}^{\dot{\alpha}}\mu^\alpha\tilde{\mu}^{\dot{\beta}}\lambda^\beta}{\langle\lambda\mu\rangle[\lambda\mu]} =$$
$$= -\frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}\frac{p^{\dot{\alpha}\beta}r^{\dot{\beta}\alpha}}{-2p\cdot r} = \frac{1}{4p\cdot r}(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}(\bar{\sigma}^\rho)^{\dot{\alpha}\beta}(\bar{\sigma}^\tau)^{\dot{\beta}\alpha}p_\rho r_\tau = \frac{1}{4p\cdot r}\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\tau]p_\rho r_\tau.$$

The trace can be calculated using the last point of exercise 1.2 and the usual trace identities of gamma matrices that every QFT student knows by heart

$$\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\tau] = \frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\tau(\mathbb{1}-\gamma^5)] = 2(\eta^{\mu\rho}\eta^{\nu\tau} + \eta^{\mu\tau}\eta^{\rho\nu} - \eta^{\mu\nu}\eta^{\rho\tau} + i\epsilon^{\mu\rho\nu\tau})$$

The last term is antisymmetric in $\mu\nu$ and $\rho\tau$ so, be it because it must be summed to a symmetric term or because it is multiplied by a symmetric tensor, gives no contribution. Combining together

$$\sum_h \epsilon_h^\mu\epsilon_h^{*\nu} = \frac{4}{4p\cdot r}[p^\mu p^\nu + p^\nu r^\mu - (pr)\eta^{\mu\nu}] = -\eta^{\mu\nu} + \frac{p^\mu r^\nu + r^\mu p^\nu}{p\cdot r}.$$

1.8 Colour-ordered Feynman rules

Starting from the standard Feynman rules we rewrite the products of structure constants

$$f^{a_1a_2e}f^{a_3a_4e} = \frac{-i}{\sqrt{2}}\mathrm{Tr}(T^{a_1},[T^{a_2},T^e])\frac{-i}{\sqrt{2}}\mathrm{Tr}(T^{a_3},[T^{a_4},T^e]) = -\frac{1}{2}\mathrm{Tr}(T^{a_1}T^{a_2}T^e - T^{a_1}T^eT^{a_2})\mathrm{Tr}(T^{a_3}T^{a_4}T^e - T^{a_3}T^eT^{a_4}),$$

the result is made up of four terms, we extract the trace and its colour-ordering from the first and then the others can be subsequently guessed. Using equation (1.136),

$$\begin{aligned}\mathrm{Tr}\left(T^{a_1}T^{a_2}T^e\right)\mathrm{Tr}\left(T^{a_3}T^{a_4}T^e\right) &= \mathrm{Tr}\left(T^{a_2}T^eT^{a_1}\right)\mathrm{Tr}\left(T^{a_4}T^eT^{a_3}\right) = (T^{a_2}T^eT^{a_1})^i_i(T^{a_4}T^eT^{a_3})^j_j = \\ &= (T^{a_2}T^{a_3})^j_i(T^{a_4}T^{a_1})^i_j - \frac{1}{N_c}(T^{a_2}T^{a_3})^i_i(T^{a_4}T^{a_1})^j_j = \mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) - \frac{1}{N_c}\delta^{a_1a_2}\delta^{a_4a_3}.\end{aligned}$$

We found the ordering 1234 (modulo cyclic permutations). Therefore, the ordered terms coming from the first term, together with their sign, are

$$\begin{aligned}&+1234\\&-1243\\&-2134\\&+2143.\end{aligned}$$

We also notice from this list that terms proportional to the Dirac delta cancel out. For example, the first cancels with the third. The second term comes from replacing 2 with 3, 3 with 4 and 4 with 1. Hence, the four colour-ordered terms are

$$\begin{aligned}&+1342\\&-1324\\&-3142\\&+3124.\end{aligned}$$

Finally, the last term gives

$$\begin{aligned}&+1423\\&-1432\\&-4123\\&+4132.\end{aligned}$$

Notice that, as expected, there are $(4-1)! = 6$ independent orderings and the three terms share pairs of the same ordered traces. Extracting, for example, the 1234 ordering (first trace of the first term and third trace of the third term), we find the desired vertex

$$-\frac{i}{2}g^2[\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - (\eta_{\mu_1\mu_2}\eta_{\mu_4\mu_3} - \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4})] = -\frac{i}{2}g^2[2\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \eta_{\mu_1\mu_2}\eta_{\mu_4\mu_3}].$$

