MATH564: MATHEMATICAL MODELING
Homework #2

Due on February 7 2020

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Problem 1

(a). This equation is linear.

$$x_n = (1-\alpha)x_{n-1} + \beta x_n$$
$$(1-\beta)x_n = (1-\alpha)x_{n-1}$$
$$x_n = \frac{1-\alpha}{1-\beta}x_{n-1}$$
$$\lambda^2 - \frac{1-\alpha}{1-\beta}\lambda = 0$$
$$\lambda_1 = 0, \quad \lambda_2 = \frac{1-\alpha}{1-\beta}$$
$$x_n = c(\frac{1-\alpha}{1-\beta})^n$$

(b). This equation is non-linear.

$$x_{n+1} = \frac{x_n}{1+x_n}$$
$$x = \frac{x}{1+x}$$
$$x + x^2 = x$$
$$x^2 = 0$$

x = 0 is a steady state.

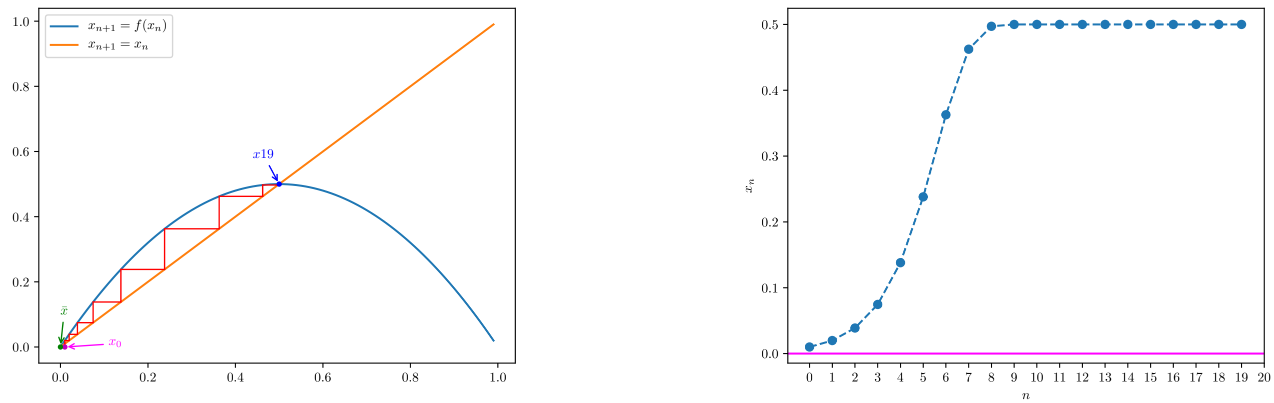
(c).

Problem 2

We use *cobwebbing method* to visualize the approximate behavior of each equation, with an initial value slight off (add perturbation) the steady state, and guess from the plot whether the steady state is stable or not. To be more confident, the conclusions will be tested by calculation.

(a). The steady state $\bar{x}$ for equation $x_{n+1} = rx_n(1 - x_n)$ is $\bar{x} = 0$. Since there's one arbitrary constant r, whether the steady state is stable or not depends on the value of r.

We first try $r = 2$:

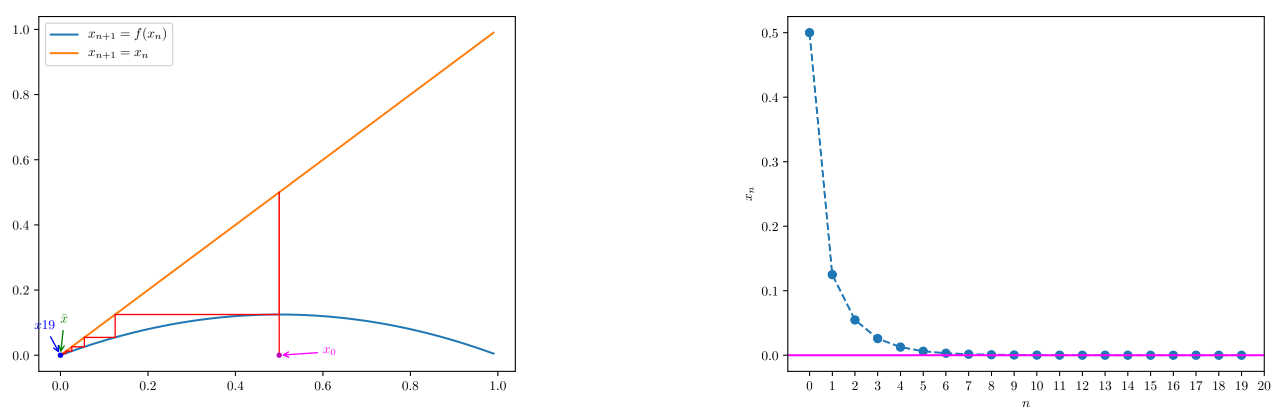


(a) x_{n+1} *v.s.* x_n (b) x_{n+1} *v.s.* n

Figure 1: The behavior of equation $x_{n+1} = 2x_n(1 - x_n)$, The steady state is marked by the horizontal magenta line.

