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Singular Value Decomposition (SVD) provides a very useful tool to compress images. This technique approximates the image on a smaller set of singular values, reducing the dimensionality of the image and, thereby, retaining the important features of an image view while discarding less important details.
There are two key parameters concerning SVD compression:

- Compression ratio (ρ): This is the amount of compression applied to an image and is mathematically represented by the rank k, which defines the number of singular values kept in the compressed image. The recommended practice is to have as much small ϱ, as it results in maximum compressibility and minimal quality loss.
- Average Relative Difference (δ): This is the measure of average difference on a pixel basis between original and compressed images. It gives a clear idea of the quality loss due to the compression process. Smaller δ means that the compression process has preserved most of the quality of the original image.

Below I will list the impact of rank(k), compression ratio, and average relative difference towards the compressed image in general:

- Rank (k): It determines the number of singular values retained when compressing the image. In SVD, the image matrix A is decomposed into 3 parts:
    - U and V are orthogonal matrices.
    - Σ is the diagonal matrix of singular values.

  $$A = U\Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T,$$

  Rank is important because retaining more singular values means keeping more information intact on the image, resulting in better quality but less compression. Therefore, in reverse, retaining less singular values lead to lower storage size but terrible image quality.
- Average Relative Difference (δ):

  $$\delta = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{|A_{i,j} - \tilde{A}_{i,j}|}{A_{i,j} + \epsilon}, \epsilon = 10^{-8}$$

    - A i,j: Pixel value in the original image.
    - Aˆi,j; Pixel value in the compressed image.
    - ϵ: Small value to avoid division by zero.(10^−8)

  Average Relative Difference is important because the smaller the value (δ) means that the quality of the compressed image is close to the original meaning minimal information loss but less compression. In reverse, the higher the value (δ) means a more significant degradation of the compressed image,resulting in a higher compression.
- Compression ratio (ρ):
    - ρ= (H*k + k +k*w)/h*w
    - H: height of the image
    - W: width of the image.
    - K:number of the singular values retained.

  What each value of ρ means:

- $0.5 < \rho < 1$: The image quality is similar to the original however it requires less storage due to compression.
- $\rho < 0.5$: High compression, significant image degradation.
- $\rho \approx 1$: Less compression, high image quality retention.

So in conclusion, if we want to have a higher quality compressed image, we can increase the rank (k) value, increase the compression ratio and lower the average relative difference. On the other hand, if we want a higher compression, we can decrease the rank (k) value, decrease the compression ratio, and increase the average relative difference.

Fast decaying singular values means that the singular values drop significantly in the first few values, whilst in the image this means that the image's information is concentrated in the first few components. Therefore, when compressed with low rank approximations, the image will retain its quality(low compression ratio and low average relative difference). On the other hand, Slow decaying singular values means that the singular values decrease gradually, whilst in the image this means that the image's information is evenly distributed in the picture, in other words the picture contains complex details. Therefore when compressed with lower rank approximation, it leads to a significant quality loss. For better quality retention, a higher rank approximation is needed, leading to higher compression rate and lower average relative difference.

For images with rapidly decaying singular values(less detail in the image), we can achieve better quality retention with less singular values(low average relative difference and high compression rate). On the other hand, images with slow decaying singular values need more singular values to achieve a similar level of quality, leading to higher average relative difference value on lower rank approximations.

Below I will explain further the effects of each image. All the image sizes are written in the format height x width.

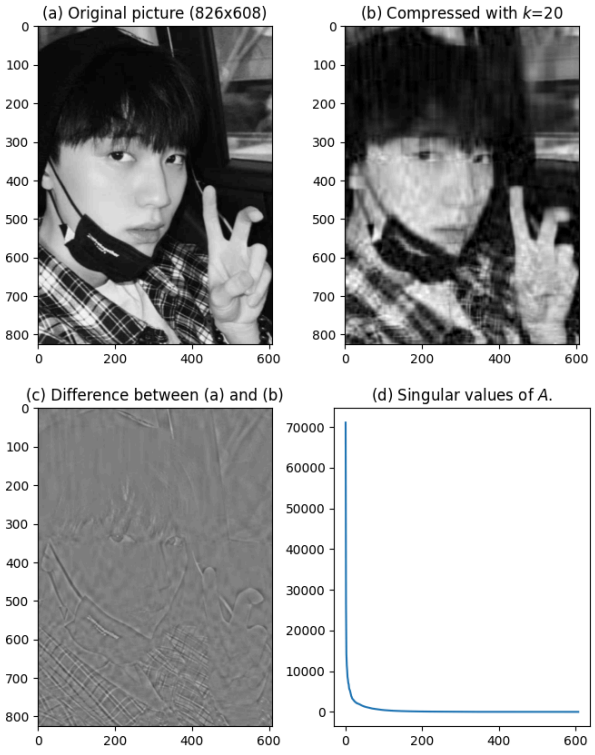
Image 1

The Image had the size 826 x 608. Below I List the SVD Compress result, the average relative difference value and the compression ratio.

Original Image

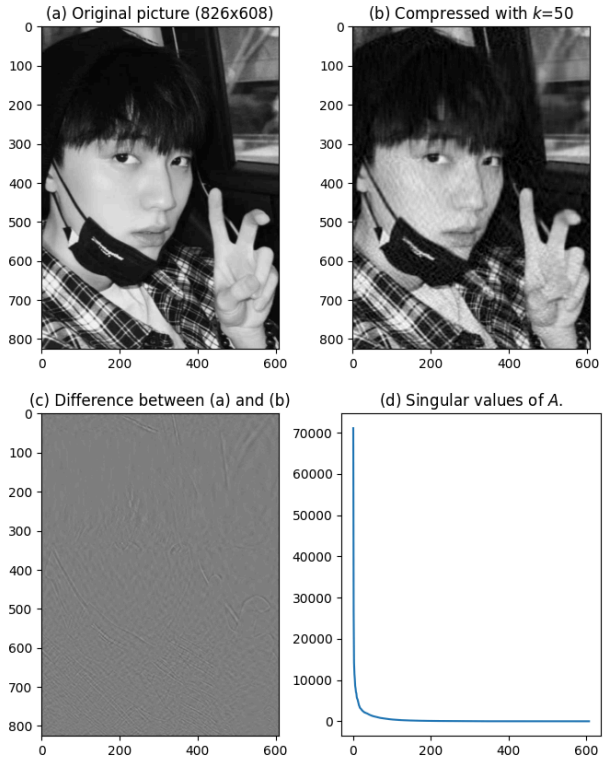
K=20

```
The size of the image is: 826 x 608
Compression ratio is: 0.057148
Average relative difference is: 0.271184
```



