Solutions to Extremal Combinatorics

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Abstract

The following document contains solutions to selected problems in the book 'Extremal Combinatorics' by Stasys Jukna. Some problems which are either trivial or require nothing more than brute force/proof mirroring have been omitted.

Contents

1 Counting

In-text. Prove:

$$\sum_{x\in Y} d(x) = \sum_{A\in\mathcal{F}} |Y \cap A| \text{ for any } Y \subseteq X. \tag{1.1}$$

$$\sum_{x\in X} d(x)^2 = \sum_{x\in\mathcal{F}} \sum_{x\in A} d(x) = \sum_{A\in\mathcal{F}} \sum_{B\in\mathcal{F}} |A \cap B|. \tag{1.2}$$

Answer. We can proceed by a counting argument. For the first part, consider the incidence matrix $M = (m_{x,a})$ of $\mathcal{F}$. Adding the rows belonging to the elements of $x \in Y$ gives us the left hand side. If we 'slice' the matrix in this way, however (removing all the rows belonging to elements $x \notin Y$) then summing via the columns we get the sum of only those elements which are both in A and in Y, ie. $|A \cap Y|$.

For the second part, it is enough to notice that if we replace each entry 1 in the incidence matrix with $d(x)$, then adding along the rows gives $\sum_{x\in X} d(x)^2$ while adding along the columns gives the term in the central equality (we add $d(x)$ for every $x \in A$, and then we sum over each of the As). The final equality follows from noticing that

$$\begin{aligned} \sum_{x\in\mathcal{F}} \sum_{x\in A} d(x) &= \sum_{A'\in\mathcal{F}} \left(\sum_{x\in A} |A \cap A'| \right) \\ &= \sum_{A\in\mathcal{F}} \sum_{B\in\mathcal{F}} |A \cap B|. \end{aligned}$$

where the second equality follows from substituting the first part.