Parerga and paralipomena

1 p_T^2 integration

In the phase space decomposition employed for the rapidity distribution, the integration on p_T^2 is computed with a Dirac delta that leaves out a factor of $|\tilde{p}_T^2|^\epsilon$. This depends on the integration variables l^2 and k'^2. Explicitly, the Dirac delta in the frame of reference where l is at rest gives

$$p_H^0 + k'^0 - \sqrt{l^2} = 0. \tag{1}$$

Because initial gluon emissions are soft in the threshold limit, this frame of reference is also the frame of reference of the partonic CoM. The equation above can be rewritten using the usual parametrization of p_H. Therefore we have the equation

$$\sqrt{m_H^2 + p_T^2}\cosh y + \sqrt{k'^2 + p_T^2\cosh^2 y + m_H^2\sinh^2 y} - \sqrt{l^2} = 0. \tag{2}$$

Solving with Mathematica with respect to p_T^2 gives

$$\tilde{p}_T^2 = \frac{\left[(k'^2)^2 - 2k'^2 m_H^2 + m_H^4 - 2k'^2 l^2 + l^4 - 2m_H^2 l^2\cosh 2y\right]\operatorname{sech}^2 y}{4l^2} \tag{3}$$

Using $\cosh 2y = 2\cosh^2 y - 1$ this reduces to

$$\tilde{p}_T^2 = \frac{(k'^2 - m_H^2 - l^2)^2\operatorname{sech}^2 y}{4l^2} - m_H^2. \tag{4}$$

The expression is different than De Ros (eq. 4.68 pag. 46) because at that point he is still using as variable p_z instead of y.
In each passage of this calculation, one can verify that in the threshold limit, be it for fixed x_1 or x_2, $\tilde{p}_T^2 \to 0$ using $k'^2 \to 0, l^2 \to s$ and $s \to m_H^2/x_2$. Therefore $|\tilde{p}_T^2|^\epsilon$ carries a factor of $[(1-x_1)(1-x^2)]^\epsilon$ that **contributes to the collinear scale**. Possibly, as shown explicitly at NLO, it also carries factor that regularize the integrals in l^2 e k'^2.

2 Rewriting Glosser and Schmidt

We shortly revise and carry on the rewriting of the GS result.
Equation 3.17 of GS is

$$\begin{aligned}
C_{sing}^{(2)} =& \delta(Q^2)\Bigg\{(11+\delta+N_c U)g_{gg} \\
&+ (N_c - n_f)\frac{N_c}{3}\left[\frac{m_H^4}{s} + \frac{m_H^4}{t} + \frac{m_H^4}{u} + m_H^2\right]\Bigg\} \\
&+ \Bigg\{\left(\frac{1}{-t}\right)\left[-P_{gg}(z_t)\log\frac{\mu_F^2 z_t}{-t} + p_{gg}(z_t)\left(\frac{\log(1-z_t)}{1-z_t}\right)_+\right]g_{gg,t}(z_t) \\
&+ \left(\frac{1}{-t}\right)\left[-2n_f P_{qg}(z_t)\log\frac{\mu_F^2}{Q_{max}^2 q} + 2n_f z_t(1-z_t)\right]g_{qg,t}(z_t) \\
&+ \left(\frac{z_t}{-t}\right)\left[\left(\frac{\log(1-z_t)}{1-z_t}\right)_+ - \log\frac{Q_T^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right] \\
&\qquad \cdot \frac{N_c^2}{2}\left[\frac{m_H^8 + s^4 + t^4 + u^4 + Q^8 + z_t z_u(m_H^8 + s^4 + Q^8 + (u/z_u)^4 + (t/z_t)^4)}{sut}\right] \\
&- \left(\frac{z_t}{-t}\right)\left(\frac{1}{1-z_t}\right)_+\frac{\beta_0}{2}N_c\left(\frac{m_H^8 + s^4 + z_t z_u((u/z_u)^4 + (t/z_t)^4)}{sut}\right) \\
&+ (t \leftrightarrow u)\Bigg\}
\end{aligned}$$

$$+ N_c^2\left[\frac{(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)(Q^2+Q_T^2)}{s^2Q^2Q_T^2}+\right.$$
$$\left.+\frac{2m_H^4((m_H^2-u)^4+(m_H^2-t)^4+t^4+u^4)}{sut(m_H^2-t)(m_H^2-u)}\right]\frac{1}{p_T^2}\log\frac{p_T^2}{Q_T^2}$$

