Calculation of the splitting function $q \to q$

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1 Context: Deep Inelastic Scattering

Consider a lepton colliding against a hadron at high energies. The experiment will result in the fragmentation of the hadron into several partons that, after hadronization, will be detetced as hadrons. With these also the outcoming lepton is detected, which may be different from the incident one if weak processes are taken into consideration. In order to keep the following discussion simple we will focus on the following event:

$$e^-(l^\mu) + p(p^\mu) \to e^-(l'^\mu) + X$$

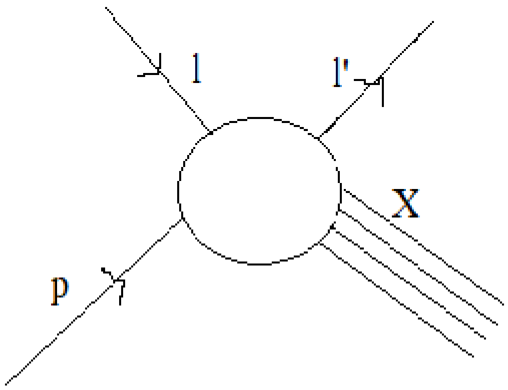
, where p denotes the proton and X the set of fragmentation products. While the outcoming lepton momentum is measured, the total momentum associated with the products isn't. Such processes are said to be *semi-inclusive*. One of their feature is the different number of indipendent cinematic variables on which the amplitude will depend with respect to ordinary processes. In this case the mass-shell relation for p_X is unknown: instead of two variables the amplitude will depend on three. Two of these may be the common Mandelstam variables $s = (l+p)^2$ and $t = q^2 = (l-l')^2$, that in this context is called $-Q^2$. Naturally the third one would be m_X^2 but for practical reason the *Bjorken* x was introduced, defined as:

$$x = \frac{Q^2}{2(pq)}.$$

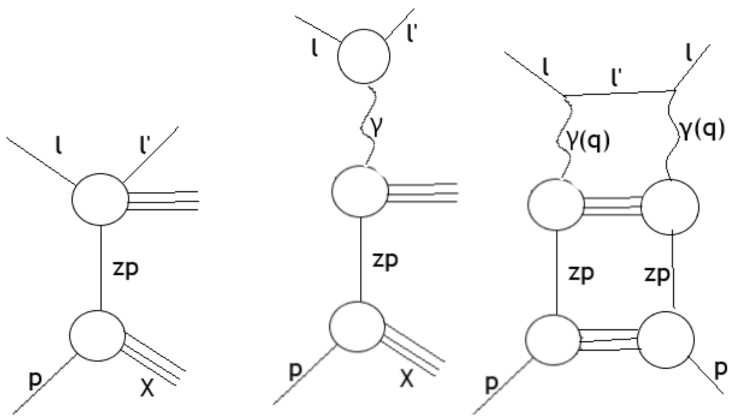
If we knew m_X^2 we would be able to calculate x in terms of Q^2 and s. To see this consider first the following calculation

$$m_X^2 = (p_X)^2 = (p+l-l')^2 = m_p^2(p+q)^2 = -Q^2 + 2(pq) = m_p^2 + 2(pq)(1-x)$$

If $m_X^2 = m_p^2$ there's no fragmentation and $x = 1$. This is the case of elastic scattering. For $x < 1$ the scattering is inelastic. Finally, we can write $(pq) = (pl) - (pl')$. The first products depends on s, the second on u, which can be written as a combination of the other two variables and the masses. This gives an equation that relates m_X^2 and x. After this preliminary, though important, consideration, let's see how the cross section is calculated. The generic diagram for the process is



According to the parton model, the lepton interacts with a parton (quark or gluon) inside the proton carrying a fraction z of its total momentum. We can therefore decompose the central blob into two.



