# Solutions to Extremal Combinatorics

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**Abstract**

The following document contains solutions to selected problems in the book 'Extremal Combinatorics' by Stasys Jukna. Some problems which are either trivial or require nothing more than brute force/proof mirroring have been omitted.

# Contents

# 1 Counting

## 1.1 Review

- The binomial theorem and binomial coefficients provide a way to count objects.
- Exact values of binomial coefficeints are hard to compute, but they can be roughly estimated. Consider the following inequalities:
  1. $1 + t < e^t$ for $t \neq 0$
  2. $1 - t > e^{-t-t^2/2}$ for $0 < t < 1$

  they can be used to prove

  $$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \qquad \sum_{i=1}^{k} \binom{n}{i} \leq \left(\frac{en}{k}\right)^k.$$

- The Stirling approximation is another approximation which shows

  $$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\alpha_n}$$

  with $1/(12n+1) < \alpha_n < 1/12n$. Similarly, an asymptotic approximation for the $k$-factorial is

  $$(n)_k = n^k e^{-k^2/2n - k^3/6n^2 + o(1)}$$

  which is valid for $k = o(n^{3/4})$.
- The stars and bars technique allows you to partition a set into ordered partitions.
- Double counting can be done using incidence matrices: in this technique you count the rows and the columns of a zero-one matrix, which should yield the same sums.
- The number of vertices in a graph with odd degree is even.
- Jensen's Inequality: a function is convex if

  $$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$$

  for $0 \leq \lambda \leq 1$. Suppose that each $\lambda_i$ is in $[0,1]$, and $\sum \lambda_i = 1$. If $f$ is convex then

  $$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

- Inclusion-exclusion provides a way to sum up all the elements in some sets that intersect.
- Derangements are permutations which don't fix any points.

## 1.2 Problems

**In-text.** Prove:

$$\sum_{x \in Y} d(x) = \sum_{A \in \mathcal{F}} |Y \cap A| \text{ for any } Y \subseteq X. \tag{1.1}$$

$$\sum_{x \in X} d(x)^2 = \sum_{x \in \mathcal{F}} \sum_{x \in A} d(x) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|. \tag{1.2}$$

**Answer.** We can proceed by a counting argument. For the first part, consider the incidence matrix $M = (m_{x,a})$ of $\mathcal{F}$. Adding the rows belonging to the elements of $x \in Y$ gives us the left hand side. If we 'slice' the matrix in this way, however (removing all the rows belonging to elements $x \notin Y$) then summing via the columns we get the sum of only those elements which are both in $A$ and in $Y$, i.e. $|A \cap Y|$.

For the second part, it is enough to notice that if we replace each entry 1 in the incidence matrix with $d(x)$, then adding along the rows gives $\sum_{x \in X} d(x)^2$ while adding along the columns gives the term in the central equality (we add $d(x)$ for every $x \in A$, and then we sum over each of the $A$s). The final equality follows from noticing that

$$\begin{aligned} \sum_{x \in \mathcal{F}} \sum_{x \in A} d(x) &= \sum_{A' \in \mathcal{F}} \left( \sum_{x \in A} |A \cap A'| \right) \\ &= \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|. \end{aligned}$$

where the second equality follows from substituting the first part.

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**Question 1.1** In how many ways can we distribute $k$ balls to $n$ boxes so that each box has at most one ball?

**Answer.** Depends on whether $k \leq n$ or $k > n$. If $k \leq n$, we can choose $\binom{n}{k}$ boxes, and the balls can be ordered in $k!$ ways. it follows that the number of ways is $\binom{n}{k} k! = (n)_k$.

If $k > n$, then there are $\binom{k}{n}$ ways we can choose the balls, and we can order them in $n!$ ways. This gives us the answer of $(k)_n$.

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**Question 1.2** Show that for every $k$ the product of any $k$ consecutive natural numbers is divisible by $k!$.

**Answer.** Let the $k$ numbers be $n+1, \ldots, n+k$. Consider the number $\binom{n+k}{k}$. This is the number of ways you can choose $k$ numbers from $n+k$ numbers, and is an integer. Expanding, we get

$$\binom{n+k}{k} = \frac{n!}{k!(n-k)!}$$

which clearly shows that $k!$ divides $(n+1)(n+2)\ldots(n+k)$.

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**Question 1.3** Show that the number of pairs $(A, B)$ of distinct subsets of $\{1, \ldots, n\}$ with $A \subset B$ is $3^n - 2^n$.

**Answer.** We can proceed as follows. Select a subset $S$ of size $k$, and then select a subset of $S$. The former can be done in $\binom{n}{k}$ ways, while the latter can be done in $2^k - 1$ ways. This gives us the sum

$$\sum_{k=0}^{n} \binom{n}{k}(2^k - 1)$$

The binomial theorem tells us that this is equal to $(2+1)^n - (1+1)^n = 3^n - 2^n$.

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**Question 1.4** Show that

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

**Answer.** We will count the number of ways to choose $k$ balls from $n$ balls. Just choosing the $k$ balls is the left hand side. Alternately, we can choose one ball first, and then choose $k-1$ balls from the remaining $n-1$ balls. However, this leads to a $k$-recounting, since each $k$ size subset is selected $k$ times (once when each element is the 'fixed' element). We are done.

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**Question 1.6** There is a set of $2n$ people: $n$ male and $n$ female. A good party is a set with the same number of male and female. How many possibilities are there to build such a good party?

**Answer.** For each $k$ there are $k$ ways to choose men and $k$ ways to choose women, so the total number of parties of size $k$ are $\binom{n}{k}^2$. Adding over all the $k$'s we get

$$\sum_{i=0}^{n} \binom{n}{i}^2 .$$

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**Question 1.7** Use Proposition 1.3 to show that

$$\sum_{i=0}^{r} \binom{n+i-1}{i} = \binom{n+r}{r}$$

**Answer.** We expand the RHS using Pascal's identity, then recursively expand one of the terms.

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**Question 1.8** Let $0 \le a \le m \le n$ be integers. Show that

$$\sum_{i=m}^{n} \binom{i}{a} = \binom{n+1}{a+1} - \binom{m}{a+1}.$$

**Answer.** Same trick as the previous question, expand $\binom{n+1}{a+1}$.

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**Question 1.9** Prove the Cauchy-Vandermonde identity:

$$\binom{p+q}{k} = \sum_{i=0}^{k} \binom{p}{i}\binom{q}{k-i}.$$

**Answer.** We count twice. The left is selecting $k$ items from $p+q$ items. Another way we can count this is to select $i$ items from $p$ items and $k-i$ items from $q$ items, then add over every $i$.

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**Question 1.10** Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

**Answer.** Start with the previous problem, then set $p = q = k = n$. It follows that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

However, we know that to choose $k$ things, we can choose $k$ or choose $n-k$, so the latter terms are equal.

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**Question 1.11** Prove the following analogy of the binomial theorem for factorials:

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k}.$$

**Answer.** Immediately follows from the same consideration as the typical proof of the binomial theorem.

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**Question 1.28** This took me way too much fucking time, unfortunately. Here's how you do it:

First, let $|E|$ be the number of edges in the graph. Note that $|B|D \geq |E| \geq |A|d$ by definition, since the *minimum* number of edges leaving $A$ is $|A|d$, and the *maximum* number of edges leaving $B$ is $|B|D$. Together with the fact given in the question this gives $|A|d = |B|D$. Furthermore, because the number of edges is sandwiched between these two, quantities, we know that $E = |A|d$. This tells us that every vertex in $A$ *must* have degree $d$.

It follows that the number of edges leaving $A_0 = |A_0|d$.