MATH564: MATHEMATICAL MODELING
Homework #1

Due on January 28 2020

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MATH PART

Problem 1

Consider the difference equation
$$x_{n+2} - 3x_{n+1} + 2x_n = 0.$$

(a). Show that the general solution to this equation is
$$x_n = A_1 + 2^n A_2$$

Now suppose that $x_0 = 10$ and $x_1 = 20$. Then A_1 and A_2 must satisfy the system of equations
$$A_1 + 2^0 A_2 = x_0 = 10,$$
$$A_1 + 2^1 A_2 = x_1 = 20.$$

(b). Solve for A_1 and A_2 and find the solution to the above *initial value problem.*

Solution

(a). The characteristic solution is
$$\lambda^2 - 3\lambda + 2 = 0$$

The characterictic roots are
$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

By the principle of linear superposition, the general solution is indeed
$$x_n = A_1 + 2^n A_2$$

(b). Solving the system of equations
$$A_1 + 2^0 A_2 = x_0 = 10,$$
$$A_1 + 2^1 A_2 = x_1 = 20.$$

gives
$$A_1 = 0, \quad A_2 = 10.$$

Therefore, the solution should be
$$x_n = 10 \cdot 2^n$$

Problem 2

Solve the following difference equations subject to the specified x values and sketch the solutions:

(a). $x_n - 5x_{n-1} + 6x_{n-2} = 0; \quad x_0 = 2, x_1 = 5.$

(c). $x_n - x_{n-2} = 0; \quad x_1 = 3, x_2 = 5.$

(e). $x_{n+2} + x_{n+1} - 2x_n = 0; \quad x_0 = 6, x_1 = 3.$

Solution

(a). The characterictic function is

$$\lambda^2 - 5\lambda + 6 = 0$$

The characterictic roots are

$$\lambda_1 = 2, \quad \lambda_2 = 3.$$

The general solution should be

$$A_1 2^n + A_2 3^n = 0.$$

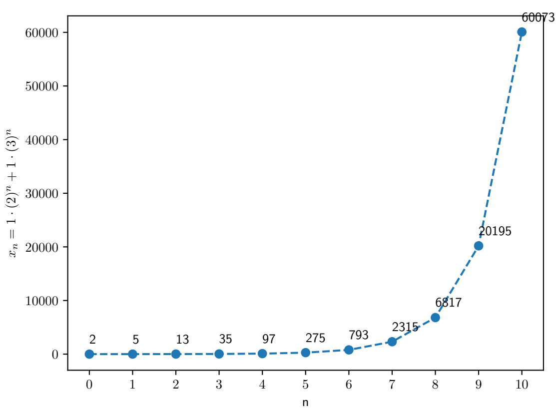
Plug in $x_0 = 2$ and $x_1 = 5$, we have

$$\begin{cases} A_1 + A_2 = 2 \\ 2A_1 + 3A_2 = 5 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ A_2 = 1 \end{cases}$$

To sum up, the specific solution should be

$$x_n = 2^n + 3^n.$$

And the plot would look like



(c). The characterictic function is

$$\lambda^2 - 1 = 0$$

The characterictic roots are

$$\lambda_1 = -1, \quad \lambda_2 = 1.$$

The general solution should be

$$A_1(-1)^n + A_2 = 0$$

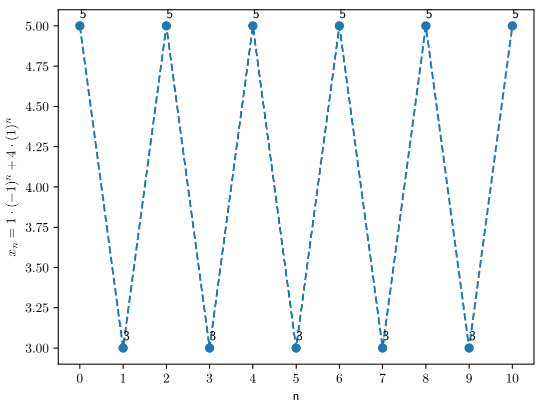
Plug in $x_1 = 3$ and $x_2 = 5$, we have

$$\begin{cases} -A_1 + A_2 = 3 \\ A_1 + A_2 = 5 \end{cases} \quad \Rightarrow \quad \begin{cases} A_1 = 1 \\ A_2 = 4 \end{cases}$$

To sum up, the specific solution should be

$$x_n = (-1)^n + 4.$$

And the plot would look like



(e). The characterictic function is

$$\lambda^2 + \lambda - 2 = 0$$

The characterictic roots are

$$\lambda_1 = 1, \quad \lambda_2 = -2.$$

The general solution should be

$$A_1 + A_2(-2)^n = 0$$

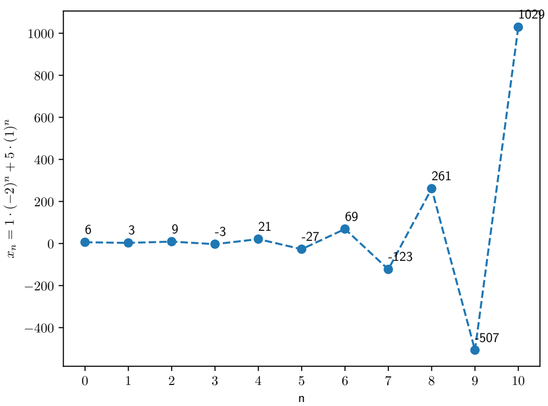
Plug in $x_0 = 6$ and $x_1 = 3$, we have

$$\begin{cases} A_1 + A_2 = 6 \\ A_1 - 2A_2 = 3 \end{cases} \Rightarrow \begin{cases} A_1 = 5 \\ A_2 = 1 \end{cases}$$

