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Quantum field theory in curved spacetime: the case of Inflation

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Introduction

Quantum field theory in curved spacetime (QFTCS) is the study of the behaviour of quantum fields in generic spacetimes. It is a generalization of standard quantum field theory in Minkowski universe. The background is assumed to be fixed while quantum fields evolve accordingly to their equation of motions. This theory has a theoretical interest per se. It is the only framework in which quantum and gravitational phenomena coexist. In this sense it may be seen as a weak limit to quantum gravity theories. Historically however, quantum field theory in curved spacetime has provoked great interests mainly for two important applications, in cosmological inflation and in black holes theory.
Fluctuations of the inflaton field during accelerated expansion are able to explain fluctuations in the temperature of the cosmic microwave background. First calculations of quantum fields in expanding spacetimes were made by Leonard Parker in his Ph.D thesis in 1966 [1], well before cosmological inflation was proposed.
In 1975 Stephen Hawking discovered that black holes should radiate energy [2]. Today this radiation bears his name. To derive it, Hawking studied fluctuations of quantum fields near the horizon of a black hole. The year later William G. Unruh, studying aspects of the Hawking radiation, derived the Unruh effect [3] starting from a preceding work by Stephen A. Fulling. This quantum phenomenon is not strictly related to curved spacetime because it concerns accelerated observers in Minkowski space, but it is nonetheless very instructive on the relative nature of particles, one of the main traits of quantum fields in curved spacetime.
In this thesis we will present the main features of the quantization of scalar fields in curved spacetime. In chapter 1 we will discuss the quantization of the harmonic oscillator with time-dependent frequency. This system has already many of the both mathematical and physical complication of QFTCS. In the following chapter we will revise quantization in Minkowski spacetime employing a generalization of the method used for the harmonic oscillator. In chapter 3 we will finally be able to formulate quantum field theory in curved spacetime. In chapter 4 we will introduce coordinates for accelerated observers, the Rindler coordinates, and derive the Unruh effect. Last chapter will be devoted to study the application of QFTCS in inflationary universe. We will review Friedmann-Lemaitre-Robertson-Walker metrics, the Friedmann equations and

the main ideas of inflationary theory. Quantization of scalar field in the de Sitter universe will be presented and we will calculate the power spectrum of field fluctuations in the context of linear perturbations. Lastly we will compare latest data from Planck collaboration and BICEP2 with theoretical results.

Conventions

We work in natural units ($\hbar = c = 1$) and use $(+,-,-,-)$ signature.

Contents

Chapter 1

Quantization of harmonic oscillator

In this section we show a method for quantizing a time-dependent harmonic oscillator. The procedure generalizes the standard decomposition of the position and momentum operators into creation and annihilation operators. Even if our aim is the study of fields, the calculations presented here has already all the main physical features that characterize quantization of fields both in flat and in curved spacetimes.
Let us consider an harmonic oscillator with unitary mass and time-dependent frequency. The lagrangian of the system is:

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2(t)q^2. \tag{1.0.1}$$

The equation of motion is found by varying the action $S[q] = \int L[q]\, dt$ with respect to q and then by determining the stationary point, having fixed initial conditions $q(0) = q_0$ and $\dot{q}(0) = \dot{q}_0$.
The equation is:

$$\ddot{q} + \omega^2(t)q = 0. \tag{1.0.2}$$

The conjugate momentum is $p = \frac{\mathrm{d}L}{\mathrm{d}\dot{q}} = \dot{q}$. To quantize the system q and p are replaced with the hermitian operators $\hat{q}$ and $\hat{p}$ and the canonical commutation relation is imposed (recall that we are working in natural units):

$$[\hat{q}, \hat{p}] = i. \tag{1.0.3}$$

Equation (1.0.2) is now to be intended as an equation for the operator q in the Heisenberg representation. The Hilbert space of the system is found by introducing the non hermitian operator $\hat{a}$ in terms of which $\hat{q}$ and $\hat{p}$ are expanded. Precisely we seek an operator a such that

$$\begin{cases} \hat{q}(t) = v(t)\hat{a} + v^*(t)\hat{a}^\dagger \\ \hat{p}(t) = \dot{v}(t)\hat{a} + \dot{v}^*(t)\hat{a}^\dagger \end{cases}, \tag{1.0.4}$$

where v is a complex-valued function called *mode function*. Notice that the equations are constructed so that $\hat{q}$ and $\hat{p}$ are real. Because the position and momentum operators satisfy the harmonic oscillator equation, v must be chosen so that it also satisfies the same equation in complex sense (i.e. real and imaginary party satisfy it separately). The explicit expression for $\hat{a}$ and $\hat{a}^\dagger$ is simply found by inverting the above system of equations:

$$\begin{cases} \hat{a} = \frac{\dot{v}^*(t)\hat{q}(t) - v^*(t)\hat{p}(t)}{v(t)\dot{v}^*(t) - \dot{v}(t)v^*(t)} \\ \hat{a}^\dagger = \frac{-\dot{v}(t)\hat{q}(t) + v(t)\hat{p}(t)}{v(t)\dot{v}^*(t) - \dot{v}(t)v^*(t)} \end{cases} . \tag{1.0.5}$$

Introducing the bracket product

$$\langle v, w \rangle = i(v^* \dot{w} - \dot{v}^* w), \tag{1.0.6}$$

the equations can be rewritten in a compact manner as

$$\begin{cases} \hat{a} = \frac{\langle v, \hat{q} \rangle}{\langle v, v \rangle} \\ \hat{a}^\dagger = -\frac{\langle v^*, \hat{q} \rangle}{\langle v, v \rangle} \end{cases} , \tag{1.0.7}$$

where all the time dependencies have been omitted because the bracket does not depend on t if v and w are solutions to equation (1.0.2), as it can be explicitly derived by computing $\partial_t \langle v, w \rangle$.
Therefore $\hat{a}$ and $\hat{a}^\dagger$ are only determined by the initial conditions for the canonical coordinates, q_0 and p_0, and for the mode function v, whose only requirement is to satisfy the equation of motion. Because the equation is linear we can normalize v so that $\langle v, v \rangle = 1$. Observe that this does not completely fix the function. The final expression for the two operators is

$$\begin{cases} \hat{a} = \langle v, \hat{q} \rangle \\ \hat{a}^\dagger = -\langle v^*, \hat{q} \rangle \end{cases} . \tag{1.0.8}$$

$\hat{a}$ is called *annihilation operator* and $\hat{a}^\dagger$ is the *creation operator*. The commutation relation between the annihilation and the creation operator is found from equation (1.0.3):

$$[\hat{q}, \hat{p}] = i \iff [\hat{a}, \hat{a}^\dagger] = \frac{1}{\langle v, v \rangle} = 1. \tag{1.0.9}$$

Where the normalization of the mode function has been used.
Now we can build a basis of the Hilbert space. First let's define the normalized state $|0\rangle$, called *vacuum state*, satisfying by definition

$$a|0\rangle = 0.$$

Given $n \in \mathbb{N}$ let $|n\rangle := (1/\sqrt{n!})(\hat{a}^\dagger)^n|0\rangle$ the *excited states*. Such states are eigenstates of the *number operator*, $\hat{N} = \hat{a}^\dagger\hat{a}$ with eigenvalue n, that is to say $\hat{N}|n\rangle = n|n\rangle$. This fact is trivial for the vacuum state. It can be shown for

$n = 1$ using the commutation relation (1.0.9) for the annihilation and creation operator, and the unit bracket normalization for v:

$$\hat{N}|1\rangle = \hat{a}^\dagger \hat{a}(\hat{a}^\dagger|0\rangle) = \hat{a}^\dagger \hat{a}^\dagger \hat{a}|0\rangle + \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger]|0\rangle = \hat{a}^\dagger|0\rangle = |1\rangle \tag{1.0.10}$$

Where the defining property for the vacuum state has been used. For $n > 1$ it can be shown by induction.
The Hilbert space depends on the creation and annihilation operators, which depend on the choice of mode function, through equations (1.0.5). Therefore the Hilbert space is, in principle, not unique. There exists however a preferred choice in the case of a time independent harmonic oscillator with frequency $\omega(t) \equiv \omega$. In this case the vacuum state is the ground state of the Hamiltonian $\hat{H}$. To show this it is sufficient write $\hat{H}$ in terms of the creation and annihilation operators using (1.0.4) and (1.0.9):

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2(t)q^2 \frac{1}{2}[\dot{v}(t)\hat{a} + \dot{v}^*(t)\hat{a}^\dagger]^2 + \frac{1}{2}\omega^2(t)[v(t)\hat{a} + v^*(t)\hat{a}^\dagger]^2 = \\
&= \frac{1}{2}\Big[(\dot{v}^2 + \omega^2(t)v^2)\hat{a}^2 + (\dot{v}^{*2} + \omega^2(t)v^{*2})(\hat{a}^\dagger)^2 + (|\dot{v}^2| + \omega^2(t)|v|^2)(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\Big] \\
&= \frac{1}{2}\Big[(\dot{v}^2 + \omega^2(t)v^2)\hat{a}^2 + (\dot{v}^{*2} + \omega^2(t)v^{*2})(\hat{a}^\dagger)^2 + (|\dot{v}^2| + \omega^2(t)|v|^2)2(\hat{N} + \frac{1}{2})\Big]
\end{aligned}$$

Acting on the vacuum state we get

$$\hat{H}|0\rangle = \frac{1}{2}(\dot{v}^{*2} + \omega^2(t)v^{*2})(\hat{a}^\dagger)^2|0\rangle + \frac{1}{2}(|\dot{v}^2| + \omega^2(t)|v|^2)|0\rangle. \tag{1.0.11}$$

Therefore the vacuum state is an eigenstate if and only if the mode function satisfies $\dot{v}^{*2} + \omega^2 v^{*2} = 0$, that is to say $\dot{v} = \pm i\omega(t)v$. For consistency we observe that solutions to this differential equation satisfy the harmonic oscillator equation, as required by the mode function, but the opposite does not necessarily holds. Substituting this relation into the bracket product we get

$$\langle v, v\rangle = \mp 2\omega|v|^2. \tag{1.0.12}$$

Because we have imposed the unit normalization, we see that the mode function must obey the equation with a minus sign:

$$\dot{v} = -i\omega(t)v, \tag{1.0.13}$$

Where the initial condition is $v(0) = 1/\sqrt{2\omega}e^{i\theta}$, following from equation (1.0.12). Modes satisfying this relation are called *positive-frequency* modes. The phase θ is set to 0, this choice doesn't have any consequence on physical results. The mode function is therefore:

$$v(t) = \frac{1}{\sqrt{2\omega}}e^{-i\omega t}, \tag{1.0.14}$$

and the Hamiltonian $\hat{H}$ becomes

$$\hat{H} = (|\dot{v}^2| + \omega^2(t)|v|^2)\left(\hat{N} + \frac{1}{2}\right) = \omega\left(\hat{N} + \frac{1}{2}\right). \tag{1.0.15}$$

The expansion for $q(t)$ is found by substituting v into (1.0.4):

$$\hat{q}(t) = \frac{1}{\sqrt{2\omega}}(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger), \tag{1.0.16}$$

thus obtaining the standard mode expansion found in classical treatments of the quantum harmonic oscillator problem.
If ω were time-dependent, the solutions to equation (1.0.13) would not be also solutions of the equation of motion (1.0.2). In this case we are not able to determine a physically preferred choice of modes. It may however occur that a set of modes satisfies equation (1.0.13) instantaneously, say at time t_0. Such modes would define a vacuum state $|0_{t_0}\rangle$ which for an observer at time t_0 is the ground state at that fixed time. In this case $|0_{t_0}\rangle$ is said to be *instantaneous ground state*.
If $\omega(t)$ is asymptotically constant at $t \to \pm\infty$, we can then build a set of modes which satisfy the positive-frequency condition in the *in* regime, that is to say for $t \to -\infty$, or in the *out* regime, at $t \to +\infty$. The vacuum states consequently defined would be instantaneous ground states in their respective time domains, while at other times they wouldn't be physically significant.
Not having in general a preferred choice of modes, we would like to know how to relate state defined by different mode functions. If we chose another pair of normalized mode functions $w(t)$ and $w^*(t)$ the position operator would be expanded as:

$$q(t) = w(t)\hat{b} + w^*(t)\hat{b}^\dagger, \tag{1.0.17}$$

with

$$\begin{cases} \hat{b} = \langle w, \hat{q}\rangle \\ \hat{b}^\dagger = -\langle w^*, \hat{q}\rangle \end{cases}. \tag{1.0.18}$$

Because $v(t)$ and $v^*(t)$ form a basis of the two dimensional vector space of solutions of equation (1.0.2), there exist two complex coefficients α and β so that

$$\begin{cases} w(t) = \alpha v(t) + \beta v^*(t) \\ w^*(t) = \beta^* v(t) + \alpha^* v^*(t) \end{cases}. \tag{1.0.19}$$

Substituting in equation (1.0.18), we find the relation between the operators

$$\begin{cases} \hat{b} = \alpha^*\hat{a} - \beta^*\hat{a}^\dagger \\ \hat{b}^\dagger = -\beta\hat{a} + \alpha\hat{a}^\dagger \end{cases}. \tag{1.0.20}$$

Given the number operator $\hat{N}_w = \hat{b}^\dagger\hat{b}$ associated to w modes, we can calculate the average of the operator in the vacuum state $|0_v\rangle$ associated to v modes using equation (1.0.20) and the canonical commutation relation for a and $a^\dagger$:

$$\langle 0_v|\hat{N}_w|0_v\rangle = \langle 0_v|(-\beta\hat{a} + \alpha\hat{a}^\dagger)(\alpha^*\hat{a} - \beta^*\hat{a}^\dagger)|0_v\rangle = \beta^2. \tag{1.0.21}$$

We see that in principle the number of excitations depend on the chosen mode. We will see that this has important consequences when applied to physical fields.

Chapter 2

Quantization of scalar field in Minkowski spacetime

In this section we quantize the scalar field in Minkowski spacetime following the same procedure adopted for the harmonic oscillator, that is the mode expansion. We will see how to generalize all the different steps in field theory context. We could quantize straightly the field via, for example, Fourier transformation, but the technique here employed will be of fundamental importance later on, both for introducing quantum field theory in curved spacetime and for the study of the Unruh effect. Only at some point we will assume an explicit form for modes and we will find the standard plane wave expansion.