Now defining new coefficients this is rewritten as

$$\begin{aligned}
C^{(2)}_{sing,rap} = \int_0^1 dq J(x_1,x_2,q)Q^2_{max} \quad & \left\{\frac{\delta(q)}{Q^2_{max}}\mathcal{V}+\right.\\
& \frac{1}{-t}\mathcal{A}_t\left[-p_{gg}\left(-\left(\frac{\ln(1-z_t)}{1-z_t}\right)_+ + \ln\frac{\mu^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right) - \beta_0\ln\frac{\mu^2 z_t}{-t}\frac{\delta(q)}{Q^2_{max}}\right]+\\
& \frac{1}{-t}\left[\mathcal{B}_{1,t}\ln\frac{1}{qQ^2_{max}}+\mathcal{B}_{2,t}\right]+\\
& \frac{z_t}{-t}\mathcal{C}_t\left[\left(\frac{\ln(1-z_t)}{1-z_t}\right)_+ - \ln\frac{Q_T^2 z_t}{-t}\left(\frac{1}{1-z_t}\right)_+\right]-\\
& \frac{z_t}{-t}\mathcal{D}_t\left(\frac{1}{1-z_t}\right)_+ + (t\leftrightarrow u)\\
& \left.+\,\mathcal{E}\right\}
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathcal{A}_t &= g_{gg,t}(z_t)\\
\mathcal{A}_u &= g_{gg,u}(z_u)\\
\mathcal{B}_{1,t} &= -2n_f P_{qg}(z_t)g_{qg,t}(z_t)\\
\mathcal{B}_{2,t} &= 2n_f z_t(1-z_t)g_{qg,t}(z_t)\\
\mathcal{B}_{1,u} &= -2n_f P_{qg}(z_u)g_{gq,t}(z_t)\\
\mathcal{B}_{2,t} &= 2n_f z_u(1-z_u)g_{gq,t}(z_t),\\
\mathcal{C}_t = \mathcal{C}_u &= \frac{N_c^2}{2}\left[\frac{m_H^8+s^4+t^4+u^4+Q^8+z_t z_u(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)}{sut}\right]\\
\mathcal{D}_t = \mathcal{D}_u &= \frac{\beta_0}{2}N_c\left(\frac{m_H^8+s^4+z_t z_u((u/z_u)^4+(t/z_t)^4)}{sut}\right)\\
\mathcal{E} &= N_c^2\left[\frac{(m_H^8+s^4+Q^8+(u/z_u)^4+(t/z_t)^4)(Q^2+Q_T^2)}{s^2Q^2Q_T^2}+\frac{2m_H^4((m_H^2-u)^4+(m_H^2-t)^4+t^4+u^4)}{sut(m_H^2-t)(m_H^2-u)}\right]\frac{1}{p_T^2}\log\frac{p_T^2}{Q_T^2}
\end{aligned}$$

notice that in the definition of the $\mathcal{B}$ coefficients, quarks and gluons functions are exchanged too. Rearranging

$$\begin{aligned}
C^{(2)}_{sing,rap} = \int_0^1 dq J(x_1,x_2,q)&\Bigg\{\\
& Q^2_{max}\left[\mathcal{E}+\frac{\mathcal{B}_{2,t}}{-t}+\frac{\mathcal{B}_{1,t}}{-t}\ln\frac{1}{qQ^2_{max}}\right]\\
& +\delta(q)\Bigg[\mathcal{A}_t\beta_0\ln t + N_c\mathcal{A}_t\ln^2 Q^2_{max} - N_c\mathcal{A}_t\ln^2 t - \mathcal{C}_t\ln Q^2_{max}\ln p^2_{T,max} + \frac{1}{2}\mathcal{C}_t\ln^2 Q^2_{max}\\
& -\frac{1}{2}\mathcal{C}_t\ln^2 t + \mathcal{C}_t\ln p^2_{T,max} - \mathcal{D}_t\ln Q^2_{max} + \mathcal{D}_t\ln t + \mathcal{C}_t\ln Q_T^2\ln t + \mathcal{V}\Bigg]\\
& +\left(\frac{1}{q}\right)_+\left[\frac{\mathcal{A}_t p_{gg}}{z_t}\ln Q^2_{max} + \mathcal{C}_t\ln Q^2_{max} - \mathcal{C}_t\ln Q_T^2 - \mathcal{D}_t\right]\\
& +\left(\frac{\ln q}{q}\right)_+\left[\frac{\mathcal{A}_t p_{gg}}{z_t}+\mathcal{C}_t\right] + (t\leftrightarrow u)\Bigg\},
\end{aligned} \tag{6}$$

where in terms proportional to $\delta(q)$ we have set $z_t = 1$ and therefore $p_{gg}(1) = 2N_c$.