On the bottom there's the non-pertubative process involving the fragmentation of the proton and the hadronization of its products. This cannot be calculated analitically but relies on experimental measures. On the top there's the interaction of the lepton with the parton carrying a fraction of the momentum of the proton. We drew the parton as a quark, but it can also be a gluon. The contribution of this interaction can be computed. For simplicity we suppose the interaction between the electron and the partons electromagnetic, neglecting the weak interactions, this is justified at high energies. (SIAMO SICURI?). This makes possible to operate yet another decomposition into a purely QCD process and a mixed one. Because the strong coupling is leading, for LO and NLO we can ignore real and virtual corrections to the QED process. Having made this considerations, we can draw the diagram for the squared amplitude. Now, the calculation of the squared amplitude involves the product of two tensors, one coming from the lepton interaction $L^{\mu\nu}$ and the other from the parton $W^{\mu\nu}$. At lowest orders the first one is just the trace $Tr[\not{l}\gamma^\mu \not{l}'\gamma^\nu]$, with other factors, such as 1/2 from the averaged spin states and the coupling. Field theory considerations lead us to write the hadronic tensor as

$$W^{\mu\nu}(p,q) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(p,q) + \frac{1}{(pq)}\left(p^\mu - \frac{(pq)}{q^2}q^\mu\right)\left(p^\nu - \frac{(pq)}{q^2}q^\nu\right) F_2(p,q),$$

where F_1 and F_2 are form factors which include both the nonperturbative and the perturbative contributions. Alternatively the tensor can be decomposed into a transverse and longitudinal form factor F_T and F_L such that $P_L^{\mu\nu}W_{\mu\nu} = F_L$ and $P_R^{\mu\nu}W_{\mu\nu} = F_R$, where $P_L = \frac{4x^2}{(pq)}p^\mu p^\nu$ and $P_T = x\left(-g^{\mu\nu} + \frac{q^\mu p^\nu + q^\nu p^\mu}{(pq)} - q^2\frac{pp^\mu p^\nu}{(pq)^2}\right)$ are projectors (DIREI DI NO). Reorganizing all the tensors we find $F_L = F_2 - 2xF_1$ and $F_T = 2xF_1$. In the simplified context we are working, we can integrate on the final lepton momentum and contract the leptonic tensor with the hadronic tensor to get the differential cross section

$$\frac{d^2\sigma}{dQ^2dx} = \frac{2\pi\alpha^2}{xQ^4}\left[(1+(1-y)^2)F_2(x,Q^2) - y^2F_L(x,Q^2)\right].$$

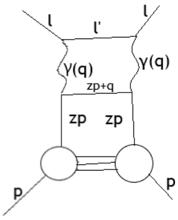
Where $y = \frac{Q^2}{xs}$ is a cinematic variable alternative to s. The factor Q^4 comes directly from the squared photon propagator. The integration on the momentum of the hadronic products from the intermediate process are into the form factors. It is worth to notice that in the center of mass reference system the first therm in parenthesis is $1+\cos^4\theta/2$ and the second is $\sin^4\theta/2$ According to the parton model we can write the from factors as

$$F(x,Q^2) = \sum_a \int_0^1 dz\, f_a(z)F_a(x/z,Q^2)$$

where the sum runs over all the partons, that is the six quarks, six antiquarks and the gluon. The first function is called *parton density* and is determined experimentally, the second function is the *partonic structure function*, and comes from the analytical calculation of the intermediate hadronic process, with the integration on the final momenta.

1.1 LO order calculation

At leading order the squared amplitude is given by the following diagram



What we should calculate now are just the two form factors. F_L should be zero because the photon couples to a quark on mass-shell, that is a particle of spin 1/2. However the process is very simple so the calculation can be done entirely, so to verify all the predicted dependencies, ignoring the non perturbative contribution of course.

$$|\mathcal{M}|^2 = 2e^4 e_q^2 \frac{s^2+u^2}{Q^4}$$

$$d\sigma = \frac{1}{2s}\int \frac{d^3l'}{(2\pi)^3 2l'_0}\frac{d^3r}{(2\pi)^3 2r_0}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(zp+l-r'-l')$$

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}[1+(1-y)^2]$$

Clearly the differential cross section depends only on two variables because we have not considered the non perturbative process, as if the proton were composed only by the interacting quark.To get back to the complete result we need to add the x dependence. Because the outcomiing quark is on-shell we have $0=(r)^2=(zp+q)^2=-Q^2+2z(pq)$ and therefore $x=z$. Finally:

$$\frac{d\sigma}{dxdQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}[1+(1-y)^2]\delta(x-z)$$

