The optical theorem

1 Formal argument

Because the eigenvalue of the Hamiltonian $\hat{\mathcal{H}}$ must be real, the operator must be Hermitian. For this reason the time evolution operator $\hat{S}$, because it is its exponential, must be unitary, that is to say $S^\dagger S = \mathbb{I}$. This relation has a simple but deep consequence on the transition matrix $\hat{T}$, defined by $S = \mathbb{I} + iT$. Indeed, substituting the definition in the unitarity condition, we have $-i(T - T^\dagger) = T^\dagger T$, or

$$\boxed{2\mathrm{Im}T = T^\dagger T}. \tag{1}$$

This is the most general form of the **optical theorem**. We see now the consequences on amplitudes.

2 Consequences on amplitudes

Let us evaluate the operatorial relation we have obtained between two generic states $\langle f|$ e $|i\rangle$ introducing a completeness relation on the Fock space[1]

$$\mathbb{I} = \sum_{n=1}^{\infty} \prod_i \int \frac{d^3 q_i}{(2E_i)(2\pi)^3} |\{q\}\rangle\langle\{q\}|. \tag{2}$$

Therefore the LHS is

$$\langle f|T^\dagger T|i\rangle = \sum_{n=1}^{\infty} \prod_i \int \frac{d^3 q_i}{(2E_i)(2\pi)^3} \langle f|T^\dagger|\{q\}\rangle\langle\{q\}|T|i\rangle. \tag{3}$$

Using the definition of amplitude $iT_{if} = (2\pi)^4\delta^{(4)}(p_f - p_i)\mathcal{M}_{if}$, we have

$$\langle f|T^\dagger T|i\rangle = \sum_{n=1}^{\infty} \prod_i \int \frac{d^3 q_i}{(2E_i)(2\pi)^3} (2\pi)^4\delta^{(4)}(p_q - p_f)\mathcal{M}^*_{fq}(2\pi)^4\delta^{(4)}(p_q - p_i)\mathcal{M}_{iq}, \tag{4}$$

where $p_q = \sum q_i$. The RHS is

$$\langle f|2\mathrm{Im}T|i\rangle = 2\mathrm{Im}[(2\pi)^4\delta^{(4)}(p_f - p_i)\mathcal{M}_{if}]. \tag{5}$$

Since $\delta^{(4)}(p_q - p_f) = \delta^{(4)}(p_q - p_i)\delta^{(4)}(p_f - p_i)$, we can simplify one Dirac delta on both sides leaving us with

$$\mathrm{Im}\mathcal{M}_{if} = \frac{1}{2}\sum_{n=1}^{\infty} \prod_i \int \frac{d^3 q_i}{(2E_i)(2\pi)^4} \mathcal{M}^*_{fq}\mathcal{M}_{iq}(2\pi)^4\delta^{(4)}(p_q - p_i). \tag{6}$$

The LHS side is the imaginary part of a given amplitudes. The RHS is the same amplitude limited only to all the intermediate contributions integrated on their on-shell momenta[2]. Diagrammatically

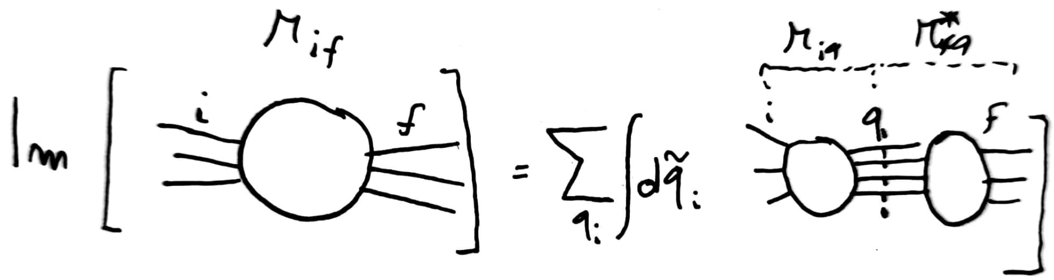

Figure 1: Diagrammatic interpretation of equation 7. The cut on the q lines means that they are on-shell

[1]The first terms explicitly are $|0\rangle\langle 0| + \int d^3\tilde{k}|k\rangle\langle k| + \int d^3\tilde{k_1}\int d^3\tilde{k_2}|k_1k_2\rangle\langle k_1k_2| + \dots$

[2]Particles of momenta q are on-shell because they come from Fock space states.

In practice, since the computation of amplitudes proceeds perturbatively, at a given order in the coupling constant g in perturbation theory, only the first terms in the Fock space states sum contribute, since any new particle in the final states carries at least one more power in g inside the new vertex.