2.1 Mode expansion

The scalar field ϕ is described by the Klein-Gordon lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi\right). \tag{2.1.1}$$

By definition, the action of the field is

$$\mathcal{S}[\phi] = \int d^4x\, \mathcal{L}[\phi]. \tag{2.1.2}$$

The physical configurations for the field are the ones that sit on a stationary point of the action $S[\phi]$. Therefore to find the equation of motion for the field the action is varied with respect to the field:

$$\frac{\delta S}{\delta\phi} = 0. \tag{2.1.3}$$

Assuming that field's first derivatives go to zero fast enough, we obtain the *Klein-Gordon equation*

$$(\partial_\mu\partial^\mu + m^2)\phi = 0. \tag{2.1.4}$$

Defining the d'Alambert operator $\Box := \partial_t^2 - \Delta = \partial_\mu \partial^\mu$ the equation is rewritten as

$$(\Box + m^2)\phi = 0. \tag{2.1.5}$$

Formally the equation obtained is a second-order hyperbolic partial differential equation.
The procedure to quantize this system will follow the same steps carried for the harmonic oscillator. We first calculate the conjugate momentum

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}. \tag{2.1.6}$$

$\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$ are replaced with the hermitian operators $\hat{\phi}(\mathbf{x}, t)$ and $\hat{\pi}(\mathbf{x}, t)$. The canonical commutation relations, generalized for fields, are imposed:

$$\begin{aligned}
&[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{x}', t)] = 0 \\
&[\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = 0 \\
&[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}').
\end{aligned} \tag{2.1.7}$$

Following equations (1.0.4) we would like to expand the field operator into creation and annihilation operators. In the case of the harmonic oscillator we had defined a complex mode function v. This was enough to expand a general real solution into the two functions v and v^*. We have shown this by explicit inversion of the system (1.0.4). We could have had concluded this a priori because the equation of motion is a second order linear ordinary differential equation and therefore the real vector space of solution is two dimensional. Since the Klein-Gordon equation is a linear partial differential equation, the space of solution is infinite dimensional and, assuming a limited spatial domain, it admits a countable basis. Therefore, in this situation, we need to define a whole set of complex mode functions $\{f_n\}$ that, together with their complex conjugate, form a complete set on to expand the scalar field operator. Precisely, for every mode we define the non hermitian operator $\hat{a}_n$ such that

$$\begin{cases} \hat{\phi}(\mathbf{x}, t) = \sum_n f_n(\mathbf{x}, t)\hat{a}_n + f_n^*(\mathbf{x}, t)\hat{a}_n^\dagger \\ \hat{\pi}(\mathbf{x}, t) = \sum_n \dot{f}_n(\mathbf{x}, t)\hat{a}_n + \dot{f}_n^*(\mathbf{x}, t)\hat{a}_n^\dagger \end{cases}, \tag{2.1.8}$$

where $\hat{a}_n$ and $\hat{a}_n^\dagger$ are again the creation and annihilation operators. We wish to find an explicit expression for them in terms of the field operator and of the mode functions as done in equation (1.0.5). To do this we need to generalize the bracket product (1.0.6). We guess the following obvious generalization:

$$(f, g)_{KG} = i \int_\Sigma (f^*(\mathbf{x}, t)\dot{g}(\mathbf{x}, t) - \dot{f}^*(\mathbf{x}, t)g(\mathbf{x}, t))d^3x, \tag{2.1.9}$$

where Σ is the domain of the equation that, for now, we consider limited. We see that it is indeed bilinear, symmetric under complex conjugation and time independent, as it can be explicitly shown:

$$\begin{aligned}
\partial_t(f,g)_{KG} = & \qquad (2.1.10)\\
&= i\int_\Sigma \partial_t(f^*\partial_t g - g\partial_t f^*)d^3x = \qquad (2.1.11)\\
&= i\int_\Sigma (f^*\partial_t^2 g - \partial_t^2 f^* g) = \qquad (2.1.12)\\
&= i\int_\Sigma (f^*(\Delta g - m^2 g) - (\Delta f^* - m^2 f^*)g) = \qquad (2.1.13)\\
&= i\int_\Sigma (f^*\Delta g - \Delta f^* g) = \qquad (2.1.14)\\
&= -i\int_\Sigma (\Delta f^* g - \Delta f^* g) + f^*\nabla g\cdot\hat{\mathbf{n}}\big|_{\partial\Sigma} - g\nabla f^*\cdot\hat{\mathbf{n}}\big|_{\partial\Sigma} = 0, \qquad (2.1.15)
\end{aligned}$$

where in the third line we have used KG equation and in the last one Green's first identity twice, where the boundary term disappears assuming that mode functions f_n vanish in $\partial\Sigma$. The product $(,)_{KG}$ is the *Klein-Gordon inner product*, even if formally speaking it is not an inner-product on the whole space of solutions of the equation, because it is positive definite only on a subset of the space. Precisely if f is positive definite, then f^* is not. Because we will always choose our mode functions to be normalized they will also be positive definite.

For the expansion of the solution to work, the mode functions must also be orthogonal, we therefore write the following conditions:

$$(f_n, f_{n'}) = \delta_{nn'} \qquad (2.1.16)$$
$$(f_n, f^*_{n'}) = 0 \qquad (2.1.17)$$
$$(f^*_n, f^*_{n'}) = -\delta_{nn'}, \qquad (2.1.18)$$

where the last two follow from the first.

Generalizing equation (1.0.8) we can now write an explicit expression for the creation and annihilation operators:

$$\begin{cases} \hat{a}_n = (f_n, \hat{\phi})_{KG} \\ \hat{a}_n^\dagger = -(f_n^*, \hat{\phi})_{KG} \end{cases} . \qquad (2.1.19)$$

These relations can be explicitly verified by substituting the field expansion into the Klein-Gordon inner product and then using linearity and orthonormality. We now calculate the commutation relation between the mode operators using (2.1.7) and (2.1.19):

$$[\hat{a}_n, \hat{a}_{n'}] = [(f_n, \hat{\phi})_{KG}, (f_{n'}, \hat{\phi})_{KG}] = -(f_n, f^*_{n'})$$
$$[\hat{a}_n, \hat{a}^\dagger_{n'}] = [(f_n, \hat{\phi})_{KG}, -(f^*_n, \hat{\phi})_{KG}] = (f_n, f^*_n). \qquad (2.1.20)$$
$$(2.1.21)$$

Using (2.1.16) we have the standard commutation relations between creation and annihilation operators:

$$
\begin{aligned}
&[\hat{a}_n, \hat{a}_{n'}] = 0 \\
&[\hat{a}^\dagger_n, \hat{a}^\dagger_{n'}] = 0 \\
&[\hat{a}_n, \hat{a}^\dagger_{n'}] = \delta_{nn'}.
\end{aligned} \tag{2.1.22}
$$

In summary, given a set of Klein-Gordon-orthonormal complex mode functions we succeeded to determine an expansion for the fields $\phi(\mathbf{x}, \mathbf{t})$ and $\pi(\mathbf{x}, \mathbf{t})$ in terms of a set of creation and annihilation operators obeying harmonic oscillator like commutation relations.

2.2 Fock space

Having defined the creation and annihilation operators, we can now build a basis for the Hilbert space of the theory. The vacuum state is a normalized state such that:

$$a_{\mathbf{n}}|0\rangle = 0\,\forall n. \tag{2.2.1}$$

This vector alone spans a trivial Hilbert space isomorphic to $\mathbb{C}$ denoted as $\mathcal{H}_0 \cong \mathbb{C}$. The *one-particle Hilbert state* $\mathcal{H}_1$ is the space spanned by the *one-particle states*, which are states constructed by applying the creation operators on the vacuum state

$$\mathcal{H}_1 := \overline{\{|1_n\rangle = a^\dagger_n|0\rangle\,\forall n\}}. \tag{2.2.2}$$

The *two-particles Hilbert space* is the space spanned by tensor products of one-particle states

$$\mathcal{H}_2 := \overline{\{|1_n\rangle \otimes |1_m\rangle = \mathcal{N} a^\dagger_n|0\rangle \otimes a^\dagger_m|0\rangle\,\forall n, m\}} \cong \mathcal{H}_1 \otimes \mathcal{H}_1, \tag{2.2.3}$$

where $\mathcal{N}$ is a normalization. In this way all *p-particles Hilbert spaces* are constructed. Explicitly a state in $\mathcal{H}_p$ has the form $|p_{n_1}\rangle \otimes |p_{n_2}\rangle \otimes |p_{n_3}\rangle \otimes \cdots \otimes |p_{n_k}\rangle = |p_{n_1}, p_{n_2}, p_{n_3}, \ldots, p_{n_k}\rangle$ where n_i are distinct indexes, $|p_{n_i}\rangle = \mathcal{N}(\hat{a}^\dagger_{n_i})^{n_{n_i}}|0\rangle$ and $p_{n_1} + p_{n_2} + \ldots p_{n_k} = p$. The whole space of the theory is the direct sum of all such spaces, and it is called *Fock space* $\mathcal{F}$. Formally:

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H}_1 \oplus (\mathcal{H}_1 \otimes \mathcal{H}_1) \oplus \cdots = \bigoplus_{p=0}^{\infty} \Big(\bigotimes_{j=0}^{p} \mathcal{H}_1 \Big). \tag{2.2.4}$$

By using the same argument applied for the harmonic oscillator, one can show that the states constructed by applying $a^\dagger_n$ repeatedly on the vacuum state are eigenstates of the number operator $\hat{N}_n := \hat{a}^\dagger_n \hat{a}_n$, so that the number operator associated to a given mode m acts on a vector in $\mathcal{H}_p$ as

$$\hat{N}_m|p_1, p_2, p_3, \ldots, p_m, \ldots\rangle = p_m|p_1, p_2, p_3, \ldots, p_m, \ldots\rangle. \tag{2.2.5}$$

2.3 Eigenstates of the Hamiltonian

The Fock space built in the previous section depends on the vacuum state and, therefore, on the creation and annihilation operators that are defined through equations (2.1.19) and on the modes function f_n, which are complex solution of the Klein-Gordon equation. We have only required the modes to be orthogonal and normalized, but, not having fixed any boundary condition, we expect in principle to have a free choice which will imply a different vacuum state. Similarly to what has been done in the discussion of the harmonic oscillator we look for modes that will give a vacuum state coincident to the ground state of the system. The Hamiltonian density of the scalar field is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}(\dot{\phi}^2 + |\nabla\phi|^2 + m^2\phi^2), \tag{2.3.1}$$

and therefore the Hamiltonian operator is

$$\hat{H} = \int d^3x\mathcal{H} = \int d^3x\frac{1}{2}(\dot{\hat{\phi}}^2 + |\nabla\hat{\phi}|^2 + m^2\hat{\phi}^2). \tag{2.3.2}$$

The expression can be simplified using Green's first identity and the fact that ϕ is a solution of the Klein-Gordon equation:

$$\hat{H} = \int d^3x\frac{1}{2}(\dot{\hat{\phi}}^2 - \hat{\phi}\Delta\hat{\phi} + m^2\hat{\phi}^2) = \tag{2.3.3}$$

$$= \int d^3x\frac{1}{2}(\dot{\hat{\phi}}^2 - \hat{\phi}\ddot{\hat{\phi}}). \tag{2.3.4}$$

As done with the harmonic oscillator we calculate $\hat{H}$ in terms of the expansion (2.1.8).

$$\begin{aligned}\hat{H} = &\int d^3x\frac{1}{2}\Big[(\sum_n \dot{f}_n\hat{a}_n + \dot{f}_n^*\hat{a}_n^\dagger)(\sum_{n'} \dot{f}_{n'}\hat{a}_{n'} + \dot{f}_{n'}^*\hat{a}_{n'}^\dagger) - \\ &- (\sum_n f_n\hat{a}_n + f_n^*\hat{a}_n^\dagger)(\sum_{n'} \ddot{f}_{n'}\hat{a}_{n'} + \ddot{f}_{n'}^*\hat{a}_{n'}^\dagger)\Big] = \\ &= \frac{1}{2}\int d^3x\Big[\sum_{nn'}(\dot{f}_n^*\dot{f}_{n'}^* - f_n^*\ddot{f}_{n'}^*)\hat{a}_n^\dagger\hat{a}_{n'}^\dagger + (\dot{f}_n\dot{f}_{n'} - f_n\ddot{f}_{n'})\hat{a}_n\hat{a}_{n'} + \\ &+ (\dot{f}_n\dot{f}_{n'}^* - f_n\ddot{f}_{n'}^*)2(\hat{a}_n^\dagger\hat{a}_{n'} + \frac{1}{2}\delta_{nn'})\Big].\end{aligned} \tag{2.3.5}$$

Applying the vacuum state to the Hamiltonian operator and imposing it to be an eigenstate, we find an equation for the modes corresponding to the one found in the case of the harmonic oscillator

$$\hat{H}|0\rangle = \frac{1}{2}\int d^3x\Big[\sum_{nn'}(\dot{f}_n^*\dot{f}_{n'}^* - f_n^*\ddot{f}_{n'}^*)\hat{a}_n^\dagger\hat{a}_{n'}^\dagger + (\dot{f}_n\dot{f}_{n'}^* - f_n\ddot{f}_{n'}^*)\delta_{nn'}\Big]|0\rangle, \tag{2.3.6}$$

where the defining property of the vacumm state and the commuation relation between creation and annihilation operators has been used. $|0\rangle$ is an eigenstate if the mode functions obey the coupled equations:

$$\int d^3x(\dot{f}_n^*\dot{f}_{n'}^* - f_n^*\ddot{f}_{n'}^*) = 0\,\forall\, n, n', \tag{2.3.7}$$

whose solution is a non trivial problem.
We see that if this condition is satisfied the Hamiltonian is:

$$\hat{H} = \int d^3x \sum_{nn'} \Big[(\dot{f}_n\dot{f}_{n'}^* - f_n\ddot{f}_{n'}^*)(\hat{a}_n^\dagger\hat{a}_{n'} + \frac{1}{2}\delta_{nn'})\Big]. \tag{2.3.8}$$

