MATH564: MATHEMATICAL MODELING
Homework #2

Due on February 7 2020

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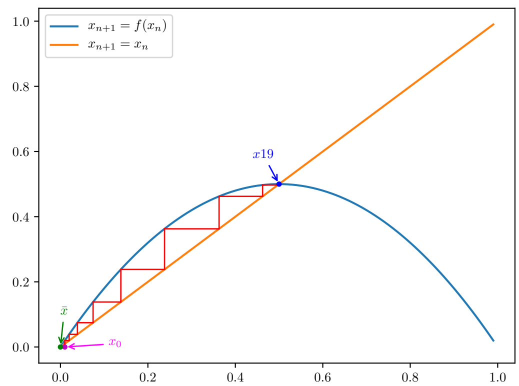
February 6, 2020

Problem 2

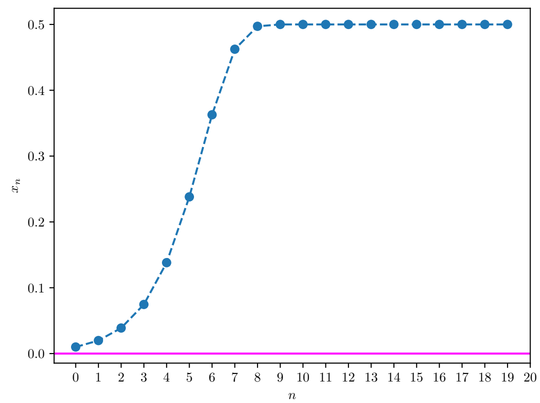
For each sub-problem, subfigure (a) displays the cobwebbing method result, with an initial value slight off (add perturbation) the steady state, and guess from the plot whether the steady state is stable or not. To be more confident, the conclusions will be tested by calculation.

(a). The steady state $\bar{x}$ for equation $x_{n+1} = rx_n(1 - x_n)$ is $\bar{x} = 0$. Since there's one arbitrary constant r, whether the steady state is stable or not depends on the value of r.

We first try $r = 2$:



(a) x_{n+1} *v.s.* x_n

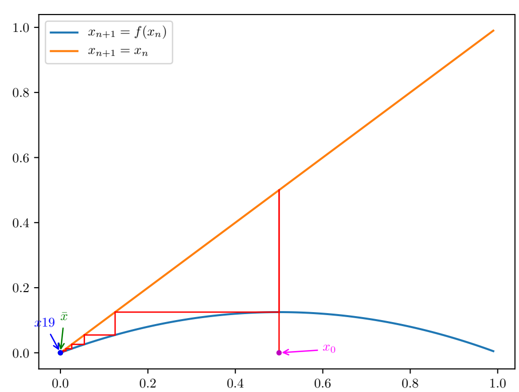


(b) x_{n+1} *v.s.* n

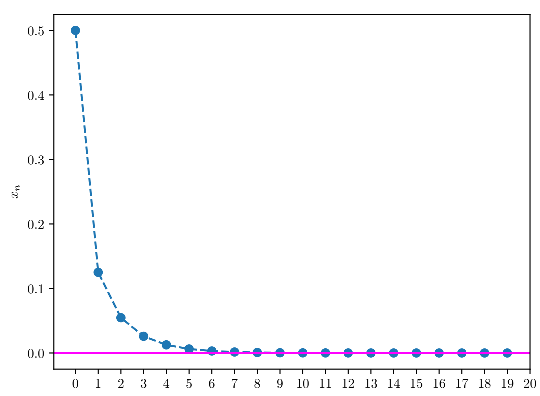
Figure 1: The behavior of equation $x_{n+1} = 2x_n(1 - x_n)$, The steady state is marked by the horizontal magenta line.

It's clear in Figure 1 that when $r = 2$, $\bar{x} = 0$ is not a stable steady state.

The next try is $r = 0.5$: It seems that when $r = 0.5$, $\bar{x} = 0$ is a stable steady state. It's intuitive to



(a) x_{n+1} *v.s.* x_n



(b) x_{n+1} *v.s.* n

Figure 2: The behavior of equation $x_{n+1} = \dfrac{x_n(1 - x_n)}{2}$, The steady state is marked by the horizontal magenta line.

Problem 2 continued on next page. . .

guess that probably $r < 1$ will make $\bar{x} = 0$ be stable.

Further exploration is shown in Figure 3 below:

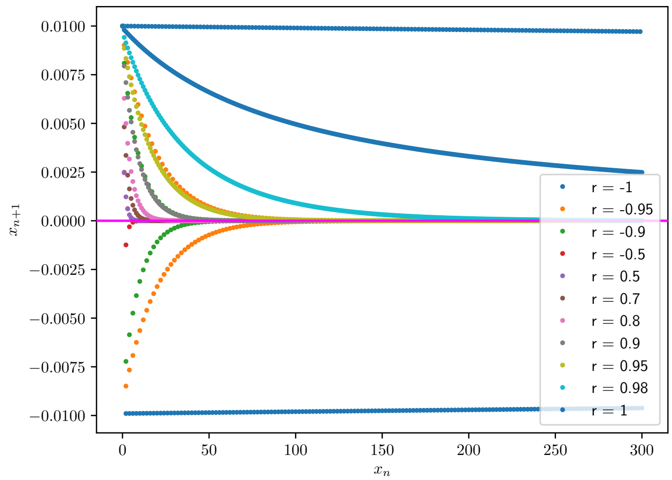


Figure 3: The behavior of equations $x_{n+1} = rx_n(1 - x_n)$, with $r \in \{-1, -0.95, -0.5, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98, 1\}$. It can be seen that the all equations except for the one with $r = -1$ converges to the steady state (marked by the magenta horizontal line).