In practice the optical theorem finds application in the case where $i = f$ (*forward scattering*). In this case the theorem is particularly meaningful since the RHS reduces to the total cross section. First, equation 6 reduces to

$$\mathrm{Im}\mathcal{M}_{ii} = \frac{1}{2}\sum_{n=1}^{\infty}\prod_{i}\int \frac{d^3 q_i}{(2E_i)(2\pi)^3}|\mathcal{M}_{iq}|^2(2\pi)^4\delta^{(4)}(p_q - p_i), \tag{7}$$

and now the RHS is manifestly proportional to the expression for the total cross section of the process $i \to \{q\}$ summed on *all possible* final states $\{q\}$. This kind of cross sections, called *inclusive*, are relevant in particle physics since often it is not possible to measure all the possible products in a particle detector, and for this reason one must sum over all the observables with all possible unmeasured final states.

Restricting to the case of two particles in the initial states and recalling the definition of the flux factor $\frac{1}{\Phi} = \frac{1}{4E_1E_2v_{rel}}$, we have the final form of the optical theorem

$$\boxed{\sigma^{tot}(2 \to \{q\}) = \frac{2}{\Phi}\mathrm{Im}\mathcal{M}(2 \to 2).} \tag{8}$$

Computationally, the origin of this result is at the core of loop amplitudes calculation. In practical term, the imaginary part applied on propagators with the ϵ prescription gives Dirac deltas that, applied on loop measures d^4k constraint the integration on such intermediate state to be on-shell, mapping loop and phase space integrals.

We can understand this explicitly considering some distributional relations. Start first from[3]

$$\lim_{\epsilon\to 0}\left[\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon}\right] = 2\pi i\delta^{(+)}(k^2 - m^2), \tag{9}$$

where $\delta^{(+)}(k^2 - m^2) := \Theta(k^0)\delta^4(k^2 - m^2)$. but the first term also is

$$\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon} = \frac{1}{k^2 - m^2 + i\epsilon} - \left(\frac{1}{k^2 - m^2 + i\epsilon}\right)^* = 2i\mathrm{Im}\frac{1}{k^2 - m^2 + i\epsilon}. \tag{10}$$

Comparing we finally get

$$2\mathrm{Im}\frac{1}{k^2 - m^2 + i\epsilon} = 2\pi\delta^{(+)}(k^2 - m^2). \tag{11}$$

Using the well-know properties of Dirac deltas

$$\int \frac{d^4k}{(2\pi)^4}2\pi\delta^{(+)}(k^2 - m^2) = \int \frac{d^3k}{(2\pi)^3(2E_i)}, \tag{12}$$

and combining together with th above identity, we are left with

$$\int \frac{d^4k}{(2\pi)^4}2\mathrm{Im}\frac{1}{k^2 - m^2 + i\epsilon} = \int \frac{d^3k}{(2\pi)^3(2E_i)}. \tag{13}$$

This is what we have anticipated: the imaginary part maps loops integrals into phase space integrals. In particles physics jargon, the imaginary part *cuts*, or constraint on-shell, intermediate states in amplitudes.

The identity can be applield to the RHS of equation 7 to rederive crudely the theorem

$$\sigma \propto \sum_{n=1}^{\infty}\prod_{i}\int \frac{d^3 q_i}{(2E_i)(2\pi)^3}|\mathcal{M}_{iq}|^2(2\pi)^4\delta^{(4)}(p_q - p_i) = \sum_{n=1}^{\infty}\prod_{i}\int \frac{d^4 q_i}{(2\pi)^4}2\mathrm{Im}\frac{1}{q_i^2 - m^2 + i\epsilon}|\mathcal{M}_{iq}|^2(2\pi)^4\delta^{(4)}(p_q - p_i), \tag{14}$$

therefore

$$\sigma(2 \to 2) \propto 2\sum_{n=1}^{\infty}\prod_{i}\int \frac{d^4 q_i}{(2\pi)^4}\mathcal{M}^*(i \to \{q\})\mathrm{Im}\frac{1}{q_i^2 - m^2 + i\epsilon}\mathcal{M}(i \to \{q\})(2\pi)^4\delta^{(4)}(p_q - p_i) = 2\mathrm{Im}M_{ii}. \tag{15}$$

Where we have extracted the imaginary part Im, from all the integrals. This is not strictly possible, since with a generic number of intermediate states, the *cutting* identity 13 must be generalized to the *Cutkosky rules*[4]. Cutting rules give an efficient method to compute total cross sections from amplitudes.