To sum up, the specific solution should be

$$x_n = (-2)^n + 5.$$

And the plot would look like



Problem 3

(a). In Section 1.3 it was shown that the general solution to equation (16a, b) is (22) provided $\lambda_1 \neq \lambda_2$. Show that if $\lambda_1 = \lambda_2 = \lambda$ then the general solution is

$$A_1\lambda^n + A_2 n\lambda^n.$$

(b). Solve and graph the solutions to each of the following equations or systems

(ii) $x_{n+2} - 2x_{n+1} + x_n = 0$,

(iii) $x_{n+1} = -3x_n - 2y_n$,
$y_{n+1} = 2x_n + y_n$.

Solution

(a). *Proof* by Mathematical Induction: We know from (16a, b) that

$$x_{n+2} = B_1 x_{n+1} + B_2 x_n$$

is true for all n, where B_1 and B_2 are some constants. Suppose that λ_0 is the repeated solution to the characteristic equation

$$\lambda^2 - B_1\lambda - B_2 = 0$$

By Vieta's formulas, we know that $B_1 = 2\lambda_0$ and $B_2 = -\lambda_0^2$.

Suppose that the general solution formula is true for all integers from 0 through k, then we have

$$\begin{aligned} x_k &= A_1\lambda_0^k + A_2 k\lambda_0^k, \\ x_{k-1} &= A_1\lambda_0^{k-1} + A_2(k-1)\lambda_0^{k-1}. \end{aligned}$$

For x_{k+1}, we have

$$\begin{aligned} x_{k+1} &= B_1 x_k + B_2 x_{k-1} \\ &= B_1(A_1\lambda^k + A_2 k\lambda^k) + B_2(A_1\lambda^{k-1} + A_2(k-1)\lambda^{k-1}) \\ &= A_1(B_1\lambda^k + B_2\lambda^{k-1}) + A_2(B_1 k\lambda^k + B_2(k-1)\lambda^{k-1}) \\ &= A_1(2\lambda^{k+1} - \lambda^{k+1}) + A_2(2k\lambda^{k+1} + (1-k)\lambda^{k+1}) \\ &= A_1\lambda^{k+1} + A_2(k+1)\lambda^{k+1} \end{aligned}$$

The truth of x_0 and x_1 is automatic since A_1 and A_2 are numbers selected intentionally to make the following equations true:

$$x_0 = A_1, \quad x_1 = (A_1 + A_2)\lambda.$$

□

Problem 3 continued on next page. . .

(b). Note: the constants A_1 and A_2 are se as 1 for both questions.

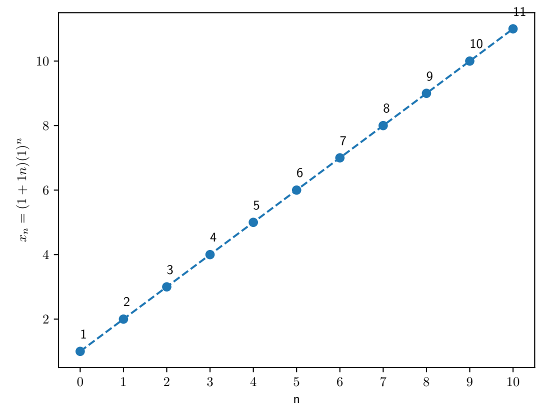
(ii) The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0.$$

So the roots should be $\lambda_1 = \lambda_2 = \lambda = 1$. The general solution would be

$$x_n = A_1 + A_2 n,$$

since the power of 1 is always 1.



(iii) The system of linear difference equation can be written in matrix form:

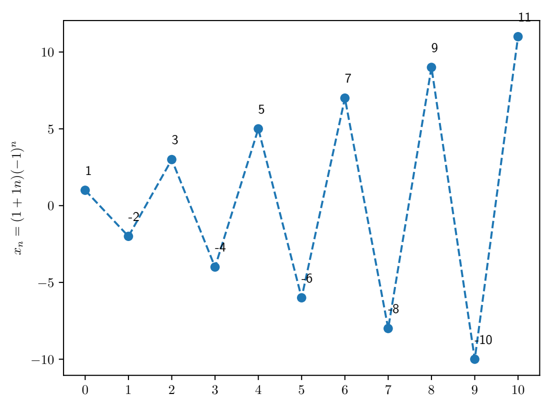
$$\begin{pmatrix} A_1\lambda^{n+1} \\ A_2\lambda^{n+1} \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} A_1\lambda^n \\ A_2\lambda^n \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix} -3-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Let determinant of the matrix of coefficients euqal to 0, it leads to

$$\mathbf{det} \begin{pmatrix} -3-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = 0 \qquad \Rightarrow \qquad \lambda^2 + 2\lambda + 1 = 0$$

So we hanve two repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda = -1$. And the general solution would be

$$x_n = (A_1 + A_2 n)(-1)^n,$$



Problem 4

In Section 1.4 we determined that there are two values λ_1 and λ_2 and two vectors $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ and $\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$ called *eigenvectors* that satisfy equation (29).