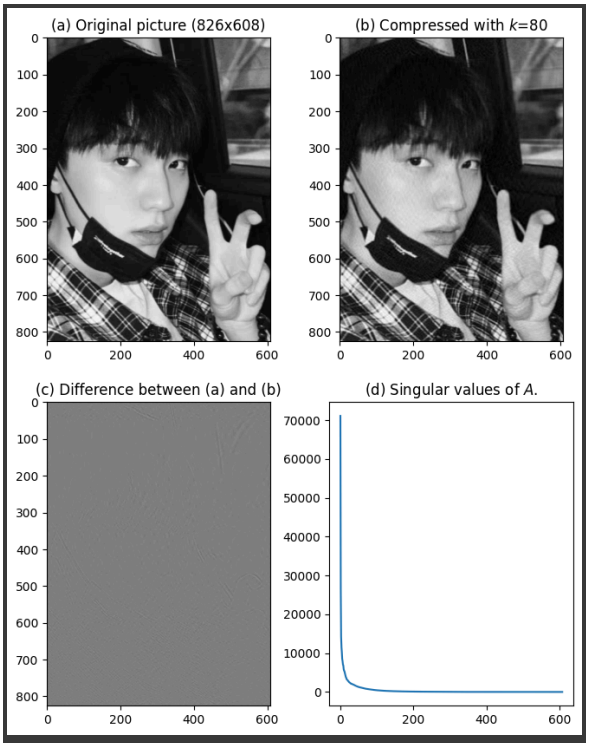
K=50

```
The size of the image is: 826 x 608
Compression ratio is: 0.128582
Average relative difference is: 0.166273
```



K=80

```
The size of the image is: 826 x 608
Compression ratio is: 0.228591
Average relative difference is: 0.089703
```



Based on the three cases of different rank values(20,50, and 80), It can be concluded that the relationship between the average relative value between is inversely proportional. This can be seen in all cases:

- K=20, the values were
    - Compression ratio: 0.06
    - Average relative difference value: 0.27
- K=50, the values were
    - Compression ratio: 0.13
    - Average relative difference value: 0.17
- K=80, the values were
    - Compression ratio: 0.23
    - Average relative difference value: 0.09

If we want the compression ratio to be less than 0.5 or 1, we can continue to decrease the rank value, which will result in higher compression. For example, in the case of K=20, we observe that the average relative difference increases to 0.27, while the compression ratio decreases to 0.06. This indicates that the image has undergone significant degradation due to higher compression, with the compressed image being further from the original both in terms of quality and compression.

If we want the average relative difference to be less than 5%, we can achieve this by increasing the rank value. For example, in the case where K=80, the compression ratio is 0.23, while the average relative difference is 0.09. This shows that the image has less compression (since a higher rank is used) but retains higher quality (as shown by the lower relative difference).

Based on the trend of the average relative difference values (0.27 -> 0.17 -> 0.09), It suggests that the singular values decay gradually, requiring more singular values for quality retention, therefore the image has slow-decaying singular values.

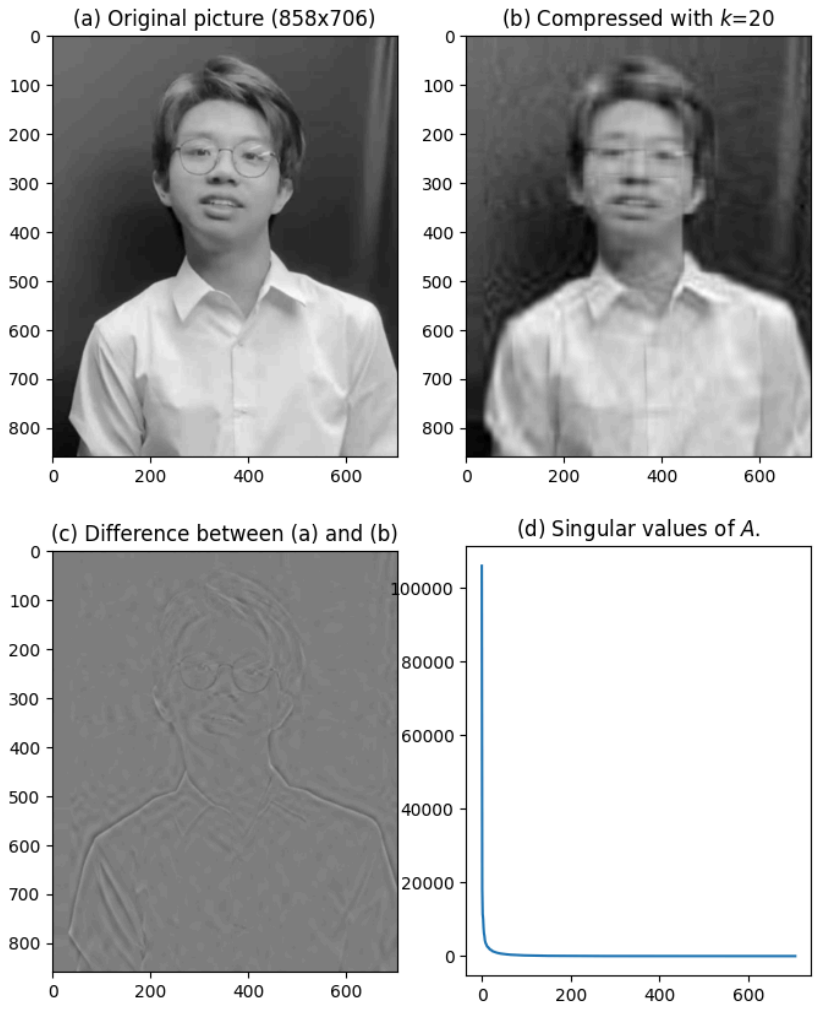
Image 2

The Image had the size 858 x 706. Below I List the SVD Compress result, the average relative difference value and the compression ratio.

