Projecy 7 Exact inference in graphical models

Team K - Minghang Li

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Problem 17 and 18 are handwritten (not energetic enough for formatting this time as I usually do) and are included using `pdfpages`. Thye don't fit under the title page so I put problem 19 at the beginning.

Problem 19: Message passing on a chain

(a) Store clique potentials in an R object

```
# use the hint!
clique_potentials <- array(dim = c(2, 2, 4),
                  dimnames = list(
                   c("0", "1"), c("0", "1"),
                   c("Psi12", "Psi23", "Psi34", "Psi45")))
```

```
X1 <- c(1/3, 2/3)
X2 <- c(4/5, 2/3)
X3 <- c(5/7, 1/3)
X4 <- c(3/5, 2/5)
X5 <- c(1/2, 7/9)
```

```
clique_potentials[ , 1, "Psi12"] <- (1 - X1) * (1 - X2)
clique_potentials[ , 2, "Psi12"] <- (1 - X1) * X2
clique_potentials[ ,  , "Psi23"] <- c(1-X3, X3)
clique_potentials[ ,  , "Psi34"] <- c(1-X4, X4)
clique_potentials[ ,  , "Psi45"] <- c(1-X5, X5)
clique_potentials
```

```
## , , Psi12
##
##         0         1
## 0 0.1333333 0.5333333
## 1 0.1111111 0.2222222
##
## , , Psi23
##
##         0         1
## 0 0.2857143 0.7142857
## 1 0.6666667 0.3333333
```

```
##
## , , Psi34
##
##     0   1
## 0 0.4 0.6
## 1 0.6 0.4
##
## , , Psi45
##
##           0         1
## 0 0.5000000 0.5000000
## 1 0.2222222 0.7777778
```

(b) Computing forward messages

```
fwd_msg <- array(dim=c(5, 2),
                 dimname=list(
                   c("X1", "X2", "X3", "X4", "X5"),
                   c("0", "1")
                 ))
fwd_msg["X1", ] <- c(1, 1)

foreach(i=2:5) %do% {
  fwd_msg[i, ] <- fwd_msg[i-1, ] %*% clique_potentials[, , i-1]
}
```

```
fwd_msg
```

```
##            0         1
## X1 1.0000000 1.0000000
## X2 0.2444444 0.7555556
## X3 0.5735450 0.4264550
## X4 0.4852910 0.5147090
## X5 0.3570253 0.6429747
```

(c) Computing backward messages

Initialize in such shape for easy computation :)

```
bkwd_msg <- array(dim=c(2, 5),
                  dimname=list(
                    c("0", "1"),
                    c("X1", "X2", "X3", "X4", "X5")
                  ))
bkwd_msg[, "X5"] <- c(1, 1)
foreach (i=4:1) %do% {
  bkwd_msg[, i] <- clique_potentials[, , i] %*% bkwd_msg[, i+1]
}
```

```
bkwd_msg
```

```
##          X1 X2 X3 X4 X5
## 0 0.6666667  1  1  1  1
## 1 0.3333333  1  1  1  1
```

(d) Compute the marginal probability distribution for eahc node

```
marginal <- array(0, dim=c(5,2),
                  dimname=list(
                    c("P(X1)", "P(X2)", "P(X3)", "P(X4)", "P(X5)"),
                    c("0", "1")))
foreach (i=1:5) %do% {
  marginal[i, ] <- fwd_msg[i, ] * bkwd_msg[, i]
}
Z <- rowSums(marginal)
marginal <- marginal / Z
```

```
marginal
```

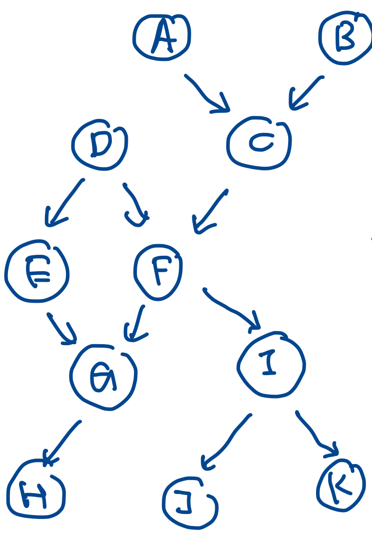
```
##               0         1
## P(X1) 0.6666667 0.3333333
## P(X2) 0.2444444 0.7555556
## P(X3) 0.5735450 0.4264550
## P(X4) 0.4852910 0.5147090
## P(X5) 0.3570253 0.6429747
```

```
## [1] "The normalising constant Z is 1"
```

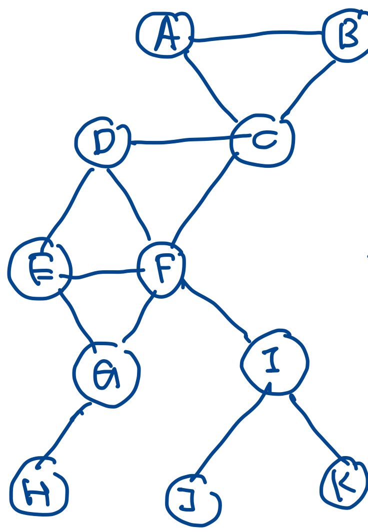
```
## P(X1) P(X2) P(X3) P(X4) P(X5)
##     1     1     1     1     1
```

SMCB Project 7.

