Evolutionary Dynamics Homework 6

Minghang Li

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Problem 1: One-dimensional Fokker-Planck equation

Consider the one-dimensional Fokker-Planck equation with constant coefficients,

$$\partial_t \psi(p,t) = -m\partial_p \psi(p,t) + \frac{v}{2}\partial_p^2 \psi(p,t), \tag{1}$$

with $p \in \mathbb{R}, v > 0$.

(a) See below.

Show that for vanishing selection, $m = 0$,

$$\psi(p,t) = \frac{1}{\sqrt{2\pi vt}} \exp\left(-\frac{p^2}{2vt}\right) \tag{2}$$

solves the Fokker-Planck equation. To which initial condition does this solution correspond?

For vanishing selection $m = 0$, the Fokker-Planck equation can be written as simply

$$\partial_t \psi(p,t) = \frac{v}{2}\partial_p^2 \psi(p,t) \tag{3}$$

Let's compute the left-hand side and the right-hand side to show that the solution is correct.

LHS of (3)

$$\begin{aligned}
\partial_t \psi(p,t) &= \partial_t \left(\frac{1}{\sqrt{2\pi vt}} \exp\left(-\frac{p^2}{2vt}\right)\right) \\
&= -\frac{1}{\sqrt{2\pi vt}\cdot 2t} \cdot \exp\left(-\frac{p^2}{2vt}\right) + \frac{1}{\sqrt{2\pi vt}} \cdot \frac{p^2}{2vt^2} \cdot \exp\left(-\frac{p^2}{2vt}\right) \\
&= \frac{p^2 - vt}{2vt^2} \cdot \frac{1}{\sqrt{2\pi vt}} \cdot \exp\left(-\frac{p^2}{2vt}\right)
\end{aligned} \tag{4}$$

RHS of (3)

$$
\begin{aligned}
\frac{v}{2}\partial_p^2\psi(p,t) &= \frac{v}{2}\partial_p\left(\partial_p\left(\frac{1}{\sqrt{2\pi vt}}\exp\left(-\frac{p^2}{2vt}\right)\right)\right) \\
&= \frac{v}{2\sqrt{2\pi vt}}\partial_p\left(\partial_p\left(\exp\left(-\frac{p^2}{2vt}\right)\right)\right) \\
&= \frac{v}{2\sqrt{2\pi vt}}\partial_p\left(\left(-\frac{p}{vt}\right)\cdot\exp\left(-\frac{p^2}{2vt}\right)\right) \\
&= \frac{v}{2\sqrt{2\pi vt}}\cdot\left(-\frac{1}{vt}\exp\left(-\frac{p^2}{2vt}\right)+\frac{p^2}{v^2t^2}\cdot\exp\left(-\frac{p^2}{2vt}\right)\right) \\
&= -\frac{1}{\sqrt{2\pi vt}\cdot 2t}\cdot\exp\left(-\frac{p^2}{2vt}\right)+\frac{1}{\sqrt{2\pi vt}}\cdot\frac{p^2}{2vt^2}\cdot\exp\left(-\frac{p^2}{2vt}\right) \\
&= \frac{p^2-vt}{2vt^2}\cdot\frac{1}{\sqrt{2\pi vt}}\cdot\exp\left(-\frac{p^2}{2vt}\right)
\end{aligned}
\tag{5}
$$

It's clear that (4) equals (5), so (2) is indeed the solution.

To investigate the initial condition, we need to examine the behavior of $\psi(p,t)$ as $t \to 0^+$. As $t \to 0^+$, we have:

- The exponential term $\exp(-p^2/(2vt))$ approaches:
  - $0, \forall p \neq 0$ becuase $-p^2/(2vt) \to -\infty$
  - 1 for $p = 0$.
- The prefactor $\frac{1}{\sqrt{2\pi vt}}$ approaches ∞ for $t \to 0^+$.

This behavior is consistent with the Dirac delta function $\delta(p)$. Hence, the initial condition is

$$\psi(p,0) = \delta(p),$$

where $\delta(p)$ is the Dirac delta function defined as

$$\delta(p) = \begin{cases} 0, & \text{if } p \neq 0, \\ \infty, & \text{if } p = 0. \end{cases}$$

This means the initial distribution is concentrated at $p = 0$, which represents the fact that the starting population is initially at a single allele frequency.