Original Image

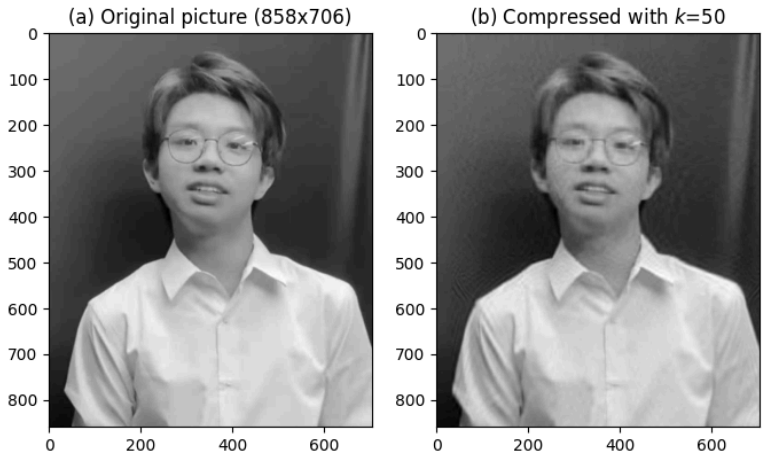
K=20

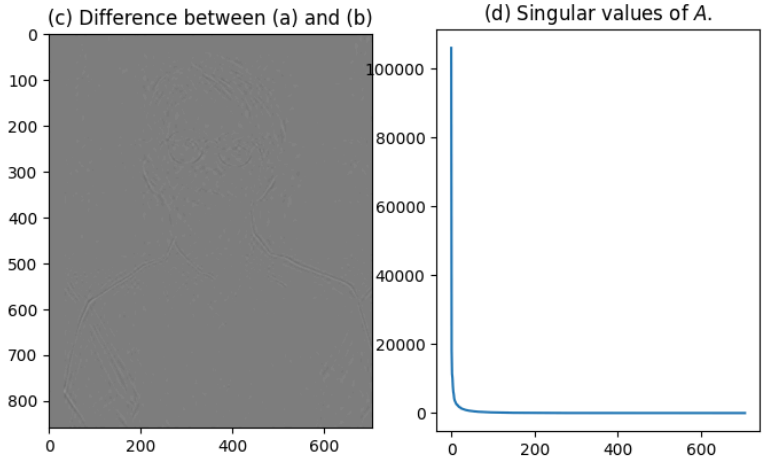
```
The size of the image is: 858 x 706
Compression ratio is: 0.051672
Average relative difference is: 0.058669
```



K=50

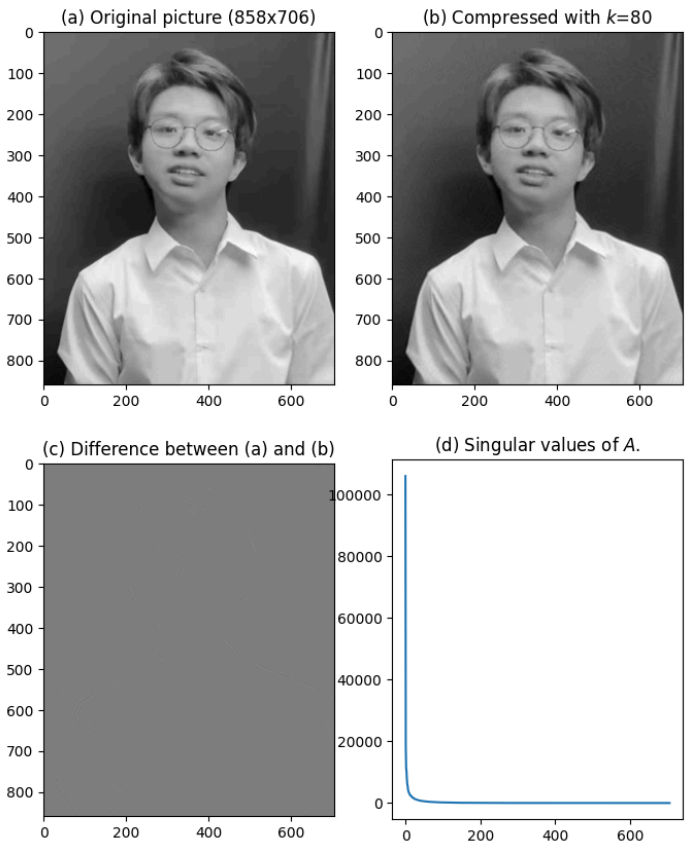
```
The size of the image is: 858 x 706
Compression ratio is: 0.129179
Average relative difference is: 0.025102
```





K=80

```
The size of the image is: 858 x 706
Compression ratio is: 0.206687
Average relative difference is: 0.014450
```



Based on the three cases of different rank values(20,50, and 80), It can be concluded that the relationship between the average relative value between is inversely proportional. This can be seen in all cases:

- K=20, the values were
    - Compression ratio: 0.05
    - Average relative difference value: 0.06
- K=50, the values were
    - Compression ratio: 0.13
    - Average relative difference value: 0.03
- K=80, the values were
    - Compression ratio: 0.2
    - Average relative difference value: 0.01

If we want the compression ratio to be less than 0.5 or 1, we can continue to decrease the rank value, which will result in higher compression. For example, in the case of K=20, we observe that the average relative difference increases to 0.06, while the compression ratio decreases to 0.05. This indicates that the image has undergone significant degradation due to higher compression, with the compressed image being further from the original both in terms of quality and compression

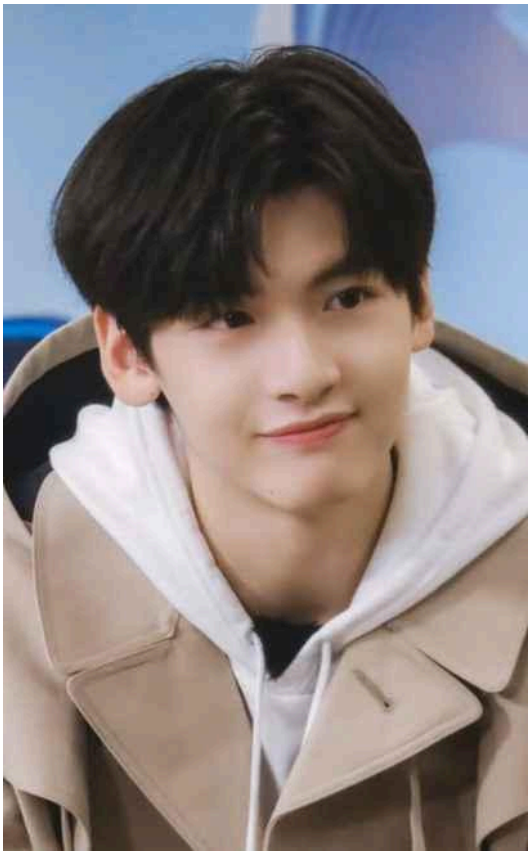
If we want the average relative difference to be less than 5%, we can achieve this by increasing the rank value. For example, in the case where K=80, the compression ratio is 0.2, while the average relative difference is 0.01. This shows that the image has less compression (since a higher rank is used) but retains higher quality (as shown by the lower relative difference).

Based on the trend of the average relative difference values (0.06 -> 0.03 -> 0.09), It suggests that the singular values decay rapidly, allowing good quality retention as it requires less singular values, therefore the image has fast-decaying singular values.