2.4 A choice for modes: Fourier ansatz

After having approached the quest for quantization in full generality it's time to choose a form for the modes to simplify the equations. We choose as domain the compact set V of $\mathbb{R}^3$, $V = [-L, L] \times [-L, L] \times [-L, L]$ and impose periodic boundary conditions. We now make the Fourier ansatz:

$$f_{\mathbf{k}}(\mathbf{x}, t) = \phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.4.1}$$

where the functions $\phi_{\mathbf{k}}(t)$ are called *Fourier modes*. From the requirement $f_{\mathbf{k}}(-L\mathbf{e}_i, t) = f_{\mathbf{k}}(L\mathbf{e}_i, t)$ we get that $\mathbf{k} = (\pi/L)\mathbf{l}$ with $\mathbf{l} \in \mathbb{Z}^3$. The index n that enumerated the mode functions can now be replaced with $\mathbf{k}$ (this is possible because $\mathbb{N}$ and $\mathbb{Z}^3$ have the same cardinality).
We now find the equation that the Fourier modes should satisfy so that the functions $f_{\mathbf{k}}$ form an orthonormal set of modes that generate a vacuum state that is eigenstate of the Hamiltonian operators. To do this we substitute the Fourier ansatz (2.4.1) into the Klein-Gordon equation (2.1.5), in the orthonormality condition (2.1.9) and in the eigenstate condition (2.3.7):

$$(\Box - m^2)f_{\mathbf{k}} = 0 \Longrightarrow \ddot{\phi}_{\mathbf{k}} + \omega^2(\mathbf{k})\phi_{\mathbf{k}} = 0 \tag{2.4.2}$$

$$i\int_\Sigma (f^*\dot{g} - \dot{f}^*g)d^3x = \delta_{\mathbf{k}\mathbf{k}'} \Longrightarrow \langle\phi_{\mathbf{k}}, \phi_{\mathbf{k}}\rangle = \frac{1}{V} \tag{2.4.3}$$

$$\int d^3x(\dot{f}_n^*\dot{f}_{n'}^* - f_n^*\ddot{f}_{n'}^*) = 0 \Longrightarrow \dot{\phi}_{\mathbf{k}} = \pm i\omega(\mathbf{k})\phi_{\mathbf{k}}, \tag{2.4.4}$$

where we have defined the frequency $\omega(\mathbf{k}) := |\mathbf{k}|^2 + m^2$ and have used the Fourier orthonormality relation $\int_V e^{i\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}')} = V\delta_{\mathbf{k}\mathbf{k}'}$. We see that the Fourier modes must satisfy the same set of equations that the mode function $v(t)$ of the harmonic oscillator had to, with only a different normalization. We can then already write the solution:

$$\phi_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega(\mathbf{k})V}}e^{-i\omega(\mathbf{k})t}. \tag{2.4.5}$$

From equation (2.4.1) we obtain the modes

$$f_{\mathbf{k}}(\mathbf{x},t) = \frac{1}{\sqrt{2\omega(\mathbf{k})V}} e^{i[-\omega(\mathbf{k})t+\mathbf{k}\cdot\mathbf{x}]}, \tag{2.4.6}$$

which, by construction, satisfy the orthonormality relations. Defining the quadrivectors $x^\mu = (t, \mathbf{x})$ and $k^\mu = (\omega, \mathbf{k})$ the exponent $-\omega(\mathbf{k})t + \mathbf{k}\cdot\mathbf{x}$ becomes $-k_\mu x^\mu$. Explicitly, the field expansion into modes is

$$\hat{\phi}(x^\mu) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega(\mathbf{k})V}} [e^{-ik_\mu x^\mu} \hat{a}_{\mathbf{k}} + e^{ik_\mu x^\mu} \hat{a}^\dagger_{\mathbf{k}}] \,. \tag{2.4.7}$$

Now we can rewrite the Hamiltonian (2.3.5) using these modes. The first two addends are 0 by equation (2.4.4). The last one can be rewritten by substituting the Fourier ansatz into the modes and rewriting the Fourier modes using with intelligence equations (2.4.2) and (2.4.4) to make equation (2.4.3) appear.

$$\begin{aligned}
\hat{H} &= \sum_{\mathbf{k}\mathbf{k}'} \frac{1}{2} \int d^3x (\dot{f}_{\mathbf{k}} \dot{f}^*_{\mathbf{k}'} - f_{\mathbf{k}} \ddot{f}^*_{\mathbf{k}'}) 2 (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}'} + \frac{1}{2}\delta_{\mathbf{k}\mathbf{k}'}) = \\
&= \sum_{\mathbf{k}\mathbf{k}'} \int d^3x (\dot{\phi}_{\mathbf{k}} \dot{\phi}^*_{\mathbf{k}'} - \phi_{\mathbf{k}} \ddot{\phi}^*_{\mathbf{k}'}) e^{-i(k-k')_\mu x^\mu} (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}'} + \frac{1}{2}\delta_{\mathbf{k}\mathbf{k}'}) = \\
&= \sum_{\mathbf{k}\mathbf{k}'} (\dot{\phi}_{\mathbf{k}} \dot{\phi}^*_{\mathbf{k}'} - \phi_{\mathbf{k}} \ddot{\phi}^*_{\mathbf{k}'}) V \delta_{\mathbf{k}\mathbf{k}'} (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}'} + \frac{1}{2}) = \\
&= \sum_{\mathbf{k}} \omega(\mathbf{k}) \langle \phi_{\mathbf{k}}, \phi_{\mathbf{k}} \rangle V (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{2}) = \\
&= \sum_{\mathbf{k}} \omega(\mathbf{k}) (\hat{N}_{\mathbf{k}} + \frac{1}{2}).
\end{aligned} \tag{2.4.8}$$

We see that not only the vacuum state is an eigenstate, but with this choice of modes it is also the ground state of the Hamiltonian. In particular the energy of the vacuum is:

$$\langle 0|\hat{H}|0\rangle = \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}} = \frac{1}{2}\sum_{\mathbf{k}} \sqrt{|\mathbf{k}|^2 + m^2}. \tag{2.4.9}$$

The series on the right hand side is divergent, which implies the presence of an infinite energy. Because experiments studying quantum systems measure differences of energy rather than absolute values, this energy is simply ignored. It is then customary to redefine the *normal ordered* Hamiltonian

$$: \hat{H} := \hat{H} - \langle 0|\hat{H}|0\rangle = \sum_{\mathbf{k}} \omega(\mathbf{k}) \hat{N}_{\mathbf{k}}, \tag{2.4.10}$$

where the vacuum energy has been subtracted.

2.5 Particle interpretation

The states of the Fock basis are all eigenstates of the Hamiltonian (2.4.8). Through Noether's theorem, classical field theory enables us to build other tensorial quantity of physical interest from the fields. One of these is the momentum $\mathbf{P}$ of the field, defined with the energy-momentum tensor

$$\hat{\mathbf{P}} = \int_V d^3x\, \hat{\mathcal{P}}^i = \int_V d^3x\, \hat{T}^{0i} = \int_V d^3x\, (-\partial_t\hat{\phi}\partial_i\hat{\phi}). \tag{2.5.1}$$

Using the field expansion (2.4.7) we get the expression

$$\hat{\mathbf{P}} = \sum_{\mathbf{k}} \mathbf{k}(\hat{N}_{\mathbf{k}} + \frac{1}{2}) = \sum_{\mathbf{k}} \mathbf{k}\hat{N}_{\mathbf{k}}, \tag{2.5.2}$$

where the second addend is 0 because the index $\mathbf{k}$ is summed over a symmetric interval.
We observe that states of the Fock basis are also eigenstates of the momentum operator. This is also true for one-particle states $|1_{\mathbf{k}}\rangle = a^{\dagger}_{\mathbf{k}}|0\rangle$, to whose we can associate a momentum $\mathbf{k}$ and an energy ω, which are the eigenvalues of $\hat{\mathbf{P}}$ and $\hat{H}$ respectively, that, by definition of ω, respect the mass-shell relation $m = \sqrt{\omega^2 - |\mathbf{k}|^2}$ with m the mass parameter of the theory. Due to this and to their discrete nature, it is natural to interpret such states as particles, as it is traditionally done. We should stress that this is only a matter of interpretation. We have already argued that the states of the Fock space depend on the choice of modes which is not unique. Physically a particle is the result of a measure done by a particle detector in a certain frame of reference. The result of this measure may or may not be the same of the average of the number operator in that given physical state. In the next chapter we'll see under which condition the two numbers are the same. We will then see an explicit example studying the Unruh effect, concerning the quantization of the scalar field in Minkowski spacetime in an accelerated frame of reference.

2.6 Continuum limit

Even if most physical phenomena are contained in a limited volume, in order to study fundamental fields we need to extend the domain of the scalar field to the whole Minkowski space. With $L \to \infty$ the condition of discretization on the wave number $\mathbf{k}$ disappears and becomes a continuous variable. Repeating the calculations with the Fourier ansatz get

$$f_{\mathbf{k}}(\mathbf{x}, t) = \frac{1}{\sqrt{2\omega(\mathbf{k})(2\pi)^3}} e^{-ik_\mu x^\mu}, \tag{2.6.1}$$

Where the integral representation of the Dirac delta $\int_{\mathbb{R}^3} e^{i(\mathbf{k}'-\mathbf{k})} = (2\pi)^3\delta^{(3)}(\mathbf{k}'-\mathbf{k})$ has been used in substitution of the Fourier orthonormality relation. We see that the modes are plane waves with the dispersion relation $\omega(\mathbf{k})^2 =$

$|\mathbf{k}|^2 + m^2$. Having now the modes indexed by a continuous variable, sums becomes integrals and the modes expansion for the field operator is

$$\hat{\phi}(x^\mu) = \int_{\mathbb{R}^3} d^3k \, \frac{1}{\sqrt{2\omega(\mathbf{k})(2\pi)^3}} [e^{-ik_\mu x^\mu} \hat{a}_{\mathbf{k}} + e^{ik_\mu x^\mu} \hat{a}^\dagger_{\mathbf{k}}]. \tag{2.6.2}$$

Plane waves are not normalizable, therefore all the relations should be intended in a distributional sense. In the orthonormality relation, as can be explicitly verified with the plan wave modes, the Kronecker delta becomes a Dirac delta:

$$(f_{\mathbf{k}}, f_{\mathbf{k}'}) = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{2.6.3}$$

$$(f^*_{\mathbf{k}}, f^*_{\mathbf{k}'}) = -\delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{2.6.4}$$

$$(f_{\mathbf{k}}, f^*_{\mathbf{k}'}) = 0, \tag{2.6.5}$$

and the Hamiltonian operator is

$$\hat{H} = \int_{\mathbb{R}^3} d^3k \, \omega(\mathbf{k}) \Big(\hat{N}_{\mathbf{k}} + \frac{1}{2} \delta^{(3}(0) \Big), \tag{2.6.6}$$

where the infinite volume term $\delta^{(3)}(0)$ is an extra divergence that appears having now accessible infinitely small wave numbers $\mathbf{k}$, whence the name *infrared divergence.* Through subtraction the divergence can be once again eliminated.

Chapter 3

Quantum field theory in curved spacetime

In this section we finally address the problem of quantization in curve spacetime, restricted to scalar fields. Firstly the formulation of classical field theory is presented, then, employing the same procedure followed in Minkowski spacetime, the scalar field is quantized through expansion in modes. We will then discuss the concept of particles and particle detectors introducing the conditon of *positive-frequency*, which will permit to single out a preferred choice of modes for certain observers. We will then show how to relate different sets of modes and operators through the *Bogoljubov transformations*, which are of key importance in applications.

3.1 Classical field theory in curved spacetime

Given a spacetime, i.e. a Lorentzian manifold, with metric $g_{\mu\nu}$, we want to define a scalar field on the spacetime with an action that generalizes through an appropriate modification the action in Minkowski spacetime (2.1.2). This is realized using the principle of *general covariance*, which is to say that the laws of physics shall be invariant under choice of coordinates. Under any diffeomorphism any law must have the same form. This means that we need to add a term $\sqrt{-g}$ which, under a transformation of coordinates, transforms so that it cancels out with the transformation term in the differential d^4x of the action. We will also replace ordinary derivatives ∂_μ with covariant derivatives ∇_μ so we will be able to take derivatives of vector fields. With this substitutions in equation (2.1.2) we obtain the action

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}\Big(\nabla_\mu\phi\nabla^\mu\phi - m^2\phi^2\Big), \tag{3.1.1}$$

with Lagrangian

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\Big(\nabla_\mu\phi\nabla^\mu\phi - m^2\phi^2\Big). \tag{3.1.2}$$

A coupling term $\xi R\phi^2$ between the field and the Ricci scalar may be added, but for our applications it will not be relevant. The equation of motion is again found by varying the action with respect to the field

$$\nabla_\mu \nabla^\mu \phi - m^2 \phi = 0, \tag{3.1.3}$$

The structure is the same of the Klein-Gordon equation. The operator $\nabla_\mu \nabla^\mu$ is the generalization of the D'Alambert operator in curved spacetime and it can be rewritten as

$$\nabla_\mu \nabla^\mu f = \frac{\partial_\rho(\sqrt{-g} g^{\rho\sigma} \partial_\sigma f)}{\sqrt{-g}}. \tag{3.1.4}$$

3.2 Quantization of fields in curved spacetime

To quantize the field we generalize the procedure used for Minkowski spacetime. The conjugate momentum is

$$\pi(\mathbf{x}, t) = \frac{\partial \phi}{\partial(\nabla_0 \phi)} = \sqrt{-g} \nabla_0 \phi. \tag{3.2.1}$$

Having replaced classical fields with field operators, we impose the commutation relations

$$\begin{aligned} &[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{x}', t)] = 0 \\ &[\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = 0 \\ &[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \end{aligned} \tag{3.2.2}$$

Given two complex solutions of the Klein-Gordon equation in curved spacetime (3.1.3), we straightforwardly generalize the Klein-Gordon inner product (2.1.9) as

$$(f_1, f_2) = i \int_\Sigma (f_1^* \nabla_\mu f_2 - \nabla_\mu f_1^* f_2) \sqrt{\gamma} n^\mu \, d^3x, \tag{3.2.3}$$

where Σ is any spacelike hypersurface of the spacetime with normal vector n^μ and γ the induced metric on hypersurface. As the Klein-Gordon inner product, it is an indefinite complex bilinear product in the space of solutions, independent of the time coordinate.

We now take as domain the whole spacetime, that we assume spatially noncompact. In close analogy with what we have done in Minkowski spacetime, we define a set of complex mode functions $\{f_\mathbf{k}(\mathbf{x}, t)\}$. Following from the noncompact domain assumption we take the set as uncountable, with index $\mathbf{k} \in \mathbb{R}^3$. The modes satisfy the orthonormality conditions

$$(f_\mathbf{k}, f_{\mathbf{k}'}) = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{3.2.4}$$

$$(f_\mathbf{k}, f^*_{\mathbf{k}'}) = 0. \tag{3.2.5}$$

Generalizing equations (2.1.19) we define the creation and annihilation operators

$$\begin{cases} \hat{a}_{\mathbf{k}} = (f_{\mathbf{k}}, \hat{\phi}) \\ \hat{a}^{\dagger}_{\mathbf{k}} = -(f^{*}_{\mathbf{k}}, \hat{\phi}) \end{cases}, \tag{3.2.6}$$

for which holds the mode expansion

$$\phi(\mathbf{x}, t) = \int_{\mathbb{R}^3} d^3k \, f_{\mathbf{k}}(t, \mathbf{x}) \hat{a}_{\mathbf{k}} + f^{*}_{\mathbf{k}}(t, \mathbf{x}) \hat{a}^{\dagger}_{\mathbf{k}}. \tag{3.2.7}$$

Using the commutation relations (3.2.2) and the definitions (3.2.6), we can calculate the usual harmonic oscillator commutation relations for the creation and annihilation operators

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = 0 \tag{3.2.8}$$

$$[\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = 0 \tag{3.2.9}$$

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{3.2.10}$$

The Hilbert space of the theory, that is the Fock space, is built in the same way as in Minkowski spacetime. In particular we define again the vacuum state as the normalized state annihilated by all the annihilation operators:

$$\hat{a}_{\mathbf{k}} |0\rangle = 0 \tag{3.2.11}$$

3.3 The positive-frequency condition

As we have already discussed in the case of Minkowski spacetime, excited states created by the creation operators are interpreted as particles. We have also observed that the states of the theory depend on the choice of the set of modes, which is in principle arbitrary. Given an experiment with a particle detector, what set of modes describes the correct particle content of the physical system that is being measured?