[3] It can be easily derived from the Sokhotski–Plemelj formula

[4] R. E. Cutkosky *Singularities and Discontinuities of Feynman Amplitudes* (1960)

2.1 An example in ϕ^4

Consider a $2 \to 2$ scattering between Klein-Gordon particles with a ϕ^4 interaction. The 1-loop amplitude of such process is

$$i\mathcal{M} = -i\lambda \left[1 + \frac{\lambda}{32\pi^2}\int_0^1 dx \ln\frac{M^2(s)}{\Lambda^2} + \ln\frac{M^2(t)}{\Lambda^2} + \ln\frac{M^2(u)}{\Lambda^2}\right], \tag{16}$$

where $M^2(s) = m^2 + sx(x-1)$. We would like to verify the optical theorem starting from this result. We know that the imaginary part of such amplitude will give the total cross section for the totally inclusive process with the same starting configuration. Because the equation is λ^2 order in perturbation theory, for a consistent result is sufficient to consider the cross section from the tree diagram of the same $2 \to 2$ process. Including loops or more products would increase the order (remember that to compute the cross section the amplitude must be squared, thus doubling the order of the result). Therefore the following must hold:

$$\frac{2}{\Phi}\mathrm{Im}\left(-\lambda \left[1 + \frac{\lambda}{32\pi^2}\int_0^1 dx \ln\frac{M^2(s)}{\Lambda^2} + \ln\frac{M^2(t)}{\Lambda^2} + \ln\frac{M^2(u)}{\Lambda^2}\right]\right) = \frac{\lambda^2}{32\pi s} \tag{17}$$

An imaginary part may arise when the argument of one of the logarithms becomes negative in the integration range [5]. We will now show that this is the case for the first one. $M^2(s) = sx^2 - sx + m^2 < 0$ has solution $\frac{1}{2}\left[1 - \sqrt{1 - \frac{4m^2}{s}}\right] < x < \frac{1}{2}\left[1 + \sqrt{1 - \frac{4m^2}{s}}\right]$. From basic kinematics we know that $s > 4m^2$, so that those square root are always real and the solution is always in the integration range, and therefore there are values of x in which $M^2(s)$ is negative and so the corresponding logarithm with a non-zero imaginary part. Because t and u are always negative the corresponding inequalities will never be satisfied in the integration range.

The optical theorem may be verified directly by integrating the negative part of the logarithm argument along the extrema of the solution of the inequality. Alternatively we can compute directly the imaginary part of the Feynman 1-loop diagram. We shall first follow this direction. The calculation of the amplitude involves 4 diagrams, the tree diagram and three 1-loop diagrams in all the three channel (s, t and u). As we have shown from the final result we know that only the s diagram does contribute to the imaginary part, that is

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k+\frac{p}{2})^2 - m^2 + i\epsilon}\frac{1}{(k-\frac{p}{2})^2 - m^2 + i\epsilon}, \tag{18}$$

where p is the sum of the incoming momenta p_1 and p_2. In the center of mass reference we have $p = (p, \vec{0})$, $p_1, p_2 = (\frac{p}{2}, \pm\vec{p})$, thus the propagator becomes

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k_0+\frac{p}{2})^2 - [|\vec{k}|^2 + m^2 + i\epsilon]}\frac{1}{(k_0-\frac{p}{2})^2 - [|\vec{k}|^2 + m^2 + i\epsilon]} = \tag{19}$$

$$= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k_0+\frac{p}{2})^2 - [E_k^2 + i\epsilon]}\frac{1}{(k_0-\frac{p}{2})^2 - [E_k^2 + i\epsilon]} = \tag{20}$$

$$= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k_0+\frac{p}{2} - \sqrt{E_k^2 + i\epsilon})(k_0+\frac{p}{2} + \sqrt{E_k^2 + i\epsilon})}\frac{1}{(k_0-\frac{p}{2} - \sqrt{E_k^2 + i\epsilon})(k_0-\frac{p}{2} + \sqrt{E_k^2 + i\epsilon})} = \tag{21}$$

$$= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k_0+\frac{p}{2} - E_k + i\epsilon)(k_0+\frac{p}{2} + E_k - i\epsilon)}\frac{1}{(k_0-\frac{p}{2} - E_k + i\epsilon)(k_0-\frac{p}{2} + E_k - i\epsilon)} \tag{22}$$