(a). Show that this equation can be written in matrix form as

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M} \begin{pmatrix} A \\ B \end{pmatrix}$$

where $\mathbf{M}$ is given by equation (27c).

(b). Show that one way of expressing the eigenvectors in terms of a_{ij} is:

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix}$$

for $a_{12} \neq 0$.

(c). Show that eigenvectors are defined only up to a multiplicative constant; i.e., if $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ, then $\alpha\mathbf{v}$ is also an eigenvector corresponding to λ for all real numbers α.

Solution

(a). Equation (29) in the book is

$$0 = A(a_{11} - \lambda) + B(a_{12})$$
$$0 = A(a_{21}) + B(a_{22} - \lambda)$$

Re-arranging the terms with λ to the left-hand side gives

$$\lambda A = a_{11}A + a_{12}B$$
$$\lambda B = a_{21}A + a_{22}B$$

which is equivalent of

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M} \begin{pmatrix} A \\ B \end{pmatrix}$$

(b). *Proof.* Suppose the eigenvalue is λ_i and the eigenvector corresponding to it is $\mathbf{v}_i = \begin{pmatrix} A_i \\ B_i \end{pmatrix}$. Then we know by the definition of eigenvector that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

Suppose $\mathbf{v}_i = \begin{pmatrix} 1 \\ \dfrac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix}$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{v}_i = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \dfrac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix} = \begin{pmatrix} \lambda_i \\ a_{21} + \dfrac{a_{22}(\lambda_i - a_{11})}{a_{12}} \end{pmatrix}$$

And

$$\lambda_i \mathbf{v}_i = \begin{pmatrix} \lambda_i \\ \dfrac{\lambda_i^2 - a_{11}\lambda_i}{a_{12}} \end{pmatrix}$$

Suppose

$$\begin{pmatrix} \lambda_i \\ a_{21} + \dfrac{a_{22}(\lambda_i - a_{11})}{a_{12}} \end{pmatrix} = \begin{pmatrix} \lambda_i \\ \dfrac{\lambda_i^2 - a_{11}\lambda_i}{a_{12}} \end{pmatrix}$$

since we have $a_{12} \neq 0$, there will be

$$\lambda_i^2 - a_{11}\lambda_i = a_{21}a_{12} + a22\lambda_i - a + 11a_{22}$$

which gives

$$\lambda_i = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

where

$$\beta = a_{11} + a_{22}, \quad \gamma = (a_{11}a_{22} - a_{21}a_{12}).$$

And that's exactly the value of λ_i.

Thus we have shown that $\mathbf{v}_i = \begin{pmatrix} 1 \\ \dfrac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix}$ satisfies $\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$, i.e. it is indeed one way of expressing the eigenvector.

$\square$

(c). *Proof.* Suppose that $\mathbf{v}$ is the eigenvector corresponding to eigenvalue λ, then it is known that

$$\mathbf{M}\mathbf{v} = \lambda \mathbf{v}$$

For $\alpha\mathbf{v}$, we know from the properties of scalar multiplication

$$\mathbf{M}\,(\alpha\mathbf{v}) = \alpha\,(\mathbf{M}\mathbf{v})$$

and

$$\lambda(\alpha\mathbf{v}) = \alpha(\lambda\mathbf{v})$$

It's clear that the two quantities are equal. Thus, by the definition of eigenvectors, $\alpha\mathbf{v}$ is also an eigenvector.

$\square$

Problem 8

The following complex numbers are expressed as $\lambda = a + bi$, where a is the real part and b is the imaginary part. Express the nubmer in polar form $\lambda = re^{i\theta}$, and use your result to compute the indicated power λ^n of this complex number. Sketch λ^n, for $n = 0, 1, 2, 3, 4$ as a function of n.