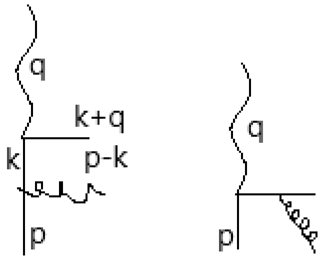
Doing this with all the quarks (at LO the gluon cannot contribute) and rearranging terms so to compare with the general form, we have:

$$\frac{d\sigma}{dxdQ^2} = \frac{2\pi\alpha^2}{xQ^4}[1+(1-y)^2]\sum_q\int_0^1 dz\, xe_q^2 f(z)\delta(x-z)$$

This is the final result for DIS at leading order. Notice the partonic strucutre function does not depend on Q^2. This result is called *Bjorken scaling.* The main insight is that there's no dependence on any other dimensional quantity than can compensate the dimension of Q^2 to get an adimensional form factor. The experimental evidence of this fact at high energy tells that the lepton interacts with point-like particles, which do not have any characteristic length $1/Q_0$. At higher order the scaling is obviously broken because renormalization introduces a new scale.

1.2 DIS at next to leading order

At NLO order the intermediate contribution has both real and virtual corrections. Ignoring for a moment the other pieces, the two diagrams for the real corrections are



Where for simplicity we have renamed zp as p. The diagrams for the virtual corrections are 3, two from the self-energy of the incoming and the outgoing quark and one correction to the vertex. The diagrams contributing to squared amplitude are the four combinations of the two real correction diagrams and the interference diagrams between the virtual corrections and the tree diagram. The first are

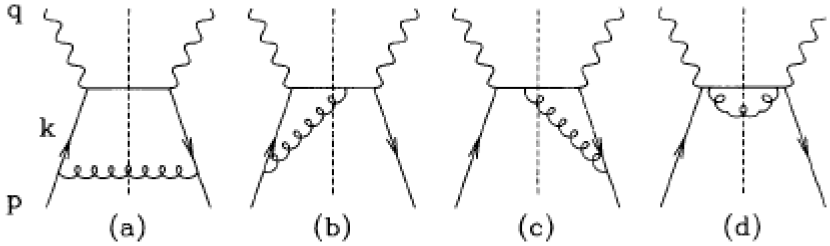


Fig. 4.7. Real gluon emission diagrams contributing to deep inelastic scattering. . It is easy to generalize to the generic high order real corrections. A ladder of emissions of these three type (emission from the incoming parton, emission from the outougoing parton, interferences).

1.3 Infrared divergences in NLO DIS

In QCD both real emission diagrams and virtual correction diagrams display infrared divergences. There are of two types, soft and collinear. We see that the first diagram does display both divergences writing the intermediate impulse k in the following way (**Sudakov parametrization**):

$$k^\mu = \xi p^\mu + \beta n^\mu + k_T^\mu$$

Where $p^2 = n^2 = (pk_T) = (nk_T) = 0$ and $(pn) = 1$. For example, in the infinite momentum frame the explicit representation is $(P, 0, 0, P), (1/2P, 0, 0, -1/2P), (0, \vec{k}_T, 0)$. A relation between the coefficient comes from requiring that $p-k$ us on-shell (see diagram). This gives $0 = (p-k)^2 = -2(pk)+k^2 = -2\beta - |\vec{k}_T|^2 + 2\xi\beta = -2(1-\xi)\beta - |\vec{k}_T|^2$. Therefore the parametrization can also be written as

$$k^\mu = \xi p^\mu - \frac{|\vec{k}_T|^2}{2(1-\xi)} n^\mu + k_T^\mu.$$

This gives $k^2 = -\frac{|\vec{k}_T|^2}{1-\xi}$. We will make us of this expressions in a moment. Now, the phase space for the first diagram is

$$\int \frac{d^3 r}{(2pi)^3 2r_0} \frac{d^3 g}{(2\pi)^3 2g_0} (2\pi)^4 \delta^{(4)}(r+g-q-p)$$

where r is the momentum of the outcoming quark and g the momentum of the outcoming gluon. By the usual property of Dirac delta this is

$$\int \frac{d^4 r}{(2\pi)^3} \frac{d^4 g}{(2\pi)^3} \delta^{(+)}(r^2) \delta^{(+)}(g^2) (2\pi)^4 \delta^{(4)}(r+g-q-p)$$