Consider the integration with respect to k_0. Such integral can be performed using the residue theorem taking into consideration a semicircular path passing through the real axis. To get the desired result the path must include the two poles under the real axis, that comes from the first and third parenthesis. However only the first one does contribute to the imaginary part, we will therefore ignore the other. The residue is

$$\mathrm{Res}[f(k_0), k_0 = E_k - \frac{p}{2} - i\epsilon] = \frac{1}{(k_0+\frac{p}{2} + E_k - i\epsilon)(k_0-\frac{p}{2} - E_k + i\epsilon)(k_0-\frac{p}{2} + E_k - i\epsilon)}\Big|_{k_0 = E_k - \frac{p}{2} - i\epsilon} = \tag{23}$$

$$= \frac{1}{2E_k p_0(p_0 - 2E_k + i\epsilon)} \tag{24}$$

Substituting back

$$i\delta\mathcal{M} = -\frac{(2\pi i)\lambda^2}{(2\pi)2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k p_0(p_0 - 2E_k + i\epsilon)}, \tag{25}$$

where the minus sign comes from the clockwise integration path. Moving to spherical coordinates the integral is

$$i\delta\mathcal{M} = -\frac{(2\pi i)4\pi}{(2\pi)^4 2}\lambda^2 \int_0^{+\infty} d|\vec{k}|\frac{|\vec{k}|^2}{2E_k p_0(p_0 - 2E_k + i\epsilon)}. \tag{26}$$

[5] Remember that $\mathrm{Log}z = \log z + i\mathrm{Arg}z$, so that if z is a negative number the imaginary part will be $i\pi$ or $-i\pi$

Now applying the coordinate change $E = \sqrt{|\vec{k}|^2 + m^2}$ and some simplifications, we are left with

$$\delta\mathcal{M} = -\frac{\lambda^2}{2(2\pi)^2}\frac{1}{p^0}\int_m^{+\infty} dE \frac{\sqrt{E^2 - m^2}}{p_0 - 2E + i\epsilon}. \tag{27}$$

In order to calculate the imaginary part we compute the discontinuity using Sokhotski-Plemelij as above

$$2i\mathrm{Im}\mathcal{M} = \delta\mathcal{M} - \delta\mathcal{M}^* = -\frac{\lambda^2}{2(2\pi)^2}\frac{1}{p^0}\int_m^{+\infty} dE\sqrt{E^2 - m^2}\left(\frac{1}{p_0 - 2E + i\epsilon} - \frac{1}{p_0 - 2E - i\epsilon}\right) \tag{28}$$

$$= \frac{\lambda^2(2\pi i)}{2(2\pi)^2}\frac{1}{p^0}\int_m^{+\infty} dE\sqrt{E^2 - m^2}\,\delta(p^0 - 2E). \tag{29}$$

Simplifying we are finally left with

$$\mathrm{Im}\mathcal{M} = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{s}}. \tag{30}$$

Multiplying by the proper phase space factor we have

$$\frac{2}{\Phi}\mathrm{Im}\mathcal{M} = \frac{\lambda^2}{32\pi s}, \tag{31}$$

that is the tree amplitude of the process. The optical theorem is thus verified.

The residue theorem first and Sokhotski-Plemelij later is equivalent to adopting the following Cutkosky rule, the simplest

$$\frac{1}{(k + \frac{p}{2})^2 - m^2 + i\epsilon} \to -2\pi i\delta\left(\left(k + \frac{p}{2}\right)^2 - m^2\right) \tag{32}$$

This in practice can be interpreted as taking the two propagators of the loop and putting their momentum on-shell. This is graphically represented as *cutting* the diagram with a dashed line, as when diagrams of squared amplitudes are drawn2. The explicit calculation using Cutkosky rules for this example follows.

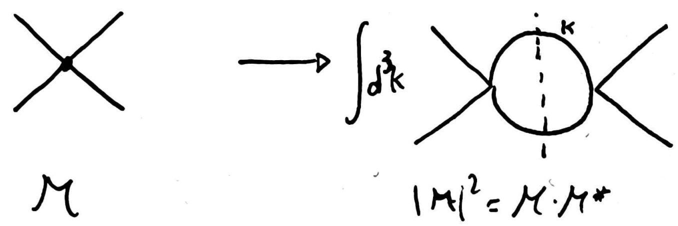

Figure 2: The squared amplitude of the tree diagram can be interpreted as a *cut* loop diagram.