It seems that for $-1 < r < 1$, or $|r| < 1$, $\bar{x} = 0$ will be a stable steady state. That is to say, even with small perturbation (here is $x' = 0.01$) present, the equation still can eventually reach the steady state.

Test

Recall

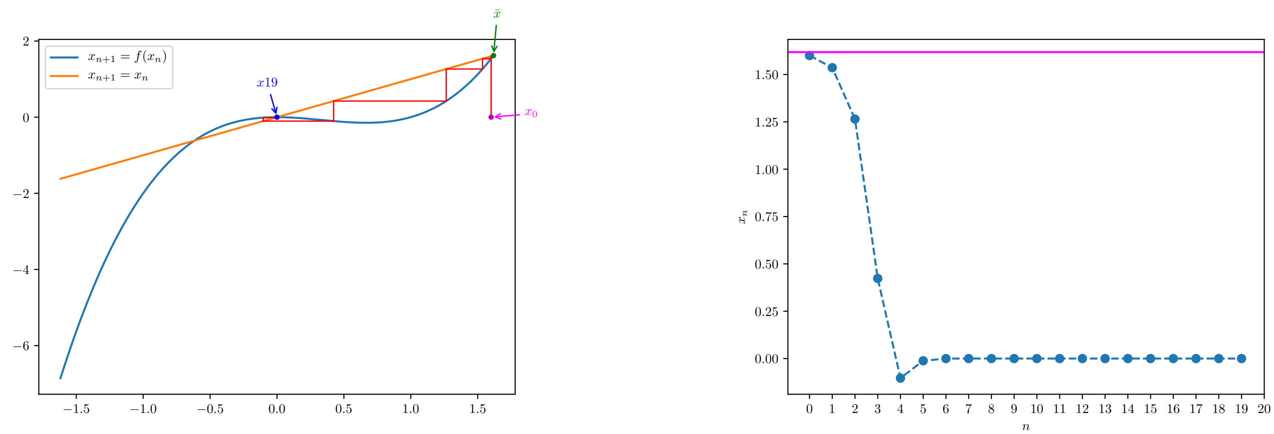
$$\boxed{\bar{x} \text{ is a stable steady state} \Longleftrightarrow \left| \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\bar{x}} \right| < 1}$$

We can derive that

$$\begin{aligned} &\left| \left.\frac{\mathrm{d}}{\mathrm{d}x}\left(rx(1-x)\right)\right|_{\bar{x}} \right| \\ &= \left| (r - 2rx)|_{\bar{x}} \right| \\ &= |r| < 1 \end{aligned}$$

It confirms with our conclusion that when $|r| < 1$, $\bar{x} = 0$ will be a stable steady state.

(b). Plotting the figure of equation $x_{n+1} = -x_n^2(1-x_n)$ with initial condition $x_0 = 1.6$ slightly off the given steady state $\bar{x} = (1+\sqrt{5})/2$ gives:



(a) x_{n+1} $v.s.$ x_n (b) x_{n+1} $v.s.$ n

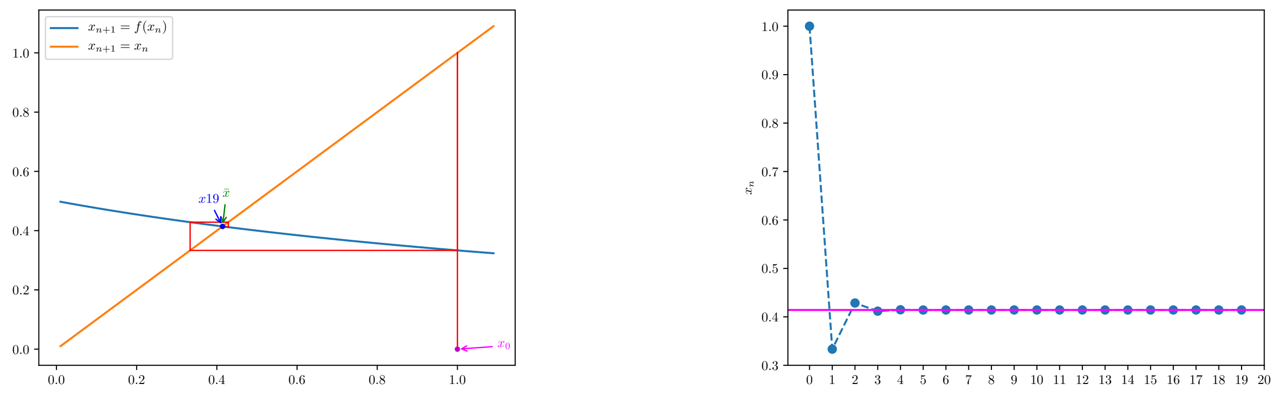
Figure 4: The behavior of equation $x_{n+1} = -x_n^2(1-x_n)$

We can see that the steady state is unstable. The calculation of condition:

$$\begin{aligned} &\left| \frac{\mathrm{d}}{\mathrm{d}x}\left(-x^2(1-x)\right)\Big|_{\bar{x}} \right| \\ &= \left| (-2x+3x^2)\big|_{\bar{x}} \right| \\ &= \frac{7+\sqrt{5}}{2} > 1 \end{aligned}$$

also confirms that $\bar{x} = (1+\sqrt{5})/2$ is not stable.