Image 3

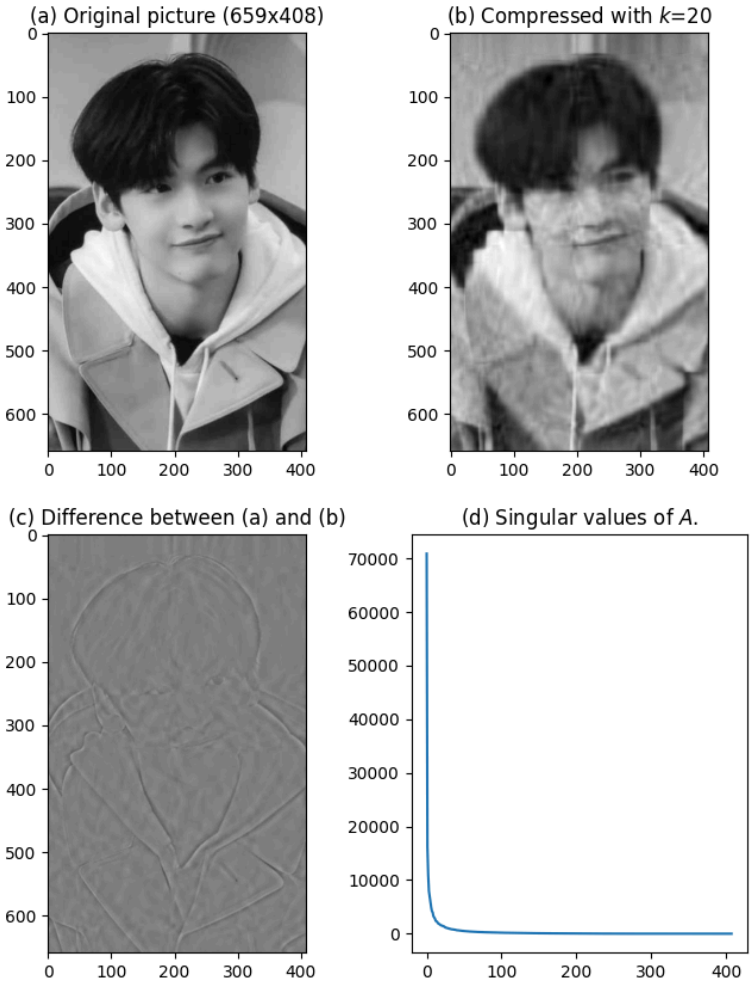
The Image had the size 659 x 408. Below I List the SVD Compress result, the average relative difference value and the compression ratio.

Original Image



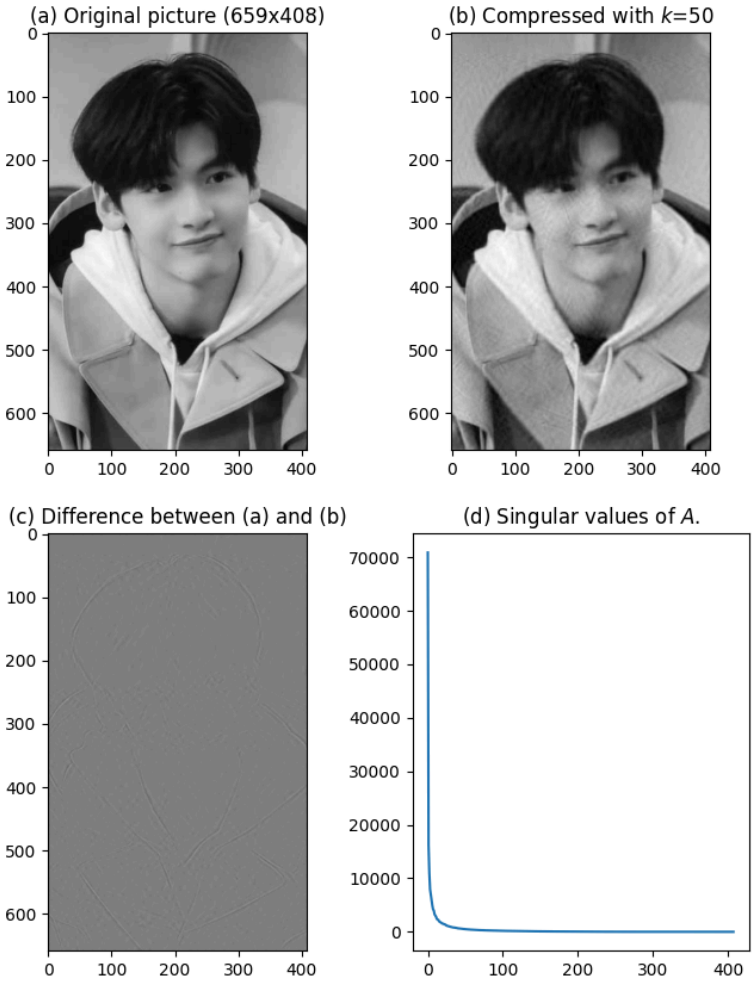
K=20

```
The size of the image is: 659 x 408
Compression ratio is: 0.079443
Average relative difference is: 0.085214
```