1.9 Independent gluon partial amplitudes

Factoring out the most obvious parity redundant amplitudes, we are left with

$$\begin{aligned}&1^+2^+3^+4^+\\&1^+2^+3^+4^-\\&1^+2^+3^-4^-\\&1^+2^-3^-4^-\\&1^+2^-3^+4^-\\&1^+2^-3^-4^+\\&1^+2^-3^+4^+\\&1^+2^+3^-4^+.\end{aligned}$$

The third and the sixth are related by cyclicity and reindexing (this passage will be from now on left implicit). So are the second, the fifth and the last two. By parity and cyclicity, the second is also related to the third. The remaining are

$$\begin{aligned}&1^+2^+3^+4^+\\&1^+2^+3^+4^-\\&1^+2^+3^-4^-\\&1^+2^-3^-4^+.\end{aligned}$$

The first two must be independent since the number of gluons with negative helicity is different. The last two can be related by the photon decoupling identity:

$$\begin{aligned}
&(++--)+(++--)+(+-+-)+(+--+)+(+--+)+(+-+-)=0\\
&2(++--)+2(+-+-)+2(+--+)=0\\
&2(+--+)+(+-+-)=0.
\end{aligned}$$

Hence the independent amplitudes are three.
Now we move on to the case of five gluons. Following the pattern arising in the four gluons case, we simply write those amplitudes with no negative helicity, one negative helicity and two negative helicities. The others are related by parity and conjugation. In the case of two negative helicities, we have the case where the two negative ones are contiguous or separated by one positive helicity. Hence the surviving amplitudes are

$$\begin{aligned}
&+++++\\
&++++-\\
&+++--\\
&++-+-\,.
\end{aligned}$$

The last two must be related by the decoupling identity, for obvious combinatorial reasons.

1.10 The $\overline{\text{MHV}}_3$ amplitude

Applying Feynman rules

$$A(1^+,2^+,3^-)=\epsilon^{\mu_1}_{1,+}\epsilon^{\mu_2}_{2,+}\epsilon^{\mu_3}_{3,-}\frac{ig}{\sqrt{2}}[(p_1-p_2)^{\mu_3}\eta^{\mu_1\mu_2}+(p_2-p_3)^{\mu_1}\eta^{\mu_2\mu_3}+(p_3-p_1)^{\mu_2}\eta^{\mu_3\mu_1}].$$

Using the same choice for the reference spinors, the first term is 0. Therefore the momenta part of the vertex will be multiplied only by the 1 and 2 polarization vectors, hence we use energy conservation to cancel p_3 and apply transversality.

$$\begin{aligned}
A(1^+,2^+,3^-)&=\epsilon^{\mu_1}_{1,+}\epsilon^{\mu_2}_{2,+}\epsilon^{\mu_3}_{3,-}\frac{ig}{\sqrt{2}}[(p_1+2p_2)^{\mu_1}\eta^{\mu_2\mu_3}-(p_2+2p_1)^{\mu_2}\eta^{\mu_3\mu_1}]\\
&=i\sqrt{2}g[(p_2\epsilon_{1,+})(\epsilon_{2,+}\epsilon_{3,-})-(p_1\epsilon_{2,+})(\epsilon_{1,+}\epsilon_{3,-})]
\end{aligned}$$

To calculate the product between the momenta and the polarization we use $(ab)=\frac{1}{2}a_{\dot{\alpha}\alpha}b^{\dot{\alpha}\alpha}$. In this way we find

$$p_2\epsilon_{1,+}=-\frac{1}{\sqrt{2}}\frac{[21]\langle 2\mu\rangle}{\langle 1\mu\rangle}\qquad\qquad p_1\epsilon_{2,+}=-\frac{1}{\sqrt{2}}\frac{[12]\langle 1\mu\rangle}{\langle 2\mu\rangle}.$$