(c). Plotting the figure of equation $x_{n+1} = 1/(2+x_n)$ with initial condition $x_0 = 0.4$ slightly off the given steady state $\bar{x} = \sqrt{2} - 1$ gives:



(a) x_{n+1} $v.s.$ x_n (b) x_{n+1} $v.s.$ n

Figure 5: The behavior of equation $x_{n+1} = 1/(2+x_n)$

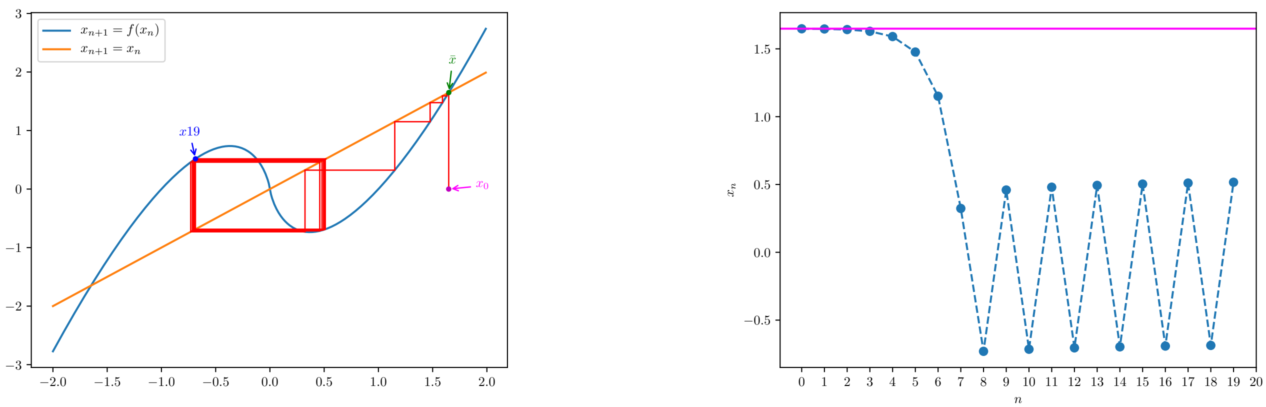
Problem 2 continued on next page. . .

We can see that the steady state is stable. The calculation of condition:

$$
\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(1/(2+x)\right)\Big|_{\bar{x}} \right| \\
&\quad = \left| \left(\frac{-1}{(2+x)^2} \right)\Big|_{\bar{x}} \right| \\
&\quad = 3 - 2\sqrt{2} < 1
\end{aligned}
$$

also confirms that $\bar{x} = \sqrt{2} - 1$ is stable.

(d). Plotting the figure of equation $x_{n+1} = x_n \ln x_n^2$ with initial condition $x_0 = 1.648$ slightly off the given steady state $\bar{x} = e^{1/2}$ gives:



(a) x_{n+1} *v.s.* x_n (b) x_{n+1} *v.s.* n

Figure 6: The behavior of equation $x_{n+1} = x_n \ln x_n^2$

We can see that the steady state is unstable. The calculation of condition:

$$
\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(x \ln x^2\right)\Big|_{\bar{x}} \right| \\
&\quad = 2.5 > 1
\end{aligned}
$$

also confirms that $\bar{x} = e^{1/2}$ is not stable.

Problem 4

(a). The relationship between λ and K is depicted below in Figure 7. It's clear in Figure 7 that $\lambda > 1$ when

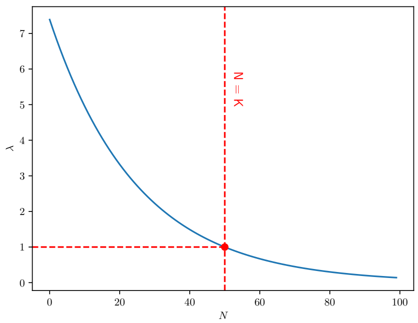


Figure 7: The relationship between $\lambda = \exp[r(1 - N_t/K)]$ and K. λ, the population's growth rate, is described as a function of K. Here K is selected to be 50 and r is selected to be 2.

$N < K$, i.e., the population can continue to grow and reproduce only if $N < K$.

(b). Plug in $\bar{N} = K$ into the equation, we'll have

$$\begin{aligned} N_t &= \bar{N} = K, \\ N_{t+1} &= N_t \exp[r(1 - N_t/K)] = K \exp[r(1 - K/K)] = K \end{aligned}$$

Therefore, for $\bar{N}$ can make $N_{t+1} = N_t$ for two successive timepoints. It is indeed the steady state.