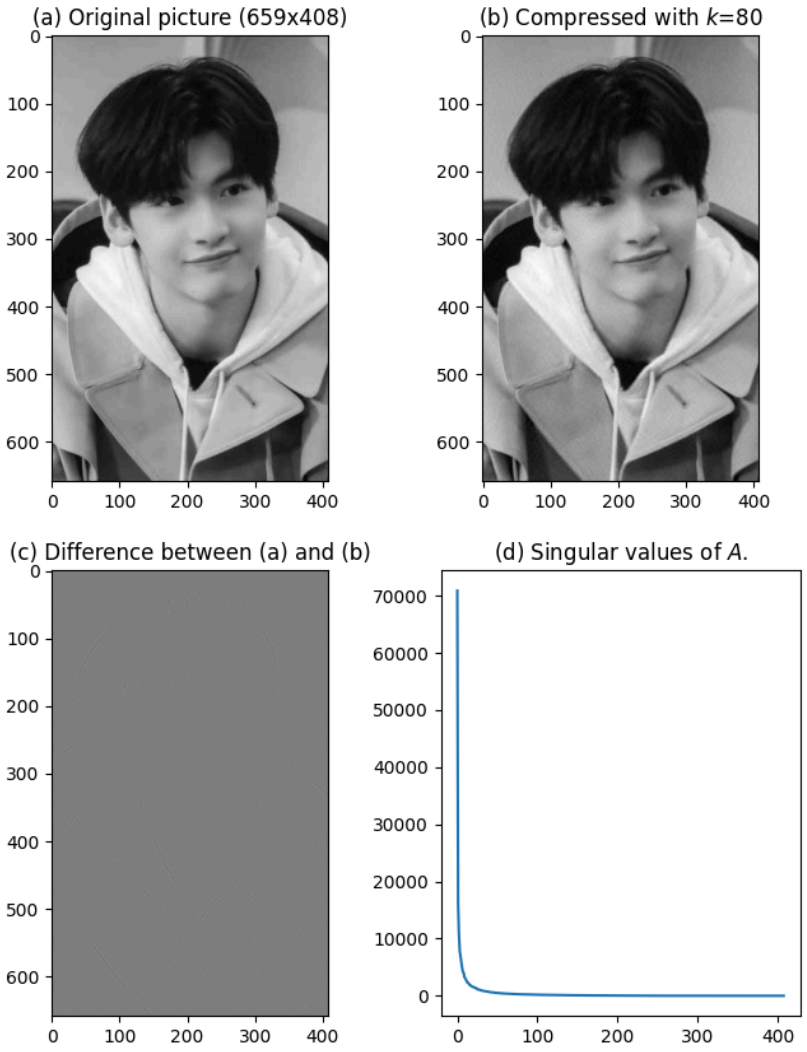
K=50

```
The size of the image is: 659 x 408
Compression ratio is: 0.198608
Average relative difference is: 0.036905
```



K=80

```
The size of the image is: 659 x 408
Compression ratio is: 0.317772
Average relative difference is: 0.022295
```



Based on the three cases of different rank values(20,50, and 80), It can be concluded that the relationship between the average relative value between is inversely proportional. This can be seen in all cases:

- K=20, the values were
    - Compression ratio: 0.08
    - Average relative difference value: 0.09
- K=50, the values were
    - Compression ratio: 0.19
    - Average relative difference value: 0.04
- K=80, the values were
    - Compression ratio: 0.32
    - Average relative difference value: 0.02

If we want the compression ratio to be less than 0.5 or 1, we can continue to decrease the rank value, which will result in higher compression. For example, in the case of K=20, we observe that the average relative difference increases to 0.09, while the compression ratio decreases to 0.08. This indicates that the image has undergone significant degradation due to higher compression, with the compressed image being further from the original both in terms of quality and compression

If we want the average relative difference to be less than 5%, we can achieve this by increasing the rank value. For example, in the case where K=80, the compression ratio is 0.32, while the average relative difference is 0.02. This shows that the image has less compression (since a higher rank is used) but retains higher quality (as shown by the lower relative difference).

Based on the trend of the average relative difference values (0.09 -> 0.04 -> 0.02), It suggests that the singular values decay gradually, requiring more singular values for quality retention, therefore the image has slow-decaying singular values.

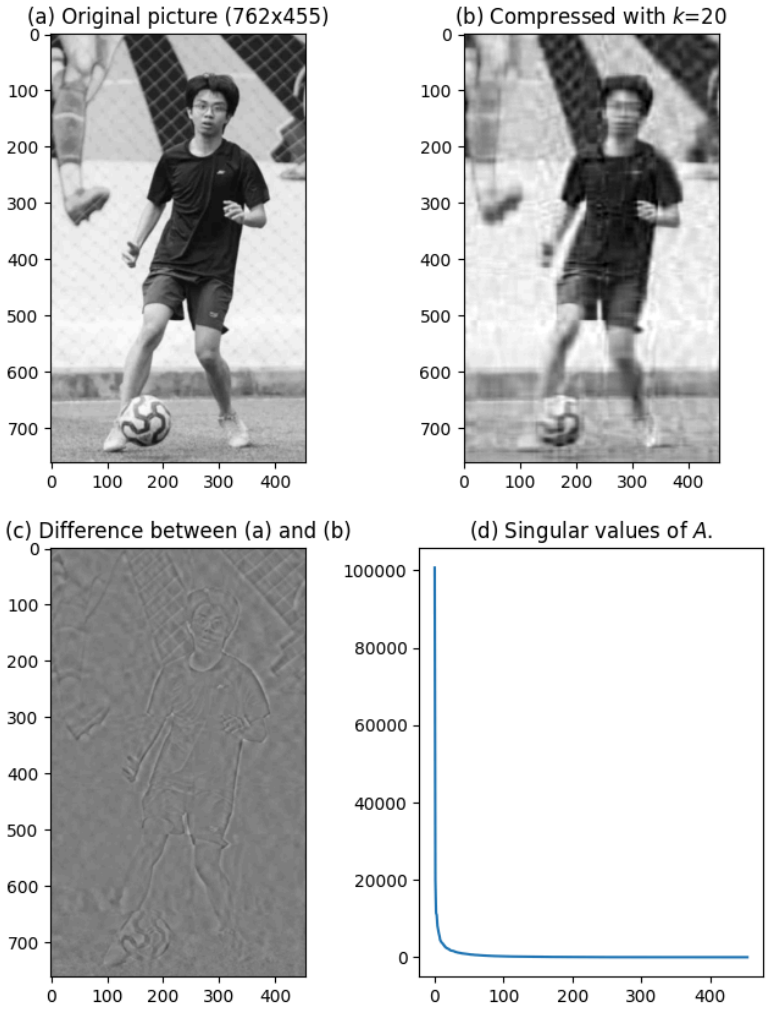
Image 4

The Image had the size 762 x 455. Below I List the SVD Compress result, the average relative difference value and the compression ratio.