To give an answer we first define the notion of *positive-frequency modes*. A mode function $f_{\mathbf{k}}$ is positive-frequency with respect to a timelike vector field K^{μ}, if the following holds:

$$\mathcal{L}_{K^{\mu}} f_{\mathbf{k}} = -i\omega_{\mathbf{k}} f_{\mathbf{k}}, \tag{3.3.1}$$

where $\mathcal{L}_{K^{\mu}}$ is the Lie derivative and ω is a positive constant. Now suppose we have a particle detector moving along a spacetime trajectory with a locally defined tangent timelike vector K^{μ}. Physical states described by p-particle states defined by the positive-frequency modes with respect to K^{μ} will correspond to physical situations in which the particle detector actually measures p particles. In other words, the "particles" $a^{\dagger}_{\mathbf{k}}|0\rangle$, defined mathematically through those modes, describe the particles that the detector actually measures.

With the positive-frequency condition we have obtained a preferred choice of modes for a given observer. If we restrict to static spacetimes we can find a preferred choice of modes independent of the observers. For a static spacetime, by definition, there exists a globally timelike Killing vector field. Positive-frequency modes with respect to that vector field will define a vacuum state that corresponds to the ground state of the Hamiltonian ([4] for the original reference). This important result tells us that in spacetimes with a time-like symmetry we have a physically preferred choice of modes independent of the observer. We indeed notice that Minkowski spacetime is static with Killing vector field ∂_t, and that plane wave modes are indeed positive-frequency modes with respect to that vector, as follows from equation (2.4.4). Moreover, because they satisfy that same condition, the particle states defined through plane wave modes do correspond with particle measurement made by particle detectors moving along the orbits of ∂_t, that is to say particle detectors in an inertial frame.
For spacetimes which are not static but are asymptotically flat, we can define modes which are only asymptotically positive-frequency in the "in" regime, that is to say for $t \to -\infty$, or in the "out" regime, for $t \to +\infty$. This is the field theory analogous of what we have briefly illustrated for time-dependent harmonic oscillators.
In and out modes respectively define an in-vacuum state and an out-vacuum state. The physical interpretation of these states will depend on the observer. For example, the in-vacuum state is the actual notion of vacuum for inertial observers in the past. Observers in the future will detect particles defined through the "out" set of modes, and consequently their notion of vacuum will be different from the "in"-vacuum. If they measured the particle content of the in-vacuum they would find a non zero result. This means that if we actually have, on average, no particles in the past, we would find particles in the future. This phenomenon is known as *particle creation* and has important application both in cosmology and in the context of the Hawking radiation.

3.4 Bogoljubov transformations

Not having a preferred choice of modes we would like to relate states defined with different sets of mode functions. Be $\{f_{\mathbf{k}}\}$ and $\{g_{\mathbf{k}}\}$ two set of orthonormal modes. The field can be expandend in terms of both:

$$\begin{cases} \hat{\phi} = \int d^3k \, \{f_{\mathbf{k}} \hat{a}_{\mathbf{k}} + f^*_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}}\} \\ \hat{\phi} = \int d^3k \, \{g_{\mathbf{k}} \hat{b}_{\mathbf{k}} + g^*_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}}\} \end{cases} , \qquad (3.4.1)$$

where the creation and annihilation operators are defined through equation (3.2.6).
Because both $\{f_{\mathbf{k}}\}$ and $\{g_{\mathbf{k}}\}$ form a complete set of modes in the space of solutions, an element of the first may be expanded as a linear combination of

the elements of the second and viceversa

$$g_{\mathbf{k}} = \int d^3p \, \{\alpha_{\mathbf{kp}} f_{\mathbf{p}} + \beta_{\mathbf{kp}} f^*_{\mathbf{p}}\}. \tag{3.4.2}$$

Relations between modes are called *Bogoljubov transformation* and $\alpha_{\mathbf{kp}}$ and $\beta_{\mathbf{kp}}$ are the Bogoljubov coefficents. They are a generalizations of the transformations (1.0.19) we have seen in the case of the harmonic oscillator. The coefficients can be find decomposing one mode onto another using the inner product (3.2.3)

$$\begin{aligned} \alpha_{\mathbf{kp}} &= (g_{\mathbf{k}}, f_{\mathbf{p}}) \\ \beta_{\mathbf{kp}} &= -(g_{\mathbf{k}}, f^*_{\mathbf{p}}), \end{aligned} \tag{3.4.3}$$

as it can be explicitly verified plugging equation (3.4.2) in these two. The inverse transformation is obtained by writing the definition of the inverse coefficients with these same formulas, with f and g exchanged. Substituting in $g_{\mathbf{k}}$ its expansion in term of $f_{\mathbf{k}}$ and using the properties of the Klein-Gordon inner product we obtain

$$f_{\mathbf{k}} = \int d^3p \, \{\alpha^*_{\mathbf{kp}} g_{\mathbf{p}} - \beta_{\mathbf{kp}} g^*_{\mathbf{p}}\}. \tag{3.4.4}$$

Bogoljubov transformation between modes induce transformations between operators. Having defined $\hat{b}_{\mathbf{k}} = (g_{\mathbf{k}}, \hat{\phi})$, substituting $g_{\mathbf{k}}$ in terms of f modes with the Bogoljubov transformation and using the linearity of the inner product, we recognize the definition of the creation and annihilation operators for f modes. The transformation between operators is then

$$\hat{b}_{\mathbf{k}} = \int d^3p \left(\alpha^*_{\mathbf{kp}} \hat{a}_{\mathbf{p}} - \beta^*_{\mathbf{kp}} \hat{a}^\dagger_{\mathbf{p}}\right). \tag{3.4.5}$$

In the same way the inverse transformation is obtained:

$$\hat{a}_{\mathbf{k}} = \int d^3p \left(\alpha_{\mathbf{kp}} \hat{b}_{\mathbf{p}} + \beta^*_{\mathbf{kp}} \hat{b}^\dagger_{\mathbf{p}}\right). \tag{3.4.6}$$

From the orthonormality relations for f and g we find normalization conditions for the Bogoljubov coefficients:

$$\int d^3p \big[\alpha_{\mathbf{kp}} \alpha^*_{\mathbf{k'p}} - \beta_{\mathbf{kp}} \beta^*_{\mathbf{k'p}}\big] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{3.4.7}$$

$$\int d^3p \big[\alpha_{\mathbf{kp}} \beta_{\mathbf{k'p}} - \beta_{\mathbf{kp}} \alpha_{\mathbf{k'p}}\big] = 0. \tag{3.4.8}$$

Now we want to study the relation between the vacuum states defined by the two sets of modes. We call particles defined by f modes *f-particles* and particles defined by g modes *g-particles.* Correspondingly we define the *f-vacuum* $|0_f\rangle$ and the *g-vacuum* $|0_g\rangle$. A particle detector measuring f-particles will find no particles in the f-vacuum and a particle detector measuring g-particles will find no particles in the g-vacuum. What is, on the other hand, the f-particles content of the g-vacuum? We need to calculate $\langle 0_g|\hat{N}_{\mathbf{k}_f}|0_g\rangle$, where $\hat{N}_{\mathbf{k}_f} = a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$. We can calculate it in term of the Bogoljubov coefficients :

$$\langle 0_g|\hat{N}_{\mathbf{k}_f}|0_g\rangle = \langle 0_g|\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}|0_g\rangle = \tag{3.4.9}$$

$$= \langle 0_g| \int d^3p \left(\alpha_{\mathbf{kp}}\hat{b}_{\mathbf{p}} - \beta^*_{\mathbf{kp}}\hat{b}^\dagger_{\mathbf{p}}\right) \int d^3p' \left(\alpha^*_{\mathbf{kp'}}\hat{b}^\dagger_{\mathbf{p'}} - \beta_{\mathbf{kp'}}\hat{b}_{\mathbf{p'}}\right)|0_g\rangle = \tag{3.4.10}$$

$$= \int d^3p\, |\beta_{\mathbf{kp}}|^2, \tag{3.4.11}$$

where the canonical commutation relation between creation and annihilation operators has been used. We can also calculate in the same way the average number of g-particles measured in the f-vacuum

$$\langle 0_f|\hat{N}_{\mathbf{k}_g}|0_f\rangle = \int d^3p\, |\beta_{\mathbf{kp}}|^2. \tag{3.4.12}$$

We observe that the result is the same.
If the two set of modes have null β coefficients the particle content of the vacuum state will be 0 in both cases. Inspecting equation (3.4.2) we see that this happens if positive-frequency modes of one set of functions are expressed in terms of solely the positive-frequency modes of the other set.
The definition of the Bogoljubov transformations holds in any spacetime.
We can use Bogoljubov coefficients to relate also different sets of modes in Minkowski space. It is easy to show that modes related by Lorentz transformation have $\beta_{\mathbf{kp}} = 0$. For different set of modes this does not necessarily hold. An example of this will be given in the next chapter.

Chapter 4

Quantum field theory in accelerated frames

In this chapter we will study the scalar field in the perspective of an accelerated observer in Minkowski spacetime. Firstly we will build the *Rindler coordinates*, which are an alternate system of coordinates for a patch of Minkowski spacetime, the *Rindler wedge*. We will then quantize the scalar field using Rindler coordinates. In this way we build two alternative set of modes for the Klein-Gordon field which, according to the positive-frequency condition, produce vacuum states for inertial and accelerated observers respectively. Calculating Bogoljubov coefficients we will show the *Unruh effect*: the accelerated observer measures a black body spectrum of particles in the vacuum state defined in an inertial frame.

4.1 Rindler coordinates

The first step is to calculate the spacetime trajectory of an accelerating object in special relativity. We need to distinguish three frames of reference. We call the frame of reference of the accelerated observer *proper frame*. The usual inertial frame in Minkowski spacetime is the *laboratory frame*. For any given instant t_0 in the laboratory frame, we define the *comoving frame* as the frame of reference of an intertial observer whose velocity in that instant is the same of the velocity of the proper frame. In the comoving frame the accelerated observer is instaneously at rest.
In special relativity an observer is said to be in constant acceleration if the acceleration measured in the comoving frame is constant. We will see that this is compatible with the fact that nothing can move faster than light.
Now we would like to find $x^\mu(\tau) = (t(\tau), \mathbf{x}(\tau))$. The four-velocity $u^\mu(\tau)$ and

the four-acceleration $a^\mu(\tau)$ are

$$u^\mu(\tau) = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}(\tau) \tag{4.1.1}$$

$$a^\mu(\tau) = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}(\tau) = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}(\tau). \tag{4.1.2}$$

For the four acceleration the usual normalization holds

$$u_\mu(\tau)u^\mu(\tau) = 1. \tag{4.1.3}$$

Deriving both sides we get

$$a_\mu(\tau)u^\mu(\tau) = 0. \tag{4.1.4}$$

This relation hold in any frame of reference. In the comoving frame $u^\mu(\tau) = (1, 0, 0, 0)$. Writing $a^\mu(\tau) = (a^t, \mathbf{a})$ and using the last equation, we have $a^t(\tau) = 0$. Therefore we obtain

$$a^\mu a_\mu = -|\mathbf{a}|^2. \tag{4.1.5}$$

The left-hand side is Lorentz-invariant, and so it holds also in the laboratory frame. We assume that the observer moves only in the x direction with $y = z = 0$. Therefore the y and z components of the four-velocity and four-acceleration are 0. To find $x^\mu(\tau)$ we can integrate the system of differential equations that we have found:

$$\begin{cases} (\dot{u}^t)^2(\tau) - (\dot{u}^x)^2(\tau) = -a^2 \\ (u^t)^2(\tau) - (u^x)^2(\tau) = 1 \end{cases} \tag{4.1.6}$$

Where the dot symbol is the derivation with respect to the proper time τ. The structure of the equations clearly resemble the equation of an hyperbola, which is parametrized by hyperbolic sines and cosines. The following ansatz is natural:

$$\begin{cases} \dot{u}^t(\tau) = a \sinh a\tau \\ \dot{u}^x(\tau) = a \cosh a\tau \end{cases} . \tag{4.1.7}$$

Integrating once:

$$\begin{cases} u^t(\tau) = \cosh a\tau + A \\ u^x(\tau) = \sinh a\tau + B \end{cases} . \tag{4.1.8}$$

Imposing $u^x(0) = 0$ and requiring the second equation to be satisfied at all times, we obtain $A = B = 0$. Integrating once again:

$$\begin{cases} t(\tau) = \frac{1}{a} \sinh a\tau + C \\ x(\tau) = \frac{1}{a} \cosh a\tau + D \end{cases} . \tag{4.1.9}$$

If $x(0) = a^{-1}$ and $t(0) = 0$, $C = D = 0$, and we obtain the final expression:

$$\begin{cases} t(\tau) = \frac{1}{a} \sinh a\tau \\ x(\tau) = \frac{1}{a} \cosh a\tau \end{cases} . \tag{4.1.10}$$

The wordline of the observer in the Minkowski diagram is an hyperbola centered in the origin, intersecting the space axis in a^{-1}. Physically at $t \to -\infty$ the accelerated observer is at $x \to +\infty$ with velocity $-c$. Positive constant acceleration makes the observer slow down until it stops at $t = 0$ and starts gaining velocity in the opposite direction. At $t \to +\infty$ the observer reaches c. Now we will work in $1 + 1$ dimensions. *Rindler coordinates* is the set of coordinates of Minkowski spacetime fitting accelerated observers. Precisely, it is the map from the couple (τ, ξ), where τ is the proper time and ξ the distance measured by the observer, and (t, x), which are the coordinate of an inertial observer. We have already derived

$$\begin{cases} t(\tau, 0) = \frac{1}{a}\sinh a\tau \\ x(\tau, 0) = \frac{1}{a}\cosh a\tau \end{cases} . \tag{4.1.11}$$

