Solutions to *Scattering Amplitudes in Quantum Field Theory*

Abstract

Personal solutions to *Scattering Amplitudes in Quantum Field Theory* by S. Badger, J. Henn, J. Plefka and S. Zoia. Work in progress.

1 Chapter 1

1.1 Manipulating spinor indices

(1) Expanding

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^\mu)^{\beta\dot{\beta}} = \epsilon_{\alpha 1}\epsilon_{\dot{\alpha}\dot{1}}(\bar{\sigma}^\mu)^{\dot{1}1} + \epsilon_{\alpha 2}\epsilon_{\dot{\alpha}\dot{1}}(\bar{\sigma}^\mu)^{\dot{1}2} + \epsilon_{\alpha 1}\epsilon_{\dot{\alpha}\dot{2}}(\bar{\sigma}^\mu)^{\dot{2}1} + \epsilon_{\alpha 2}\epsilon_{\dot{\alpha}\dot{2}}(\bar{\sigma}^\mu)^{\dot{2}2}$$

Using the definition of the ϵ tensor this is

$$\begin{cases} (\sigma^\mu)_{1\dot{1}} = (\bar{\sigma}^\mu)^{\dot{2}2} \\ (\sigma^\mu)_{1\dot{2}} = -(\bar{\sigma}^\mu)^{\dot{1}2} \\ (\sigma^\mu)_{2\dot{1}} = -(\bar{\sigma}^\mu)^{\dot{2}1} \\ (\sigma^\mu)_{2\dot{2}} = (\bar{\sigma}^\mu)^{\dot{1}1} \end{cases}$$

We see that clearly $\sigma^0 = \bar{\sigma}^0$ and $\sigma^i = -\bar{\sigma}^1$, keeping in mind that Pauli matrices are traceless, and for this reason $\bar{\sigma}^{\dot{1}1} = -\bar{\sigma}^{\dot{2}2}$.

(2) Obvious from the metric signature $\eta_{\mu\nu} = (+1, -1, -1, -1)$.

(3) Use the standard manipulation

$$\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu] = \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu] + \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu] = \frac{1}{2}\mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu] + \frac{1}{2}\mathrm{Tr}[\bar{\sigma}^\nu\sigma^\mu] = \frac{1}{2}\mathrm{Tr}[\{\sigma^\mu, \bar{\sigma}^\nu\}]$$

Now, when at least one of the two has Greek index 0, we clearly have e.g. $\{\sigma^0, \bar{\sigma}^\nu\} = 2\bar{\sigma}^\nu$, when both have Latin indices, then we can use the anticommutation relations between Pauli matrices $\{\sigma^i, \bar{\sigma}^j\} = -\{\sigma^i, \sigma^j\} = -2\delta^{ij}$. Combining together we get the desired identity, keeping in mind that the 1/2 factor multiplying Tr cancels the extra 2 factor coming from the trace of the omitted spinorial identity matrix.

(4) Multiply and trace both sides by the Pauli matrix $\bar{\sigma}^\rho$

$$(\sigma^\mu)_{\alpha\dot{\alpha}}\mathrm{Tr}[\bar{\sigma}^\rho\sigma_\mu] = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^\rho)^{\dot{\beta}\beta}.$$

Using the trace relation derived in the previous point on the LHS and contracting indices on the RHS we get an identity.

1.2 Massless Dirac equation and Weyl spinors

(a) Start from the helicity relations. Notice that

$$P_+\psi = \frac{1+\gamma_5}{2}\psi = \psi \rightarrow \gamma_5\psi = \psi$$

From the eq. (1.24) this implies that, defining $\psi = (\chi, \xi)$, $\xi = \chi$. In this way all the helicity relations are readily verified. Using this property for u_+, we can simplify Dirac equation $\gamma^\mu k_\mu \psi = 0$ focusing only on the first two components of the equation. These are

$$\begin{pmatrix} k^0 - k^3 & -(k_1 - ik_2) \\ -(k_1 + k_2) & k^0 + k^3 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

Notice that the determinant of the matrix is 0 (using $k^2 = 0$ since $m = 0$), therefore we can focus only on the first equation and fix a normalization. In particular

$$\chi_1 = \frac{k_1 - ik_2}{k^0 - k^3}\chi_2 = \frac{\sqrt{k^+}e^{-i\phi(k)}}{\sqrt{k^-}}\chi_2$$

Choosing $\chi_2 = \sqrt{k^-}e^{i\phi(k)}/\sqrt{2}$ we find u_+. Analogously we find u_-.

(b) First, using $\{\gamma_0, \gamma_5\} = 0$ we notice that $P_+\gamma_5 = \gamma_5 P_-$. Because, in addition, $P_+^\dagger = P_+$ and $P_-^\dagger = P_-$, the helicity relations for the conjugate expression are the same with $+$ and $-$ exchanged.