Original Image

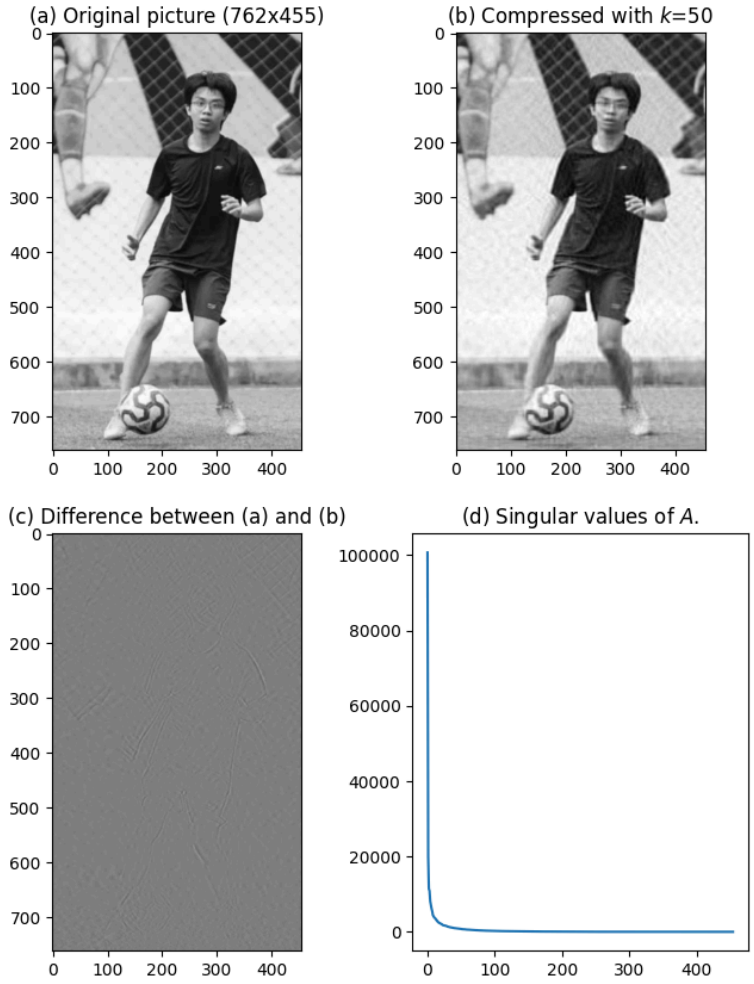
K=20

```
The size of the image is: 762 x 455
Compression ratio is: 0.070260
Average relative difference is: 0.099602
```



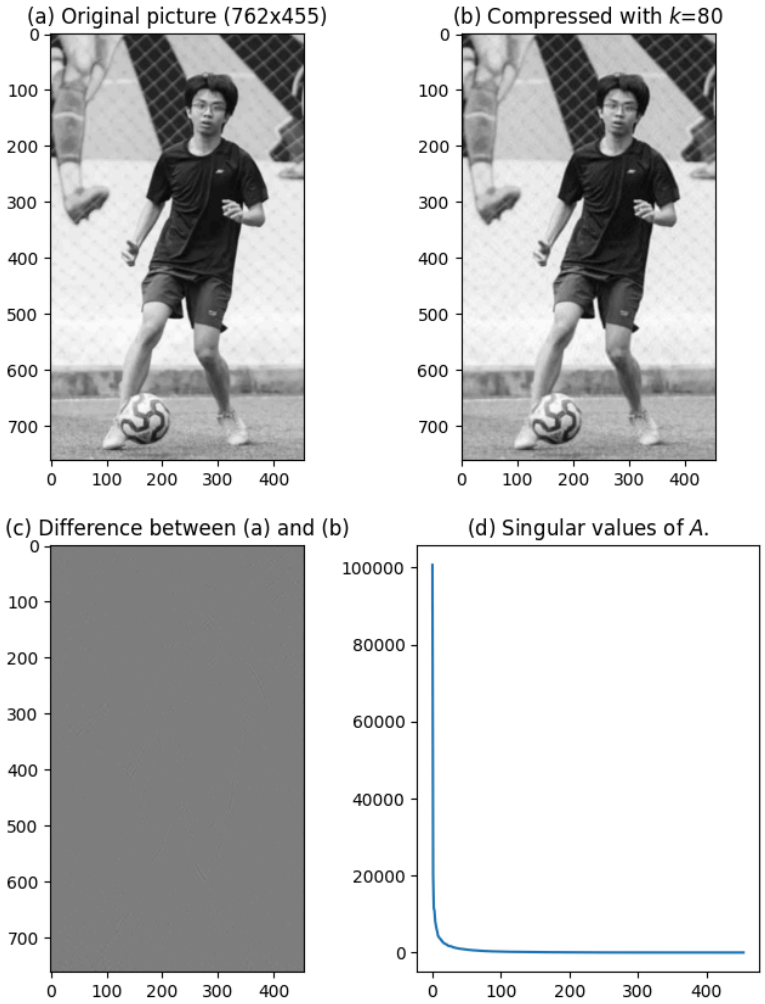
K=50

```
The size of the image is: 762 x 455
Compression ratio is: 0.175651
Average relative difference is: 0.043364
```



K=80

```
The size of the image is: 762 x 455
Compression ratio is: 0.281042
Average relative difference is: 0.023236
```



Based on the three cases of different rank values(20,50, and 80), It can be concluded that the relationship between the average relative value between is inversely proportional. This can be seen in all cases:

- K=20, the values were
    - Compression ratio: 0.07
    - Average relative difference value: 0.10
- K=50, the values were
    - Compression ratio: 0.18
    - Average relative difference value: 0.04
- K=80, the values were
    - Compression ratio: 0.28
    - Average relative difference value: 0.02

If we want the compression ratio to be less than 0.5 or 1, we can continue to decrease the rank value, which will result in higher compression. For example, in the case of K=20, we observe that the average relative difference increases to 0.10, while the compression ratio decreases to 0.07. This indicates that the image has undergone significant degradation due to higher compression, with the compressed image being further from the original both in terms of quality and compression

If we want the average relative difference to be less than 5%, we can achieve this by increasing the rank value. For example, in the case where K=80, the compression ratio is 0.28, while the average relative difference is 0.02. This shows that the image has less compression (since a higher rank is used) but retains higher quality (as shown by the lower relative difference).

Based on the trend of the average relative difference values (0.10 -> 0.04 -> 0.02), It suggests that the singular values decay gradually, requiring more singular values for quality retention, therefore the image has slow-decaying singular values.