To obtain the complete transformation, we need to know how the observer in the proper frame measures distances. At a given τ, suppose to have a ruler of length ξ in the proper frame, whose endpoints are in $(\tau, 0)$ and (τ, ξ). In the comoving frame, that is inertial, the endpoints have coordinates $(0, 0)$ and $(0, \xi)$. At a given time τ the relation between the laboratory frame and the proper frame is

$$x^\mu(\tau, \xi) = \begin{pmatrix} t(\tau) \\ x(\tau) \end{pmatrix} + \Lambda^{-1}(v) s^\mu(\xi), \tag{4.1.12}$$

where s^μ is the vector in the laboratory frame and Λ is the inverse Lorentz transformation with associated velocity $v := u^x(\tau)/\gamma$, that is the velocity of the comoving frame. Explicitly the inverse transformation is

$$\Lambda^{-1}(v) = \begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix} = \begin{pmatrix} u^t(\tau) & u^x(\tau) \\ u^x(\tau) & u^t(\tau) \end{pmatrix}, \tag{4.1.13}$$

where we have used $u^t(\tau) = \gamma$. In the case we are studying $s^\mu = (0, \xi)$ and therefore

$$\begin{cases} t(\tau, \xi) = t(\tau) + \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}\xi \\ x(\tau, \xi) = x(\tau) + \frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}\xi \end{cases} . \tag{4.1.14}$$

Using equations (4.1.11), we get the expression for Rindler coordinates:

$$\begin{cases} t(\tau, \xi) = \frac{1+a\xi}{a}\sinh a\tau \\ x(\tau, \xi) = \frac{1+a\xi}{a}\cosh a\tau \end{cases} . \tag{4.1.15}$$

Computing $x^2(\tau, \xi) - t^2(\tau, \xi)$ we see that lines of constant ξ, with $\xi > -a^{-1}$ are branches of hyperbolas intersecting the x-axis in $a^{-1}(1 + a\xi)$. If $\xi = -a^{-1}$ the curve is a degenerate hyperbola with a right angle in the origin. Physically this is the trajectory of an observer on which an infinite force is exerted at $\tau = 0$. This means that an accelerated observer cannot measure distances in direction opposite to his acceleration with a ruler with length greater than a^{-1} because that measurement would imply an infinite strength. It is said

that the observer perceives an *horizon* at $\xi = -a^{-1}$. In Minkowski coordinates this is the surface $t = \pm x$ with $x > 0$. This means that the coordinates that we have obtained are only valid for $-a^{-1} < \xi < +\infty$ and $\tau \in \mathbb{R}$. Because lines of constant ξ are right branches of hyperbolas, we get that only the $x > |t|$ region is covered. We say that the Rindler coordinates are an *incomplete* system of coordinates which cover only the *right Rindler wedge* of Minkowski spacetime.
It is convenient to define another space coordinate $\tilde{\xi}$ that ranges through all real values. We define $\tilde{\xi} = \log(1 + a\xi)/a$ so that $1 + a\xi = \exp a\tilde{\xi}$

$$\begin{cases} t(\tau, \tilde{\xi}) = \frac{1}{a} e^{a\tilde{\xi}} \sinh(a\tau) \\ x(\tau, \tilde{\xi}) = \frac{1}{a} e^{a\tilde{\xi}} \cosh(a\tau) \end{cases} . \tag{4.1.16}$$

In these coordinates the metric is

$$g' = J^T(\tau, \tilde{\xi}) \eta J(\tau, \tilde{\xi}), \tag{4.1.17}$$

where J is the Jacobian of the Rindler transformation (4.1.16) and the matrix η is the metric in Minkowski space. Explicitly:

$$J = e^{a\tilde{\xi}} \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) \\ \sinh(a\tau) & \cosh(a\tau) \end{pmatrix} . \tag{4.1.18}$$

J is proportional to a Lorentz transformation and therefore, using the defining property of the Lorentz group, g' is

$$g' = J^T(\eta, \xi) \eta J(\eta, \xi) = e^{2a\xi} \eta, \tag{4.1.19}$$

and we can write the invariant line element as

$$ds^2 = e^{2a\xi}(d\eta^2 - d\tilde{\xi}^2). \tag{4.1.20}$$

In the following section we will use this set of coordinates. For simplicity we redefine $\tilde{\xi}$ as ξ.

4.2 Unruh Effect

We will now study the scalar field in the perspective of an accelerated observer. To do this we will start from the standard Klein-Gordon action and change its coordinates into Rindler coordinates. For simplicity we will still work in $1 + 1$ dimensions and we will consider a massless field. The action is

$$S[\phi] = \int dt dx \, \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \int dt dx \, (\partial_t^2 \phi - \partial_x^2 \phi). \tag{4.2.1}$$

We now rewrite the action in Rindler coordinates:

$$S[\phi] = \int d\tau d\xi |(Jx^\mu)(\tau, \xi)| \, \tilde{g}^{\mu\nu} \tilde{\partial}_\mu \phi \tilde{\partial}_\nu \phi, \tag{4.2.2}$$

where $|(Jx^\mu)(\tau,\xi)|$ is the Jacobian of the coordinate transformation and $\tilde{g}^{\mu\nu}$ is the transformed metric (4.1.19). These two cancel out exactly and we get

$$S[\phi] = \int d\tau d\xi \eta^{\mu\nu}\tilde{\partial}_\mu\phi\tilde{\partial}_\nu\phi = \int d\tau d\xi \,(\partial_\tau^2\phi - \partial_\xi^2\phi). \tag{4.2.3}$$

The field equations we obtain are standard wave equations in 1 spatial dimension

$$\begin{aligned} \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} &= 0 \\ \frac{\partial^2\phi}{\partial \tau^2} - \frac{\partial^2\phi}{\partial \xi^2} &= 0. \end{aligned} \tag{4.2.4}$$

We have by now thoroughly understood how fields must be quantized. We take the positive-frequency modes obeying condition (3.3.1) with respect to ∂_t and ∂_τ respectively. The solution was already determined in the section on quantization in Minkowski. The modes are plane waves with appropriate normalization. The field expansion using the two coordinate system is:

$$\hat{\phi}(t,x) = \int_{\mathbb{R}} dk \,\frac{1}{\sqrt{2|k|(2\pi)}}[e^{-i(|k|t-kx)}\hat{a}_k + e^{i(|k|t-kx)}\hat{a}_k^\dagger] \tag{4.2.5}$$

$$\hat{\phi}(\tau,\xi) = \int_{\mathbb{R}} dk \,\frac{1}{\sqrt{2|k|(2\pi)}}[e^{-i(|k|\tau-k\xi)}\hat{b}_k + e^{i(|k|\tau-k\xi)}\hat{b}_k^\dagger], \tag{4.2.6}$$

where we have used $\omega = |k|$. The two sets of modes satisfy the same field equation in different coordinates. If we change coordinates to one of the two sets of modes they will, by costruction, satisfy the same equation. Specifically if we take the plane waves in Minkowski spacetime and write (t,x) in terms of (τ,ξ) using the Rindler coordinate transformation (4.1.16), we obtain a set of modes satisfying equation (4.2.4). We have thus succeed to build explicitly two set of modes in the same spacetime (or precisely in the same subset of a spacetime, the Rindler wedge):

$$f_k(\tau,\xi) = \frac{e^{-i(|k|t(\tau,\xi)-kx(\tau,\xi))}}{\sqrt{2|k|(2\pi))}} \qquad g_k(\tau,\xi) = \frac{e^{-i(|k|\tau-k\xi)}}{\sqrt{2|k|(2\pi)}} \tag{4.2.7}$$

One can explicitly verify that f_k modes do satisfy equation (4.2.4). If, for example, $k>0$, we have

$$f_k(\tau,\xi) = \frac{1}{\sqrt{2|k|(2\pi)}}\exp\Big(\frac{ik}{a}e^{a(\xi-\tau)}\Big), \tag{4.2.8}$$

which is in the form $A(\xi \pm \tau)$, that is always solution of the one dimensional wave equation (4.2.4).

The two sets of modes define the operators $\hat{a}_k$ and $\hat{b}_k$, by which the vacuum states $|0_\mathrm{M}\rangle$, called *Minkowski vacuum*, and $|0_\mathrm{R}\rangle$, the *Rindler vacuum*, are respectively constructed. Following the discussion in the last chapter we know that the first vacuum state is the one measured by a detector in an inertial frame, while the second vacuum state is the one measured by a detector moving in an accelerated frame. This is true because they are positive-frequency modes with respect to the tangent vector to their wordlines, ∂_t and ∂_τ. Now suppose our spacetime to be in the state $|0_{\rm M}\rangle$. An inertial observer will measure no particles at all. Using equation (3.4.12) we know that an accelerated observer will measure N_k particles with momentum k in the Minkowski vacuum, where N_k is

$$N_k = \langle 0_{\rm M}|b_k^\dagger b_k|0_{\rm M}\rangle = \int d^3p\, |\beta_{kp}|^2, \tag{4.2.9}$$

and β_{kp} is the coefficient of the Bogoljubov transformation between f_k and g_p. We can determine the coefficient using equation (3.4.3). The computation however is not trivial at all. We need to use some tricks to do the calculation. We will follow [5]. Firstly we use *lightcone coordinates*

$$\begin{cases} u(\tau,\xi) = \tau - \xi \\ v(\tau,\xi) = \tau + \xi \end{cases}, \begin{cases} \tilde{u}(t,x) = t - x \\ \tilde{v}(t,x) = t + x \end{cases}. \tag{4.2.10}$$

Taking the sum and the difference of equations (4.1.16), we obtain the relation between the two pairs of lightcone coordinates

$$\begin{cases} \tilde{u}(u) = -a^{-1}e^{-au} \\ \tilde{v}(v) = a^{-1}e^{av} \end{cases}. \tag{4.2.11}$$

We notice that the u and $\tilde{u}$ coordinates do not mix with v and $\tilde{v}$ coordinates. Modes can now be rewritten in a simpler form

$$f_k(u,v) = \begin{cases} \frac{e^{-ik\tilde{u}(u)}}{\sqrt{2k(2\pi))}} \,\text{if}\, k>0 \\ \frac{e^{ik\tilde{v}(v)}}{\sqrt{2|k|(2\pi))}} \,\text{if}\, k<0 \end{cases} \quad g_k(u,v) = \begin{cases} \frac{e^{-iku}}{\sqrt{2k(2\pi))}} \,\text{if}\, k>0 \\ \frac{e^{ikv}}{\sqrt{2|k|(2\pi))}} \,\text{if}\, k<0 \end{cases} \tag{4.2.12}$$

Mode expansions (4.2.5) and (4.2.6) can be rewritten in terms of lightcone coordinates by splitting the integral over $\mathbb{R}$ in two integrals over $(-\infty, 0)$ and $(0,\infty)$, and changing the sign of the variable in the first. In this way we obtain

$$\phi(\tilde{u}(u)),\tilde{v}(v)) = \int_0^\infty \frac{dk}{\sqrt{2\pi 2k}}\left[e^{-ik\tilde{u}}\hat{a}_k + e^{ik\tilde{u}}\hat{a}_k^\dagger + e^{-ik\tilde{v}}\hat{a}_{-k} + e^{ik\tilde{v}}\hat{a}_{-k}^\dagger\right] \tag{4.2.13}$$

$$\phi(u,v) = \int_0^\infty \frac{dk}{\sqrt{2\pi 2k}}\left[e^{-iku}\hat{b}_k + e^{iku}\hat{b}_k^\dagger + e^{-ikv}\hat{b}_{-k} + e^{ikv}\hat{b}_{-k}^\dagger\right], \tag{4.2.14}$$

where in the right hand side we have omitted $\tilde{u}$ and $\tilde{v}$ dependence on u and v. Both expansion have a piece depending only on one coordinate or on the other

$$\phi(\tilde{u}(u),\tilde{v}(v)) = A(\tilde{u}(u)) + B(\tilde{v}(v)) \quad \phi(u,v) = C(u) + D(v), \tag{4.2.15}$$

then the equation

$$A(\tilde{u}(u)) + B(\tilde{v}(v)) = C(u) + D(v), \tag{4.2.16}$$

can be separated into two

$$A(\tilde{u}(u)) = C(u) \quad B(\tilde{v}(v)) = D(v). \tag{4.2.17}$$

We will now focus on the first one. If we write it explicitly

$$\int_0^\infty \frac{dk}{\sqrt{2\pi 2k}} \left[e^{-ik\tilde{u}}\hat{a}_k + e^{ik\tilde{u}}\hat{a}_k^\dagger\right] = \int_0^\infty \frac{dp}{\sqrt{2\pi 2p}} \left[e^{-ipu}\hat{b}_p + e^{ipu}\hat{b}_p^\dagger\right]. \tag{4.2.18}$$

We now Fourier transform both sides with $e^{ip'u}$ plane waves. The right hand side becomes

$$\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \int_0^\infty \frac{dp}{\sqrt{2\pi 2p}} \left[e^{-ipu}\hat{b}_p + e^{ipu}\hat{b}_p^\dagger\right] = \int_0^\infty \frac{dp}{\sqrt{2p}} \left[\delta(p-p')\hat{b}_p + \delta(p+p')\hat{b}_p^\dagger\right]. \tag{4.2.19}$$

If $p' > 0$ only the first Dirac delta remains and we are left with $\frac{1}{\sqrt{2p'}}\hat{b}_{p'}$. Following calculations will always implicitly assume $p' > 0$. The left-hand side is more complicated. By defining the auxiliary function

$$F(k,p) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{ipu - ik\tilde{u}} \tag{4.2.20}$$

we can write it as

$$\int_0^{+\infty} \frac{dk}{\sqrt{2k}} \left[F(k,p')\hat{a}_k + F(-k,p')\hat{a}_k^\dagger\right]. \tag{4.2.21}$$

Putting altogether

$$\frac{1}{\sqrt{2p'}}\hat{b}_{p'} = \int_0^{+\infty} \frac{dk}{\sqrt{2k}} \left[F(k,p')\hat{a}_k + F(-k,p')\hat{a}_k^\dagger\right]. \tag{4.2.22}$$

By analogy with the Bogoljubov transformation (3.4.5) we find that, if k is positive, the Bogoljubov coefficients are

$$\alpha_{pk}^* = \sqrt{\frac{p}{k}} F(k,p) \tag{4.2.23}$$

$$-\beta_{pk}^* = \sqrt{\frac{p}{k}} F(-k,p), \tag{4.2.24}$$

where, for simplicity, p' has be redefined as p. If k is negative, Bogoljubov coefficients are 0. If p is negative, the opposite holds: the coefficients are nonzero only if k is negative.

