Solutions to Extremal Combinatorics

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Abstract

The following document contains solutions to selected problems in the book 'Extremal Combinatorics' by Stasys Jukna. Some problems which are either trivial or require nothing more than brute force/proof mirroring have been omitted.

Contents

1 Counting

1.1 Review

- The binomial theorem and binomial coefficients provide a way to count objects.
- Exact values of binomial coefficients are hard to compute, but they can be roughly estimated. Consider the following inequalities:
  1. $1+t<e^t$ for $t \neq 0$
  2. $1-t>e^{-t-t^2/2}$ for $0<t<1$

  they can be used to prove

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \qquad \sum_{i=1}^{k}\binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

- The Stirling approximation is another approximation which shows

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}e^{\alpha_n}$$

  with $1/(12n+1)<\alpha_n<1/12n$. Similarly, an asymptotic approximation for the k-factorial is

$$(n)_k = n^k e^{-k^2/2n-k^3/6n^2+o(1)}$$

  which is valid for $k=o(n^{3/4})$.
- The stars and bars technique allows you to partition a set into ordered partitions.
- Double counting can be done using incidence matrices: in this technique you count the rows and the columns of a zero-one matrix, which should yield the same sums.
- The number of vertices in a graph with odd degree is even.
- Jensen's Inequality: a function is convex if

$$f(\lambda a+(1-\lambda)b) \leq \lambda f(a)+(1-\lambda)f(b)$$

  for $0 \leq \lambda \leq 1$. Suppose that each λ_i is in $[0,1]$, and $\sum \lambda_i = 1$. If f is convex then

$$f\left(\sum_{i=1}^{n}\lambda_i x_i\right) \leq \sum_{i=1}^{n}\lambda_i f(x_i).$$

- Inclusion-exclusion provides a way to sum up all the elements in some sets that intersect.
- Derangements are permutations which don't fix any points.

1.2 Problems

In-text. Prove:

$$\sum_{x\in Y} d(x) = \sum_{A\in\mathcal{F}} |Y\cap A| \text{ for any } Y\subseteq X. \tag{1.1}$$

$$\sum_{x\in X} d(x)^2 = \sum_{x\in\mathcal{F}}\sum_{x\in A} d(x) = \sum_{A\in\mathcal{F}}\sum_{B\in\mathcal{F}} |A\cap B|. \tag{1.2}$$

Answer. We can proceed by a counting argument. For the first part, consider the incidence matrix $M=(m_{x,a})$ of $\mathcal{F}$. Adding the rows belonging to the elements of $x\in Y$ gives us the left hand side. If we 'slice' the matrix in this way, however (removing all the rows belonging to elements $x\notin Y$) then summing via the columns we get the sum of only those elements which are both in A and in Y, i.e. $|A\cap Y|$.

For the second part, it is enough to notice that if we replace each entry 1 in the incidence matrix with $d(x)$, then adding along the rows gives $\sum_{x\in X} d(x)^2$ while adding along the columns gives the term in the central equality (we add $d(x)$ for every $x\in A$, and then we sum over each of the As). The final equality follows from noticing that

$$\begin{aligned}\sum_{x\in\mathcal{F}}\sum_{x\in A} d(x) &= \sum_{A'\in\mathcal{F}}\left(\sum_{x\in A}|A\cap A'|\right)\\ &= \sum_{A\in\mathcal{F}}\sum_{B\in\mathcal{F}}|A\cap B|.\end{aligned}$$

where the second equality follows from substituting the first part.

Question 1.1 In how many ways can we distribute k balls to n boxes so that each box has at most one ball?

Answer. Depends on whether $k\le n$ or $k>n$. If $k\le n$, we can choose $\binom{n}{k}$ boxes, and the balls can be ordered in $k!$ ways. it follows that the number of ways is $\binom{n}{k}k! = (n)_k$.

If $k>n$, then there are $\binom{k}{n}$ ways we can choose the balls, and we can order them in $n!$ ways. This gives us the answer of $(k)_n$.

Question 1.2 Show that for every k the product of any k consecutive natural numbers is divisible by $k!$.

Answer. Let the k numbers be $n+1, \dots, n+k$. Consider the number $\binom{n+k}{k}$. This is the number of ways you can choose k numbers from $n+k$ numbers, and is an integer. Expanding, we get

$$\binom{n+k}{k} = \frac{n!}{k!(n-k)!}$$

which clearly shows that $k!$ divides $(n+1)(n+2)\dots(n+k)$.

Question 1.3 Show that the number of pairs (A, B) of distinct subsets of $\{1, \dots, n\}$ with $A \subset B$ is $3^n - 2^n$.