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k + \frac{p}{2})^2 - m^2 + i\epsilon}\frac{1}{(k - \frac{p}{2})^2 - m^2 + i\epsilon} = \tag{33}$$

$$= \frac{\lambda^2}{2}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\frac{1}{k_1^2 - m^2 + i\epsilon}\frac{1}{k_2^2 - m^2 + i\epsilon}(2\pi)^4\delta(k^1 + k^2 - p), \tag{34}$$

applying Cutkosky

$$2i\mathrm{Im}i\mathcal{M} = \frac{\lambda^2}{2}\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}(-2\pi i)\delta(k_1^2 - m^2)(-2\pi i)\delta(k_2^2 - m^2)(2\pi)^4\delta^{(4)}(k^1 + k^2 - p). \tag{35}$$

Through standard manipulations we obtain

$$2\mathrm{Im}\mathcal{M} = \frac{1}{2}\int\frac{d^3k_1}{(2\pi)^3 2E_1}\int\frac{d^3k_2}{(2\pi)^3 2E_2}\lambda^2(2\pi)^4\delta^{(4)}(k_1 + k_2 - p) \tag{36}$$

This is exactly the optical theorem as stated in 8 $\mathcal{O}(\lambda^2)$, where λ^2 is simply the squared amplitude of the tree level diagram and the integrals before constitute the ingredients to calculate the cross section on the final state particles (which, at this perturbative level, are only 2).

In this calculation we have only used the properties of the propagator without considering the details of the interaction, this is what maked the Cutkosky rules particularly powerful.

3 Optical theorem and decays

Consider the optical theorem for a single incoming particle $\sigma(1 \to \{q\}) = \frac{2}{\Phi}\mathrm{Im}\mathcal{M}(1 \to 1)$. On the left-hand side there is exactly the total cross section for a decay while on the right hand side the imaginary part of the propagator of the particle decaying. This leads to our first observation: a particle decays if its propagator has an imaginary part. In free theories the imaginary part of the propagator is given by the ϵ prescription, so that the imaginary is just a Dirac delta and the cross sectoin trivial. In interacting theories the full propagator may acquire a non trivial imaginary part. Employing the OPI decomposition, the full propagator is

$$G^2(p) = \frac{1}{p^2 - m^2 - M^2(p^2) + i\epsilon} \tag{37}$$

where $-iM^2(p^2)$ is the sum of the OPI contributions. Knowing that the real part gives the mass of the particle we can decompose it so that we are left with

$$G^2(p) = \frac{1}{p^2 - m^2_{phys} - i\mathrm{Im}(M^2(p^2)) + i\epsilon}. \tag{38}$$

Comparing with the relativistic generalization of the Breit-Wigner formula

$$\frac{1}{p^2 - m^2_{phys} + im\Gamma + i\epsilon} \tag{39}$$

where Γ is the decay rate, we find

$$m\Gamma = \mathrm{Im}(-M^2(m^2_{phys})) \tag{40}$$

where the imaginary part is calculated in the physical mass because that is where the Breit-Wigner formula is valid (*Narrow width approximation*). Now we can apply to the imaginary part of the OPI amplitude the optical theorem. This gives

$$\Gamma = \frac{1}{2m}\sum_f \int df\, |\langle f|1\rangle|^2, \tag{41}$$

where the full expression of the sum on Fock space states has been omitted for clearness. We have thus obtained a formula of the decay rate. This is what one could naively calculate applying the formula for the cross section on the original process $\sigma(1 \to q)$, if in such case the asymptotical in and out states formalism were well-defined.

A Calculation of the imaginary directly from the loop integral

We can calculate the imaginary part of the loop amplitude

$$i\mathcal{M} = -i\lambda\left[1 + \frac{\lambda}{32\pi^2}\int_0^1 dx \ln\frac{M^2(s)}{\Lambda^2} + \ln\frac{M^2(t)}{\Lambda^2} + \ln\frac{M^2(u)}{\Lambda^2}\right] \tag{42}$$

directly from this result.

We have shown that the first logarithm has an imaginary part when x ranges between $\frac{1}{2}\left[1 - \sqrt{1 - \frac{4m^2}{s}}\right]$ and $\frac{1}{2}\left[1 + \sqrt{1 - \frac{4m^2}{s}}\right]$, and this is the only term carrying imaginary parts. Therefore we can restrict both $\mathcal{M}$ and the integration range to

$$i\delta\mathcal{M} = -i\frac{\lambda^2}{32\pi^2}\int_{x_{min}}^{x_{max}} dx \ln\frac{M^2(s)}{\Lambda^2}. \tag{43}$$

The imaginary part of the logarithm is simply π, so that the intgral is trivially

$$\mathrm{Im}\mathcal{M} = \frac{\lambda^2}{32\pi^2}\pi(x_{max} - x_{min}) = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{s}}, \tag{44}$$

which is exactly what we have obtained in equation 30.