(c) Using $\gamma_0^2 = \mathbb{1}$ the transformation matrix becomes

$$U = \frac{1}{\sqrt{2}}(\mathbb{1} - i\gamma^1\gamma^2\gamma^3) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = \frac{1}{\sqrt{2}}(\mathbb{1} - \gamma^0\gamma^5) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}.$$

Applying the chiral transformation on the gamma matrices we see that

$$\gamma_C^0 = \gamma_D^5 \gamma_C^i = \gamma_D^i \qquad\qquad \gamma_C^5 = -\gamma_D^0.$$

Finally, applying U on $u_+ = (\chi, \chi)$ and on $u_- = (\chi, -\chi)$ we find $u_{+,C} = (0, \sqrt{2}\chi)$ and $u_{-,C} = (\sqrt{2}\chi, 0)$.

(d) In the chiral representation $(1 - \gamma_5) = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$. Consider the RHS

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\right],$$

since in the chiral representation, all matrices are off-diagonal, the last product is

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\begin{pmatrix} 0 & 0 \\ \bar{\sigma}^\tau & 0 \end{pmatrix}\right]$$

Rerunning the same argument we find

$$\frac{1}{2}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau(\mathbb{1} - \gamma_5)] = \mathrm{Tr}\left[\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\tau & 0 \\ 0 & 0 \end{pmatrix}\right] = \mathrm{Tr}[\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\tau].$$

1.3 SU(N_c) identities

(a) Trivial.

(b) Apply equation (1.46) twice on each term. On the first, for example, we get

$$[T^a, [T^b, T^c]] = [T^a, f^{bce}T^e] = f^{bce}[T^a, T^e] = f^{bce}f^{aeg}T^g,$$

where all coefficients have been omitted. Now, collecting T^g on all three terms, we find the desired expression using that T^a matrices are a basis of the algebra.

(c) Following the hint, we take a generic matrix M and expand it on a base of the corresponding vector space. Clearly, the dimension of the space is N_c^2, and since the dimension of the algebra is $N_c^2 - 1$, it takes one more linear independent matrix to get a basis. Since the identity matrix is not traceless, we can take this linear independent matrix to be $\mathbb{1}$. Therefore, in general

$$M = \sum_a c_a T^a + d\mathbb{1}.$$

where, tracing M with the different elements of the basis we find for the coefficients of the expansion

$$c_a = \mathrm{Tr}[T^a M] \qquad\qquad \mathrm{Tr}[M] = N_c d.$$

Substituting back in the expansion we get

$$M = \sum_a \mathrm{Tr}[T^a M]T^a + \frac{\mathrm{Tr}[M]}{N_c}\mathbb{1}.$$

Now we rewrite the expression in component form with rows i_1 and columns j_1

$$(M)_{i_1}^{j_1} = \sum_a (T^a)_k^l (M)_l^k \, (T^a)_{i_1}^{j_1} + \frac{1}{N_c}(M)_k^k \, \delta_{i_1}^{j_1}.$$

The expression is evaluated on the canonical basis of the vector space, made up of matrices with only a non-zero term in the $i_2 - j_2$ position, that in component form are $(E_{i_2 j_2})_{i_1}^{j_1} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1}$. Substituting

$$\delta_{i_1}^{j_2}\delta_{i_2}^{j_1} = \sum_a (T^a)_{i_1}^{j_1}(T^a)_{i_2}^{j_2} + \frac{1}{N_c}(\delta)_{i_2}^{j_2}\,\delta_{i_1}^{j_1}.$$

This is the completeness relation.

1.4 Casimir operators (and another identity)

(a) To prove that it is a Casimir operator we calculate the commutator with a generic generator T^b.

$$\sum_a [T^a T^a, T^b] = \sum_a [T^a, T^b] T^a + T^a [T^a, T^b] = \sum_{a,c} i\sqrt{2} f^{abc} (T^c T^a + T^a T^c) = 0,$$

where in the last line we used that the product of an antisymmetric tensor with a symmetric tensor is 0.