(a). $1 + i$

(b). $1 - i$

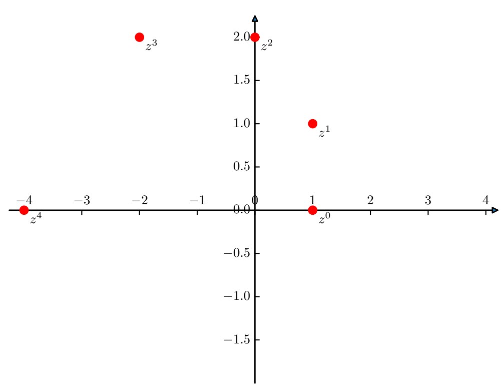
(c). $10i$

(d). $-1 + \sqrt{3}i$

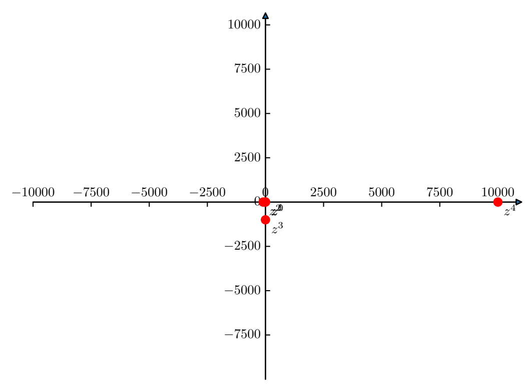
(e). $-\frac{1}{2} - \frac{i}{2}$

Solution

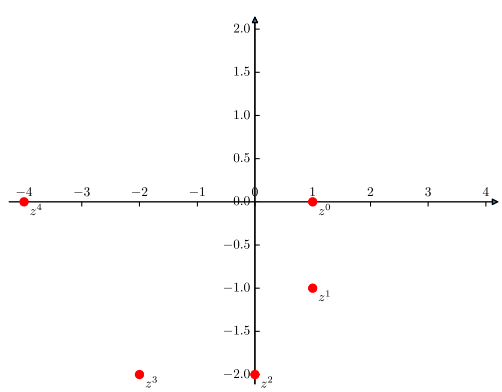
(a). **Polar form**: $\begin{aligned} z^n &= (\sqrt{1^2+1^2}e^{i\tan^{-1}(1)})^n \\ &= (\sqrt{2})^n e^{i\pi n/4} \end{aligned}$.



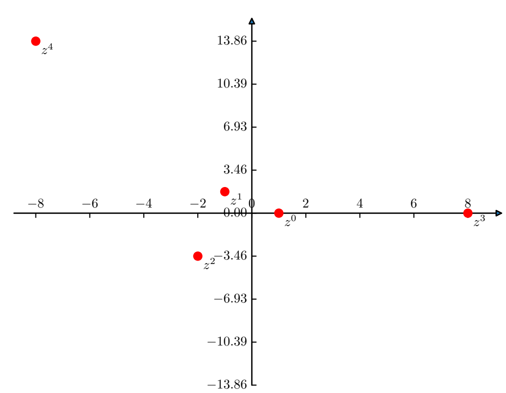
(c). **Polar form**: $\begin{aligned} z^n &= (\sqrt{0^2+10^2}e^{i\tan^{-1}(10/0)})^n \\ &= 10^n e^{i\pi n/2} \end{aligned}$.



(b). **Polar form**: $\begin{aligned} z^n &= (\sqrt{1^2+(-1)^2}e^{i\tan^{-1}(-1)})^n \\ &= (\sqrt{2})^n e^{-i\pi n/4} (\cos > 0 \& \sin < 0) \end{aligned}$

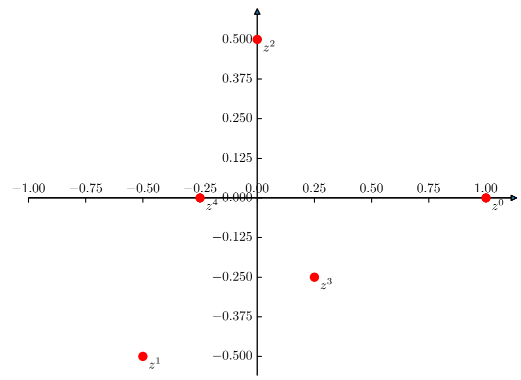


(d). **Polar form**: $\begin{aligned} z^n &= (\sqrt{(-1)^2+(\sqrt{3})^2}e^{i\tan^{-1}(-\sqrt{3})})^n \\ &= 2^n e^{2i\pi n/3} \end{aligned}$



Problem 8 continued on next page...

(e). **Polar form**:
$$\begin{aligned} z^n &= (\sqrt{(-1/2)^2 + (-1/2)^2} e^{i \tan^{-1}(1)})^n \\ &= (\frac{1}{\sqrt{2}})^n e^{i5\pi n/4} (\sin \& \cos < 0) \end{aligned}.$$



Problem 9

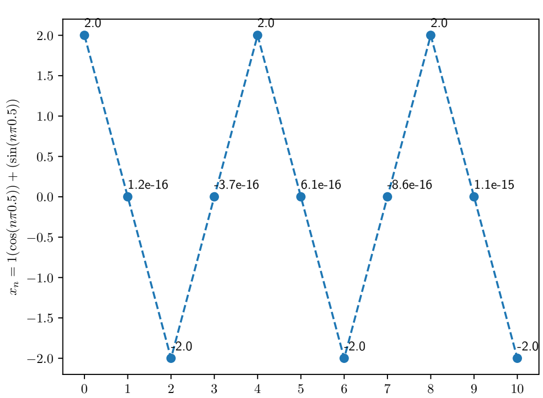
Complex eigenvalues. Solve and graph the solutions to the following difference equations.