(b) Construct a solution for constant selection, $m \neq 0$, by substituting $z = p - mt$ for p in (1) . What is the mean and variance?

Let $z(p,t) = p - mt$, then we have $\partial_p z(p,t) = 1$, and $\partial_t z(p,t) = -m$. Substitute $z(p,t)$ into $\psi(p,t)$ to obtain $\psi(z,t)$, using the chain rule, we have

$$
\begin{aligned}
\frac{\partial\psi(p,t)}{\partial t} &= \frac{\partial\psi(z,t)}{\partial z}\frac{\partial z}{\partial t}+\frac{\partial\psi(z,t)}{\partial t}\frac{\cancel{\partial t}}{\cancel{\partial t}} \\
&= -m\frac{\partial\psi(z,t)}{\partial z}+\frac{\partial\psi(z,t)}{\partial t}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\frac{\partial\psi(p,t)}{\partial p} &= \frac{\partial\psi(z,t)}{\partial z}\frac{\partial z}{\partial p} \\
&= \frac{\partial\psi(z,t)}{\partial z}
\end{aligned}
\tag{7}
$$

Using (6) and (7), we can rewrite the Fokker-Planck equation:

$$
\begin{aligned}
\frac{\partial\psi(p,t)}{\partial t} &= -m\frac{\partial\psi(p,t)}{\partial p} + \frac{v}{2}\frac{\partial^2\psi(p,t)}{\partial p^2} \\
\cancel{-m\frac{\partial\psi(z,t)}{\partial z}} + \frac{\partial\psi(z,t)}{\partial t} &= \cancel{-m\frac{\partial\psi(z,t)}{\partial z}} + \frac{v}{2}\frac{\partial^2\psi(z,t)}{\partial z^2} \\
\frac{\partial\psi(z,t)}{\partial t} &= \frac{v}{2}\frac{\partial^2\psi(z,t)}{\partial z^2}
\end{aligned}
\tag{8}
$$

Voila! This equation (8) has exactly the same form as

Problem 2: Diffusion approximation of the Moran process

DeriveadiffusionapproximationfortheMoranprocessoftwospecies.Assumethefirstspecieshasa smallse- lectiveadvantages. (a) Thegeneraldefinitionforthedriftcoefficientis M(p)=E[X(t)−X(t−1)|X(t−1)=i]/N, wherep=i/NandX(t)denotestheabundanceofthefirstallele.Evaluatethisexpressionforthe Moranprocesswithselection.Showtha Fisherprocessfrom thelecture,dividedbyN. (1point) (b)ByasimilarargumentcalculatethediffusioncoefficientV(p). (1point) (c)Useyour results from(a) and(b) in the forwardKolmogorovequationtopresent adiffusion equa- tionfortheMoranmodel. (1point) (d)Nowassumethats≪1.Approximateyourresultsfrom(a)and(b)andusethegeneralexpression forthefixationprobabilityρ(p0)toshowthatthefixationprobabilityisgivenby: (1point) ρ(p0=1/N)= 1−e−s 1−e−Ns . (3) (e) Takethelimittoderivearesultforthefixationprobabilityofaneutralallele,s=0.Evaluate(3) forN=10andN=1000forbothpositive,s=2%,andnegativeselection,s=−2%,respectively. Compareyourre- sultswithρ1oftheexactMoranprocess. (1point) 1 Problem3:Absorptiontimeinthediffusionapproximation Inthediffusionapproximationofaprocesswithabsorbingstates0and1theexpectedfixationtime, conditione- donabsorptioninstate1,reads: τ1(p0)=2(S(1)−S(0)) 1 p0 ρ(p)(1−ρ(p)) e−A(p)V(p) dp+1−ρ(p0) ρ(p0) p0 0 ρ(p)2 e−A(p)V(p)dp , whereρ(p)denotesthefixationprobability,A(p)= p 0 2M(p)/V(p)dp,andS(p)= p 0 exp(−A(p))dp. (a)Calculatetheconditionalexpectedwaitingtimeforfixation,τ1(p0),ofanalleleoffrequencyp0 intheneutralWright-Fisherprocess.Approximatetheresultforsmallp0. (1point) (b)Computeτ0, thecondi- tionalexpectedwaitingtimeuntilextinction(absorptioninstate0) inthe neutralWright-Fisherprocess.Alsoderivetheuncondition τuntileither fixationorextinction