3 Integration of the $\delta(q)$ terms

Terms multiplied by $\delta(q)$ are straightforward to compute using $\delta(q)z_t = \delta(q)z_u = 1$. The relevant coefficients become

$$\mathcal{A}_t = \mathcal{A}_u = g_{LO}$$
$$\mathcal{C}_t = \mathcal{C}_u = N_c g_{LO}$$
$$\mathcal{D}_t = \mathcal{D}_u = \frac{\beta_0}{2} g_{LO}$$

We plug this into the equation above.

$$\begin{aligned} C^{(2)}_{sing,rap} = &\Bigg[g_{LO}\beta_0 \ln t + N_c g_L O \ln^2 Q^2_{max} - N_c g_{LO} \ln^2 t - N_c g_{LO} \ln Q^2_{max} \ln p^2_{T,max} + \frac{1}{2} N_c g_{LO} \ln^2 Q^2_{max} \\ & - \frac{1}{2} N_c g_{LO} \ln^2 t + N_c g_{LO} \ln p^2_{T,max} - \frac{\beta_0}{2} g_{LO} \ln Q^2_{max} + \frac{\beta_0}{2} g_{LO} \ln t + N_c g_{LO} \ln Q^2_T \ln t + \mathcal{V} \Bigg] J(x_1, x_2, 0) \\ & + \int_0^1 dq J(x_1, x_2, q) \Bigg\{ \\ & Q^2_{max} \left[\mathcal{E} + \frac{\mathcal{B}_{2,t}}{-t} + \frac{\mathcal{B}_{1,t}}{-t} \ln \frac{1}{q Q^2_{max}} \right] \\ & + \left(\frac{1}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} \ln Q^2_{max} + \mathcal{C}_t \ln Q^2_{max} - \mathcal{C}_t \ln Q^2_T - \mathcal{D}_t \right] \\ & + \left(\frac{\ln q}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} + \mathcal{C}_t \right] + (t \leftrightarrow u) \Bigg\}, \end{aligned} \tag{7}$$

Simplifying

$$\begin{aligned} C^{(2)}_{sing,rap} = &\Bigg[- \frac{3}{2} N_c g_{LO} \ln^2 t + \frac{3}{2} N_c g_{LO} \ln^2 Q^2_{max} - N_c g_{LO} \ln Q^2_{max} \ln Q^2_T + N_c g_{LO} \ln Q^2_T \ln t \\ & + \frac{3}{2} \beta_0 g_{LO} \ln t - \frac{1}{2} \beta_0 g_{LO} \ln Q^2_{max} + \mathcal{V} \Bigg] J(x_1, x_2, 0) \\ & + \int_0^1 dq J(x_1, x_2, q) \Bigg\{ \\ & Q^2_{max} \left[\mathcal{E} + \frac{\mathcal{B}_{2,t}}{-t} + \frac{\mathcal{B}_{1,t}}{-t} \ln \frac{1}{q Q^2_{max}} \right] \\ & + \left(\frac{1}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} \ln Q^2_{max} + \mathcal{C}_t \ln Q^2_{max} - \mathcal{C}_t \ln Q^2_T - \mathcal{D}_t \right] \\ & + \left(\frac{\ln q}{q}\right)_+ \left[\frac{\mathcal{A}_t p_{gg}}{z_t} + \mathcal{C}_t \right] + (t \leftrightarrow u) \Bigg\}, \end{aligned} \tag{8}$$

4 General properties of the integration in q

The integrand is singular both for $q \to 0$ and for $q \to 1$. The second is carried by terms proportional to $1/p_T^2$. Both extrema are regularized by plus distributions. The plus distributions centred around 0 come from the phase space factor $(Q^2)^{-\epsilon}$, while the plus distributions around 1 come from $|p_T^2|^{-\epsilon}$. The latter distributions are not explicitly written in the result since GS is only interested in the doubly differential distribution.
The generic term to integrate is

$$\int_0^1 dq f(x_1, x_2, q) \left(\frac{1}{q}\right)_+ \left(\frac{1}{1-q}\right)_+ = \int_0^1 dq \frac{f(x_1, x_2, q) - q f(x_1, x_2, 1) - (1-q) f(x_1, x_2, 0)}{q(1-q)}.$$

The integration can be simplified by expanding the integrand with respect to x_1 around 1 and retaining the leading terms in the singly soft limit $x_1 \to 1$. Since the integrand is rational this can be done by the dominated convergence theorem.