(c). The condition of stability for the steady state depends on:

$$\begin{aligned} &\left| \frac{\mathrm{d}}{\mathrm{d}N} \left(N \exp[r(1 - N/K)]\right) \bigg|_{\bar{N}} \right| \\ &\quad = \left| \left(\exp[r(1 - N/K)] - \frac{rN}{K} \exp[r(1 - N/K)] \right) \bigg|_{\bar{N}} \right| \\ &\quad = |1 - r| \end{aligned}$$

To make the steady state stable, we need

$$|1 - r| < 1$$

Solving this inequity gives

$$0 < r < 2$$

(d). See explanation in the figure legends. We've known from (c) that the stability only depends on r but not K, hence I only chose to depict various r values. In every picture the K is selected to be 200. And the initial value is selected to be 5. A r value smaller than 0 is not biologically sensible, hence not included in the simulation.

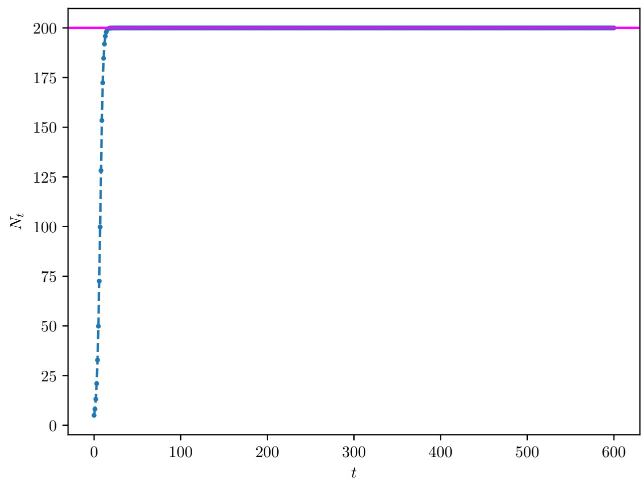


Figure 8: N_t behavior when $r = 0.5$. It's clear that the steady state is stable, which confirms the stability condition $0 < r < 2$

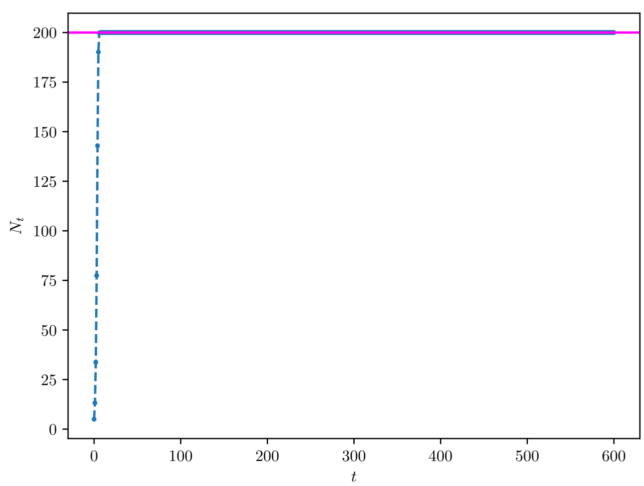


Figure 9: N_t behavior when $r = 1.0$. It's clear that the steady state is still stable, which confirms the stability condition $0 < r < 2$

Problem 4 continued on next page...

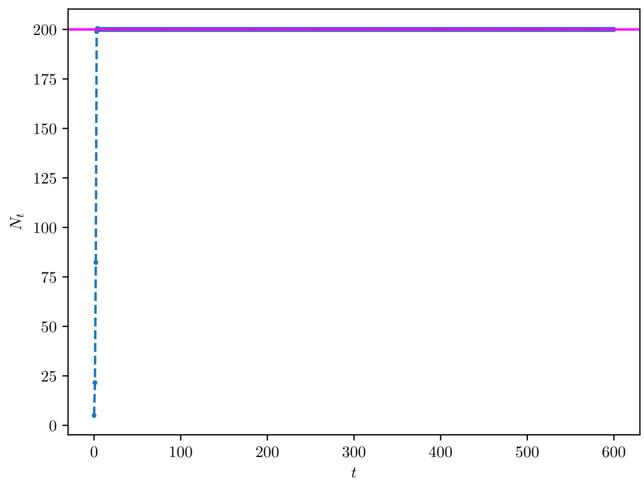


Figure 10: N_t behavior when $r = 1.5$. It's clear that the steady state is still stable, which confirms the stability condition $0 < r < 2$