Because we want to calculate the particle content of vacuum we only need to evaluate the integral of the square modulus of the β coefficients, as in equation (4.2.9). Instead of calculating directly the integral we use the normalization (3.4.7)

$$\int_0^{+\infty} dp \left[\sqrt{\frac{pp'}{k^2}} F(k,p) F^*(k,p') - \sqrt{\frac{pp'}{k^2}} F(-k,p) F^*(-k,p') \right] = \delta^{(3)}(p-p') \tag{4.2.25}$$

We now use the following property of the auxiliary function F:

$$F(k,p) = F(-k,p) e^{\frac{\pi p}{a}}. \tag{4.2.26}$$

To show this we write F expressing $\tilde{u}$ in terms of u with equation (4.2.11)

$$F(k,p) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} \exp\left(ipu + \frac{ik}{a} e^{-au} \right). \tag{4.2.27}$$

Using residue theorem we can rewrite this integral. We define the closed curve $\gamma(R) := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$\gamma_1 = t \qquad t \in [-R, R] \tag{4.2.28}$$
$$\gamma_2 = R - \pi a^{-1} + \pi a^{-1} e^{-i\theta} \qquad \theta \in [0, \pi/2] \tag{4.2.29}$$
$$\gamma_3 = -t - i\pi a^{-1} + R - \pi a^{-1} \qquad t \in [0, 2R - 2\pi a^{-1}] \tag{4.2.30}$$
$$\gamma_4 = -R + \pi a^{-1} + \pi a^{-1} e^{-i(\theta + \pi/2)} \qquad \theta \in [0, \pi/2], \tag{4.2.31}$$

with $R \in \mathbb{R}$. Because the F is analytical, from the residue theorem follows

$$\oint_\gamma \frac{dz}{2\pi} \exp\left(ipz + \frac{ik}{a} e^{-az} \right) = 0. \tag{4.2.32}$$

For $R \to \infty$ the integrals on γ_2 and γ_4 vanish. We can see this by studying how the integrand behaves for $z \to \pm\infty - i\alpha$, where $\alpha \in [0, \pi a^{-1}]$. Imaginary parts of the exponential are just oscillation. The only real part appears in the second term. Explicitly if we let $z = t - i\alpha$

$$Re\left(\frac{ik}{a} e^{-az} \right) = Re\left(\frac{ik}{a} e^{-at + ia\alpha} \right) = -\frac{k}{a} e^{-at} \sin(a\alpha) \tag{4.2.33}$$

If $t \to -\infty$, the real part goes to $-\infty$, which means that the integrand decays exponentially and therefore converges. If $t \to +\infty$ we need to add a regularization term, as e^{-bz^2}. Calculating the integral and then letting the regularization parameter b go to 0, it converges.
We are then left with the integrals on γ_1 and γ_3. Equation (4.2.32) becomes

$$\int_{\gamma_1} \frac{dz}{2\pi} \exp\left(ipz + \frac{ik}{a} e^{-az} \right) = \int_{-\gamma_2} \frac{dz}{2\pi} \exp\left(ipz + \frac{ik}{a} e^{-az} \right). \tag{4.2.34}$$

In the limit $R \to +\infty$, the left hand side is $F(k,p)$, while the right hand side becomes an integral on the real line shifted down by $i\pi a^{-1}$. We calculate it with the substitution $z = t - i\pi a^{-1}$

$$\int_{-\infty}^{+\infty} \frac{dt}{2\pi} \exp\left[ipt + \frac{\pi p}{a} + \frac{ik}{a} e^{-a(t - i\pi/a)}\right] = e^{\frac{\pi p}{a}} \int_{-\infty}^{+\infty} \frac{dt}{2\pi} \exp\left[ipt - \frac{ik}{a} e^{-at}\right]. \tag{4.2.35}$$

On the right hand side we have $e^{\frac{p\pi}{a}} F(-k,p)$. With this relation, the equation (4.2.25) becomes

$$\int_0^{+\infty} \frac{\sqrt{pp'}}{k} \left[e^{\frac{\pi(p+p')}{a}} F(-k,p) F^*(-k,p') - F(-k,p) F^*(-k,p')\right] = \delta^{(3)}(p - p'). \tag{4.2.36}$$

Letting $p = p'$ and rearranging we are left with

$$\int_0^{+\infty} dp \frac{p}{k} F(-k,p) F^*(-k,p) = \left[e^{\frac{2\pi p}{a}} - 1\right]^{-1} \delta^{(3)}(0). \tag{4.2.37}$$

On the left hand side have precisely the factor

$$\int d^3p\, |\beta_{\mathbf{kp}}|^2 = \left[e^{\frac{2\pi p}{a}} - 1\right]^{-1} \delta^{(3)}(0). \tag{4.2.38}$$

Then the average number of particles with momentum p that an accelerated observer measures in Minkowski vacuum, is

$$N_p = \frac{\delta^{(3)}(0)}{e^{2\pi p/a} - 1}. \tag{4.2.39}$$

The Dirac delta term is the divergent volume factor. If we did the calculations in finite volume that would be the total volume V of the space. We then rewrite the result in terms of density of particles

$$n_p = \frac{1}{e^{2\pi p/a} - 1}. \tag{4.2.40}$$

Defining the *Unurh temperature*

$$T = \frac{a}{2\pi}, \tag{4.2.41}$$

we write the density of particles as

$$n_p = \frac{1}{e^{p/T} - 1}. \tag{4.2.42}$$

We can see explicitly that, even if an inertial observer measures no particles, an accelerated observer measures a Bose-Einstein distribution of particle with temperature (4.2.41). This is precisely what is meant by Unruh effect. In the previous chapter we noticed that equation (3.4.12) is symmetric. Therefore

we can also conclude that if an accelerated observer measures no particles, an inertial observer will measure the same Bose-Einstein distribution with temperature (4.2.41). We have shown this for a massless field in 1+1 dimensions, but it can also be derived for fields with mass in 3+1 dimensions.
We would like to emphasize that what is most remarkable about Unruh effect is that it shows that the concept of particle, even in flat spacetime, does not have an absolute meaning, but depends on the wordline of the detector by which we do we our measurements. What ultimately does have an absolute physical reality are fields.
Experimental tests of the Unruh effect are in practice hard to realize. We can evaluate the order of magnitude of the Unruh temperature by restoring SI units. Left hand side of equation (4.2.41) is an acceleration in SI units. The factor of conversion from an acceleration to a temperature must have dimension $s^2 m^{-1} K$. This is realized by the combination $\hbar / c k_{\mathrm{B}}$. Therefore the temperature in SI units is

$$T = \frac{\hbar}{c k_{\mathrm{B}}} \frac{a}{2\pi} \approx (4 \cdot 10^{-21} s^2 m^{-1} K) a. \tag{4.2.43}$$

Consider for example a glass of water. In an accelerated frame it would be in a thermal bath of particles with given temperature, with which would eventually reach thermodynamic equilibrium. If we wanted water to boil, we need $T = 373K$. This temperature requires an acceleration $a \approx 3 \cdot 10^{22}\, m s^{-2}$, which is far out from experimental limits. Several experimental setups have been however proposed to test the effect. A review can be found in [6].

Chapter 5

Quantum effects in Inflation

In this final chapter we will see a physical application of quantum field theory in curved spacetime. In the first section we will briefly revise some basics of cosmology and the FLRW metric. We will then discuss, without going into mathematical details, the initial conditions problems that arises in standard cosmological models with the observation of the CMB. After introducing the Inflation theory, we will show how to setup the quantization of a scalar field in FLRW metrics and study the case of the de Sitter universe. We will then present the calculation of linear perturbations to study the coupling of the inflaton field with the metric. The approximate action will give the action of the scalar field studied in the previous section. Without going into full details, in the last section several measurable quantities will be defined and compared with latest experimental results.

5.1 FLRW metric and cosmology

Experimental observations show that on large scales the universe is homogeneous and isotropic. Spacetimes with this symmetries are described by the *Friedmann-Lemaitre-Robertson-Walker (FLRW) metric*

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j, \tag{5.1.1}$$

where $a(t)$ is a generic function called *scale factor* and δ_{ij} is metric of the spacelike hypersurfaces, that we have assumed to be flat. The scale factor is determined by imposing the metric $g_{\mu\nu}$ to be a solution of the Einstein's field equations with a given energy-momentum tensor $T_{\mu\nu}$

$$G_{\mu\nu} - g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}. \tag{5.1.2}$$

The energy-momentum tensor needs to be compatible with the symmetries of the spacetime. Because the metric is homogeneous and isotropic, $T_{\mu\nu}$ must have the form of the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = -pg_{\mu\nu} + u_\mu u_\nu (p + \rho). \tag{5.1.3}$$

In the rest frame of the fluid we have

$$\begin{cases} T^0_0 = \rho \\ T^i_j = -p\delta^i_j \end{cases}, \tag{5.1.4}$$

where ρ is the density of the fluid and p is the pressure. Types of matter in the universe are classified by their *equation of state*, which is the relation $p = p(\rho)$ between density and pressure.
Separating the 00 component and the spatial components of the Einstein's field equations, and calculating the Einstein tensor for the FLRW metric, we obtain the *Friedmann equations*

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{5.1.5}$$

$$\frac{a''}{a} = -\frac{8\pi G}{6}(\rho + 3p). \tag{5.1.6}$$

It is convenient to define the *Hubble parameter* $H = a'/a$, in terms of which the Friedmann equations become

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \tag{5.1.7}$$

$$H' + H^2 = -\frac{8\pi G}{6}(\rho + 3p). \tag{5.1.8}$$

The first equation is enough to determine $a(t)$ from the density $\rho(t)$. The density itself however depends on the the metric of the spacetime. From the continuity equation, $\nabla_\mu T^{\mu\nu} = 0$, we can derive a formal relation between these observables

$$\rho' + 3H(p + \rho) = 0. \tag{5.1.9}$$

Writing the equation of state as $p = w\rho$, assuming w constant, we find an equation for $\rho(a)$ given an initial condition $\rho(t_0)$

$$\rho(t) = \rho(t_0)a^{-3(1+w)}. \tag{5.1.10}$$

Substituting into the first Friedmann equation and solving, we find the time dependence of the scale factor in terms of w, assuming that the cosmological constant Λ can be neglected

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \tag{5.1.11}$$

where we have included all constants in t_0.

For non-relativistic matter $w = 0$ and $a(t) \propto t^{2/3}$, while for relativistic matter, or radiation, $w = 1/3$ and $a(t) \propto t^{1/2}$. We will refer to both as ordinary matter. For a universe with no matter and no radiation, but positive cosmological constant, we have $\rho = 0$. Therefore the Hubble parameter is constant. Imposing $a(0) = 1$, integration of the first Friedmann equation gives

$$a(t) = e^{\sqrt{\Lambda/3}t}, \tag{5.1.12}$$

with $H = \sqrt{\Lambda/3}$. Explicitly FLRW metric with such scale factor is

$$ds^2 = dt^2 - e^{2tH}(dx^2 + dy^2 + dz^2), \tag{5.1.13}$$

which is the metric of the *de Sitter universe*.
A complete physical model should include all the possible contributions in the Friedmann equations. However the results we have obtained do have an approximate validity when any of those contributions dominate on the others.

5.2 Big Bang and initial conditions problem

Both for matter and radiation we notice that Friedmann equations give an ever-expanding universe with an initial singularity $a(t) \to 0$ for $t \to 0$. The moment in which the scale factor vanishes is known as *Big Bang*. One of the greatest experimental evidence for the Big Bang is the *cosmic microwave background* (CMB), an electromagnetic radiation that is diffused through the whole observable universe. It is the afterglow radiation of the Big Bang that is has been "stretched" by the subsequent expansion to infrared wavelength. It has been broadly studied since its discovery by Arno Penzias and Robert Wilson, and it has been measured from all directions. In particular the frequency spectrum of the radiation corresponds almost perfectly with a black-body thermal distribution in equilibrium with temperature of 2.73 K. There are also small fluctuations $\frac{\delta T}{T} \approx 10^{-5}$, whose origin we will explain with quantum fluctuations.
However in standard cosmological models, that is to say with ordinary matter, there's no reason why the radiation should be in thermal equilibrium. This is because in such models the *particle horizon*, which is the maximum distance that light rays can have travelled in a time t after the Big Bang, is finite and grows monotonically with time. This implies that CMB should be composed of casually disconnected patches. An homogeneous distribution is explained only by admitting a fine-tuned initial conditions on the distributions of radiation. This is known as the *horizon problem*.
A second fine-tuning problem arises in considering the *curvature parameter* Ω, which is related to the curvature of the universe. Specifically, if $\Omega = 1$ the universe is flat. Modern measurements show that $0.98 < \Omega < 1.08$. It can be shown however that for standard cosmological models Ω is an *unstable fixed point*, which means that if $\Omega(t_0) = 1$ it will stay 1, while for slight deviations it will grow or decrease too quickly to be compatible for present measurements. This implies that even in very remote times the universe had to be

extremely flat. Because the curvature parameter is related with initial conditions on the velocity of particles that composed the universe, this is again a fine-tuning problem on initial conditions and it is known as *flatness problem.*

5.3 The theory of Inflation

Inflation is a short period, $10^{-34}\div 10^{-32}$ second circa, of accelerated expansion after the Big Bang, starting approximately 10^{-36} seconds after the singularity. Historically it has been introduced to solve the initial conditions problems. It can be shown that a phase of accelerated expansion implies a divergent particle horizon, which makes possible, in the past, a casual connection between the different regions of the CMB that appeared disconnected in standard cosmological models. Moreover, during inflation, the flatness condition $\Omega = 1$ becomes an *attractor*, which means that the universe evolves towards flatness, and no fine-tuning for initial conditions is required. Details on initial conditions problem and how inflation solves them can be found in [7].
During inflation the scale factor satisfies

$$\frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} > 0. \tag{5.3.1}$$

Friedmann equations tells which kind of matter can generate such scale factor. In particular, from equation (5.1.6), we see that

$$\rho + 3p < 0, \tag{5.3.2}$$

but ordinary matter obeys the so-called *strong-energy condition* $\rho + 3p > 0$ and therefore cannot produce inflation. The source for this mechanism is assumed to be a scalar field ϕ, the *inflaton*, with lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{5.3.3}$$

where $V(\phi)$ is an unknown potential. We will now derive under which conditions the field produces inflation. The energy momentum tensor is

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - g^\mu_\nu\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right]. \tag{5.3.4}$$

The classical field is defined on an homogeneous and isotropic background, therefore the spatial derivatives are 0. Pressure and density are derived from equations (5.1.4)

$$\rho_\phi = \frac{1}{2}\phi'^2 + V(\phi) \tag{5.3.5}$$

$$p_\phi = \frac{1}{2}\phi'^2 - V(\phi). \tag{5.3.6}$$

Imposing condition (5.3.2) we obtain

$$\phi'^2 < V(\phi), \tag{5.3.7}$$

known as *first slow-roll condition.*
In practice, for inflation to happen, the scalar field needs to be slowly-varying with time and in an high-energy state. The total energy must also be such to make all the other contributions to the Friedmann equation (5.1.5) negligible. However, for inflation to end, these contributions must eventually become dominant, which means that the field will need to reach the depth of its potential well. The initial energy of the field and the shape of the potential must be determined from the experimental data. For simple calculations polynomial potential $m^2\phi^n$ are often assumed.
From equations (5.3.5) and (5.3.6), we can rewrite Friedmann equations (5.1.5) and (5.1.6) for the scalar field

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\phi'^2 + V(\phi)\right] \tag{5.3.8}$$

$$H' = 8\pi G\left(-\frac{1}{2}\phi'^2\right). \tag{5.3.9}$$

By differentiating both sides of the first equation with respect to time and substituting the second equation, we get the equation for the dynamics of the field

$$\phi'' + 3H\phi' = -V_{,\phi}\,. \tag{5.3.10}$$

Together with equation (5.3.8) they form the complete set of equations for the evolution of the FLRW metric in a inflaton-dominated universe. If we want inflation to last long enough the second term of equation (5.3.10), which acts like a friction, must be dominant

$$|\phi''| < |3H\phi'|, \tag{5.3.11}$$

this is the *second slow-roll condition.*
A particularly interesting limit in inflationary theory is the *de Sitter limit*, in which we consider both left-hand sides of both slow-roll conditions (5.3.7) and (5.3.11) to be completely negligible. Equation (5.3.8) becomes

$$H^2 = \frac{8\pi G}{3}V(\phi). \tag{5.3.12}$$

Because ϕ evolves very slowly, $V(\phi)$ is practically constant and therefore also the Hubble parameter is constant. As we have seen in the first section, this means that the spacetime is de Sitter. This is valid only in an approximate regime because, as we said, inflation must end: the inflaton will ultimately need to roll down the potential curve until ordinary matter becomes dominant.