(a). $x_{n+2} + x_n = 0$,

(b). $x_{n+2} - x_{n+1} + x_n = 0$,

Solution

(a). The characterictic equation is
$$\lambda^2 + \lambda = 0,$$
with the complex conjugate roots $\lambda = 0 \pm i$. Thus $a = 0$ and $b = 1$, so that
$$\begin{aligned} r &= \sqrt{a^2 + b^2} = 1, \\ \theta &= \tan^{-1}(1/0) = \pi/2. \end{aligned}$$
Thus the real-valued solution is
$$x_n = C_1 \cos(n\pi/2) + C_2 \sin(n\pi/2)$$



(b). The characterictic equation is

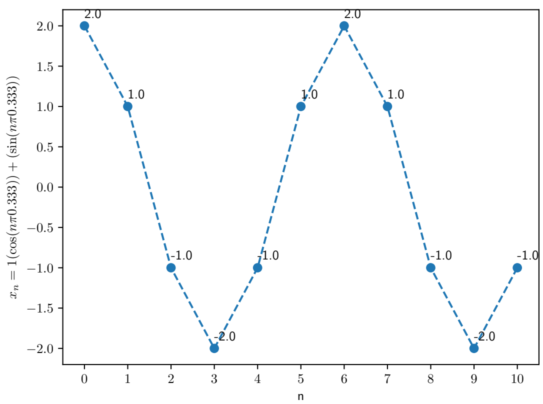
$$\lambda^2 - \lambda + 1 = 0,$$

with the complex conjugate roots $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Thus $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$, so that

$$r = \sqrt{a^2 + b^2} = 1,$$
$$\theta = \tan^{-1}(b/a) = \pi/3.$$

Thus the real-valued solution is

$$x_n = C_1 \cos(n\pi/3) + C_2 \sin(n\pi/3)$$



Problem 10

(a). Consider the growth of an aphid population described in Section 1.1. If the fractional mortality of aphids is 80% and the sex ratio (ratio of females to the total number of aphids) is 50%, what minimum fecundity f is required to prevent extinction.

(b). Establish a general condition on the fecundity of aphids to guarantee population growth given a fixed survivorship and a known sex ratio.

Solution

(a). Extinction will be prevented if the growth rate $\lambda = rf(1-m) > 1$. Solving the inequity by plugging in given values gives $f = 10$, i.e. a fecundity of 10 progeny per female.

(b). In general, we need $f \geq \frac{1}{r(1-m)}$

BIO PART

Problem 16: *Red blood cell production*

(a). We have

$$R_{n+1} = (1-f)R_n + M_n, \tag{1}$$
$$M_{n+1} = \gamma f R_n \tag{2}$$

Equation (2) can be re-written into

$$M_n = \gamma f R_{n-1} \tag{3}$$

Substitute (3) into (1), we'll get

$$R_{n+1} = (1-f)R_n + \gamma f R_{n-1}$$

(b). From (a), it's easy to see that the characteristic function is

$$\lambda^2 - (1-f)\lambda - \gamma f$$

Hence, the eigenvalues are indeed given by

$$\lambda_{1,2} = \frac{(1-f) \pm \sqrt{(1-f)^2 + 4\gamma f}}{2}$$

Since f is definitely a fraction value between 0 and 1 (it's horrible to think that your spleen removes 100% of your RBCs in circulation), and γ is larger than 0, so that there will be RBC produced.

Observe the boundary stated above, which makes the model biologically reasonable , we know

$$\sqrt{(1-f)^2 + 4\gamma f} > \sqrt{(1-f)^2} = (1-f)$$

Therefore, we should have a positive and a negative eigenvalue, i.e., $\lambda_1 > 0$ and $\lambda_2 < 0$.

(c). Suppose the positive eigenvalue $\lambda_1 = 1$, then

$$\begin{aligned}
\frac{(1-f) + \sqrt{(1-f)^2 + 4\gamma f}}{2} &= 1 \\
\sqrt{(1-f)^2} + 4\gamma f &= 1 + f \\
(1-f)^2 + 4\gamma f &= (1+f)^2 \\
4\gamma f &= 4f \\
\gamma &= 1
\end{aligned}$$

(d). From (c) we know $\gamma = 1$, so

$$\sqrt{(1-f)^2} + 4\gamma f = 1 + f$$

Then

$$\lambda_2 = -2f/2 = -f$$

And the solution

$$R_n = A\lambda_1^n + B\lambda_2^n = A + B(-f)^n$$

The solution will oscillate around constant A, and since f is a fraction the oscillation amplitude will decrease as n increases. This means that the RBC number in circulation will eventually reach an equilibrium.

Problem 17: *Annual plant propagation*

(a). The model for annual plants was condensed into a single equation (15) for p_n, the number of plants. Show that it can also be written as a single equation in S_n^1.

Solution: We have

$$p_n = \alpha s_n^1 + \beta s_n^2 \tag{4}$$
$$\overline{s_n^1} = (1-\alpha)s_n^1 \tag{5}$$
$$\overline{s_n^2} = (1-\alpha)s_n^2 \tag{6}$$
$$s_n^0 = \gamma p_n \tag{7}$$
$$s_{n+1}^1 = \sigma s_n^0 \tag{8}$$
$$s_{n+1}^2 = \sigma \overline{s_n^1} \tag{9}$$

Make several substitutions:

$$(7) \to (8) \qquad S_{n+1}^1 = \sigma\gamma p_n \tag{10}$$
$$(5) \to (9) \qquad S_n^2 = \sigma(1-\alpha)S_{n-1}^1 \tag{11}$$
$$(4) \to (10) \qquad S_{n+1}^1 = \sigma\gamma(\alpha s_n^1 + \beta s_n^2) \tag{12}$$
$$(11) \to (12) \qquad S_{n+1}^1 = \sigma\gamma(\alpha s_n^1 + \beta\sigma(1-\alpha)S_{n-1}^1) \tag{13}$$

And equation (13) should be the answer.