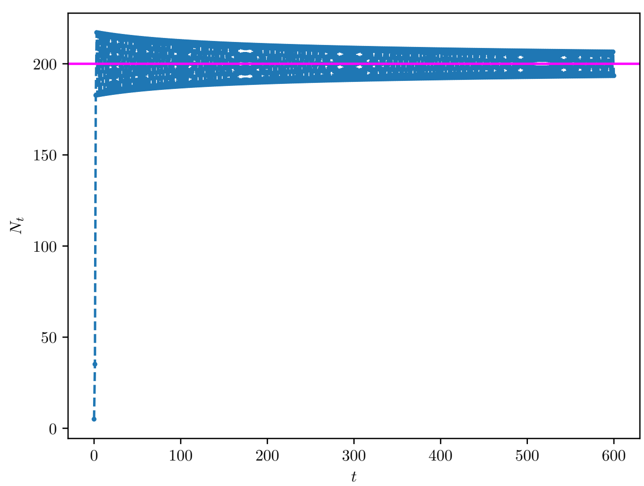
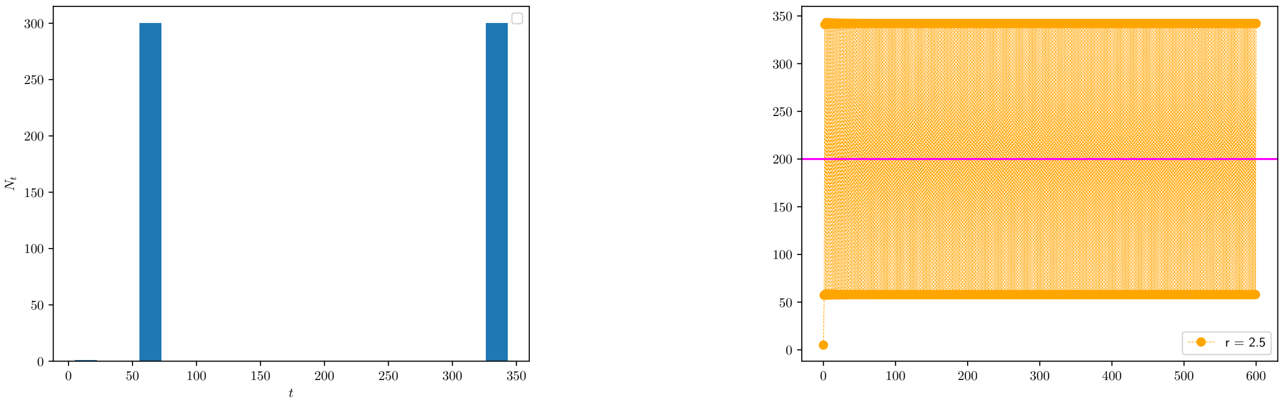
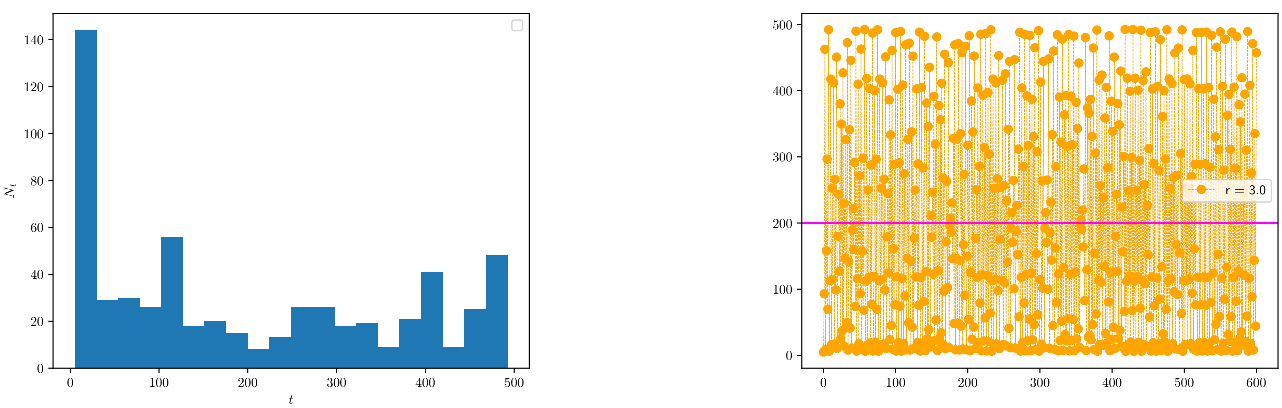


Figure 11: N_t behavior when $r = 2$, which is the boundary value for stability. Moderate oscillation observed around steady state.



(a) N_t behavior (b) Histogram

Figure 12: N_t behavior when $r = 2.5$, seemingly chaotic but actually still maintains oscillation around the steady state. The histogram can confirm that most N_t values fall in the most conspicuous 2 bars (and there's a very short bar between 0-50! Those are points before reaching steady state).

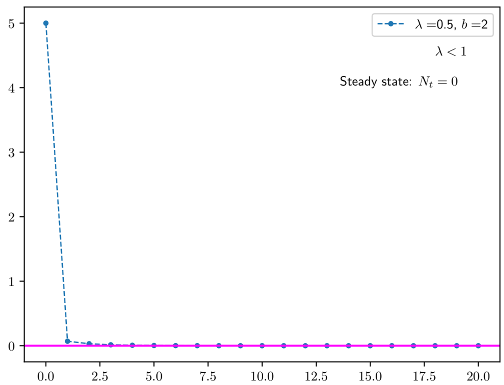


(a) N_t behavior (b) Histogram

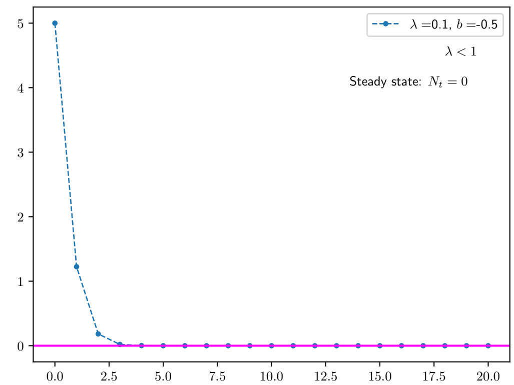
Figure 13: N_t behavior when $r = 3$, truly chaotic. The chaos status can be confirmed from the histogram — it's rather uniformly distributed compared to the situation when $r = 2.5$.