The contractions between polarization can be calculated according to eqq. 1.159. Combining

$$\begin{aligned}
A(1^+,2^+,3^-)&=i\sqrt{2}g\left[-\frac{1}{\sqrt{2}}\frac{[21]\langle 2\mu\rangle}{\langle 1\mu\rangle}\left(-\frac{\langle 3\mu\rangle[2\mu]}{\langle 2\mu\rangle[3\mu]}\right)+\frac{1}{\sqrt{2}}\frac{[12]\langle 1\mu\rangle}{\langle 2\mu\rangle}\left(-\frac{\langle 3\mu\rangle[1\mu]}{\langle 1\mu\rangle[3\mu]}\right)\right]=\\
&=ig\frac{[12]\langle 3\mu\rangle}{\langle 1\mu\rangle\langle 2\mu\rangle[3\mu]}(\langle 2\mu\rangle[2\mu]+[1\mu]\langle 1\mu\rangle)
\end{aligned}$$

We could now apply Fierz identity twice as in the example, or the conservation of energy formula (eq. 1.117) in the case $a=b=\mu$. This gives

$$\langle 2\mu\rangle[2\mu]+\langle 1\mu\rangle[1\mu]=-\langle\mu 2\rangle[2\mu]-\langle\mu 1\rangle[1\mu]=\langle\mu 3\rangle[3\mu]=-\langle 3\mu\rangle[3\mu].$$

Substituting

$$A(1^+,2^+,3^-)=-ig\frac{[12]\langle 3\mu\rangle^2}{\langle 1\mu\rangle\langle 2\mu\rangle}=-ig\frac{[12]^2}{[23][31]}.$$

In the last passage, we used three-point momentum conservation as in equation 1.176.

1.11 Four-point quark-gluon scattering

We consider two distinct diagrams. The u-channel diagram is not included, since it would be the same as the t-channel with 4 and 3 exchanged, contributing to $A(1^-_q,2^+_q,4^+,3^-)$. Notice that the gluon vertex in the first diagram involves the momenta p_3, p_4 and p_3+p_4, we therefore expect that by choosing as reference momenta $r_3=p_4$ and $r_4=p_3$ the diagram vanishes. This is true since with this choice all the Lorentz invariant products that arise from the vertex are zero

$$\epsilon_{3,-}\cdot\epsilon_{4,+}=\frac{\langle 34\rangle[44]}{\langle 43\rangle[34]}=0\qquad\qquad p_3\cdot\epsilon_{4,+}=-\frac{\sqrt{2}}{2}\frac{\langle 43\rangle[33]}{\langle 43\rangle}=0\qquad\qquad p_4\cdot\epsilon_{3,-}=0.$$

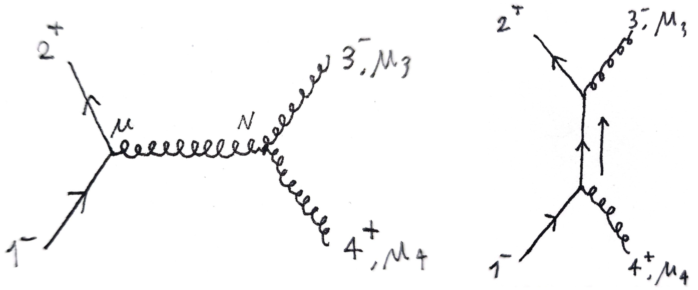

Figure 1: Diagrams

Consider now the second diagram. The direction of the momentum q flowing in the internal leg is shown by the arrow. Writing the conservation of momentum equation on the superior vertex we get $+q - p_2 - p_3 = 0$, thus $q = p_2 + p_3$. Applying Feynman rules (for external fermions use eqq. 1.105-106) the amplitude is

$$A^{tree}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = \left(\frac{ig}{\sqrt{2}}\right)^2 [2|\gamma_{\mu_3} \frac{i(\not{p}_2 + \not{p}_3)}{s_{23}} \gamma_{\mu_4} |1\rangle \epsilon^{\mu_3}_{3,-} \epsilon^{\mu_4}_{4,+} = -\frac{ig^2}{2s_{23}} [2|\not{\epsilon}_{3,-}(\not{p}_2 + \not{p}_3)\not{\epsilon}_{4,+}|1\rangle$$

