Evolutionary Dynamics Homework 8

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Problem 1: Weak selection

Consider a population of size N engaged in a two-player evolutionary game. The population consists of two types of individuals, A and B, with frequencies x_A and $x_B = 1 - x_A$, respectively. The fitness of the two types depends on the payoffs from the game and is given by the payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

The average payoffs (fitness) of individuals of type A and B, denoted by f_A and f_B, respectively, are given by:

$$\begin{aligned} f_A &= ax_A + b(1 - x_A) = ax_A + bx_B \\ f_B &= cx_A + d(1 - x_A) = cx_A + dx_B \end{aligned}$$

(a) Write down the replicator equation for the change in the frequency x_A of type A in the population overtime.

Consider a population of size N, then under the current settings we have $x_A N$ individuals of type A and $(1 - x_A)N$ individuals of type B.

(b) Now assume weak selection, where the fitness is given by $1 + \delta f_i$, where f_i is the payoff for type i and δ is a small selection strength parameter. Linearize the replicator equation to first order in δ.

Let $x := (x_A, x_B)$. We reenote the fitness of type A as F_A

$$\dot{x}_A = x_A[F_A - \phi]$$

(c) Determine the condition for evolutionary stability (ESS) of type A. What condition must the payoff parameters a, b, c, d satisfy for A to be evolutionary stable? How does this depend on the choice of δ?

(d) Analyze condition for evolutionary stability.

Suppose A and B represent two strategies in the classic Hawk-Dove ganme, with the follwoing payoff matrix:

	Hawk	Dove
Hawk	$\frac{V-C}{2}$	V
Dove	0	$\frac{V}{2}$

where V is the value of the contested resource and C is the cost of fighting. Analyze the condition for evolutionary stability under weak selection for this game.

Problem 2: Strong selection

Consider the two-strategy game

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

(a) In an *infinite* population with replicator dynamics, decide for all games of this type whether strategies A and B are dominant, coexisting or bi-stable, based on the two varaibles $\xi = a - c$ and $\zeta = d - b$.

Regurgitating lecture slides:

- $\xi > 0, \zeta > 0$: A and B are bi-stable ($a > c$, $d > b$)
- $\xi < 0, \zeta < 0$: A and B are coexisting ($a < c$, $d < b$)
- $\xi > 0, \zeta < 0$: A is dominant ($a > c$, $d < b$)
- $\xi < 0, \zeta > 0$: B is dominant ($a < c$, $d > b$)
- $\xi = 0, \zeta = 0$: A and B are neutral ($a = c$, $d = b$)

(Problem (2) continued)

Now consider a population of *finite size* N that evolves according to an unstructured Moran process. Suppose the fitness of A and B are given respectively by

$$f_i = \frac{a(i-1) + b(N-i)}{N-1}$$
$$g_i = \frac{ci + d(N-i-1)}{N-1}.$$

Note that this corresponds to limit of strong selection $w = 1$, as compared to the lecture. We want to classify the evolutionary stability of A and B as a function of the population size N and the payoff values a, b, c, and d. To this end, we analyze the difference in fitness $h_i = f_i - g_i$.

(b) Analyze the relationship between h_i, ξ, and ζ.

Show that

$$h_i = \xi' \frac{i}{N-1} - \zeta' \frac{N-i}{N-1}$$

with

$$\xi' = \xi - \frac{a-d}{n} \quad \text{and} \quad \zeta' = \zeta + \frac{a-d}{N}.$$

Proof.

$$
\begin{aligned}
h_i &= f_i - g_i \\
&= \frac{a(i-1)+b(N-i)}{N-1} - \frac{ci + d(N-i-1)}{N-1} \\
&= \frac{ai - a + bN - b - ci - dN + di + d}{N-1} \\
&= \frac{(a-c)i - (d-b)(N-i) - (a-d)}{N-1} \\
&= \frac{(a-c)i - (d-b)(N-1) - (a-d)\cdot(N-i+i)/N}{N-1} \\
&= \frac{i}{N-1}\cdot\left((a-c) - \frac{a-d}{N}\right) - \frac{N-i}{N-1}\cdot\left((d-b) + \frac{a-d}{N}\right) \\
&= \xi' \frac{i}{N-1} - \zeta' \frac{N-i}{N-1}
\end{aligned}
$$

□

(c) Show that for $\xi' > 0 > \zeta'$, strategy A is dominant. Derive a criterion for the dominance of B.

Given $\xi' > 0 > \zeta'$, we have $h_i > 0$ since it becomes the sum of two positive numbers, which means that A is dominant by the definition of h_i ($f_i > g_i$).

For B to be dominant, we must have $f_i < g_i$ i.e. $h_i < 0$. This means that $\xi' < 0$ and $\zeta' > 0$.

Problem 3: Hawk-Dove game example

Consider an (infinite) population where individuals can adopt one of two strategies: Hawk (H) or Dove (D). They payoff matrix for interactions between individuals using these strategies is given by:

	Hawk	Dove
Hawk	$\frac{V-C}{2}$	V
Dove	0	$\frac{V}{2}$

where V is the value of the resource $V > 0$ and C is the cost of the conflict. Assume $V < C$. Which (if any) of the following statements is true regarding the conditions for an Evolutionarily Stable Strategy (ESS) in this game? Please provide a proper rationale for your answer.

A) Strategy Hawk (H) is an ESS if $(V-C)/2 > 0$ and $V \geq V/2$

B) Strategy Dove (D) is an ESS if $V/2 > 0$ and $0 \geq V$

C) Strategy Hawk (H) is an ESS if $(V-C)/2 > V$ and $V \geq V/2$

D) Strategy Dove (D) is an ESS if $V/2 > 0$ and $0 \geq (V-C)/2$

Regurgitating the lecture slides:

S_k is an ESS if $\forall i \neq k$, either

- $a_{kk} > a_{ik}$, or
- $a_{kk} = a_{ik}$ and $a_{ki} > a_{ii}$

In the Hawk-Dove game settings, if we want strategy H to be an ESS, we must have

- $(V-C)/2 > V \to V < -C$

- $(V - C)/2 = V \to V = -C$ and $V > V/2$ (trivially true)

We can see that when $C > V > 0$, neither of the conditions are satisfied, so strategy H is never an ESS.

For strategy D to be an ESS, we must have

- $V/2 > 0$, or
- $V/2 = 0 \to V = 0$ and $(V - C)/2 < 0$.

We can see that when $C > V > 0$, the first condition is satisfied, so strategy D is an ESS.

Following the retionale, the correct answer is $\boxed{\text{D}}$.