MATH564: MATHEMATICAL MODELING
Homework #3

Due on February 18 2020

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Problem 7: *(a)*

$$
\begin{aligned}
N_{t+1} &= \lambda N_t e^{-aP_t} \\
P_{t+1} &= cN_t(1 - e^{-aP_t})
\end{aligned}
$$

Plug in $n_t = acN_t$ and $p_t = aP_t$,

$$
\begin{aligned}
\frac{n_{t+1}}{ac} &= \lambda \frac{n_t}{ac} e^{-a\frac{p_t}{a}} \\
\frac{p_{t+1}}{a} &= c\frac{n_t}{ac}(1 - e^{-a\frac{p_t}{a}})
\end{aligned}
$$

After canceling a and c, the equations look like

$$
\begin{aligned}
n_{t+1} &= \lambda n_t e^{-p_t} \\
p_{t+1} &= n_t(1 - e^{-p_t})
\end{aligned}
$$

There are only one parameter, λ, in this system of equations.

Problem 10

(a). If $m > 1$, $f(N_t, P_t) = e^{-\frac{1}{aP_t}}$, where aP_t is raised to some power. As P_t grows, $f(N_t, P_t)$ decreases, which has no biological significance.

(b).

$$N_{t+1} = \lambda N_t e^{-(aP_t)^{1-m}}$$
$$P_{t+1} = N_t(1 - e^{-(aP_t)^{1-m}})$$

(c). Since m represents a decrease in search efficiency for the parasites due to large density, introducing this parameter allows lower q values, thus, more elevated stable states. Lower q values also allow the system to be stable with a wide range of r values.

Steady State Analysis:

$$\bar{N} = \lambda \bar{N}(\exp(a\bar{P}^{1-m}))$$
$$\bar{P} = \bar{N}(1 - \exp(-(a\bar{P})^{1-m}))$$

Solving for $\bar{N}$ and $\bar{P}$,

$$\bar{P} = \frac{ln(\lambda)^{\frac{1}{1-m}}}{a}$$
$$\bar{N} = \frac{\bar{P}}{1-\lambda}$$

Taking the partial derivatives for stability analysis,

$$a_{11} = 1 \tag{1}$$
$$a_{12} = -\frac{\lambda^2}{\lambda - 1} ln(\lambda)(1-m) \tag{2}$$
$$a_{21} = \frac{\lambda - 1}{\lambda} \tag{3}$$
$$a_{22} = \frac{ln(\lambda)}{1-m}\frac{\lambda}{\lambda - 1} \tag{4}$$
$$\gamma = a_{11}a_{22} - a_{12}a_{21} = ln\lambda(1-m)(\lambda + \frac{1}{\lambda - 1}) \tag{5}$$

Introducing m, thus, makes γ smaller and creates a more stable steady state.

(d). Since there are less parasites within the system, the average number of encounter per host should increase. We can modify $f(N_t, P_t)$ as such:

$$f(N_t, P_t) = e^{-(aP_t)^c}$$

where c represents the relative degree of parasites' degree of sparsity, and $c > 1$.

Problem 18

Recall the denotation in this problem:

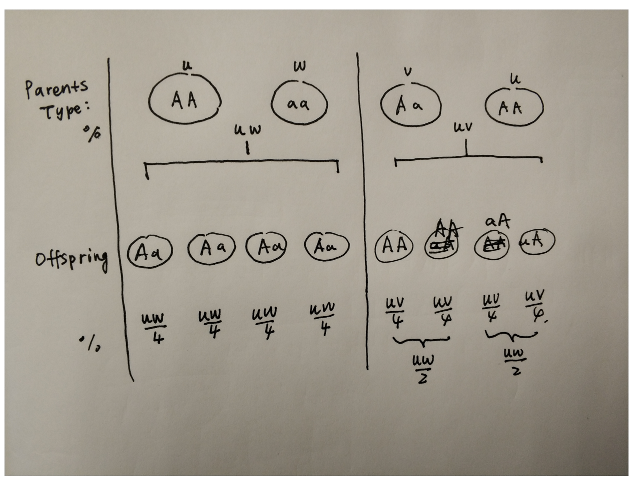
$$\begin{aligned} p &= \text{frequency of allele } A \\ q &= \text{frequency of allele } a \\ u &= \text{frequency of } AA \text{ genotype} \\ v &= \text{frequency of } Aa \text{ genotype} \\ w &= \text{frequency of } aa \text{ genotype} \end{aligned}$$

(a). From the definition of u, v and w, it's clear that $u+v+w=1$. The frequency of allele $A = u + \frac{1}{2}v$ and the frequency of allele $a = \frac{1}{2}v + w$. It's easy to see that $p + q = u + \frac{1}{2}v + \frac{1}{2}v + w = 1$.

(b). Table 3.1 is filled up and presented below as Table 1. The new values are marked in red.

	Genotype	Frequency %	Fathers		
			AA	Aa	aa
			u	v	w
Mothers	AA	u	u^2	uv	uw
	Aa	v	uv	v^2	vw
	aa	w	uw	vw	w^2

(c). The figure is shown as below.



(d). Table 3.2 is filled up and presented below as Table 2. The new values are marked in red.