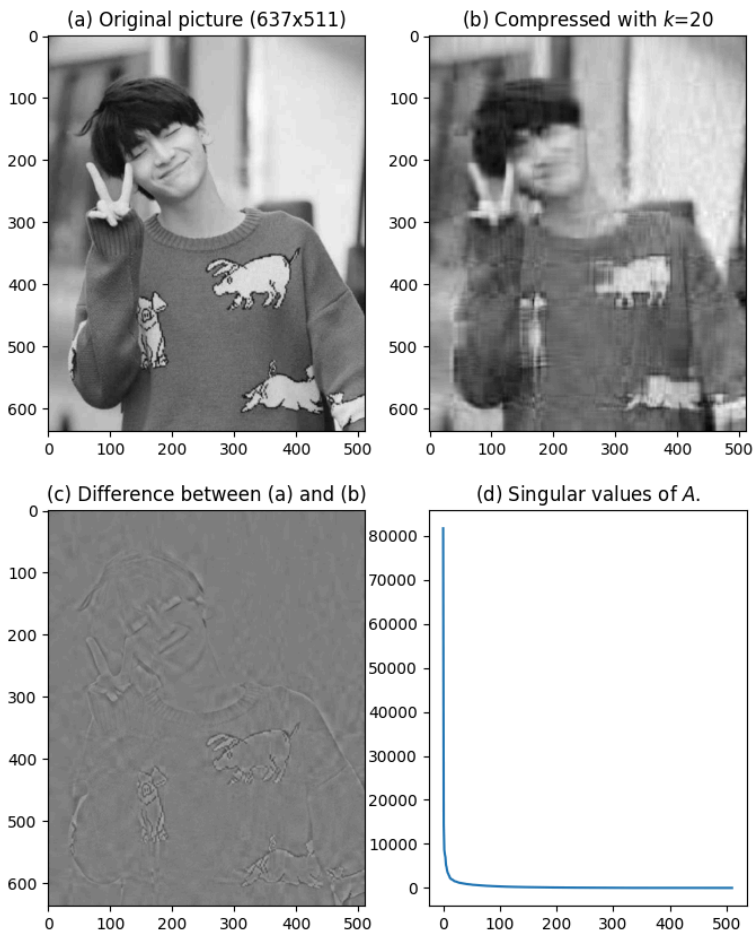
Image 5

The Image had the size 637 x 511. Below I List the SVD Compress result, the average relative difference value and the compression ratio.

Original Image

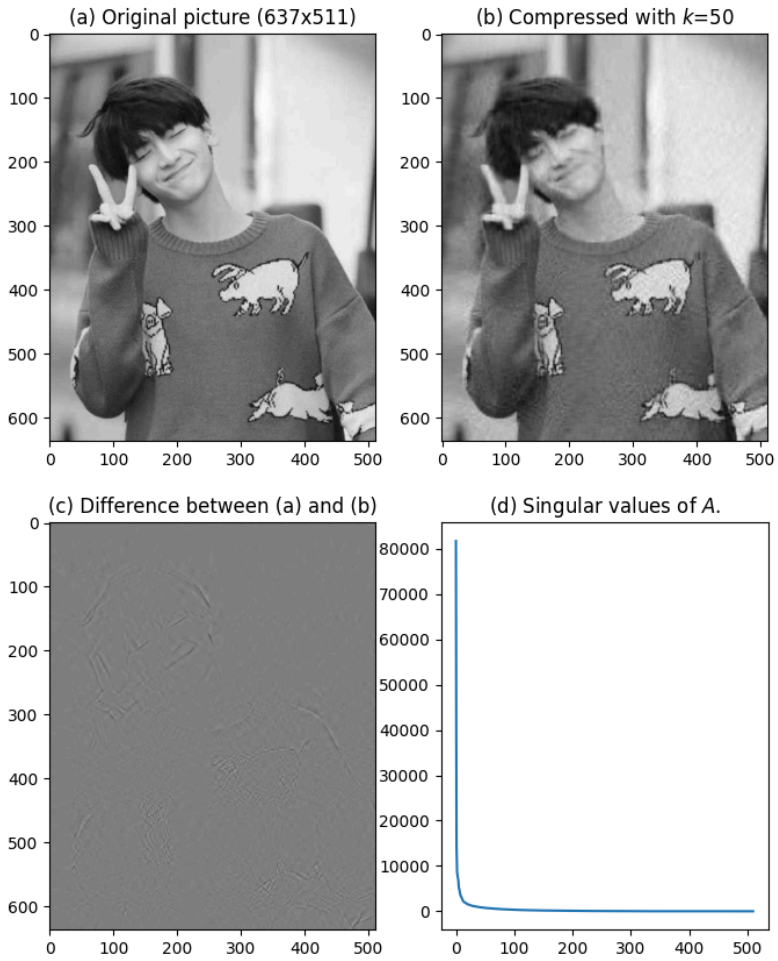
K=20

```
The size of the image is: 637 x 511
Compression ratio is: 0.070598
Average relative difference is: 0.083645
```



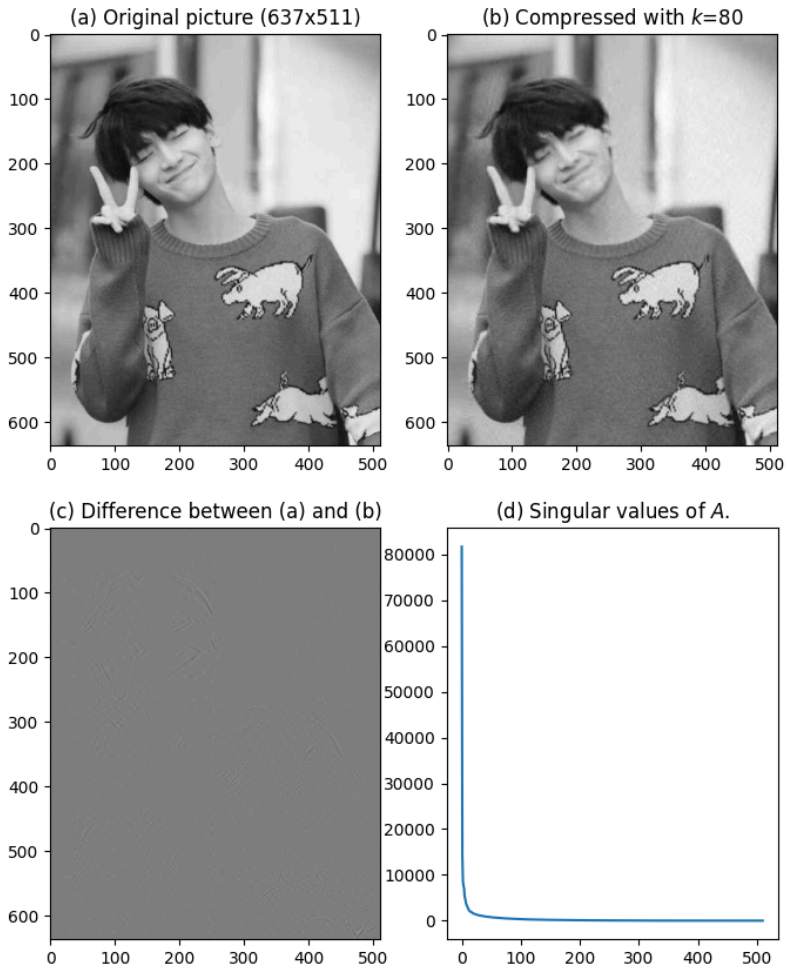
K=50

```
The size of the image is: 637 x 511
Compression ratio is: 0.176494
Average relative difference is: 0.048342
```



K=80

```
The size of the image is: 637 x 511
Compression ratio is: 0.282390
Average relative difference is: 0.032116
```



