Evolutionary Dynamics Homework 8

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The author was having a fever while composing the homework this week. Might have written nonsense.

Problem 1: Weak selection

Consider a population of size N engaged in a two-player evolutionary game. The population consists of two types of individuals, A and B, with frequencies x_A and $x_B = 1 - x_A$, respectively. The fitness of the two types depends on the payoffs from the game and is given by the payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

The average payoffs (fitness) of individuals of type A and B, denoted by f_A and f_B, respectively, are given by:

$$\begin{aligned} f_A &= ax_A + b(1 - x_A) = ax_A + bx_B \\ f_B &= cx_A + d(1 - x_A) = cx_A + dx_B \end{aligned}$$

(a) Write down the replicator equation for the change in the frequency x_A of type A in the population overtime.

Let $x := (x_A, x_B)$, we have

$$\begin{aligned} \frac{dx_A}{dt} &= x_A \left(f_A(x) - \phi(x) \right) \\ &= x_A \left(f_A(x) - x_A f_A(x) - (1 - x_A) f_B(x) \right) \\ &= x_A (1 - x_A) \left(f_A(x) - f_B(x) \right) \\ &= x_A (1 - x_A) \left(ax_A + b(1 - x_A) - cx_A - d(1 - x_A) \right) \\ &= x_A (1 - x_A) \left((a - b - c + d) x_A + (b - d) \right) \end{aligned}$$

(b) Now assume weak selection, where the fitness is given by $1 + \delta f_i$, where f_i is the payoff for type i and δ is a small selection strength parameter. Linearize the replicator equation to first order in δ.

Let $x := (x_A, x_B)$. Denote fitness as F_i and average fitness as Φ. We have

$$
\begin{aligned}
\frac{dx_A}{dt} &= x_A[F_A(x) - \Phi(x)] \\
&= x_A\left[(1+\delta f_A(x)) - (x_A(1+\delta f_A(x)) + x_B(1+\delta f_B(x))\right] \\
&= x_A\left[(1-x_A)(1+\delta f_A(x)) - (1-x_A) - (1-x_A)\delta f_B(x)\right] \\
&= \delta x_A(1-x_A)\left[f_A(x) - f_B(x)\right] \\
&= \delta x_A(1-x_A)\left[(a-b-c+d)x_A + (b-d)\right]
\end{aligned}
$$

(c) Determine the condition for evolutionary stability (ESS) of type A. What condition must the payoff parameters a, b, c, d satisfy for A to be evolutionary stable? How does this depend on the choice of δ?

A is ESS if the following two conditions hold:

1. Selection protects against invasion by B: a single B mutant has lower fitness in a A population, i.e. $F_B < F_A$.
2. Selection protects against replacement – according to the slides it means $x^* > 1/3$.

The two conditions translate to

1. $b > d$
2. $(d-b)/(a-b-c+d) > 1/3 \rightarrow d > (a+2b-c)/2$

Seems like δ doesn't affect the condition for ESS.

(d) Analyze condition for evolutionary stability.

Suppose A and B represent two strategies in the classic Hawk-Dove ganme, with the following payoff matrix:

	Hawk	Dove
Hawk	$\frac{V-C}{2}$	V
Dove	0	$\frac{V}{2}$

where V is the value of the contested resource and C is the cost of fighting. Analyze the condition for evolutionary stability under weak selection for this game.

Apply what we derived in the previous question, for Hawk to be ESS, we must have

1. $V > 0$
2. $V/2 > (V-C)/4 + V \rightarrow C > 3V$

Problem 2: Strong selection

Consider the two-strategy game

$$
\begin{array}{cc}
 & \begin{array}{cc} A & B \end{array} \\
\begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{array}
$$

(a) In an *infinite* population with replicator dynamics, decide for all games of this type whether strategies A and B are dominant, coexisting or bi-stable, based on the two varaibles $\xi = a - c$ and $\zeta = d - b$.