Problem 8: *Graph for 8(c)*

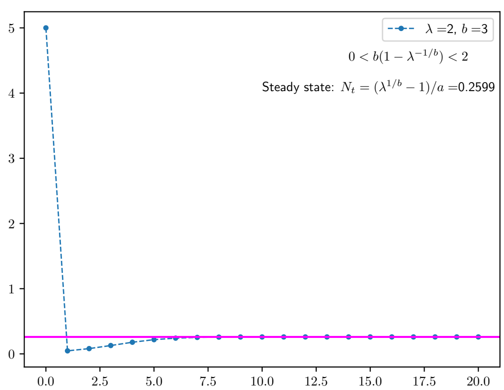
Since the stability criteria don't include a, to make plotting easier, the constant a is set to be 1. All the initial values are set to be 5, just for convenience. The steady states were marked by a magenta horizontal line in each picture.



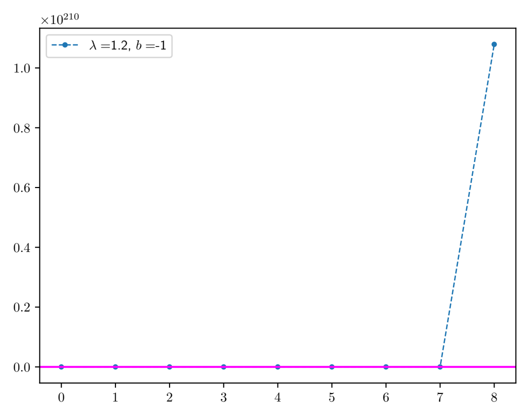
(a) $\lambda < 1$, satisfying the stability condition for steady state $\bar{N} = 0$



(b) $\lambda < 1$, satisfying the stability condition for steady state $\bar{N} = 0$



(c) $0 < b(1 - \lambda^{-1/b}) < 2$, satisfying the stability condition for steady state $\bar{N} = (\lambda^{1/b} - 1)/a$



(d) No stable steady state, resulting in steep growth (This can be observed from the y axis tick unit: $1.0 * 10^{210}$. And N_t after $t = 8$ even causes a RuntimeWarning saying the values are not finite!)

Figure 14: Graphs for 4 different sets of λ and b

Problem 16: *Graph for 16(f)*

From the name I assume that it will result in oscillation of cases: something that might look like this: However, I can only plot figures like this:

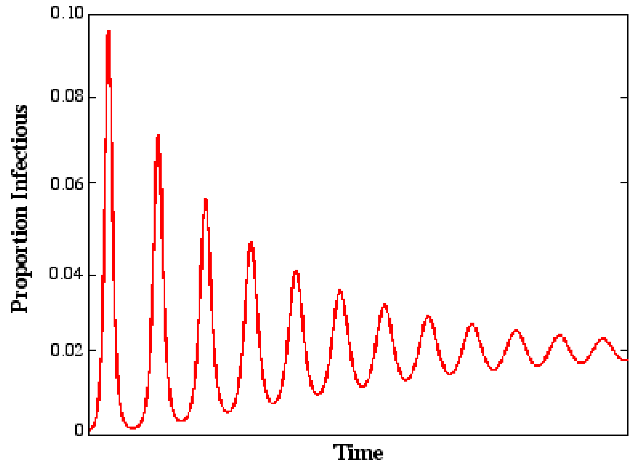
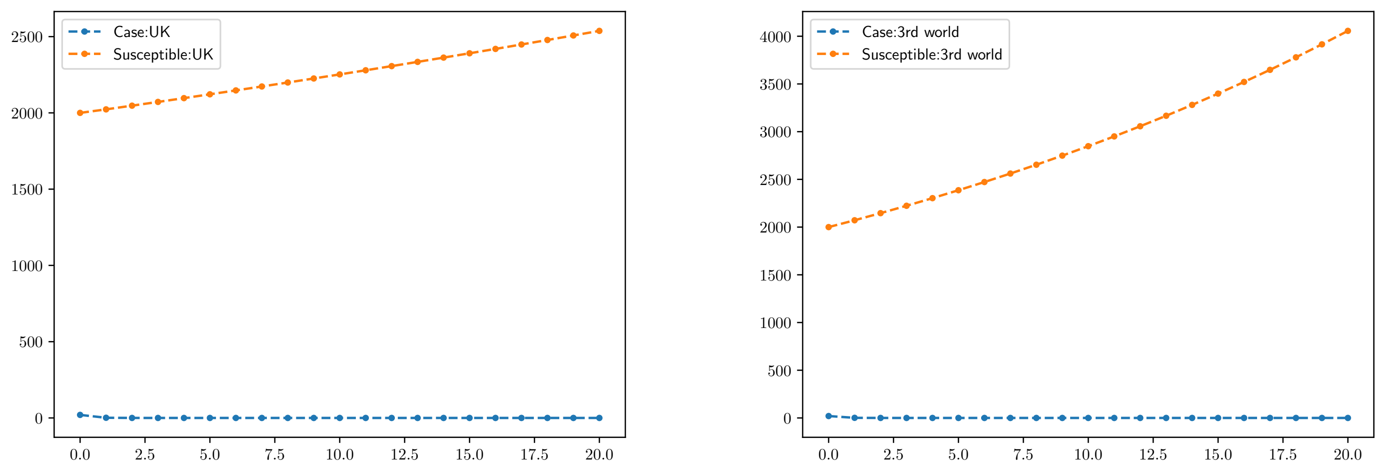


Figure 15: Credit to this website: https://plus.maths.org/content/mathematics-diseases



(a) Model using statistics for UK

(b) Model using statistics for 3rd world

Figure 16: Graphs for 16(f) but not as expected

Problem 17: *Blood CO_2 and Ventilation*

(a). The old model supposes that the loss of CO_2 is only related to ventilation. However, CO_2 can also diffuse spontaneously from high concentration region to low concentration region. The new model includes this factor.