4.1 Coefficients in the singly soft limit

The gluon fusion LO in terms of x_1 and x_2 at $q = 0$ is

$$g_{LO}(x_1, x_2, 0) = N_c \frac{(1 + x_1^4 x_2^4)(x_1 + x_2)^4 + (1 - x_1^2)^4 x_2^4 + (1 - x_2^2)^4 x_1^4}{x_1^2 x_2^2 (x_1 + x_2)^2 (1 - x_1)^2 (1 - x_2)^2}.$$

In the limit $x_1 \to 1$, this is

$$g_{LO}(1, x_2, 0) = N_c \frac{(1+x_2)(1-x_2+x_2^2)^2}{x_2^2} \frac{1}{(1-x_1)(1-x_2)}.$$

The Jacobian at $q = 0$ is

$$J(x_1, x_2, 0) = \frac{2(1+x_1x_2)x_1x_2}{(x_1+x_2)^2}$$

In the limit $x_1 \to 1$

$$J(1, x_2, 0) = \frac{2x_2}{1+x_2}$$

Hence,

$$J(x_1, x_2, q) g_{LO}(x_1, x_2, q) \xrightarrow[q=0]{x_1 \to 1} = N_c \frac{2(1-x_2+x_2^2)^2}{x_2} \frac{1}{(1-x_1)(1-x_2)}.$$

Now consider q-dependent quantities in the singly soft limit, the first leading terms for some variables are.

$$\begin{aligned}
t &= -m^2 \frac{(1-x_1)[2+q(1-x_2)]}{(1+x_2)} + \mathcal{O}(1-x_1) \\
p_T^2 &= m^2 \frac{2(1-q)(1-x_1)(1-x_2)}{1+x_2} + \mathcal{O}((1-x_1)^0) \\
\frac{p_T^2}{t} &= \frac{2(1-q)(1-x_2)}{2+q(1-x_2)} + \mathcal{O}((1-x_1)^0) \\
z_t &= \frac{[2+q(1-x_2)]x_2}{2x_2+q(1-x_2)(1+2x_2)} + \mathcal{O}((1-x_1)^0) \\
J &= \frac{2x_2}{(1+x_1)} + \mathcal{O}((1-x_1)^0)
\end{aligned}$$

4.2 Integration of terms proportional to $\left(\frac{1}{q}\right)_+$

4.2.1 Integration of $\frac{\mathcal{A}_u p_{gg}}{z_t}$

Consider first

$$I(x_1, x_2) = \int_0^1 J(x_1, x_2, q) \frac{\mathcal{A}_t(x_1, x_2, q) p_{gg}(z_t(x_1, x_2, q))}{z_t(x_1, x_2, q)} \left(\frac{1}{q}\right)_+ \left(\frac{1}{1-q}\right)_+ \ln Q_{max}^2.$$

Expanding with respect to x_1 and integrating gives

$$I(x_1, x_2) = m^2 N_c^2 \frac{f(x_2)}{1-x_1} [\ln(1-x_1)(1-x_2) - \ln x_2].$$

Notice that even if $\mathcal{A}_t$ is proportional to $[(1-x_1)(1-x_2)]^{-1}$, the integral is singular only for $x_1 \to 1$.

4.2.2 Integration of $\mathcal{C}_t$

$$I(x_1, x_2) = \int_0^1 J(x_1, x_2, q) \mathcal{C}_t(x_1, x_2, q) \left(\frac{1}{q}\right)_+ \ln \frac{Q_{max}^2(x_1, x_2, q)}{Q_T^2(x_1, x_2, q)}.$$

The log term is

$$\ln \frac{Q_{max}^2(x_1, x_2, q)}{Q_T^2(x_1, x_2, q)} = \ln \frac{1+x_2}{q - x_2 q + 2x_2} + \wr((1-x)^0).$$

Integration gives

$$I(x_1, x_2) = m^2 N_c \frac{g(x_2)}{(1-x_1)(1-x_2)}$$

The function g contains dilogarithms.

4.2.3 Integration of $\mathcal{D}_t$

$$I(x_1, x_2) = \int_0^1 J(x_1, x_2, q) \mathcal{D}_t(x_1, x_2, q) \left(\frac{1}{q}\right)_+$$

gives

$$I(x_1, x_2) = m^2 N_c \frac{h(x_2)}{(1-x_1)(1-x_2)}$$

4.3 Integration of terms proportional to $\left(\frac{\ln q}{q}\right)_+$

4.3.1 Integration of the fourth term

$$I(x_1, x_2) = \int_0^1 dq J(x_1, x_2, q) \frac{\mathcal{A}_t(x_1, x_2, q) p_{gg}(z_t(x_1, x_2, q))}{z_t(x_1, x_2, q)} \left(\frac{\ln q}{q}\right)_+$$

4.3.2 Integration of the fifth term

$$I(x_1, x_2) = \int_0^1 dq J(x_1, x_2, q) \mathcal{C}_t(x_1, x_2, q) \left(\frac{\ln q}{q}\right)_+$$

4.4 Integration of terms proportional to Q^2_{max}