ETH zürich

D BSSE

Computational Biology Group

Evolutionary Dynamics

Exercise 7

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Problem 1: Lotka-Volterra equation

The Lotka-Volterra equation is a famous example of theoretical ecology. Originally, it describes the dynamics of prey fish and predators. Let x denote the abundance of prey and y the number of predators. The dynamics is then given by

$$\begin{aligned} \dot{x} &= x(a-by) \\ \dot{y} &= y(-c+dx) \end{aligned} \tag{1}$$

with positive coefficients a, b, c, and d.

(a) What are the fixed points (x^*, y^*) of this system? **(1 point)**

(b) Use a linear stability analysis to determine the nature of the non-trivial fixed point. Describe the resulting dynamics qualitatively.
Consider the following steps:
Calculate the Jacobian of the right-hand-side of (1) and evaluate your expression at the fixed point (x^*, y^*). Then compute its eigenvalues. The real part of the eigenvalues determines whether the fixed point is attractive, whereas the imaginary part indicates oscillatory behaviour. **(2 points)**

(c) Now consider the general Lotka-Volterra equation for n species y_i with real coefficients r_i, b_{ij}:

$$\dot{y}_i = y_i\Big(r_i + \sum_{j=1}^{n} b_{ij} y_j\Big). \tag{2}$$

Show that (2) can be derived from a replicator equation with $n+1$ strategies x_i. **(2 points)**

Problem 2: Reactive strategies

Consider the Prisoner's Dilemma game. Imagine the game is played iteratively, and in each round the players choose a strategy based on the move of the opponent in the previous round. In particular, a *reactive strategy* $S(p,q)$ consists of the following moves: Cooperate with probability p if the opponent has cooperated in the round before; if he has defected, cooperate with probability q. The probabilities of defecting are then given by $1-p$, if the opponent has cooperated, and $1-q$ if he has defected. If both players have reactive strategies $S_1(p_1,q_1)$ and $S_2(p_2,q_2)$, the resulting dynamics are described by a Markov process, because in each round the new strategies are chosen in a probabilistic way based on the strategies in the previous round. The state-space of this Markov Chain is $\{CC, CD, DC, DD\}$. Here CD denotes that player one cooperates and player two defects. The transition matrix of the Markov chain is given by:

$$M = \begin{array}{c} \\ CC \\ CD \\ DC \\ DD \end{array} \begin{array}{c} \begin{array}{cccc} CC & CD & DC & DD \end{array} \\ \begin{pmatrix} p_1 p_2 & p_1(1-p_2) & (1-p_1)p_2 & (1-p_1)(1-p_2) \\ q_1 p_2 & q_1(1-p_2) & (1-q_1)p_2 & (1-q_1)(1-p_2) \\ p_1 q_2 & p_1(1-q_2) & (1-p_1)q_2 & (1-p_1)(1-q_2) \\ q_1 q_2 & q_1(1-q_2) & (1-q_1)q_2 & (1-q_1)(1-q_2) \end{pmatrix} \end{array}.$$

(a) Show that M is a stochastic matrix. **(2 points)**

(b) Because M is regular, there exists a unique stationary distribution x. Define $r_1 = p_1 - q_1$, $r_2 = p_2 - q_2$, and set

$$s_1 = \frac{q_2 r_1 + q_1}{1 - r_1 r_2}, \qquad \text{and} \quad s_2 = \frac{q_1 r_2 + q_2}{1 - r_1 r_2},$$

and let

$$x = (s_1 s_2, s_1(1-s_2), (1-s_1)s_2, (1-s_1)(1-s_2)).$$

Show that x is the stationary distribution to the Markov chain with transition matrix M. *Note: It will be sufficient to show that the first component of x solves $x_1 = \sum_j x_j M_{j1}$; the other components follow by an analogous calculation which you don't need to do.* **(2 points)**

(c) Suppose player one plays the strategy $S_1(1,0)$, against an arbitrary reactive strategy $S_2(p_2, q_2)$. What is the name of strategy $S_1(1,0)$? Show that the *long run* expected payoff for the first player is always identical to the opponent's payoff. **(1 point)**