(b) Consider first C_F. Using the same trick as exercise 1.3 we show that

$$C_F = \frac{\mathrm{Tr}[T_F^a T_F^a]}{N_c}.$$

The trace can be computed in two ways. First, employing the orthonormality relation (1.47), we can find straightforwardly $\mathrm{Tr}\,[T^a T^a] = N_c^2 - 1$. Secondly, using the completeness relation. The trace in component form is $\mathrm{Tr}[T_F^a T_F^a] = (T_F^a T_F^a)_i^i = (T_F^a)_i^j (T_F^a)_j^i$. This suggests naturally applying the completeness relation

$$\mathrm{Tr}[T_F^a T_F^a] = (T_F^a)_i^j (T_F^a)_j^i = \delta_i^i \delta_j^j - \frac{1}{N_c} \delta_i^j \delta_j^i = N_c^2 - \frac{1}{N_c} N_c = N_c^2 - 1.$$

Therefore

$$C_F = \frac{N_c^2 - 1}{N_c}.$$

Analogously, for the adjoint representation

$$C_A = \frac{\mathrm{Tr}\,[T_A^a T_A^a]}{N_c^2 - 1}.$$

Notice that there's no analogous completeness relation in the adjoint representation since T_A^a are only $N_C^2 - 1$ in a space with dimension $(N_c^2 - 1)(N_c^2 - 1)$. However, we can still calculate the trace using the definition of the adjoint representation

$$\mathrm{Tr}\,[T_A^a T_A^a] = -2\mathrm{Tr}\,[f^a f^a] = -2 f^{abc} f^{abc}$$

Using the definition of the structure constants we have

$$\mathrm{Tr}\,[T_A^a T_A^a] = \mathrm{Tr}\left[T^a [T^b, T^c]\right] \mathrm{Tr}\left[T^a [T^b, T^c]\right],$$

where the generators on the RHS are now in the fundamental representation. Now we can simplify the expression using the antisymmetry in abc, however, it is also instructive to proceed with the calculation from this point using only the completeness relation. This makes it possible to derive another identity among structure constants. First, by expanding commutators and renaming indices, we get

$$\mathrm{Tr}\,[T_A^a T_A^a] = 2\left\{\mathrm{Tr}\left[T^a T^b T^c\right] \mathrm{Tr}\left[T^a T^b T^c\right] - \mathrm{Tr}\left[T^a T^b T^c\right] \mathrm{Tr}\left[T^a T^c T^b\right]\right\},$$

which, collecting and rearranging in component form, is

$$\mathrm{Tr}\,[T_A^a T_A^a] = 2 (T^a)_{i_1}^{j_1} (T^a)_{i_2}^{j_2} \left[(T^b)_{j_1}^{k_1} (T^b)_{j_2}^{k_2} (T^c)_{k_1}^{i_1} (T^c)_{k_2}^{i_2} - (T^b)_{j_1}^{k_1} (T^b)_{k_2}^{i_2} (T^c)_{k_1}^{i_1} (T^c)_{j_2}^{k_2}\right].$$

Applying the completeness relation on the two factors inside the square brackets, expanding and collecting separately factors proportional to $\delta_{j_1}^{i_1} \delta_{j_2}^{i_2}$ and to $\delta_{j_1}^{i_2} \delta_{j_2}^{i_1}$, we are left with

$$\mathrm{Tr}\,[T_A^a T_A^a] = 2 (T^a)_{i_1}^{j_1} (T^a)_{i_2}^{j_2} \left[-N_c \delta_{j_1}^{i_2} \delta_{j_2}^{i_1} + \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}\right].$$

Using the completeness relation also on the first product and expanding we get

$$\mathrm{Tr}\,[T_A^a T_A^a] = 2 N_c (N_c^2 - 1),$$

and, hence,

$$C_A = 2 N_c.$$

Notice that by collecting N_c in the last square bracket we have again a completeness identity. This gives the following identity

$$\mathrm{Tr}\left[T^a [T^b, T^c]\right] \mathrm{Tr}\left[T^a [T^b, T^c]\right] = -2 N_c \mathrm{Tr}\left[T^a T^b\right] \mathrm{Tr}\left[T^a T^b\right].$$

Writing the LHS in terms of the structure constants we find

$$-2\mathrm{Tr}\left[T^a T^d\right] \mathrm{Tr}\,[T^a T^e]\, f^{bcd} f^{bce} = -2 N_c \mathrm{Tr}\left[T^a T^b\right] \mathrm{Tr}\left[T^a T^b\right].$$

which gives the identity

$$f^{abc} f^{abd} = N_c \delta^{cd}.$$

1.5 Spinor identities

(a) From the definitions in the chiral representation we have immediately

$$[i|\gamma^\mu|j\rangle = \begin{pmatrix}0 & (\tilde{\lambda}_i)_{\dot{\alpha}}\end{pmatrix}\begin{pmatrix}0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & \end{pmatrix}\begin{pmatrix}(\lambda_j)_\beta \\ 0\end{pmatrix} = (\tilde{\lambda}_i)_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\lambda_j)_\beta.$$

(b) Analogous.

(c) Using $(\sigma^\mu)_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon\dot{\alpha}\dot{\beta}(\bar{\sigma}^\mu)^{\dot{\beta}}\beta$ the proof is straightforward.