To calculate explicitly the amplitude in terms of helicity spinors we write slashed momenta as in eq. 1.107 and use the following analogue expressions for slashed polarization vectors

$$\not{\epsilon}_{i,+} = -\sqrt{2}\frac{|i]\langle\mu| + |\mu\rangle[i|}{\langle i\mu\rangle}, \qquad \not{\epsilon}_{i,-} = \sqrt{2}\frac{|\mu]\langle i| + |i\rangle[\mu|}{[i\mu}.$$

Thus

$$[2|\not{\epsilon}_{3,-} = \sqrt{2}\frac{[24]}{[34]}\langle 3|, \qquad \not{\epsilon}_{4,+}|1\rangle = -\sqrt{2}\frac{\langle 31\rangle}{\langle 43\rangle}|4].$$

Substituting

$$A^{tree}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = \frac{ig^2}{s_{23}}\frac{[24]\langle 31\rangle}{[34]\langle 43\rangle}\langle 3|(|2\rangle[2| + |2]\langle 2| + |3\rangle[3| + |3]\langle 3|)|4] = ig^2\frac{[24]\langle 31\rangle\langle 32\rangle[24]}{\langle 23\rangle[32][34]\langle 43\rangle}.$$

Since we know how the result must finally look like we start by multiplying and dividing for spinorial products to single out the final denominator

$$A^{tree}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = -ig^2\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[24]\langle 31\rangle\langle 32\rangle[24]\langle 12\rangle\langle 41\rangle}{[32][34]}.$$

To simplify the result we have both the conservation of energy relation used in the previous exercise and the relations implied by the equivalent definitions of the Mandelstam invariants. For example $s_{12} = s_{34}$ implies $\langle 12\rangle[12] = \langle 34\rangle[34]$. Firstly, using conservation of energy, $[24]\langle 12\rangle = -\langle 13\rangle[34]$

$$A^{tree}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = ig^2\frac{1}{\langle 13\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 31\rangle\langle 32\rangle[24]\langle 41\rangle}{[32]}.$$

Similarly, $\langle 32\rangle[24] = \langle 31\rangle[14]$

$$A^{tree}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = -ig^2\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 13\rangle\langle 31\rangle\langle 31\rangle[14]\langle 41\rangle}{[32]}.$$

Finally, using the Mandelstam relation $[14]\langle 41\rangle = [32]\langle 23\rangle$

$$A^{tree}_{\bar{q}qgg}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+) = -ig^2\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{\langle 13\rangle\langle 31\rangle\langle 31\rangle[32]\langle 23\rangle}{[32]} = -ig^2\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.$$

We now check the consistency with the helicity assignments. The denominators give each a factor of $+1$. We show this explicitly for a generic helicity

$$\begin{aligned} -\frac{1}{2}\lambda_i^\alpha\frac{\partial}{\partial\lambda_i^\alpha}\frac{1}{\langle ai\rangle\langle ib\rangle} &= \frac{1}{2}\lambda_i^\alpha\frac{\partial}{\partial\lambda_i^\alpha}\frac{1}{\langle ia\rangle\langle ib\rangle} = \frac{1}{2}\lambda_i^\alpha\frac{\partial}{\partial\lambda_i^\alpha}\frac{1}{\lambda_1^\beta\lambda_1^\gamma\lambda_{\beta,a}\lambda_{\gamma,b}} \\ &= \frac{1}{2}\lambda_i^\alpha\left[-\frac{\delta^{\alpha\beta}}{\lambda_i^{\alpha 2}\lambda_{\beta,a}\langle ib\rangle} - \frac{\delta^{\alpha\gamma}}{\langle ia\rangle\lambda_i^{\alpha 2}\lambda_{\gamma,b}}\right] = -\frac{1}{2}2\frac{1}{\langle ia\rangle\langle ib\rangle} = \frac{1}{\langle ai\rangle\langle ib\rangle}. \end{aligned}$$

The denominators are more straightforward. Care with sign is however always needed.

$$\hat{h_1}A_4 = -\frac{1}{2}3 + 1 = -\frac{1}{2}$$
$$\hat{h_2}A_4 = -\frac{1}{2} + 1 = \frac{1}{2}$$
$$\hat{h_3}A_4 = -\frac{1}{2}4 + 1 = -1$$
$$\hat{h_4}A_4 = +1$$