(b). Seeds produced this year (year n) which survived the winter and will germinate next year (year $(n+1)$)

(c). We know from the book that

$$\lambda_{1,2} = \frac{\sigma\gamma\alpha}{2}(1 \pm \sqrt{1+\delta})$$

where

$$\delta = \frac{4}{\gamma}\frac{\beta}{\alpha}\left(\frac{1}{\alpha} - 1\right)$$

We know that δ is a positive quantity since $\alpha < 1$. So we have a positive eigenvalue and a negative eigenvalue.

If we want the population to increase in size, we should have the positive eigenvalue $\lambda_1 > 1$.

Plug in $\alpha = \beta = 0.001$ and $\sigma = 1$, we'll have

$$\begin{aligned}\delta &= \frac{4}{\gamma}(\frac{1}{0.001} - 1) \approx \frac{4000}{\gamma} \\ \lambda_1 &= \frac{0.001\gamma}{2}(1+\sqrt{1+\delta})\end{aligned} \quad \to \quad \lambda_1 = \frac{\gamma}{2000}(1+\sqrt{1+\frac{4000}{\gamma}}) > 1$$

Solving the inequity gives

$$\gamma > 500$$

(d). In case (1), the parameters are:

$$\alpha = 0.5, \quad \beta = 0.25, \quad \gamma = 2.0, \quad \sigma = 0.8,$$

we can compute that

$$\begin{aligned}\delta &= \frac{4}{\gamma}\frac{\beta}{\alpha}\left(\frac{1}{\alpha} - 1\right) = 1 \\ \lambda_1 &= \frac{\sigma\gamma\alpha}{2}(1+\sqrt{1+\delta}) = 0.4(1+\sqrt{2}) \\ &\approx 0.97 < 1\end{aligned}$$

So the population decreases.

An in case (2), the parameters are:

$$\alpha = 0.6, \quad \beta = 0.3, \quad \gamma = 2.0, \quad \sigma = 0.8,$$

we can compute that

$$\begin{aligned}
\delta &= \frac{4}{\gamma}\frac{\beta}{\alpha}\left(\frac{1}{\alpha} - 1\right) = \frac{2}{3} \\
\lambda_1 &= \frac{\sigma\gamma\alpha}{2}(1 + \sqrt{1+\delta}) = 0.48(1 + \sqrt{5/3}) \\
&\approx 1.10 > 1
\end{aligned}$$

So the population increases.

(e). Consider the positive eigenvalue as

$$\lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2}$$

where

$$a = \alpha\sigma\gamma, \quad b = \beta\sigma^2(1-\alpha)\gamma$$

If we want $\lambda_1 > 1$, then

$$\begin{aligned}
\frac{a + \sqrt{a^2+4b}}{2} &> 1 \\
\sqrt{a^2+4b} &> 2 - a \\
a^2 + 4b &> a^2 - 4a + 4 \\
a + b &> 1
\end{aligned}$$

(Note: If $(2-a) < 0$ then the inequity will always be trivially true, thus it's not of too much interest. We only consider $(2-a) > 0$ here.) Substitute back $a = \alpha\sigma\gamma$ and $b = \beta\sigma^2(1-\alpha)\gamma$, we'll have

$$\gamma > \frac{1}{\alpha\sigma + \beta\sigma^2(1-\alpha)}$$

Problem 18: *Blood CO_2 and ventilation*

(a). We know from the book that

$$C_{n+1} = C_n - \mathcal{L}(V_n, C_n) + m, \tag{14}$$
$$V_{n+1} = \mathcal{S}(C_n). \tag{15}$$

From the question description we know that

$$\mathcal{L}(V_n, C_n) = \beta V_n, \qquad V_{n+1} = \alpha C_n$$

Make some simple substitution, we'll have

$$C_{n+1} = C_n - \alpha\beta C_{n-1} + m$$

i.e.

$$C_{n+1} - C_n + \alpha\beta C_{n-1} = m$$

(b). (1) Suppose $C_n = m/\alpha\beta$, then

$$C_{n+1} - C_n + \alpha\beta C_{n-1} = m/\alpha\beta - m/\alpha\beta + \alpha\beta \cdot \frac{m}{\alpha\beta} = m$$

Hence, $C_n = m/\alpha\beta$ is a particular solution.