Based on the three cases of different rank values(20,50, and 80), It can be concluded that the relationship between the average relative value between is inversely proportional. This can be seen in all cases:

- K=20, the values were
    - Compression ratio: 0.07
    - Average relative difference value: 0.08
- K=50, the values were
    - Compression ratio: 0.17
    - Average relative difference value: 0.04
- K=80, the values were
    - Compression ratio: 0.28
    - Average relative difference value: 0.03

If we want the compression ratio to be less than 0.5 or 1, we can continue to decrease the rank value, which will result in higher compression. For example, in the case of K=20, we observe that the average relative difference increases to 0.08, while the compression ratio decreases to 0.07. This indicates that the image has undergone significant degradation due to higher compression, with the compressed image being further from the original both in terms of quality and compression

If we want the average relative difference to be less than 5%, we can achieve this by increasing the rank value. For example, in the case where K=80, the compression ratio is

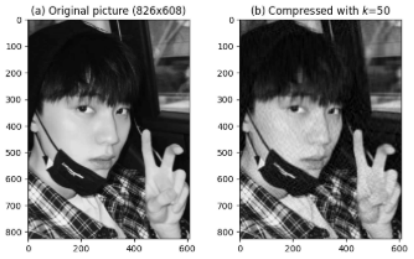
0.28, while the average relative difference is 0.03. This shows that the image has less compression (since a higher rank is used) but retains higher quality (as shown by the lower relative difference).

Based on the trend of the average relative difference values (0.08 -> 0.04 -> 0.03), It suggests that the singular values decay gradually, requiring more singular values for quality retention, therefore the image has slow-decaying singular values.

Question 3

1. What kind of pictures can be compressed easily by SVD (smaller relative difference)? And why?
    a. Images with smooth gradients(smooth transitions between colors) or simple patterns (including basic shapes and/or uniform textures) tend to have smaller relative differences, in other words easier to compress by SVD.
    b. This is because these types of images have fast-decaying singular values which means they have less detail, making it easier to compress because there are less singular values, which reduces both compression ratio and average relative difference, resulting in minimal quality loss and effective compression.
2. Which means they have low ρ and δ? Try to find its relation with the distribution of singular values of the original pictures.
    a. Compression performs better when singular values decay more rapidly as it decreases many singular values that will yield a better approximation of the image. The lower compression ratio ρ and relative difference δ that is achieved means allowing a lower number of components to represent the image and better quality of compression can be achieved.
    b. If the opposite occurs, that is, the singular values decay slowly this implies that there is a lot of detail and complexity in the image and a greater number of components are required for proper reproduction, which results in higher values of ρ and δ. This means that there is greater loss of information caused by compression which means that the image approximation represents not much of the original image.
    c. Compression is improved with faster decay of the singular values; faster decay means the more important parts of the image can be identified with smaller amounts of singular values thus requiring less distortion.
    d. If decay occurs at a slow rate, more singular values will be needed, resulting in a lower-quality image since more storage and higher rank approximations are required which causes a loss in quality of the compressed image.
    e. As example, I will be comparing Image 1 and Image 2:

```
The size of the image is: 826 x 608
Compression ratio is: 0.128582
Average relative difference is: 0.166273
```




```
The size of the image is: 858 x 706
Compression ratio is: 0.129179
Average relative difference is: 0.025102
```

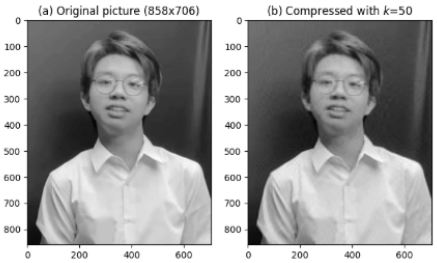
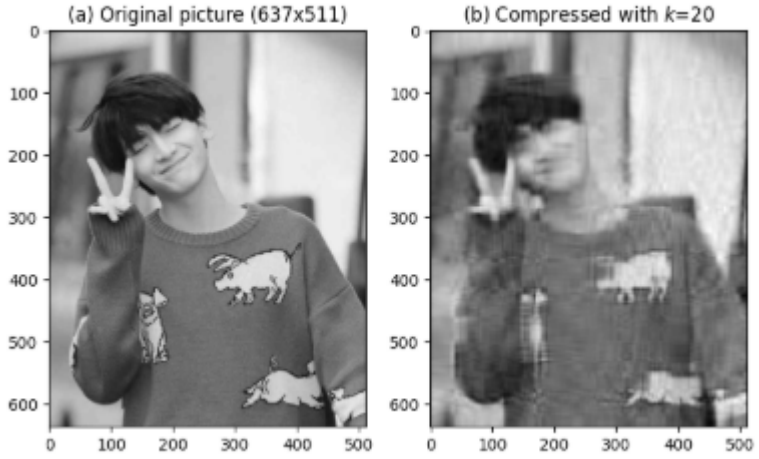


Image 1 has higher compression ratio and average relative difference value(0.13 and 0.17), which means that Image 1's singular values have slower decaying rate, meaning that there is more detail in the picture, leading to less quality retention during the compression when using the approximation of K= 50. In contrast, image 2 has lower compression ratio and average relative difference value (0.13 and 0.02), this means that image 2's singular value has a slightly faster decaying rate than image 1, which means image 2 contains less detail in the picture, leading to better quality retention after the compression.

3. Which part of the pictures have the largest difference of the compressed images? And why?
    a. Compression often impacts regions having fine details the most, such as sharp edges or text; because these parts are high frequency areas and areas that require more singular values to be correctly represented. When the compression is done using low-rank approximations (fewer singular values), these high frequency regions are poorly approximated, which leads to a blurring effect, or image distortion or loss of sharpness.
    b. Compression artifacts impact sharp edges in the photograph and text on a page even more resulting in the compressed file having blurry or pixelated sections, so these are some examples of such areas with higher differences. Because of their high frequency nature, text is especially affected by compression artifacts.
    c. As for a blue sky or properly colored wall, they maintain their shape well despite compression as well as smooth or uniform regions due to their low frequency components which tend to dissipate quickly meaning quality is preserved even with fewer singular values used in the parts.
    d. As an example we can look at these 2 pictures, where one is the original and the other is the one that has been approximated with K= 20.



We can see that the simple shapes and textures like the wall, window and shirt maintained its overall shape without being blurred. But for more complex shapes and textures like the shirt pattern and his face, they are blurred and pixelated due to poor approximation(as these are complex patterns and shapes need more singular values). Therefore, due to the poor approximation by the low-rank of 20, the compressed image has a very pixelated quality.