2 Chapter 2

2.1 The vanishing splitting function $\mathrm{Split}_+^{\mathrm{tree}}(x, a^+, b^+)$

Consider the parametrization $|5\rangle = \sqrt{x}|P\rangle$, $|6\rangle = \sqrt{1-x}|P\rangle$, then the collinear limit for the given amplitude is

$$A_6(1^-,2^-,3^+,4^+,5^+,6^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} \xrightarrow[5||6]{} i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 4P\rangle\langle P1\rangle}\frac{1}{\sqrt{x(1-x)}\langle 56\rangle}$$

We recognize $A_5(1^-,2^-,3^+,4^+,P^+)$, hence we have found $\mathrm{Split}_-^{\mathrm{tree}}(x,5^+,6^+)$. If we proceed through the factorization theorem (eq. 2.3) we find

$$A_6(1^-,2^-,3^+,4^+,5^+,6^+) \xrightarrow[P_{56}^2\to 0]{} A_5(1^-,2^-,3^+,4^+,P^+)\mathrm{Split}_-^{\mathrm{tree}}(x,5^+,6^+) + A_5(1^-,2^-,3^+,4^+,P^-)\mathrm{Split}_+^{\mathrm{tree}}(x,5^+,6^+)$$

Since $A_5(1^-,2^-,3^+,4^+,P^-)$ is not zero, we must deduce by comparison that $\mathrm{Split}_+^{\mathrm{tree}}(x,5^+,6^+) = 0$.

2.2 Soft function in the spinor helicity formalism

Consider first the gluon case with q^+

$$S_{YM}^{[0]}(a,q^+,b) = \frac{g}{\sqrt{2}}\left(\frac{\epsilon_{q,+}p_a}{qp_a} - \frac{\epsilon_{q,+}p_b}{qp_b}\right) = \frac{g}{\sqrt{2}}\left(\sqrt{2}\frac{[qa]\langle\mu a\rangle}{\langle aq\rangle\langle q\mu\rangle[qa]} - \sqrt{2}\frac{[qb]\langle\mu b\rangle}{[qb]\langle bq\rangle\langle q\mu\rangle}\right).$$

Setting $\mu = a$ the expression simplifies to

$$S_{\mathrm{YM}}^{[0]}(a,q^+,b) = -g\frac{\langle ab\rangle}{\langle bq\rangle\langle qa\rangle} = g\frac{\langle ab\rangle}{\langle aq\rangle\langle qb\rangle}.$$

Consider now the graviton case. We use $\epsilon_{++}^{\mu\nu} = \epsilon_+^\mu\epsilon_+^\nu$ with reference momenta x and y respectively.

$$S_{\mathrm{GR}}^{[0]}(q^{++},1,\ldots,n) = k\sum_{a=1}^{n}\frac{\epsilon_{q,++}^{\mu\nu}p_{\mu,a}p_{\nu,a}}{p_a q} = k\sum_{a=1}^{n}\frac{[qa]^2\langle xa\rangle\langle ya\rangle}{\langle qx\rangle\langle qy\rangle\langle aq\rangle[qa]} = k\sum_{a=1}^{n}\frac{[qa]\langle xa\rangle\langle ya\rangle}{\langle xq\rangle\langle yq\rangle\langle aq\rangle}.$$

2.3 A $\bar{q}qgg g$ amplitude from collinear and soft limits

First, it is interesting to check the factorization properties of the known amplitude $A_{\bar{q}qgg}(1_{\bar{q}}^-,2_q^+,3^-,4^+)$. First we check the collinear limit

$$A_{\bar{q}qgg}(1_{\bar{q}}^-,2_q^+,3^-,4^+) = -ig^2\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \xrightarrow[3||4]{} -ig^2\frac{\langle 1P\rangle^3\langle 2P\rangle}{\langle 12\rangle\langle 2P\rangle\langle P1\rangle}\frac{x^2}{\sqrt{x(1-x)}\langle 34\rangle} = A(1_{\bar{q}}^+,2_q^-,P^-)\mathrm{Split}_+^{\mathrm{tree}}(3^-,4^+).$$