Applying conservation of momentum, that is the four delta:

$$\int \frac{d^4q}{(2\pi)^3}\frac{d^4k}{(2\pi)^3}\delta((k+q)^2)\delta((p-k)^2)(2\pi)^4$$

So that

$$W_{\mu\nu} = \int \frac{d^4q}{(2\pi)^3}\frac{d^4k}{(2\pi)^3}\delta((k+q)^2)\delta((p-k)^2)(2\pi)^4|\mathcal{M}|^2_{\mu\nu}$$

Using Sudakov parametrization this is

$$W_{\mu\nu} = \int \frac{d^4q}{(2\pi)^3}\frac{d\xi d\beta d^2k_T}{(2\pi)^3}\delta((k+q)^2)\delta((-2(1-\xi)\beta + -|\vec{k}_T|^2)^2)(2\pi)^4|\mathcal{M}|^2_{\mu\nu} =$$

$$W_{\mu\nu} = \int \frac{d^4q}{(2\pi)^3}\frac{d\xi d\beta d^2k_T}{(2\pi)^3}\delta((k+q)^2)\frac{1}{2(1-\xi)}\delta\left(\beta + \frac{|\vec{k}_T|^2}{2(1-\xi)}\right)(2\pi)^4|\mathcal{M}|^2_{\mu\nu}$$

Using $q^\mu = (pq)n^\mu + q^\mu_T$ (PERCHE' VALE?) the other delta Dirac is

$$\delta((k+q)^2) = \delta\left(-Q^2 - \frac{|\vec{k}^2_T|}{1-\xi}) + 2\xi(pq) + (q_T k_T)\right) = \frac{1}{2(pq)}\delta\left(\xi - x + \left(-\frac{|\vec{k}^2_T|}{1-\xi}) + 2\xi(pq) + (q_T k_T)\right)\frac{1}{2(pq)}\right)$$

where we have used the first one and have made explicit the variable xi. Notice that in the limit $k \to 0$ this reduces to the known $\delta(\xi - x)$. Simplyfing all π we are left with

$$W_{\mu\nu} = \frac{1}{16\pi^2(pq)}\int d^4q\frac{d\xi}{1-\xi}d\beta d^2k_T\delta\left(\beta + \frac{|\vec{k}_T|^2}{2(1-\xi)}\right)\delta\left(\xi - x + \left(-\frac{|\vec{k}^2_T|}{1-\xi}) + 2\xi(pq) + (q_T k_T)\right)\frac{1}{2(pq)}\right)|\mathcal{M}|^2_{\mu\nu}$$

The first Dirac delta cancels the β integration. Because we will now be interested only in the properties of the real emission we reorganize the expression obtained so far:

$$W_{\mu\nu} = \frac{1}{16\pi^2(pq)}\int d^4q\,\delta\left(\xi - x + \left(-\frac{|\vec{k}^2_T|}{1-\xi}) + 2\xi(pq) + (q_T k_T)\right)\frac{1}{2(pq)}\right)\iiint \frac{d\xi}{1-\xi}d^2k_T|\mathcal{M}|^2_{\mu\nu}\delta\left(\beta + \frac{|\vec{k}_T|^2}{2(1-\xi)}\right)$$

Now let's return to our task, that is the analysis of collinear and soft singularities of these diagrams. Collinear singularities occur when $k_T\ to 0$ and soft singularities when $\xi \to 0$. How does the squared amplitude depend on these parameters? The answer of this question depends on what gauge we are working in. With a **covariant gauge** the propagator of the gluon has simpler form but requires the introduction of ghost fields. With an **axial** or **physical gauge** ghost are not necessary. For the following dicussion we choose the latter, imposing $n^\mu A_\mu = 0$. Because n is light-like, this is also called *light-cone gauge*. With this choice the sum on polarization is

$$\sum \varepsilon(g)\varepsilon^*(g) = -g^{\mu\nu} + \frac{n^\mu g^\nu + n^\nu g^\mu}{(ng)}$$

and the polarization vector satisfies $g^\mu\varepsilon(g)_\mu = 0$ and $n^\mu\varepsilon(g)_\mu = 0$. Now, being the sum on polarizations and spin implicit, the squared amplitude of the first diagram is

$$|\mathcal{M}|^2_{\mu\nu} = \frac{1}{2}e_q^2 g^2 \mathrm{Tr}[\not{p}\gamma^\rho\not{k}\gamma_\mu\not{t}\gamma_\nu\not{k}\gamma^\sigma\not{p}]\left(-g_{\rho\sigma} + \frac{n^\rho g^\sigma + n^\sigma g^\rho}{(ng)}\right)\frac{1}{k^4}C_F$$