Type of Parents	Genotype	Offspring Genotype Frequencies		
		AA	Aa	aa
$AA \times AA$	u^2	u^2	0	0
$AA \times Aa$	$2uv$	uv	uv	0
$AA \times aa$	$2uw$	0	$2uw$	0
$Aa \times Aa$	v^2	$v^2/4$	$v^2/2$	$v^2/4$
$Aa \times aa$	$2vw$	0	vw	vw
$aa \times aa$	w^2	0	0	w^2
	Total	$(u^2+uv+v^2/4)$	$(uv+2uw+vw+v^2/2)$	$(w^2+vw+v^2/4)$

(e). It can be seen clearly in the table above that the frequencies are governed by

$$u_{n+1} = u_n^2 + u_n v_n + \frac{1}{4}v_n^2, \tag{6}$$
$$v_{n+1} = u_n v_n + 2u_n w_n + \frac{1}{2}v_n^2 + v_n w_n, \tag{7}$$
$$w_{n+1} = \frac{1}{4}v_n^2 + v_n w_n + w_n^2. \tag{8}$$

(f).

$$\begin{aligned} u_{n+1} + v_{n+1} + w_{n+1} &= (u_n^2 + u_n v_n + \frac{1}{4}v_n^2) + (u_n v_n + 2u_n w_n + \frac{1}{2}v_n^2 + v_n w_n) + (\frac{1}{4}v_n^2 + v_n w_n + w_n^2) \\ &= u_n^2 + 2u_n v_n + v_n^2 + 2v_n w_n + 2u_n w_n + w_n^2 \\ &= (u_n + v_n)^2 + 2(u_n + v_n)w_n + w_n^2 \\ &= (u_n + v_n + w_n)^2 \end{aligned}$$

According to the original setting we know that $u_n + v_n + w_n = 1$. Henceforth, we've proven that $u_{n+1} + v_{n+1} + w_{n+1} = 1$.

(g). At steady states $(\bar{u}, \bar{v}, \bar{w})$, we have

$$\bar{u} = \bar{u}^2 + \bar{u}\bar{v} + \frac{1}{4}\bar{v}^2, \tag{9}$$
$$\bar{v} = \bar{u}\bar{v} + 2\bar{u}\bar{w} + \frac{1}{2}\bar{v}^2 + \bar{v}\bar{w}, \tag{10}$$
$$\bar{w} = \frac{1}{4}\bar{v}^2 + \bar{v}\bar{w} + \bar{w}^2. \tag{11}$$

Divided $\bar{u}$ on both sides of (9) (since $\bar{u}$ is not very likely if not impossible to be 0 this is doable), we can get

$$1 = \bar{u} + \bar{v} + \frac{\bar{v}^2}{\bar{u}}$$

From (f) we know that $\bar{u} + \bar{v} + \bar{w} = 1$, so

$$\begin{aligned} \bar{u} + \bar{v} + \bar{w} &= \bar{u} + \bar{v} + \frac{\bar{v}^2}{\bar{u}} \\ \bar{w} &= \frac{\bar{v}^2}{\bar{u}} \\ \bar{u} &= \bar{v}^2/\bar{w} \end{aligned}$$

□

(h). From (f) we know that $w_{n+1} = 1-u_{n+1}-v_{n+1}$. It can also be written as $w_n = 1-u_n-v_n$. Substituting this back into (6) and (7) gives:

$$u_{n+1} = u_n^2 + u_n v_n + \frac{1}{4}v_n^2, \tag{12}$$

$$\begin{aligned} v_{n+1} &= u_n v_n + 2u_n(1-u_n-v_n) + \frac{1}{2}v_n^2 + v_n(1-u_n-v_n) \\ &= -2u_n^2 - 2u_n v_n - \frac{v_n^2}{2} + v_n + 2u_n \end{aligned} \tag{13}$$

(i). From (12) it's trivial to see that

$$u_{n+1} = \left(u_n + \frac{v_n}{2}\right)^2$$

And we have

$$\begin{aligned} v_{n+1} &= (u_n + \frac{1}{2}v_n)[2 - 2(u_n + \frac{1}{2}v_n)] \\ &= 2u_n + v_n - 2(u_n + \frac{1}{2}v_n)^2 \\ &= 2u_n + v_n - 2u_n^2 - 2u_n v_n - \frac{v_n^2}{2}, \end{aligned}$$

which is exactly (13).

(j). Let X represents $u_n + \frac{v_n}{2}$, then:

$$\begin{aligned} u_{n+1} + \frac{v_{n+1}}{2} &= X^2 + X(1-X) \\ &= X \end{aligned} \qquad i.e., \left(u_{n+1} + \frac{v_{n+1}}{2}\right) = \left(u_n + \frac{v_n}{2}\right)$$

(k). Recall that

$$\begin{aligned} p_n &= u_n + \frac{v_n}{2} \\ q_n &= \frac{v_n}{2} + w_n \end{aligned}$$

So,

$$\begin{aligned} p_{n+1} &= u_{n+1} + \frac{v_{n+1}}{2} \\ q_{n+1} &= \frac{v_{n+1}}{2} + w_{n+1} \end{aligned}$$

We've known from (j) that

$$\left(u_{n+1} + \frac{v_{n+1}}{2}\right) = \left(u_n + \frac{v_n}{2}\right)$$

which means $p_{n+1} = p_n$. From (a) we know that $p+q=1$. Since p doesn't change accross generations, q also trivially won't change. I don't understand why we need to use $p^2 + 2pq + q^2 = 1$ to prove this.