(b). The system of equations for C_n and V_n is

$$C_{n+1} = C_n - \beta V_n C_n + m \tag{1}$$
$$V_{n+1} = \alpha C_n. \tag{2}$$

Writing it as a single equation that only depends on C_n:

$$C_{n+1} = C_n - \alpha\beta C_n C_{n-1} + m \tag{3}$$

(c). Suppose $\bar{C} = C_{n+1} = C_n = C_{n-1}$, substituting $\bar{C}$ into (3) gives:

$$\begin{aligned} \bar{C} &= \bar{C} - \alpha\beta\bar{C}^2 + m \\ \bar{C}^2 &= m/\alpha\beta \\ \bar{C} &= \sqrt{m/\alpha\beta} = \bar{V}/\alpha \end{aligned}$$

To determine whether it's stable or not, we need to make the absolute value of the roots of the characteristic function smaller than 1:

$$\lambda^2 - b\lambda + c = 0$$

where

$$\begin{aligned} b &= \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} + \left.\frac{\partial g}{\partial V}\right|_{\bar{C},\bar{V}} \\ c &= \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial V}\right|_{\bar{C},\bar{V}} - \left.\frac{\partial f}{\partial V}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial C}\right|_{\bar{C},\bar{V}} \end{aligned}$$

Here

$$\begin{aligned} f(C,V) &= C - \beta CV + m \\ g(C,V) &= \alpha C \end{aligned}$$

So

$$\begin{aligned} b &= (1 - \beta\bar{V}) = 1 - \sqrt{\alpha\beta m} \\ c &= -\alpha(-\beta\bar{C}) = \sqrt{\alpha\beta m} \end{aligned}$$

Recall that we derived in the book that if both eigenvalues have magnitude less than 1, then

$$\begin{cases} 2 > 1 + c > |b|, & b^2 - 4c > 0 \\ 1 > c > |b/2|^2, & b^2 - 4c < 0 \end{cases}$$

Either way, when $c = \sqrt{\alpha\beta m} < 1$, both inequities hold. So if we want the steady state to be stable, we need to have

$$m\alpha\beta < 1$$

(d). We can have oscillation when the characteristic equation has complex eigenvalues. To make this happen, we need to have

$$b^2 - 4c = (1 - \sqrt{m\alpha\beta})^2 - 4\sqrt{m\alpha\beta} < 0$$

Substitute $x = \sqrt{m\alpha\beta}$, then we should have

$$x^2 - 6x + 1 < 0$$

Problem 17 continued on next page. . .

Solving this inequity gives

$$3 - 2\sqrt{2} < x < 3 + 2\sqrt{2}$$

i.e., when we have$|x - 3| < 2\sqrt{2}$, where $x = \sqrt{m\alpha\beta}$, there will be oscillation in V_n and C_n.

(e). The system of equations will look like

$$C_{n+1} = C_n - \beta V_n C_n + m \tag{4}$$

$$V_{n+1} = \frac{V_{\max} C_n^{\ell}}{K^{\ell} + C_n^{\ell}} \tag{5}$$

It's trivial to show that if we write (5) as

$$V_n = \frac{V_{\max} C_{n-1}^{\ell}}{K^{\ell} + C_{n-1}^{\ell}} \tag{6}$$

and plug (6) to (4), then we'll have

$$C_{n+1} = C_n - \beta \frac{V_{\max} C_{n-1}^{\ell} C_n}{K^{\ell} + C_{n-1}^{\ell}} + m \tag{7}$$

(f). The relationship between $\mathcal{S}(C)$ and ℓ is shown in Figure 17. K is chosen to be 50.

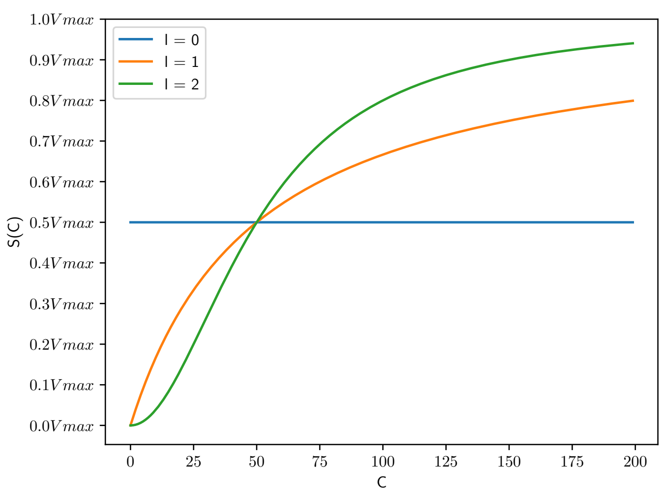


Figure 17: The relationship between $\mathcal{S}(C)$ and ℓ.