(2) Let $m = 0$, then it'll turn into a homogeneos problem. The characteristic equation would be

$$\lambda^2 - \lambda + \alpha\beta = 0$$

with eigenvalues

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4\alpha\beta}}{2},$$

which is the *complementary/homogeneous solution.*
Combined the complementary solution with the particular solution, the general solution would be

$$C_n = \frac{m}{\alpha\beta} + C_1 \left(\frac{1 + \sqrt{1 - 4\alpha\beta}}{2} \right)^n + C_2 \left(\frac{1 - \sqrt{1 - 4\alpha\beta}}{2} \right)^n$$

(c). (1) Assume that $4\alpha\beta < 1$, then $\alpha\beta < 1/4$. This implies

$$\mathcal{L}(V_n, C_n) = \beta V_n = \alpha\beta C_{n-1} < \frac{C_{n-1}}{4}$$

i.e., the amount of CO_2 loss / breathed out will not be larger than $\frac{1}{4}$ of the blood CO_2 amount. And, since $4\alpha\beta < 1$, then $0 < \sqrt{1 - 4\alpha\beta} < 1$. Let $\sqrt{1 - 4\alpha\beta} = \delta$, then

$$\lambda_1 = \frac{1 + \delta}{2} \in (\frac{1}{2}, 1)$$
$$\lambda_2 = \frac{1 - \delta}{2} \in (0, \frac{1}{2})$$

Therefore, the absolute value of each eigenvalue $|\lambda_i| < 1$, implying

$$\lim_{n \to \infty} (C_n) = \frac{m}{\alpha\beta}$$

since the power of a fraction will diminish towards 0. Under this scenario, a steady state will eventually be established regardless of the initial conditions. And the steady ventilation rate would be

$$V_n = \alpha \frac{m}{\alpha\beta} = \frac{m}{\beta}$$

(2) Assume that $4\alpha\beta > 1$, then $1 - 4\alpha\beta < 0$, which means that the characteristic equation would have complex eigenvalues. The conjugated complex eigenvalues $\lambda = a \pm bi$ have

$$a = \frac{1}{2}, \quad b = \frac{\sqrt{4\alpha\beta - 1}}{2}$$

i.e.

$$r = \sqrt{a^2 + b^2} = \sqrt{\alpha\beta}, \quad \theta = \tan^{-1}(b/a) = \tan^{-1}(\sqrt{4\alpha\beta - 1})$$

Then

$$C_n = r^n (C_1 \cos(n\theta) + C_2 \sin(n\theta)) = (\sqrt{\alpha\beta})^n \sin(n\theta + \phi)$$

where $\tan\phi = C_1/C_2$. It is clear that the solution will oscillate at frequency $f = \theta/2\pi$

If $\alpha\beta \geq 1$, then the oscillation will increase in magnitude. When $\alpha\beta = 1$, the oscillation frequency would be $f = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$.

This result could correspond to Cheyne-Stokes respiration, which is a disorder characterized by recurrent oscillation between apnea and hyperpnea (May, 1978 and Naughton, 1998).

(d). For now the ventilation rate is proportional to blood CO_2 concentration. In real biological systems, body will likely adjust the rate of breath according to C_n, just like the relationship between air friction and velocity. It's suitable to suppose that

$$\mathcal{S}(C_n) = \alpha C_n^2$$

Then the equation will turn into

$$C_{n+1} = C_n - \alpha\beta C_{n-1}^2 + m \quad C_{n+1} - C_n + \alpha\beta C_{n-1}^2 = m$$

Suppose $C_{n+1} = C_n$, then we can derive a steady-state where

$$C_n = \sqrt{\frac{m}{\alpha\beta}}$$

It would also be interesting to think about the equilibrium shifting between bicarbonates and CO_2 and the formation of carbaminohemoglobin($HbCO_2$). But deriving a nice math equation with respect to these factors is out of our ability.

Problem 20

Take p_n^1 as the number of newborns, then it should be compute by

$$\begin{aligned} p_{n+1}^1 &= \sum_{i=1}^{m} \begin{pmatrix} \text{number of births} \\ \text{in age class } i \\ \text{in last year } (n) \end{pmatrix} \begin{pmatrix} \text{number of individuals} \\ \text{in age class } i \\ \text{in last year } (n) \end{pmatrix} \\ &= \sum_{i=1}^{m} \alpha_i p_n^i \end{aligned}$$

And for each age class $1 < i < m$, it can be computed by

$$p_{n+1}^i = \sigma_{i-1} p_n^{i-1}$$

Since m is the oldest age class, p_n^m will not survive to next year. The system of equations can be written in the form of

$$\begin{aligned} p_{n+1}^1 &= \alpha_1 p_n^1 + \alpha_2 p_n^2 + \alpha_3 p_n^3 + \cdots + \alpha_{m-1} p_n^{m-1} + \alpha_m p_n^m \\ p_{n+1}^2 &= \sigma_1 p_n^1 \\ p_{n+1}^3 &= \qquad \sigma_2 p_n^2 \\ &\vdots \\ p_{n+1}^m &= \qquad\qquad\qquad \sigma_{m-1} p_n^{m-1} \end{aligned}$$

It's clear that it can be written as the matrix form:

$$\mathbf{P}_{n+1} = \mathbf{A}\mathbf{P}_n$$

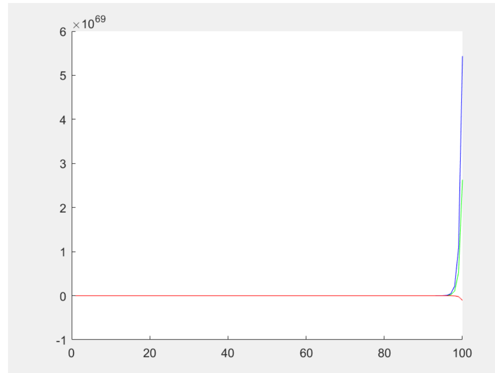
where

$$\mathbf{P}_k = \begin{pmatrix} p_k^1 \\ p_k^2 \\ \vdots \\ p_k^m \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} & \alpha_m \\ \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{m-1} & 0 \end{pmatrix}$$