Regurgitating lecture slides:

- $\xi > 0, \zeta > 0$: A and B are bi-stable ($a > c$, $d > b$)
- $\xi < 0, \zeta < 0$: A and B are coexisting ($a < c$, $d < b$)
- $\xi > 0, \zeta < 0$: A is dominant ($a > c$, $d < b$)
- $\xi < 0, \zeta > 0$: B is dominant ($a < c$, $d > b$)
- $\xi = 0, \zeta = 0$: A and B are neutral ($a = c$, $d = b$)

(Problem (2) continued)

Now consider a population of *finite size* N that evolves according to an unstructured Moran process. Suppose the fitness of A and B are given respectively by

$$f_i = \frac{a(i-1) + b(N-i)}{N-1}$$
$$g_i = \frac{ci + d(N-i-1)}{N-1}.$$

Note that this corresponds to limit of strong selection $w = 1$, as compared to the lecture. We want to classify the evolutionary stability of A and B as a function of the population size N and the payoff values a, b, c, and d. To this end, we analyze the difference in fitness $h_i = f_i - g_i$.

(b) Analyze the relationship between h_i, ξ, and ζ.

Show that

$$h_i = \xi' \frac{i}{N-1} - \zeta' \frac{N-i}{N-1}$$

with

$$\xi' = \xi - \frac{a-d}{n} \quad \text{and} \quad \zeta' = \zeta + \frac{a-d}{N}.$$

Proof.

$$\begin{aligned}
h_i &= f_i - g_i \\
&= \frac{a(i-1) + b(N-i)}{N-1} - \frac{ci + d(N-i-1)}{N-1} \\
&= \frac{ai - a + bN - b - ci - dN + di + d}{N-1} \\
&= \frac{(a-c)i - (d-b)(N-i) - (a-d)}{N-1} \\
&= \frac{(a-c)i - (d-b)(N-1) - (a-d) \cdot (N-i+i)/N}{N-1} \\
&= \frac{i}{N-1} \cdot \left((a-c) - \frac{a-d}{N}\right) - \frac{N-i}{N-1} \cdot \left((d-b) + \frac{a-d}{N}\right) \\
&= \xi' \frac{i}{N-1} - \zeta' \frac{N-i}{N-1}
\end{aligned}$$

□

(c) Show that for $\xi' > 0 > \zeta'$, strategy A is dominant. Derive a criterion for the dominance of B.

Given $\xi' > 0 > \zeta'$, we have $h_i > 0$ since it becomes the sum of two positive numbers, which means that A is dominant by the definition of h_i ($f_i > g_i$).

For B to be dominant, we must have $f_i < g_i$ i.e. $h_i < 0$. This means that $\xi' < 0$ and $\zeta' > 0$.

Problem 3: Hawk-Dove game example

Consider an (infinite) population where individuals can adopt one of two strategies: Hawk (H) or Dove (D). They payoff matrix for interactions between individuals using these strategies is given by:

	Hawk	Dove
Hawk	$\frac{V-C}{2}$	V
Dove	0	$\frac{V}{2}$

where V is the value of the resource $V > 0$ and C is the cost of the conflict. Assume $V < C$. Which (if any) of the following statements is true regarding the conditions for an Evolutionarily Stable Strategy (ESS) in this game? Please provide a proper rationale for your answer.

A) Strategy Hawk (H) is an ESS if $(V - C)/2 > 0$ and $V \geq V/2$

B) Strategy Dove (D) is an ESS if $V/2 > 0$ and $0 \geq V$

C) Strategy Hawk (H) is an ESS if $(V - C)/2 > V$ and $V \geq V/2$

D) Strategy Dove (D) is an ESS if $V/2 > 0$ and $0 \geq (V - C)/2$

Regurgitating the lecture slides:

S_k is an ESS if $\forall i \neq k$, either

- $a_{kk} > a_{ik}$, or
- $a_{kk} = a_{ik}$ and $a_{ki} > a_{ii}$

In the Hawk-Dove game settings, if we want strategy H to be an ESS, we must have

- $(V - C)/2 > V \to V < -C$
- $(V - C)/2 = V \to V = -C$ and $V > V/2$ (trivially true)

We can see that when $C > V > 0$, neither of the conditions are satisfied, so strategy H is never an ESS.

For strategy D to be an ESS, we must have

- $V/2 > 0$, or
- $V/2 = 0 \to V = 0$ and $(V - C)/2 < 0$.

We can see that when $C > V > 0$, the first condition is satisfied, so strategy D is an ESS.

Following the retionale, the correct answer is $\boxed{\text{D}}$.