Notes on special functions

There are work-in-progress personal notes on past and ongoing studies on special functions for personal interest and research.

1 Gamma function

The **Gamma function** on real numbers is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

It may be defined as a continuation of the factorial function $n!$ from $\mathbb{N}$ to $\mathbb{R}$: notice that $\Gamma(1) = 1$ and the recursion relation

$$\Gamma(x) = (x-1)\Gamma(x-1),$$

therefore $\Gamma(n) = (n-1)!$. This gives an easy way to compute its integer value. The half-integer value can be computed easily using

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi}\frac{(2n)!}{4^n n!}$$

from a recursion relation.

1.1 Beta function

We define the **Euler's Beta function**

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

which is linked to the Gamma function by

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This relation can be shown by writing explicitly the RHS with the definition of the Gamma function as a double integral. The result can be derived by writing the double integral in the region $\mathbb{R}^+ \times \mathbb{R}^+$ from cartesian coordinates to polar coordinates.

1.2 Digamma function

The logarithmic derivative of the Gamma function defines the **Digamma function**

$$\psi(x) = \frac{d \log \Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Using the definition of the Gamma function we get the integral representation

$$\psi(x) = \frac{1}{\Gamma(x)} \int_0^{+\infty} t^{x-1} e^{-t} \log t \, dt.$$

The recursion relation of the Gamma function implies the similar recursion for the Digamma

$$\psi(x+1) = \psi(x) + \frac{1}{x},$$

hence for integer values

$$\psi(n+1) = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 + \psi(1) = H_n + \psi(1),$$

where H_n is the n-th harmonic number. For this reason, we could also define the Digamma function as a continuation of the harmonic function $H(n)$ to real numbers, translated by $\psi(1) = -\gamma_E$, called Euler's constant. Hence, for integer value the function is

$$\psi(n) = -\gamma_E + \sum_{k=1}^{n-1} \frac{1}{k},$$

while for real values, this generalizes to the series representation of the Digamma function

$$\psi(x) = -\gamma_E - \sum_{k=0}^{+\infty} \left(\frac{1}{x+k} - \frac{1}{1+k} \right).$$

The properties of the Digamma function give an expansion of the logarithm of the Gamma function around 1

$$\log \Gamma(1+x) = -\gamma_E x + \sum_{n=2}^{+\infty} \frac{(-1)^n x^n}{n} \zeta_n.$$

2 Proofs

2.1 Half-integer values of the Gamma function

We evaluate $\Gamma(n+\frac{1}{2})$ using the substitution $t = s^2$

$$\Gamma\left(n+\frac{1}{2}\right) = \int_0^{+\infty} t^{n-\frac{1}{2}} e^{-t} dt = 2 \int_0^{+\infty} s^{2n} e^{-s^2} ds.$$

This is a standard Gaussian integral that can be computed by recursion. Using integration by parts,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{4}{2n+1} \int_0^{+\infty} s^{2n+2} e^{-s^2} ds = 2\frac{2}{2n+1} \Gamma(n+\frac{3}{2})$$

we get

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{2n-1}{2} \Gamma\left(n-\frac{1}{2}\right).$$

Since

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{+\infty} e^{-s^2} ds = \int_{-\infty}^{+\infty} e^{-s^2} ds = \sqrt{\pi},$$

we have

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)\ldots 3\cdot 1}{2^n}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Finally, notice that

$$\frac{(2n-1)!!}{2^n} = \frac{(2n)!}{4^n n!},$$

that can be shown by distributing the extra 2^n factor on the RHS on the even terms of the factorial at the numerator.

2.2 The Beta function identity

The product of two Gamma functions is

$$\Gamma(x)\Gamma(y) = \int_0^{+\infty} u^{x-1} e^{-x} du \int_0^{+\infty} v^{y-1} e^{-y} dv = 4 \int_0^{+\infty} da \int_0^{+\infty} db\, a^{2x-1} b^{2y-1} e^{-(a^2+b^2)},$$

where we have defined $a = u^2$ and $b = v^2$. Changing to polar coordinates $(dadb = rdrd\theta)$

$$\Gamma(x)\Gamma(y) = 4 \int_0^{+\infty} r^{2x+2y-1} e^{-r^2} dr \int_0^{\pi/2} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta = \Gamma(x+y) 2 \int_0^{\pi/2} (\cos^2\theta)^{x-\frac{1}{2}} (1-\cos^2\theta)^{y-\frac{1}{2}} d\theta.$$

Defining $t = \cos^2\theta$ we finally find $\Gamma(x)\Gamma(y) = \Gamma(x+y)\beta(x,y)$.

2.3 Recursion of Digamma function

From the recursion relation of the Gamma function we have

$$\log \Gamma(1+x) = \log \Gamma(x) + \log x.$$

Differentiation gives

$$\psi(1+x) = \psi(x) + \frac{1}{x},$$

and therefore

$$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}.$$

2.4 Series representation of the Digamma function

Applying the recursion relation we notice that

$$\psi(x+n) - \psi(1+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k} - \psi(1) - \sum_{k=0}^{n-1} \frac{1}{1+k} = \psi(x) + \gamma_E + \sum_{k=0}^{n-1} \left(\frac{1}{x+k} - \frac{1}{1+k} \right).$$

We take the limit for $n \to \infty$. The LHS can be evaluated using Stirling's approximation $\Gamma(1+x) = \sqrt{2\pi x} x^x e^{-x} \left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right]$. Taking the logarithm and then the derivative we find

$$\psi(1+x) = \left(\frac{1}{2x} + \log x \right) \left[1 + \mathcal{O}\left(\frac{1}{x} \right) \right].$$

Therefore,

$$\psi(x+n) - \psi(1+n) = \frac{1}{2(x+n-1)} + \log(x+n-1) - \frac{1}{n} = \log n + \mathcal{O}\left(\frac{1}{n}\right) = \log\left(1 + \frac{x-1}{n}\right) + \mathcal{O}\left(\frac{1}{n}\right) = 0,$$

and the desired representation follows.

2.5 Logarithmic expansion of the Gamma function

Consider the Taylor series

$$\log\Gamma(1+x) = \sum_{k=0}^{\infty} \frac{x^n}{n!} \frac{d^n}{dx^n} \log\Gamma(1+x)\Big|_{x=0}.$$

The first coefficient is $\log\Gamma(1) = 0$. The second is $\psi(1) = -\gamma_E$. The others can be calculated using the series representation of the Digamma function. For example

$$\frac{d^2}{dx^2} \log\Gamma(1+x) = \frac{d}{dx}\psi(1+x) = -\frac{d}{dx}\left(\frac{1}{1+x+k} - \frac{1}{1+k} \right) = \sum_{k=0}^{+\infty} \frac{1}{(x+k)^2}.$$

At $x = 0$ this is ζ_2. Clearly, further derivatives gives $(-1)^n (n-1)! \zeta_n$, and therefore

$$\log\Gamma(1+x) = -\gamma_E x + \sum_{n=2}^{+\infty} \frac{(-1)^n x^n}{n} \zeta_n.$$

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