Generalizing trace monoids

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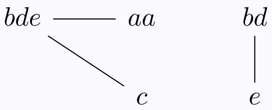
- Example:

$$ebdaac = bdeaac$$
$$bdeaac = aabdec$$
$$aabdec = aacbde$$



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- Traces $\longleftrightarrow$ executions up to concurrency

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     - ★ Normal form which faithfully represents the original trace language.

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- Is regularity preserved in simpler but related languages?
- What other results can we obtain that give us insight into the languages using generalized traces?

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- The irregularity arises from the fact that these words can be internally transformed into different words.
- Not a problem in the non-generalized case, since letters are discrete!
- Can we make strings in $\tilde{\mathbb{I}} = \{w \mid \exists v \text{ such that } (w, v) \in \mathbb{I}\}$ behave like letters?

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Partitioning Condition

We can partition Σ into $\Sigma_1 \cup \Sigma_2$, such that words in Σ_1^* can only commute with words in Σ_2^* (vice versa). Furthermore, no $v' \in \tilde{\mathbb{I}}$ is a prefix or suffix of $v \in \tilde{\mathbb{I}}$.

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If $u, v \in \tilde{\mathbb{I}}$, then u cannot be a **prefix** nor **suffix** of v.

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Result

We regain regularity of $\mathcal{L}_{x \prec y}$!

Prefix-Suffix Condition: Proof Sketch

- Consider words such that $w = w_1 w_2 w_3 \dots w_n$ where $w_i \in \tilde{\mathbb{I}}$.

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- We can apply the same proof as the non-generalized case, essentially a homomorphism to the letter case.

Partitioning + Weak Prefix-Suffix Condition: Proof Sketch

- We can partition any word into $w = u_1v_1u_2v_2 \dots u_nv_n$ where $u_i \in \Sigma_1^*, v_i \in \Sigma_2^*$.

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- We then check if each u_i and v_i can be broken into $\{u_{ij}\}_{j \in [k]}, \{v_{ij}\}_{j \in [k']}$, where the elements of each decomposition are in $\tilde{\mathbb{I}}$

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- Essentially a reduction from the generalized case to the non-generalized case
- Can we apply other methods to understand regularity in this context without imposing additional structure to $\mathbb{I}$?

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- $\{w_1^n w_2^m \mid w_2^n w_1^m \equiv w_2^m w_1^n\}$ is regular for any $w_1, w_2 \in \Sigma^*$.

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Idea of the proof

- $L_v \cap w^*$ is a submonoid of $w^* \cong \mathbb{N}$, hence, it is of the form $(w^d)^* \setminus E$ for some d and some finite set E.

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- This yields the regularity of $\{w_1^n w_2^m \mid w_1^n w_2^m \equiv w_2^m w_1^n\}$.

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We can combine these approaches (restricting $\mathbb{I}$ and simplifying the languages):

Corollary

If $\mathbb{I} \subseteq \Sigma_1^* \times \Sigma_2^* \cup \Sigma_2^* \times \Sigma_1^*$, then $L_{a \prec b} \cap a^* b^*$ is regular.

Theorem

Suppose $\mathbb{I} \subseteq \Sigma_1^* \times \Sigma_2^* \cup \Sigma_2^* \times \Sigma_1^*$, $|\mathbb{I}| < \infty$ and $a \in \Sigma_1$.

- $L_a := \{u \in \Sigma_2^* \mid au \equiv ua\}$ is regular
- $L_{a,v} := \{u \in \Sigma_2^* \mid vau \equiv vua\}$ is regular
- $L_{v,a} := \{u \in \Sigma_2^* \mid uav \equiv auv\}$ is regular

Proof

The idea is to use the fact that 2NFA's (nondeterministic Turing machines that cannot write) recognize regular languages. We can construct a 2NFA that can simulate the swaps using its head to keep track of the position of the a.

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- It turns out that $Lex(\Sigma^*)$ is regular iff S is regular.
- It would be great to find a condition on $\mathbb{I}$ that would guarantee that S is regular/finite.

Questions?