(d)

$$\langle i|\gamma^\mu|i] = (\tilde{\lambda}_i)^{\dot{\alpha}}(\sigma^\mu)_{\dot{\alpha}\alpha}(\lambda_i)^\alpha = (\sigma^\mu)_{\alpha\dot{\alpha}}p_i^{\dot{\alpha}\alpha} = (\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\alpha}p_\nu = 2\eta^{\mu\nu}p_\nu = 2p_i^\mu,$$

where we have used the definition of the helicity spinor and the trace relation from exercise 1.1.

(e) Expanding

$$\langle\lambda_1\lambda_2\rangle\lambda_3^\alpha + \langle\lambda_3\lambda_1\rangle\lambda_2^\alpha + \langle\lambda_2\lambda_3\rangle\lambda_1^\alpha = \epsilon_{\beta\gamma}(\lambda_1^\alpha\lambda_2^\beta\lambda_3^\gamma + \lambda_1^\gamma\lambda_2^\alpha\lambda_3^\beta + \lambda_1^\beta\lambda_2^\gamma\lambda_3^\alpha).$$

Using the antisymmetry of $\epsilon_{\beta\gamma}$, this is explicitly

$$\begin{aligned}\langle\lambda_1\lambda_2\rangle\lambda_3^\alpha + \langle\lambda_3\lambda_1\rangle\lambda_2^\alpha + \langle\lambda_2\lambda_3\rangle\lambda_1^\alpha = {}& \lambda_1^\alpha\lambda_2^1\lambda_3^2 + \lambda_1^2\lambda_2^\alpha\lambda_3^1 + \lambda_1^1\lambda_2^2\lambda_3^\alpha \\ & - \lambda_1^\alpha\lambda_2^2\lambda_3^1 + \lambda_1^1\lambda_2^\alpha\lambda_3^2 + \lambda_1^2\lambda_2^1\lambda_3^\alpha\end{aligned}$$

Assigning $\alpha = 1$ and $\alpha = 2$ one can explicitly verify that all terms cancel out.

(f) Employing the first point we have

$$[i|\gamma^\mu|j\rangle[k|\gamma_\mu|l\rangle = (\tilde{\lambda}_i)_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\lambda_j)_\alpha(\tilde{\lambda}_k)_{\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta}(\lambda_l)_\beta. \tag{1}$$

Rearranging and using the usual completeness relation we are left with

$$[i|\gamma^\mu|j\rangle[k|\gamma_\mu|l\rangle = (\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_k)_{\dot{\beta}}(\lambda_j)_\alpha(\lambda_l)_\beta 2\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}. \tag{2}$$

Contracting we get the desired identity.

1.6 Lorentz generators in the spinor-helicity formalism

(a) Consider the Lorentz generators in the real space acting on a test function $f(x)$ of the Schwartz space

$$M^{\mu\nu}f(x) = i\left(x^\mu\frac{\partial}{\partial x^\nu} - x^\nu\frac{\partial}{\partial x^\mu}\right)f(x).$$

Now let us compute the Fourier transform

$$\int[M^{\mu\nu}f(x)]e^{-ipx}d^4xi\int\left[\left(x^\mu\frac{\partial}{\partial x_\nu} - x^\nu\frac{\partial}{\partial x_\mu}\right)f(x)\right]e^{-ipx}d^4x,$$

using integration by parts (recall that since f is rapidly decreasing the border term is zero) we have

$$\begin{aligned}\int[M^{\mu\nu}f(x)]e^{-ipx}d^4x &= -i\int\left[\frac{\partial}{\partial x_\nu}(x^\mu e^{-ipx}) - \frac{\partial}{\partial x_\mu}(x^\nu e^{-ipx})\right]f(x)d^4x \\ &= -i\int\left[\delta^\mu_\nu e^{-ipx} - \delta^\nu_\mu e^{-ipx} + (-ip^\nu)x^\mu e^{-ipx} - (ip^\mu)x^\nu e^{-ipx}\right]f(x)d^4x \\ &= -i\int\left[p^\nu(-ix^\mu)e^{-ipx} - p^\mu(-ix^\nu)e^{-ipx}\right]f(x)d^4x \\ &= -i\int\left[p^\nu\frac{\partial}{\partial p_\mu} - p^\mu\frac{\partial}{\partial p_\nu}\right]e^{-ipx}f(x)d^4x \\ &= i\left[p^\mu\frac{\partial}{\partial p_\nu} - p^\nu\frac{\partial}{\partial p_\mu}\right]\tilde{f}(p),\end{aligned}$$

where we finally see explicitly $\tilde{M}^{\mu\nu}$.