CODE PART

Problem 6

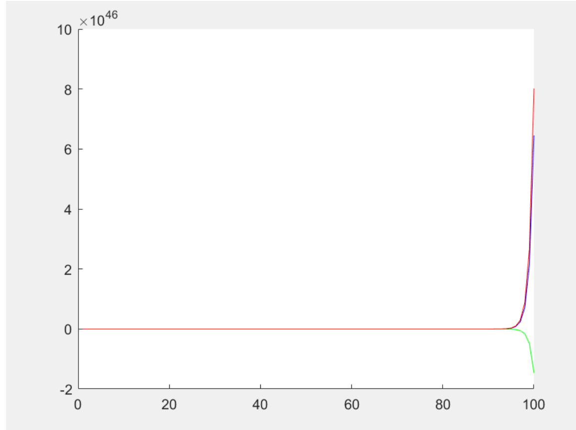
(a). As seen in this graph of 6a, each initial condition produces a unique exponential mapping, however, all the mappings follow the same pattern and are unstable.



$$x_{n+2} - 7x_{n+1} + 10x_n = 0$$

```
clear all
eig1=5;
eig2=2;
x(1)=0;
x(2)=1;
x(3)=0.5;
x(4)=-0.1;
x(5)=2;
x(6)=3;
n=linspace(1,500);
for k=1:2:6
    syms A B
    [solA,solB] = solve(
        A*eig1^x(k) + B*eig2^x(k) == x(k),
        A*eig1^x(k+1) + B*eig2^x(k+1) == x(k+1));
    n=[1:1:100];
    if k == 1
        hold on
        g=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,g,'g')
    elseif k == 3
        hold on
        b=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,b,'b')
    else
        hold on
        r=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,solA.*eig1.^n + solB.*eig2.^n,'r')
    end
end
```

(c). As seen in this graph of 6c, each initial condition produces a unique exponential mapping, however, all the mappings follow the same pattern and are unstable.



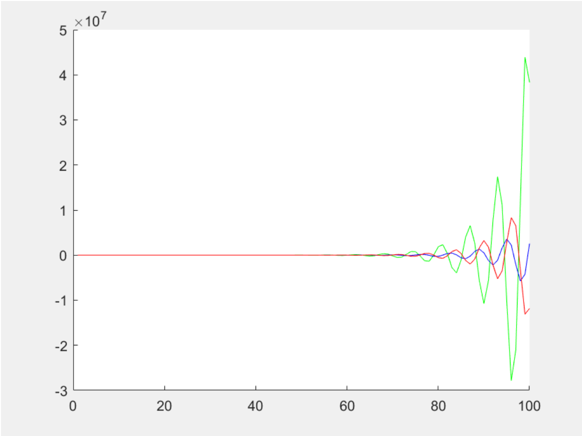
$$x_{n+2} - \sigma_1 x_{n+1} + \sigma_2 x_n = 0$$

```
clear all
s1= -1;
s2= -2;
beta= 3;
eig1=(1/2)*(-s1+(sqrt(s1^2 -4*s2*beta)))
eig2=(1/2)*(-s1-(sqrt(s1^2 -4*s2*beta)))
x(1)=7;
x(2)=1;
x(3)=0.5;
x(4)=-0.1;
x(5)=2;
x(6)=3;
n=linspace(1,500);

for k=1:2:6
    syms A B
    [solA,solB] = solve(
        A*eig1^x(k) + B*eig2^x(k) == x(k),
        A*eig1^x(k+1) + B*eig2^x(k+1) == x(k+1));
    n=[1:1:100];
    if k == 1
        hold on
        g=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,g,'g')
    elseif k == 3
        hold on
        b=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,b,'b')
    else
        hold on
        r=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,solA.*eig1.^n + solB.*eig2.^n,'r')
    end
end
```

(f). As seen in this graph of 6f, each initial condition produces a unique oscillating mapping, however, all the mappings follow the same pattern and are unstable.

$$x_{n+2} - \frac{5}{4} x_{n+1} + \frac{11}{8} x_n = 0$$



```
clear all
eig1=(1/2)*(1.25+sqrt(1.5625-5.5))
eig2=(1/2)*(1.25-sqrt(1.5625-5.5))
x(1)=7;
x(2)=1;
x(3)=0.5;
x(4)=-0.1;
x(5)=2;
x(6)=3;
n=linspace(1,500);

for k=1:2:6
    syms A B
    [solA,solB] = solve(
        A*eig1^x(k) + B*eig2^x(k) == x(k),
        A*eig1^x(k+1) + B*eig2^x(k+1) == x(k+1));
    n=[1:1:100];
    if k == 1
        hold on
        g=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,g,'g')
    elseif k == 3
        hold on
        b=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,b,'b')
    else
        hold on
        r=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,solA.*eig1.^n + solB.*eig2.^n,'r')
    end
end
```