Consider now the soft limit for $|4\rangle = \sqrt{\delta}|q\rangle$

$$A_{\bar{q}qgg}(1_{\bar{q}}^-,2_q^+,3^-,4^+) = -ig^2\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \xrightarrow[\delta\to 0]{} -ig^2\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\frac{1}{\delta}\frac{\langle 31\rangle}{\langle 3q\rangle\langle q1\rangle} = A(1_{\bar{q}}^+,2_q^-,P^-)S_{\mathrm{YM}}^{[0]}(3^-,q^+,1_{\bar{q}}^-).$$

Notice that in the last limit, we have only used the analytic structure of the denominator since no helicity spinor involving 4 appears in the numerator.
To guess $A_{\bar{q}qggg}(1_{\bar{q}}^-,2_q^+,3^-,4^+,5^+)$ we first make use of factorization properties through the factorization theorem.

$$A_{\bar{q}qggg}(1_{\bar{q}}^-,2_q^+,3^-,4^+,5^+) \xrightarrow[3||4]{} A(P^-,5^+,1_{\bar{q}}^-,2_q^+)\mathrm{Split}_+^{\mathrm{tree}}(3^-,4^+) = -ig^3\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}.$$

According to our observation above, the amplitude we found is well-behaved under the soft limit of the gluons 4 and 5. To confirm this is the best guess we check the collinear limit for gluons 4 and 5 through factorization

$$A_{\bar{q}qggg}(1_{\bar{q}}^-,2_q^+,3^-,4^+,5^+) \xrightarrow[4||5]{} A(1_{\bar{q}}^-,2_q^+,3^-,P^+)\mathrm{Split}_-^{\mathrm{tree}}(4^+,5^+) = -ig^3\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}.$$

Therefore,

$$A^{\text{tree}}_{\bar{q}qg\ldots g}(1^-_{\bar{q}}, 2^+_q, 3^-, 4^+, \ldots, n^+) = -ig^{n-2}\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}.$$

The amplitude $A^{\text{tree}}_{\bar{q}qg\ldots g}(1^-_{\bar{q}}, 2^+_q, 3^+, 4^+, \ldots, n^+)$ must be always 0, first because of eq. 1.165, secondly because factorization always leads to amplitude 2.36, since no negative helicity can appear or the splitting function will have only positive helicities, hence be 0.

2.4 The six-gluons split-helicity NMHW amplitude

2.5 Soft limit of the six-gluon split-helicity amplitude

2.6 Mixed-helicity four-point scalar-gluon amplitude

The only possible singular momentum region is $P = p_1 + p_2$. P must be scalar for the same reason as the amplitude calculated in the textbook. Hence, applying BCFW

$$A_4(1^+, 2_\phi, 3_{\bar{\phi}}, 4^-) = A_3(\hat{1}^+, 2_\phi, -\hat{P}_{\bar{\phi}})\frac{i}{P^2 - m^2}A_3(\hat{P}_\phi, 3_{\bar{\phi}}, \hat{4}^-)$$

Using equations 2.78 and 2.79 we have

$$A_4(1^+, 2_\phi, 3_{\bar{\phi}}, 4^-) = i\frac{\langle r_1|\hat{P}|\hat{1}]}{\langle r_1\hat{1}\rangle}\frac{i}{(p_1+p_2)^2 - m^2}(-i)\frac{\langle\hat{4}|\hat{p}_3|r_4]}{[\hat{4}r_4]}.$$

We choose for the reference momenta of the polarization vectors $r_1 = \hat{p}_4$ and $r_4 = \hat{p}_1$ and use the following identities

$$\langle r_1|\hat{P}|\hat{1}] = -\langle 4|p_3|1], \qquad \langle r_1\hat{1}\rangle = \langle 41\rangle,$$
$$\langle\hat{4}|\hat{p}_3|r_1] = \langle\hat{4}|p_3|\hat{1}] = \langle 4|p_3|1], \qquad [\hat{4}\hat{1}] = [41].$$

Substituting

$$A_4(1^+, 2_\phi, 3_{\bar{\phi}}, 4^-) = -i\frac{\langle 4|p_3|1]^2}{\langle 41\rangle[41][(p_1+p_2)^2 - m^2]} = i\frac{\langle 4|p_3|1]^2}{(p_1+p_4)^2[(p_1+p_2)^2 - m^2]}.$$

