Evolutionary Dynamics

Exercise 6

Prof. Dr. Niko Beerenwinkel
Johannes Gawron
Michael Schneider
Norio Zimmermann

31th October 2024

Problem 1: One-dimensional Fokker-Planck equation
Consider the one-dimensional Fokker-Planck equation with constant coefficients,

$$\partial_t \psi(p,t) = -m\partial_p \psi(p,t) + \frac{v}{2}\partial_p^2 \psi(p,t), \tag{1}$$

with $p \in \mathbb{R}$, $v > 0$.

(a) Show that for vanishing selection, $m = 0$, **(1 point)**

$$\psi(p,t) = \frac{1}{\sqrt{2\pi vt}} \exp\left(-\frac{p^2}{2vt}\right) \tag{2}$$

solves the Fokker-Planck equation. To which initial condition does this solution correspond?

(b) Construct a solution for constant selection, $m \neq 0$, by substituting $z = p - mt$ for p in (1). What is the mean and variance? **(2 points)**

Problem 2: Diffusion approximation of the Moran process
Derive a diffusion approximation for the Moran process of two species. Assume the first species has a small selective advantage s.

(a) The general definition for the drift coefficient is

$$M(p) = \mathrm{E}[X(t) - X(t-1) \mid X(t-1) = i]/N,$$

where $p = i/N$ and $X(t)$ denotes the abundance of the first allele. Evaluate this expression for the Moran process with selection. Show that this yields the result for the Wright-Fisher process from the lecture, divided by N. **(1 point)**

(b) By a similar argument calculate the diffusion coefficient $V(p)$. **(1 point)**

(c) Use your results from (a) and (b) in the forward Kolmogorov equation to present a diffusion equation for the Moran model. **(1 point)**

(d) Now assume that $s \ll 1$. Approximate your results from (a) and (b) and use the general expression for the fixation probability $\rho(p_0)$ to show that the fixation probability is given by: **(1 point)**

$$\rho(p_0 = 1/N) = \frac{1 - e^{-s}}{1 - e^{-Ns}}. \tag{3}$$

(e) Take the limit to derive a result for the fixation probability of a neutral allele, $s = 0$. Evaluate (3) for $N = 10$ and $N = 1000$ for both positive, $s = 2\%$, and negative selection, $s = -2\%$, respectively. Compare your results with ρ_1 of the exact Moran process. **(1 point)**

Problem 3: Absorption time in the diffusion approximation

In the diffusion approximation of a process with absorbing states 0 and 1 the expected fixation time, conditioned on absorption in state 1, reads:

$$\tau_1(p_0) = 2(S(1)-S(0))\left(\int_{p_0}^{1}\frac{\rho(p)(1-\rho(p))}{e^{-A(p)}V(p)}\mathrm{d}p + \frac{1-\rho(p_0)}{\rho(p_0)}\int_0^{p_0}\frac{\rho(p)^2}{e^{-A(p)}V(p)}\mathrm{d}p\right),$$

where $\rho(p)$ denotes the fixation probability, $A(p) = \int_0^p 2M(p)/V(p)\mathrm{d}p$, and $S(p) = \int_0^p \exp(-A(p))\mathrm{d}p$.

(a) Calculate the *conditional expected waiting time for fixation*, $\tau_1(p_0)$, of an allele of frequency p_0 in the neutral Wright-Fisher process. Approximate the result for small p_0. **(1 point)**

(b) Compute τ_0, the conditional expected waiting time until *extinction* (absorption in state 0) in the neutral Wright-Fisher process. Also derive the unconditioned expected waiting time $\bar{\tau}$ until *either fixation or extinction*. **(1 point)**