Rearranging and applying momentum conservation

$$|\mathcal{M}|^2_{\mu\nu} = \frac{1}{2}e_q^2 g^2 \mathrm{Tr}[\not{k}\gamma^\sigma \not{p}\gamma^\rho \not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu]\left(-g_{\rho\sigma} + \frac{n^\rho(p-k)^\sigma + n^\sigma(p-k)^\rho}{(np)-(nk)}\right)\frac{1}{k^4}C_F$$

Applying Sudakov parametrization on the gluon propagator we are left with

$$|\mathcal{M}|^2_{\mu\nu} = \frac{1}{2}e_q^2 g^2 \mathrm{Tr}[\not{k}\gamma^\sigma \not{p}\gamma^\rho \not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu]\left(-g_{\rho\sigma} + \frac{n^\rho(p-k)^\sigma + n^\sigma(p-k)^\rho}{1-\xi}\right)\frac{1}{|\vec{k}_T|^4}(1-\xi)^2 C_F$$

Putting all together in the relevant part of $W_{\mu\nu}$

$$W_{\mu\nu} \propto \int d\xi d^2k_T\, \frac{1}{2}e_q^2 g^2 \mathrm{Tr}[\not{k}\gamma^\sigma \not{p}\gamma^\rho \not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu]\left(-g_{\rho\sigma} + \frac{n^\rho(p-k)^\sigma + n^\sigma(p-k)^\rho}{1-\xi}\right)\frac{1}{|\vec{k}_T|^4}(1-\xi)\delta\left(\beta + \frac{|\vec{k}_T|^2}{2(1-\xi)}\right)C_F$$

Now, tranverse momentum in the denominator may suggest the presence of a power singularity. However the factors at numerator cancel a power of two, leaving a logarithmic singularity. We can see this at amplitude level. This contains a factor

$$\not{k}\not{\varepsilon}(p-k)u(p)$$

Applying $\not{a}\not{b} + \not{b}\not{a} = 2(ab)$ this is

$$2(\varepsilon(p-k)k)u(p) - \not{\varepsilon}(p-k)\not{k}u(p)$$

In the collinear limit $k_T \to 0$ we have $k \to \xi p$, therefore the first term vanishes for the transversality and the second for the masslessness of the quark. This tells us that the numerator is $\mathcal{O}(k_T)$, so that the amplitude squared has a $|\vec{k}_T|^{-2}$ dependency, that is a logarithmic singularity under d^2k_T integration. Of course the interference diagram will only depend on $|\vec{k}_T|^{-1}$, thus is not singular. Obiously the squared amplitude of the second diagram does not diverge. It is worth to remember that we referring to singularity arising from the emission of a gluon collinear to the incoming quark, we are not referring to the usual infrared singularity occurring when the outocoming quark and the gluon are collinear. We know that this can be treated, together with soft singularities, with consistent summation of virtual corrections. We will see this explicitly for this diagram. However already power counting shows that such singularity, which is collinear, is not present in the interference diagrams.

1.4 Isolating the logarithmic singularity

Let's go back to the intermediate result obtained for $W^{\mu\nu}$. As we have discussed this is the only one to give rise to logarithmic singularity. We will make this one explicit by isolating only the terms with a $|\vec{k}_T|^2$ dependence (no more) from the numerator of the squared amplitude. Consder the product

$$\mathrm{Tr}[\not{k}\gamma^\sigma \not{p}\gamma^\rho \not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu]\left(-g_{\rho\sigma} + \frac{n^\rho(p-k)^\sigma + n^\sigma(p-k)^\rho}{1-\xi}\right)$$

Multiply

$$\mathrm{Tr}\left[\not{k}\left(2\not{p} + \frac{\not{n}\not{p}(\not{p}-\not{k}) + (\not{p}-\not{k})\not{p}\not{n}}{1-\xi}\right)\not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right]$$