2.7 Conformal algebra

We show the commutation relation by using their distributional nature.

$$[d, p^{\alpha\dot{\alpha}}]f(p) = \left[\frac{1}{2}\lambda^\beta\partial_\beta + \frac{1}{2}\tilde{\lambda}^{\dot{\beta}}\partial_{\dot{\beta}}, \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}\right]f(p)$$

Where we have already cancelled the finite term in d, since it obviously commuted. Consider the first factor

$$\begin{aligned}\left[\frac{1}{2}\lambda^\beta\partial_\beta, \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}\right]f(p) &= \frac{1}{2}\lambda^\beta\partial_\beta(\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}f(p)) - \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}\frac{1}{2}\lambda^\beta\partial_\beta f(p)\\ &= \frac{1}{2}\lambda^\beta\tilde{\lambda}^{\dot{\alpha}}\delta^\alpha_\beta f(p) + \frac{1}{2}\lambda^\beta\tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha\partial_\beta f(p) - \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}\frac{1}{2}\lambda^\beta\partial_\beta f(p) = \frac{1}{2}p^{\alpha\dot{\alpha}}f(p).\end{aligned}$$

Combining with the second factor we get

$$[d, p^{\alpha\dot{\alpha}}] = p^{\alpha\dot{\alpha}}.$$

Now consider

$$[d, k_{\alpha\dot{\alpha}}]f(p) = \left[\frac{1}{2}\lambda^\beta\partial_\beta + \frac{1}{2}\tilde{\lambda}^{\dot{\beta}}\partial_{\dot{\beta}}, \partial_\alpha\partial_{\dot{\alpha}}\right]f(p).$$

The first factor gives

$$\left[\frac{1}{2}\lambda^\beta\partial_\beta, \partial_\alpha\partial_{\dot{\alpha}}\right]f(p) = \frac{1}{2}\lambda^\beta\partial_\beta\partial_\alpha\partial_{\dot{\alpha}}f(p) - \frac{1}{2}\partial_\alpha\partial_{\dot{\alpha}}(\lambda^\beta\partial_\beta f(p)) = -\frac{1}{2}\partial_\alpha\partial_{\dot{\alpha}}f(p) = -\frac{1}{2}k_{\alpha\dot{\alpha}}f(p).$$

Combining with the second factor we get

$$[d, k_{\alpha\dot{\alpha}}] = -k_{\alpha\dot{\alpha}}.$$

Now consider

$$[d, m_{\alpha\gamma}]f(p) = \left[\frac{1}{2}\lambda^\beta\partial_\beta + \frac{1}{2}\tilde{\lambda}^{\dot{\beta}}\partial_{\dot{\beta}}, \frac{1}{2}\lambda_\alpha\partial_\gamma + \alpha\leftrightarrow\gamma\right]f(p).$$

Focusing on the first factors

$$\frac{1}{4}\left[\lambda^\beta\partial_\beta(\lambda_\alpha\partial_\gamma f(p)) - \lambda_\alpha\partial_\gamma(\lambda^\beta\partial_\beta f(p))\right] = \frac{1}{4}\left[\lambda_\alpha\partial_\gamma f + \lambda^\beta\lambda_\alpha\partial_\beta\partial_\gamma f(p) - \lambda_\alpha\partial_\gamma f(p) - \lambda_\alpha\lambda^\beta\partial_\gamma\partial_\beta f(p)\right] = 0,$$

and so the others.
Finally,

$$[k_{\alpha\dot{\alpha}}, p^{\beta\dot{\beta}}]f(p) = \left[\partial_\alpha\partial_{\dot{\alpha}}, \lambda^\beta\tilde{\lambda}^{\dot{\beta}}\right] f(p) = \partial_\alpha\partial_{\dot{\alpha}}(\lambda^\beta\tilde{\lambda}^{\dot{\beta}} f(p)) - \lambda^\beta\tilde{\lambda}^{\dot{\beta}}\partial_\alpha\partial_{\dot{\alpha}} f(p) = \partial_\alpha(\lambda^\beta\delta^{\dot{\beta}}_{\dot{\alpha}} f(p)) + \partial_\alpha(\lambda^\beta\tilde{\lambda}^{\dot{\beta}}\partial_{\dot{\alpha}} f(p)) - \lambda^\beta\tilde{\lambda}^{\dot{\beta}}\partial_\alpha\partial_{\dot{\alpha}} f(p) =$$
$$= \delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}} f(p) + \delta^{\dot{\beta}}_{\dot{\alpha}}\lambda^\beta\partial_\alpha f(p) + \delta^\beta_\alpha\tilde{\lambda}^{\dot{\beta}}\partial_{\dot{\alpha}} f(p)$$