5.4 Field quantization in FLRW metrics

We will now study how quantum fluctuations of the inflaton determine the fluctuations observed in the CMB. Before getting to the actual physics, we need to study in general how to quantize a scalar field in an expanding universe described by a fixed FLRW background (5.1.1) with generic scale factor $a(t)$. The action of the scalar field follows from equation (3.1.1), where $\sqrt{-g} = a^3(t)$

$$S = \int d^4x a^3(t) \left[(\phi')^2 - \frac{1}{a(t)^2} \nabla\phi \cdot \nabla\phi - m^2\phi^2 \right]. \tag{5.4.1}$$

We define a new time coordinate, the conformal time η, such that $d\eta = dt/a(t)$. The metric now takes the simplified form

$$ds^2 = a^2(t(\tau))(d\eta^2 - dx^2 - dy^2 - dz^2) = a^2(t(\tau))\eta_{\mu\nu}dx^\mu dx^\nu. \tag{5.4.2}$$

The time dependence of the scale factor will now be dropped. With this change of coordinates the action becomes

$$S[\phi] = \frac{1}{2}\int d^4x\, a^4(a^{-2}\eta^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2) = \frac{1}{2}\int d^4x\, a^2 \left[(\nabla_\eta\phi)^2 - |\nabla\phi|^2 - a^2m^2\phi^2 \right] \tag{5.4.3}$$

It's convenient to introduce the field $\chi = a\phi$:

$$S[\chi] = \frac{1}{2}\int d^4x \left[\dot{\chi}^2 + \chi^2\frac{\dot{a}^2}{a^2} - 2\chi\dot{\chi}\frac{\dot{a}}{a} - |\nabla\chi|^2 - m^2a^2\chi^2 \right], \tag{5.4.4}$$

where the dot derivative denotes covariant differentiation with respect to conformal time η. Adding the total derivative term $\frac{\mathrm{d}}{\mathrm{d}\eta}(\frac{\chi^2\dot{a}}{a})$, the action can be written as

$$S[\chi] = \frac{1}{2}\int d^4x \left[\dot{\chi}^2 - |\nabla\chi|^2 - \left(m^2a^2 - \frac{\ddot{a}}{a} \right)\chi^2 \right]. \tag{5.4.5}$$

By varying the action we get the equation of the field

$$\ddot{\chi} - \Delta\chi + \left(m^2a^2 - \frac{\ddot{a}}{a} \right)\chi = 0, \tag{5.4.6}$$

where Δ is the standard euclidean Laplace operator. As usual we would like to find a set of orthonormal complex mode functions $\{f_\mathbf{k}\}$ satisfying equation (5.4.6). We notice that spatial derivatives are found only in the Laplace operator, this is a consequence of the homogeneity and isotropy of the FLRW metric. As we have done in Minkowski spacetime we apply the Fourier ansatz (2.4.1) on modes

$$f_\mathbf{k}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} v_\mathbf{k}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.4.7}$$

Where the $(2\pi)^{-3/2}$ factor has been added for later simplifications. The equation for Fourier modes is

$$\ddot{v}_\mathbf{k} + \left(|\mathbf{k}|^2 + m^2a^2 - \frac{\ddot{a}}{a} \right) v_\mathbf{k} = 0. \tag{5.4.8}$$

This is the equation of an harmonic oscillator with time-dependent frequency. As we have shown in chapter 1, in general there's no preferred choice of mode functions.
We will now restrict to the de Sitter universe (5.1.13)

$$ds^2 = dt^2 - e^{2tH}(dx^2 + dy^2 + dz^2). \tag{5.4.9}$$

The conformal time is

$$\eta(t) = \int_{+\infty}^{t} e^{-tH} dt = -\frac{1}{H} e^{-tH}. \tag{5.4.10}$$

Whence the scale factor in terms of the conformal time is $a(\tau) = -\frac{1}{H\eta}$ and the metric becomes:

$$ds^2 = \frac{1}{H^2\eta^2}(d\eta^2 - dx^2 - dy^2 - dz^2). \tag{5.4.11}$$

The action of a scalar field in de Sitter is in the same form of (5.4.3). Defining the auxiliary field $\chi = -\frac{1}{H\eta}\phi$ we obtain the action (5.4.5). We will now restrict to a massless field. Rewriting equation (5.4.8) we get the equation of the Fourier modes in de Sitter for massless scalar fields

$$\ddot{v}_{\mathbf{k}}(\eta) + \left(|\mathbf{k}|^2 - \frac{2}{\eta^2}\right) v_{\mathbf{k}}(\eta) = 0. \tag{5.4.12}$$

This is a complex ordinary differential equation. The general solution is a complex combination of two linear independent solutions:

$$v_{\mathbf{k}}(\tau) = A\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + B\left(1 + \frac{i}{k\eta}\right) e^{ik\eta}, \tag{5.4.13}$$

where $k = |\mathbf{k}|$. Modes must be normalized with the inner product (3.2.3). As we have already shown in Minkowski spacetime in finite volume, see equation (2.4.3), this implies for Fourier modes to be normalized as

$$\langle v_{\mathbf{k}}, v_{\mathbf{k}} \rangle = 1, \tag{5.4.14}$$

where no volume factor appears on the right hand side because of the $(2\pi)^{-3/2}$ factor that has been added in the definition of the Fourier ansatz. Normalization implies on the coefficients A,B the constraint

$$|A|^2 - |B|^2 = \frac{1}{2k}. \tag{5.4.15}$$

As expected we see that there's a freedom in the choice of modes.
According to what we have discussed in chapter 1, if the harmonic oscillator were time-independent, we would be able to find a set of modes whose vacuum would be an eigenstate of the Hamiltonian, imposing modes to be positive-frequency. If the oscillator has a time dependent frequency, modes

can be chosen to satisfy the positive-frequency condition only at a given time. In that case the vacuum state would coincide with the ground state only in that given instant. We have mentioned how this also extends to fields in curved spacetime in section 3.3.

In de Sitter spacetime we notice that, for $\eta \to -\infty$, the equation for Fourier modes become the equation for the standard harmonic oscillator. At that time we can define an instantaneous ground state through the positive-frequency mode of the harmonic oscillator. We therefore impose the following initial condition:

$$\lim_{\eta \to -\infty} v_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}. \tag{5.4.16}$$

This implies choosing $B = 0$. Using the normalization condition we obtain the *Bunch-Davies mode*

$$v_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{5.4.17}$$

The full mode is then

$$f_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2k(2\pi)^3}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta + i\mathbf{k}\cdot\mathbf{x}}. \tag{5.4.18}$$

The final mode expansion of the field is

$$\chi(\mathbf{x}, \eta) = \int \frac{d^3k}{\sqrt{2k(2\pi)^3}} \left[\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta + i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}} + \left(1 + \frac{i}{k\eta}\right) e^{ik\eta - i\mathbf{k}\cdot\mathbf{x}} \hat{a}^{\dagger}_{\mathbf{k}}\right], \tag{5.4.19}$$

where the creation and annihilation operators have been defined accordingly to equation (3.2.6). The vacuum of the theory is, as always, defined as the state that is annihilated by all the annihilation operators. In this way, Bunch-Davies modes define the *Bunch-Davies* vacuum $|0_{\mathrm{BD}}\rangle$, that is, by construction, also the ground state at $\eta \to -\infty$. This vacuum is the relevant one for calculations in cosmology.

5.4.1 Calculation of field fluctuations

For applications in cosmology it is useful to calculate the average on the vacuum state of the amplitude of the field

$$\langle 0|\phi(\eta, \mathbf{x})\phi(\eta, \mathbf{x})|0\rangle. \tag{5.4.20}$$

Using the auxiliary field $\chi = a(\eta)\phi$ and its mode expansion:

$$= \frac{1}{a^2(\eta)} \langle \chi(\eta, \mathbf{x})\chi(\eta, \mathbf{x})\rangle = \tag{5.4.21}$$

$$= \frac{1}{a^2(\eta)(2\pi)^3} \int d^3k \int d^3k' \langle 0|(v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}} + v^*_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}^{\dagger}_{\mathbf{k}})(v_{\mathbf{k}'} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}'} + v^*_{\mathbf{k}'} e^{-i\mathbf{k}'\cdot\mathbf{x}} \hat{a}^{\dagger}_{\mathbf{k}'})|0\rangle =$$

$$= \frac{1}{a^2(\eta)(2\pi)^3} \int d^3k \int d^3k' v_{\mathbf{k}}(\eta) v^*_{\mathbf{k}'}(\eta) e^{-\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}')} \langle [a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}]\rangle = \tag{5.4.22}$$

$$= \frac{1}{a^2(\eta)(2\pi)^3} \int d^3k\, |v_{\mathbf{k}}(\eta)|^2. \tag{5.4.23}$$

Fourier modes depend only on the modulo of the wave vector, according to equation (5.4.8), therefore we can write the integral in spherical coordinates, with $d^3k = 4\pi k^2 dk$. It is also customary to write $k^2 dk$ as $k^3 d\ln k$. With this substitutions, the average amplitude becomes

$$\langle 0|\phi(\eta,\mathbf{x})\phi(\eta,\mathbf{x})|0\rangle = \frac{1}{a^2(\eta)}\int d\ln k\,\frac{4\pi}{(2\pi)^3}k^3\,|v_{\mathbf{k}}(\eta)|^2 = \int d\ln k\,\frac{k^3}{2\pi^2}\,\frac{|v_{\mathbf{k}}(\eta)|^2}{a^2(\eta)}. \tag{5.4.24}$$

By definition the integrand is the *power spectrum*

$$\Delta_\phi^2(k,\eta) = \frac{k^3}{2\pi^2}\frac{|v_{\mathbf{k}}(\eta)|^2}{a^2(\eta)}. \tag{5.4.25}$$

If $v_{\mathbf{k}}(\eta)$ is the Bunch-Davies mode, the spectrum becomes

$$\Delta_\phi^2(k,\eta) = \frac{H^2}{4\pi^2}\left(1 + k^2\eta^2\right). \tag{5.4.26}$$

In terms of the scale factor $a(\eta) = -(H\eta)^{-1}$ this is

$$\Delta_\phi^2(k,\eta) = \frac{H^2}{4\pi^2}\left(1 + \frac{k^2}{H^2 a^2}\right). \tag{5.4.27}$$

5.5 Inflation and fluctuations

Inflation provides a mechanism to generate the fine-tuned initial conditions of the energy density compatible with the main features of CMB. This is realized by introducing the inflaton, a scalar field with unknown potential. We have by now treated the field classically, but in principle we must also consider its quantum fluctuations around the classical evolution. This means that locally the field may have different values and therefore different energy. This implies that inflation may end later or sooner in different regions of spacetime. This phenomenon gives a natural explanation for the small fluctuations in the CMB over the homogeneous background. We would like to quantify the relation between field fluctuations and the observed dishomogeneity. Because fluctuations in the CMB are very small, $\Delta T/T \approx 10^{-5}$, we will work in linear perturbation theory. Every quantity X, be it a scalar, a vector or a tensor, is splitted as

$$X(t,\mathbf{x}) = \bar{X}(t) + \delta X(t,\mathbf{x}), \tag{5.5.1}$$

where $\bar{X}(t)$ is the average value on the homogeneous background of the classical theory, and $\delta X(t,\mathbf{x})$ is the small deviation, so that $\delta X \ll \bar{X}$.
Perturbation theory always implies a comparison between magnitude of observables, which is not possible in a coordinate-free language. Therefore, the decomposition (5.5.1) will always depend on the chosen coordinate system. It is also said that (5.5.1) is *gauge* dependent. A different choice of coordinates may eliminate completely the perturbation or create fictitious ones.