Using $\not{k}\not{k} = k^2$ and $\not{p}\not{p} = 0$

$$\mathrm{Tr}\left[\not{k}\left(2\not{p} - \frac{\not{n}\not{p}\not{k} + \not{k}\not{p}\not{n}}{1-\xi}\right)\not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right] = \mathrm{Tr}\left[\left(2\not{k}\not{p}\not{k} - k^2\frac{\not{k}\not{n}\not{p} + \not{p}\not{n}\not{k}}{1-\xi}\right)\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right]$$

Using $k^2 = -\frac{|\vec{k}_T|^2}{1-\xi}$

$$\mathrm{Tr}\left[\left(2\not{k}\not{p}\not{k} + |\vec{k}_T|^2 \frac{\not{k}\not{n}\not{p} + \not{p}\not{n}\not{k}}{(1-\xi)^2}\right)\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right]$$

From the first term we extract the $|\vec{k}_T|^2$ dependence using Sudakov parametrization

$$\not{k}\not{p}\not{k} = -\not{p}\not{k}\not{k} + 2(pk)\not{k} = \frac{|\vec{k}_T|^2}{1-\xi}\not{p} - 2\frac{|\vec{k}_T|^2}{2(1-\xi)}\not{k} = \frac{|\vec{k}_T|^2}{1-\xi}(\not{p}-\not{k}) = \frac{|\vec{k}_T|^2}{1-\xi}\left((1-\xi)\not{p} - \frac{|\vec{k}_T|^2}{2(1-\xi)})\not{n} + \not{k}_T\right) \approx |\vec{k}_T|^2\not{p}$$

The neglected terms contribute to the finite parts of the amplitude. Similarly for the second term

$$\not{k}\not{n}\not{p} + \not{p}\not{n}\not{k} = -\not{k}\not{p}\not{n} - \not{p}\not{k}\not{n} + 2(nk)\not{p} + 2(pn)\not{k} = -2(kp)\not{n} + 2(nk)\not{p} + 2(pn)\not{k} = \frac{|\vec{k}_T|^2}{1-\xi}\not{n} + 2\xi\not{p} + 2\not{k} = 4\xi\not{p} + 2\not{k}_T \approx 4\xi\not{p}$$

Finally

$$\mathrm{Tr}\left[\left(2|\vec{k}_T|^2\not{p} + |\vec{k}_T|^2\frac{4\xi\not{p}}{(1-\xi)^2}\right)\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right] = 2|\vec{k}_T|^2\left(1+\frac{2\xi}{(1-\xi)^2}\right)\mathrm{Tr}\left[\not{p}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right] = 2|\vec{k}_T|^2\frac{1+\xi^2}{(1-\xi)^2}\mathrm{Tr}\left[\not{p}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right]$$

Putting back again in the hadronic tensor

$$W_{\mu\nu}|_{\mathbf{div}} \propto \int d\xi d^2k_T\,\frac{1}{2}e_q^2g^2 2|\vec{k}_T|^2\frac{1+\xi^2}{(1-\xi)^2}\mathrm{Tr}\left[\not{p}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right](1-\xi)\frac{1}{|\vec{k}_T|^4}C_F$$

Rearranging

$$\int\frac{d\xi}{\xi}\frac{d^2k_T}{|\vec{k}_T|^2}\left(2C_F\frac{1+\xi^2}{(1-\xi)}\right)\frac{1}{2}e_q^2g^2\mathrm{Tr}\left[\xi\not{p}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right]$$

We have multiplied and divided by ξ so that in the trace instead of $\xi\not{p}$ we can write $\not{k}$ (this is valid in the limit we are working in)

$$\int\frac{d\xi}{\xi}\frac{d^2k_T}{|\vec{k}_T|^2}\left(2C_F\frac{1+\xi^2}{(1-\xi)}\right)\frac{1}{2}e_q^2g^2\mathrm{Tr}\left[\not{k}\gamma_\mu(\not{q}+\not{k})\gamma_\nu\right]$$

The final result gives an integral that is both divergent for ξ, that is a soft singularity, and for k_T, the predicted collinear logarithmic singularity. The trace is the same of the tree diagram process multiplied by the **splitting function**:

$$P_{qq}^{(0)}(\xi) = 2C_F\frac{1+\xi^2}{1-\xi}.$$