This can be written as in equation 2.107. One can show this by components separating the cases where $\alpha = \beta$ and $\alpha \neq \beta$ using

$$\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta \longrightarrow \lambda_1 = \epsilon_{1\beta}\lambda^\beta = \lambda^2.$$

2.8 Inversion and special conformal transformations

a) The proof is straightforward

$$IP^\mu I(x) = IP\frac{x^\mu}{x^2} = I\frac{x^\mu - a^\mu}{(x-a)^2} = \frac{\frac{x^\mu}{x^2} - a^\mu}{(\frac{x^\mu}{x^2} - a)^2} = \frac{x^\mu - x^2 a^\mu}{1 - 2a^\mu x_\mu + a^2 x^2}.$$

b) Following the hint, we calculate the determinant of the transformation as

$$\det J_{I'} J_P J_I = \det J_{I'} \det J_P \det J_I = \det J_{I'} \det J_I = \det J_{I'} J_I,$$

where we have used that the determinant of any translation is 1 and the prime reminds us that we are evaluating the transformation at a new point. First we compute the Jacobian of an inversion

$$(J_I)^\mu_\nu = \frac{\partial}{\partial x^\nu}\frac{x^\mu}{x^\rho x^\sigma \eta_{\rho\sigma}} = \frac{\delta^\mu_\nu x^2 - x^\mu x^\sigma \delta^\rho_\nu \eta_{\rho\sigma} - x^\mu x^\rho \delta^\sigma_\nu \eta_{\rho\sigma}}{x^4} = \delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2}.$$

Hence the jacobian of the second inversion is

$$(J_{I'})^\mu_\nu = \delta^\mu_\nu - 2\frac{(x^\mu - a^\mu)(x_\nu - a_\nu)}{(x-a)^4\frac{(x-a)^2}{(x-a^4)}} = \delta^\mu_\nu - 2\frac{x^\mu x_\nu + a^\mu a_\nu - 2x^{(\mu}a_{\nu)}}{(x-a)^2}.$$

We are interested in its first-order approximation in a^μ

$$(J_{I'})^\mu_\nu = \delta^\mu_\nu - 2[x^\mu x_\nu + a^\mu a_\nu - 2x^{(\mu}a_{\nu)}][x^2 - 2(ax) + a^2]^{-1} = \delta^\mu_\nu - 2\frac{[x^\mu x_\nu + a^\mu a_\nu - 2x^{(\mu}a_{\nu)}]}{x^2}\left[1 + \frac{2(ax)}{x^2}\right] =$$
$$= \delta^\mu_\nu - \frac{2}{x^2}\left[x^\mu x_\nu + \frac{2(ax)x^\mu x_\nu}{x^2} - 2x^{(\mu}a_{\nu)}\right] = \delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2} - 4\frac{(ax)x^\mu x_\nu}{x^4} + 4\frac{x^{(\mu}a_{\nu)}}{x^2}.$$

The product of the Jacobians is

$$(J_{I'})^\mu_\rho (J_I)^\rho_\nu = \left[\delta^\mu_\rho - 2\frac{x^\mu x_\rho}{x^2} - 4\frac{(ax)x^\mu x_\rho}{x^4} + 4\frac{x^{(\mu}a_{\rho)}}{x^2}\right]\left[\delta^\rho_\nu - 2\frac{x^\rho x_\nu}{x^2}\right]$$
$$= \delta^\mu_\nu - 4\frac{(ax)x^\mu x_\nu}{x^4} + 4\frac{x^{(\mu}a_{\nu)}}{x^2} + 8\frac{x^\mu x_\nu (ax)}{x^4}$$