Details on the following calculations can be found in [7] and [8]. We write the perturbed field as

$$\phi(t,\mathbf{x}) = \bar{\phi}(t) + \varphi(t,\mathbf{x}), \tag{5.5.2}$$

while the form for the most general perturbed line element is not trivial. In the context of the ADM formalism one can shown that it can be written as

$$ds^2 = N^2 dt^2 - h_{ij}(N^i dt + dx^i)(N^j dt + dx^j), \tag{5.5.3}$$

where $N(t,\mathbf{x})$ is the *lapse function*, $N^i(t,\mathbf{x})$ is the *shift vector* and $h_{ij}(t,\mathbf{x})$ can be interpreted geometrically as the induced metric on the three dimensional hypersurfaces of constant time. Now, the action of the theory is[1]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\right]. \tag{5.5.4}$$

By varying the action with respect to N and N^i, we get constraint equations for the lapse function and the shift function. To simplify calculations we choose the *spatially flat gauge*, where $h_{ij} = a^2\delta_{ij}$. Physically this means that we are choosing coordinates where spatial hypersurfaces are conformally flat. The action then is explicitly

$$\frac{1}{2}\int d^4xN\left[R + \frac{\phi'^2}{N^2} - 2\frac{N^i}{N^2}\phi'\partial_i\phi + \frac{N^iN^j}{N^2}\partial_i\phi\partial_j\phi - 2V(\phi)\right]. \tag{5.5.5}$$

When varying the action one must be careful to keep in mind that R also depends on N and N^i. The constraint equations are

$$R^{(3)} - 2V - a^2\nabla\phi\cdot\nabla\phi - N^{-2}\left[E_{ij}E^{ij} - E^2 - \phi'^2 + 2N^i\phi'\partial_i\phi - N^iN^j\partial_i\phi\partial_j\phi\right] = 0$$
$$\nabla_i\left[N^{-1}(E^i_j - E\delta^i_j)\right] = 0,$$

Where the following definitions have been made

$$E_{ij} = \frac{1}{2}(2aa'\delta^{ij} - \nabla_iE_j - \nabla_jE_i) \tag{5.5.6}$$
$$E = a^2\delta^{ij}E_{ij}, \tag{5.5.7}$$

and we have used $R = R^{(3)} + N^{-2}(E^{ij}E_{ij} - E^2)$, where $R^{(3)}$ is the curvature of spatial hypersurfaces. We now consider linear perturbations of the lapse and shift functions according to equation (5.5.1)

$$N(t,\mathbf{x}) = 1 + \alpha(t,\mathbf{x}) \qquad N_i(t,\mathbf{x}) = \partial_i\beta. \tag{5.5.8}$$

We see that if the first order terms α and β are 0, we get the ordinary FLRW metric. If we substitute this expansion into the constraint equations, together with the field linearization (5.5.2), we find

$$\alpha = \frac{\bar{\phi}'}{2H}\varphi \qquad \partial^2\beta = \frac{(\bar{\phi}')^2}{2H^2}\frac{\mathrm{d}}{\mathrm{dt}}\left(-\frac{H}{\bar{\phi}'}\varphi\right), \tag{5.5.9}$$

[1] We are working in geometrized units $8\pi G = 1$

Plugging this relation into the action, applying integration by parts and keeping only second order terms, we get the approximate action

$$S_{(2)} = \int d^4x\, a^3 \left[(\varphi')^2 - \frac{1}{a^2}\nabla\varphi \cdot \nabla\varphi - [V'' - 2(3H' - (H')^2 - \frac{H'\bar{\phi}''}{H\phi'})]\bar{\phi}^2 \right]. \tag{5.5.10}$$

To simplify it we will now apply the de Sitter limit. The Hubble parameter is constant, therefore $H' = 0$, and, because ϕ is approximately constant, $V''(\phi(t)) = 0$. One may worry about divergences from the $\bar{\phi}'$ in the denominator. However, from the second slow-roll condition (5.3.11), we see that the ratio $\frac{\bar{\phi}''}{3H\bar{\phi}'}$ is always limited. The action becomes

$$S_{(2)} = \int d^4x\, a^3 \left[(\varphi')^2 - \frac{1}{a^2}\nabla\varphi \cdot \nabla\varphi \right] \tag{5.5.11}$$

where $a(t) = e^{tH}$. Comparing with equation (5.4.1), this is exactly the action of a scalar field in de Sitter spacetime. Picking the Bunch-Davies modes we obtain that the power spectrum of field φ at time η is

$$\Delta^2_\varphi(k,\eta) = \frac{H^2}{4\pi^2}\left(1 + k^2\eta^2\right). \tag{5.5.12}$$

It is useful to express the result in terms of the curvature perturbation on hypersurfaces with costant density. This quantity is *gauge-invariant*, which means that it cannot be removed by a gauge choice. In spatially-flat gauge it is defined as

$$\zeta = -\frac{H}{\bar{\phi}'}\varphi. \tag{5.5.13}$$

Its Fourier modes $\zeta_{\mathbf{k}}$ have the very important property to be constant when $k < aH$ or, equivalently during inflation, when $k\eta > -1$. This means that for a given $k = |\mathbf{k}|$ mode,

$$\Delta^2_\zeta(k,\eta) = \left(\frac{H}{\bar{\phi}'}\right)^2 \frac{H^2}{4\pi^2}\left(1 + \frac{k^2}{H^2a^2}\right) \tag{5.5.14}$$

will be constant when that condition is satisfied. In an approximate regime we can take $k \ll aH$ and therefore

$$\Delta^2_\zeta(k) = \left(\frac{H}{\bar{\phi}'}\right)^2 \frac{H^2}{4\pi^2}. \tag{5.5.15}$$

It is important to notice that at initial times, that is $\eta \to -\infty$ or $a \to 0$, the spectrum is k-dependent and, as expected by the definition we have given of the Bunch-Davies mode, it is the same of a quantum field in Minkowski. Thanks to inflation the spectrum becomes k-independent. After inflation, the condition $k < aH$ will eventually be violated and $\zeta_{\mathbf{k}}$ will evolve according to Einstein's field equation with the initial condition that has been produced by

inflation. The results is then finally confronted with data extracted by processing the CMB.
An important measurable quantity is the *spectral index*

$$n_s = 1 + \frac{\mathrm{d} \ln \Delta_\zeta^2}{\mathrm{d} \ln k}. \tag{5.5.16}$$

The spectrum that we have derived does not depend on k, therefore $n_s = 1$. This implies a scale-invariant CMB. If we didn't consider the de Sitter limit we would get a different spectral index depending on the chosen potential. For most models the spectrum has the form

$$\Delta_\zeta^2(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1}. \tag{5.5.17}$$

Without knowing which model is the correct one, the theory of inflation is not able to predict the exact value of the spectral index, but it does say that it must be different but close to 1. This is exactly what the latest measures by the Planck collaboration [9] have shown

$$n_s = 0.9603 \pm 0.0073. \tag{5.5.18}$$

Perturbation theory gives also rise to *tensor perturbations*, which are gravitational waves. We can write a perturbed metric similar to (5.5.3), and, with simpler mathematical passages, obtain again the action of a scalar field for each polarization mode, which can be quantized in the same way. The general form of the power spectrum is

$$\Delta_t^2 = A_t \left(\frac{k}{k_*}\right)^{n_t}, \tag{5.5.19}$$

where n_t is the tensor spectral index. We define the *tensor-to-scalar ratio*

$$r_k = \frac{\Delta_t^2(k)}{\Delta_\zeta^2(k)} \tag{5.5.20}$$

where $\Delta_\zeta^2(k)$ is taken to be the actual measured value. Usually a pivot scale for k is taken, such as $k_* = 0.05 Mpc^{-1}$. Tensor perturbation are harder to detect directly. It can be however shown that they induce a certain polarization pattern in the CMB. Because this pattern has not been yet detected we have an upper bound for the power spectrum of tensor perturbations. Latest estimates [10] give

$$r_{0.05} < 0.036. \tag{5.5.21}$$

Tensor-to-scalar ratio gives also an estimate of the order of magnitude of the energy scale of the inflation, that is the initial high-energy state at which the inflaton started to roll down. This is because $\Delta_t^2 \propto H^2$, which is, by equation (5.3.12), approximately $V(\phi_0)$. The energy scale is $V^{1/4} := V^{1/4}(\phi_0)$, where

we put 1/4 because V is an energy density (that is $[E]^4$ in natural units).
Current measures give

$$V^{1/4} \lessapprox 10^{16}\text{GeV}, \tag{5.5.22}$$

which is far bigger than the energy probed at the LHC.
Combining the experimental constraints with theoretical calculations, we can rule out inflationary models which do not give correct predictions. Image 5.1 shows regions compatible with the constraints and shows the predictions of different inflationary models.

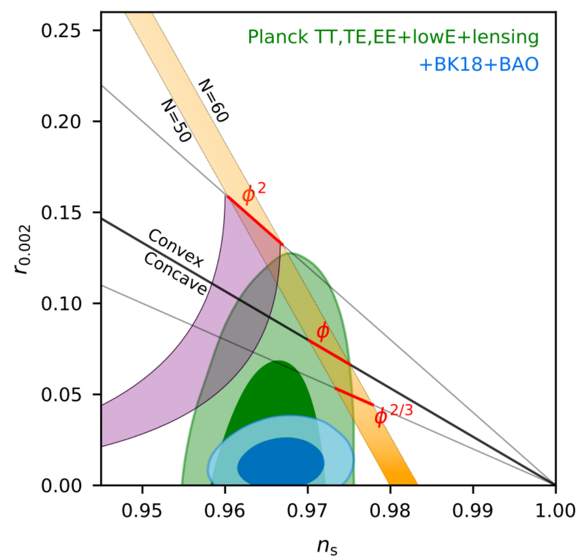

Figure 5.1: Experimental constraints on inflationary models on the tensor-to-scalar ratio and spectral index plane. The pivot point chosen for calculating the ratio is $k_* = 0.002 Mpc^{-1}$. Theoretical predictions, in red, are taken assuming different durations of inflation, measured by the *e-folds number* N. Purple region represents *natural inflation*, where the potential is a sinusoid. The image is from [10].

Conclusions

In this thesis we have shown how to quantize scalar fields in curved spacetime by employing the method of mode expansion. We have started by introducing mode expansion in the context of harmonic oscillators with time-dependent frequency and we have shown how to single out a preferred mode function in the case of the time-independent oscillator. To work with general cases we defined the concept of instantaneous ground state, and mentioned how to relate different choices of mode function.

We have then moved to field theory and have applied the method of mode expansion to quantize a scalar field in Minkowski spacetime, by introducing a set of mode functions properly normalized with the Klein-Gordon inner product. Defining creation and annihilation operators, we have defined the vacuum state and the Hilbert space of the theory, the Fock space. Making use of homogeneity and isotropy, we have assumed a Fourier form for modes and we have found the standard plane wave decomposition of the scalar field. We have shown that the Fock basis is a set of eigenstates for the Hamiltonian and momentum operator and we have seen how this leads to the particle interpretation of field excitations.

We have easily generalized this method for fields in curved spacetime and have correspondingly defined the vacuum state. By introducing the positive-frequency condition with respect to a given timelike vector field (3.3.1), we have discussed under which conditions vacuum states built from those modes give physically preferred vacua. With the Bogoljubov transformations we have shown how to relate different sets of modes and operators, and we have seen that two distinct vacua have the same particle content only if positive-frequency mode of one set are proportional to positive-frequency modes of the other one.

We have explicitly constructed two distinct set of modes for a massless scalar field in 1+1 Minkowski spacetime by studying the quantization of scalar field in Rindler coordinates. With the Bogoljubov coefficients we have shown that the vacuum state measured by an inertial observer is a thermal state for an accelerated observer with temperature given by equation (4.2.41).

Finally, we have quantized a scalar field in de Sitter spacetime defining the Bunch-Davies vacuum, and have computed the power spectrum of the field amplitude (5.4.27). We have applied this calculation to scalar perturbations in inflationary context using the spatially-flat gauge and applying the de Sitter limit. We have then obtained the power spectrum of the gauge-invariant curvature perturbation ζ (5.5.15) and evaluated it in the limit $k \ll aH$. We have found a scale-invariant spectrum, which means that the scalar spectral index is 1. We have also argued that in stronger limits than de Sitter, the spectral index should be different but not far from 1. We have seen that this is compatible with experimental measures from the CMB. We have mentioned that inflation produces also tensor perturbations, and have defined the tensor-to-scalar ratio from the ratio of the tensor and scalar power spectra. Experimental evidence gives only an upper-bound for the tensor-to-scalar ratio. Combining measurements and theoretical predictions we have mentioned how we can rule out distinct models for the potential of the inflaton (Figure 5.1).

Bibliography

[1] Leonard Parker and Jose Navarro-Salas. Fifty years of cosmological particle creation. Feb 2017.

[2] S Hawking. Particle creation by black holes. *Communications in Mathematical Physics*, 43:199–220, Aug 1975.

[3] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14:870–892, Aug 1976.

[4] A. Ashtekar and A Magnon. Quantum fields in curved space-time. *Proc. R. Soc. Lond. A*, 345:375–394, Nov 1975.

[5] Viatcheslav Mukhanov and Sergei Winitzki. *Introduction to Quantum Effects in Gravity*. Cambridge University Press, 2007.

[6] Luís C. B. Crispino, Atsushi Higuchi, and George E.A. Matsas. The unruh effect and its applications. *Reviews of modern physics*, 80:787–838, Jul 2008.

[7] Daniel Baumann. Tasi lectures on inflation. arXiv:0907.5424 [hep-th], 2009.

[8] Juan Maldacena. Non-gaussian features of primordial fluctuations in single field inflationary models. *Journal of High Energy Physics*, 2003(05):013–013, May 2003.

[9] Planck collaboration. Planck 2013 results. xvi. cosmological parameters. *A& A*, 571:A16, Nov 2014.

[10] P. A. R Ade, Z Ahmed, M Amiri, D Barkats, R. Basu Thakur, C. A Bischoff, D Beck, J. J Bock, H Boenish, E Bullock, V Buza, J. R Cheshire, J Connors, J Cornelison, M Crumrine, A Cukierman, E. V Denison, M Dierickx, L Duband, M Eiben, S Fatigoni, J. P Filippini, S Fliescher, N Goeckner-Wald, D. C Goldfinger, J Grayson, P Grimes, G Hall, G Halal, M Halpern, E Hand, S Harrison, S Henderson, S. R Hildebrandt, G. C Hilton, J Hubmayr, H Hui, K. D Irwin, J Kang, K. S Karkare, E Karpel, S Kefeli, S. A Kernasovskiy, J. M Kovac, C. L Kuo,

K Lau, E. M Leitch, A Lennox, K. G Megerian, L Minutolo, L Moncelsi, Y Nakato, T Namikawa, H. T Nguyen, R O'Brient, R. W Ogburn, S Palladino, T Prouve, C Pryke, B Racine, C. D Reintsema, S Richter, A Schillaci, R Schwarz, B. L Schmitt, C. D Sheehy, A Soliman, T. St Germaine, B Steinbach, R. V Sudiwala, G. P Teply, K. L Thompson, J. E Tolan, C Tucker, A. D Turner, C Umiltà, C Vergès, A. G Vieregg, A Wandui, A. C Weber, D. V Wiebe, J Willmert, C. L Wong, W. L. K Wu, H Yang, K. W Yoon, E Young, C Yu, L Zeng, C Zhang, and S Zhang. Improved constraints on primordial gravitational waves using planck , wmap, and bicep/ keck observations through the 2018 observing season. *Physical review letters*, 127(15):151301–151301, Oct 2021.