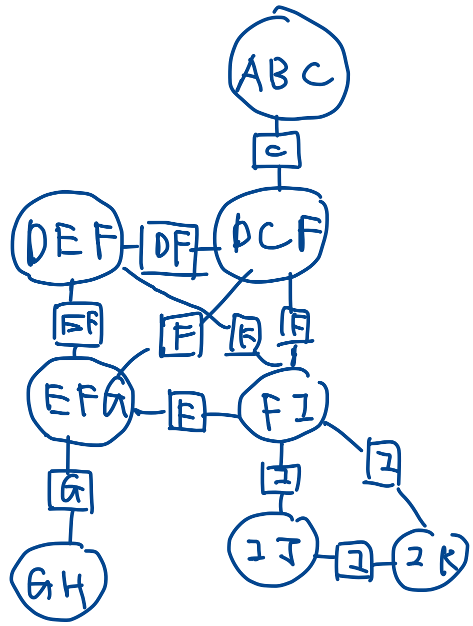
Problem 17. Junction tree



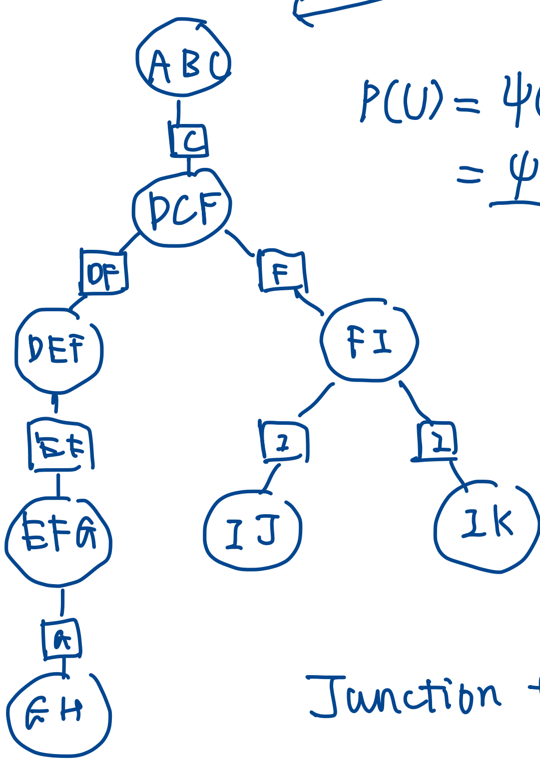
DAG



Moral graph



Junction graph.



$$P(U) = \psi(U)$$

$$= \frac{\psi(ABC)\psi(DCF)\psi(DEF)\psi(EFG)\psi(GH)\psi(FI)\psi(IJ)\psi(IK)}{\psi(C)\psi(DF)\psi(F)\psi(EF)\psi(G)\psi(I)^2}$$

Junction tree

Problem 18

(a). forward message:

$$\mu_\alpha(x_n) = \sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n)\,\mu_\alpha(x_{n-1})$$

with initialization

$$\mu_\alpha(x_1) = 1 \quad \left(\text{so that } \mu_\alpha(x_2) = \sum_{x_1} \psi_{1,2}(x_1, x_2) \cdots\right)$$

backward message

$$\mu_\beta(x_n) = \sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1})\,\mu_\beta(x_{n+1})$$

with initialization

$$\mu_\beta(x_5) = 1 \quad \left(\text{so that } \mu_\beta(x_4) = \sum_{x_5} \psi_{x_4,x_5}(x_4, x_5) \ldots\right)$$

(b). $O(NK^2)$

Each node K values. Forward $(n-1)K^2$, backward $(N-n)K^2$.

Total: $(N-1)K^2 \rightarrow O(NK^2)$

(c). If values re stored, after doing one message passing in forward direction and one message passing in backward direction, we can directly "read off" all probabilities. The total calculation is $2(N-1)K^2$. or $O(NK^2)$

In the general case, it will be $N(N-1)K^2$ or $O(N^2K^2)$