Answer. We can proceed as follows. Select a subset S of size k, and then select a subset of S. The former can be done in $\binom{n}{k}$ ways, while the latter can be done in $2^k - 1$ ways. This gives us the sum

$$\sum_{k=0}^{n} \binom{n}{k}(2^k - 1)$$

The binomial theorem tells us that this is equal to $(2+1)^n - (1+1)^n = 3^n - 2^n$.

Question 1.4 Show that

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

Answer. We will count the number of ways to choose k balls from n balls. Just choosing the k balls is the left hand side. Alternately, we can choose one ball first, and then choose $k-1$ balls from the remaining $n-1$ balls. However, this leads to a k-recounting, since each k size subset is selected k times (once when each element is the 'fixed' element). We are done.

Question 1.6 There is a set of $2n$ people: n male and n female. A good party is a set with the same number of male and female. How many possibilities are there to build such a good party?

Answer. For each k there are k ways to choose men and k ways to choose women, so the total number of parties of size k are $\binom{n}{k}^2$. Adding over all the k's we get

$$\sum_{i=0}^{n} \binom{n}{i}^2.$$

Question 1.7 Use Proposition 1.3 to show that

$$\sum_{i=0}^{r} \binom{n+i-1}{i} = \binom{n+r}{r}$$

Answer. We expand the RHS using Pascal's identity, then recursively expand one of the terms.

Question 1.8 Let $0 \leq a \leq m \leq n$ be integers. Show that

$$\sum_{i=m}^{n} \binom{i}{a} = \binom{n+1}{a+1} - \binom{m}{a+1}.$$

Answer. Same trick as the previous question, expand $\binom{n+1}{a+1}$.

Question 1.9 Prove the Cauchy-Vandermonde identity:

$$\binom{p+q}{k} = \sum_{i=0}^{k} \binom{p}{i}\binom{q}{k-i}.$$

Answer. We count twice. The left is selecting k items from $p+q$ items. Another way we can count this is to select i items from p items and $k-i$ items from q items, then add over every i.

Question 1.10 Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Answer. Start with the previous problem, then set $p = q = k = n$. It follows that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

However, we know that to choose k things, we can choose k or choose $n-k$, so the latter terms are equal.

Question 1.11 Prove the following analogy of the binomial theorem for factorials:

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}.$$

Answer. Immediately follows from the same consideration as the typical proof of the binomial theorem.

Question 1.28 This took me way too fucking long. Here's how you do it:

First, let $|E|$ be the number of edges in the graph. Note that $|B|D \geq |E| \geq |A|d$ by definition, since the *minimum* number of edges leaving A is $|A|d$, and the *maximum* number of edges leaving B is $|B|D$. Together with the fact given in the question this gives $|A|d = |B|D$. Furthermore, because the number of edges is sandwiched between these two, quantities, we know that $E = |A|d$. This tells us that every vertex in A *must* have degree d. Now take B_0 to be the set such that every neighbor has $\alpha D/2$ vertices to A_0.

It follows that the number of edges leaving A_0 is $|A_0|d = \alpha|A|d = \alpha|B|D$. Of these, some go to B_0, while others go outside of B_0. It follows that $|E_{A_0}| = |E_{A_0 \to B_0}| + |E_{A_0 \to B \setminus B_0}|$. We know that $|E_{A_0 \to B_0}| \leq \alpha|B_0|D/2$. Suppose that $|E_{A_0 \to B \setminus B_0}| \geq |E_{A_0 \to B_0}|$, which implies $|E_{A_0}| = \alpha|B|D \leq \alpha|B_0|D/2 + \alpha|B_0|D/2 = \alpha|B_0|D$, which is preposterous since $|B_0| \leq |B|$. This proves (iii), that $|E_{A_0 \to B \setminus B_0}| \leq |E_{A_0 \to B \setminus B_0}|$, which implies that more than half of the edges leaving A_0 go to B_0.

The first part now easily follows. The number of edges going from A_0 to B_0 is exactly the number of edges going from B_0 to A_0, which we know is greater than

$\alpha|B|D/2$ (which is half the number of edges leaving A_0). The edges leaving B_0 include this number *and more*. This means that $|B_0|D \geq \alpha|B|D/2$, which gives us (i), ie. $|B_0| \geq \alpha|B|/2$.

Very annoying problem. It's not conceptually difficult but there's a number of very shitty moving parts. Happy I solved it, finally.

Question 1.29 This was also mildly annoying, but I got it after a few minutes of serious thought. You apply Jensen's inequality to the parameters $f(x) = x^{t/s}$, $\lambda_i = 1/n$, and $x_i = a_i^s$. The inequality follows from the fact that f is convex iff $t \geq s$.