(g). For steady state of C_n and V_n, we have

$$\bar{C} = \bar{C} - \beta \frac{V_{\max} \bar{C}^{\ell+1}}{K^{\ell} + \bar{C}^{\ell}} + m \tag{8}$$

Rearrange (8) we can derive the following equation (9), which is what the question wanted

$$\boxed{\frac{m}{\beta} = \frac{V_{\max} \bar{C}^{\ell+1}}{K^{\ell} + \bar{C}^{\ell}} = \bar{C}\bar{V}} \tag{9}$$

(h). When $\ell = 1$,

$$C_{n+1} = C_n - \beta V_n C_n + m$$
$$V_{n+1} = \frac{V_{\max} C_n}{K + C_n}$$

and

$$C_{n+1} = C_n - \beta \frac{V_{\max} C_{n-1} C_n}{K + C_{n-1}} + m \tag{10}$$

Suppose $\bar{C} = C_{n+1} = C_n = C_{n-1}$ and plug it into (10),

$$\bar{C} - \beta \frac{V_{\max} \bar{C}^2}{K + \bar{C}} + m = \bar{C} \tag{11}$$
$$\frac{\bar{C}^2}{K + \bar{C}} = \frac{m}{\beta V_{\max}} \tag{12}$$
$$\bar{C}^2 - \frac{m}{\beta V_{\max}} \bar{C} - \frac{m}{\beta V_{\max}} K = 0 \tag{13}$$

Let $\delta = \dfrac{m}{\beta V_{\max}}$, then

$$\bar{C} = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 + 4K\delta}\right)$$
$$\bar{V} = \frac{V_{\max}\bar{C}}{K + \bar{C}}$$

We need to make the roots for the characteristic equation have a magnitude < 1 again:

$$\lambda^2 - b\lambda + c = 0$$

where

$$b = \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} + \left.\frac{\partial g}{\partial V}\right|_{\bar{C},\bar{V}}$$
$$c = \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial V}\right|_{\bar{C},\bar{V}} - \left.\frac{\partial f}{\partial V}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial C}\right|_{\bar{C},\bar{V}}$$

Here

$$f(C, V) = C - \beta C V + m$$
$$g(C, V) = \frac{V_{\max} C}{K + C}$$

So

$$b = (1 - \beta\bar{V}) = 1 - \frac{\beta V_{\max}\bar{C}}{K + \bar{C}} \tag{14}$$
$$c = \frac{\beta V_{\max} K \bar{C}}{(K + \bar{C})^2} \tag{15}$$

Use (12) to make (14) and (15) look prettier:

$$b = 1 - \frac{m}{\bar{C}}$$
$$c = \frac{m^2 K}{\beta V_{\max} \bar{C}^3}$$

Recall again that we derived in the book that if both eigenvalues have magnitude less than 1, then

$$2 > 1 + c > |b|$$

Therefore we should have

$$\frac{m^2 K}{\beta V_{\max} \bar{C}^3} < 1$$

Note that if $\bar{C} = \delta - \sqrt{\delta^2 + 4K\delta}$, it will be < 0 since $\delta^2 + 4K\delta > \delta^2$, which will make the inequity trivially true (it's not biologically meaningful anyway since negative blood CO_2 concentration seems deadly). So the inequity will become

$$\frac{m^2 K}{\beta V_{\max}} < \bar{C}^3 = \left(\frac{m}{\beta V_{\max}} + \sqrt{\left(\frac{m}{\beta V_{\max}}\right)^2 + \frac{4Km}{\beta V_{\max}}} \right)^3$$

(It looks a bit difficult to simplify for me. So I'll leave it be.)

(i). If we want oscillation to happen, then the characteristic equation should have complex roots, i.e.:

$$b^2 - 4c = \left(1 - \frac{m}{\bar{C}}\right)^2 - \frac{4m^2 K}{\beta V_{\max} \bar{C}^3} < 0$$

Consider $l > 1$, with $\frac{\partial f}{\partial C}$, $\frac{\partial f}{\partial V}$ and $\frac{\partial g}{\partial V}$ unchanged, we'll have

$$\begin{aligned} \frac{\partial g}{\partial C} &= \frac{\ell V_{\max} C^{\ell-1}(K^\ell + C^\ell) - \ell V_{\max} C^\ell C^{\ell-1}}{(K^\ell + C^\ell)^2} \\ &= \frac{\ell V_{\max} K^\ell C^{\ell-1}}{(K^\ell + C^\ell)^2} \end{aligned}$$

This will make

$$c = \frac{\ell \beta V_{\max} K^\ell \bar{C}^\ell}{(K^\ell + \bar{C}^\ell)^2}$$

For oscillation to happen, we still need to have

$$b^2 - 4c < 0$$

Work load distribution

Hannah Wu: Problem 1, 8(except (c)), 11, 16 (except (f))

Minghang Li: Problem 2, 4, 8(c), 16(f), 17 (basically everything concerning plotting)

Note from Minghang

Answers to (g), (h) and (i) might not be correct, but it is the best I can do.