It's clear in Figure 1 that when $r = 2$, $\bar{x} = 0$ is not a stable steady state.

The next try is $r = 0.5$: It seems that when $r = 0.5$, $\bar{x} = 0$ is a stable steady state. It's intuitive to



(a) x_{n+1} *v.s.* x_n (b) x_{n+1} *v.s.* n

Figure 2: The behavior of equation $x_{n+1} = \dfrac{x_n(1 - x_n)}{2}$, The steady state is marked by the horizontal magenta line.

guess that probably $r < 1$ will make $\bar{x} = 0$ be stable.

Further exploration is shown in Figure 3 below:

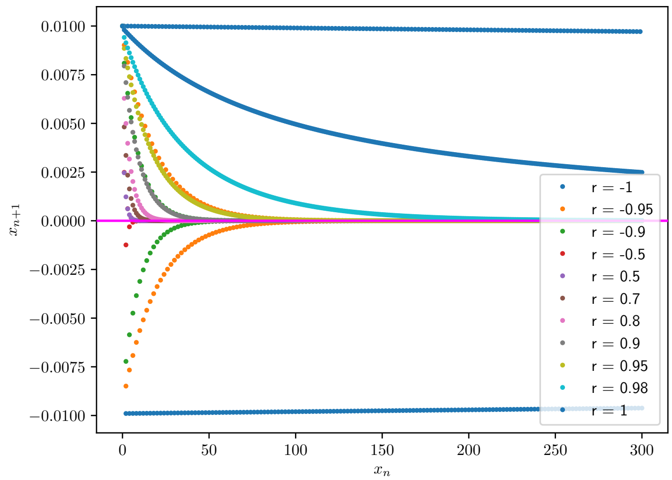


Figure 3: The behavior of equations $x_{n+1} = rx_n(1 - x_n)$, with $r \in \{-1, -0.95, -0.5, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98, 1\}$. It can be seen that the all equations except for the one with $r = -1$ converges to the steady state (marked by the magenta horizontal line).

It seems that for $-1 < r < 1$, or $|r| < 1$, $\bar{x} = 0$ will be a stable steady state. That is to say, even with small perturbation (here is $x' = 0.01$) present, the equation still can eventually reach the steady state.

Test

Recall

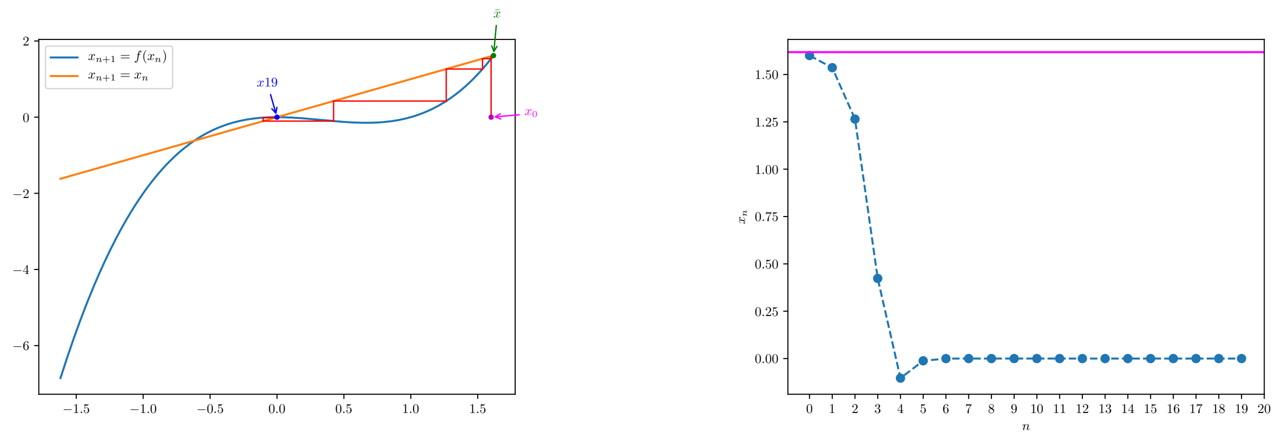
$$\boxed{\bar{x} \text{ is a stable steady state} \Longleftrightarrow \left| \left. \frac{\mathrm{d}f}{\mathrm{d}x} \right|_{\bar{x}} \right| < 1}$$

We can derive that

$$\begin{aligned} &\left| \left. \frac{\mathrm{d}}{\mathrm{d}x} \left(rx(1-x) \right) \right|_{\bar{x}} \right| \\ &= \left| \left(r - 2rx \right)\right|_{\bar{x}} \right| \\ &= |r| < 1 \end{aligned}$$

It confirms with our conclusion that when $|r| < 1$, $\bar{x} = 0$ will be a stable steady state.

(b). Plotting the figure of equation $x_{n+1} = -x_n^2(1 - x_n)$ with initial condition $x_0 = 1.6$ slightly off the given steady state $\bar{x} = (1+\sqrt{5})/2$ gives:



(a) x_{n+1} *v.s.* x_n

(b) x_{n+1} *v.s.* n

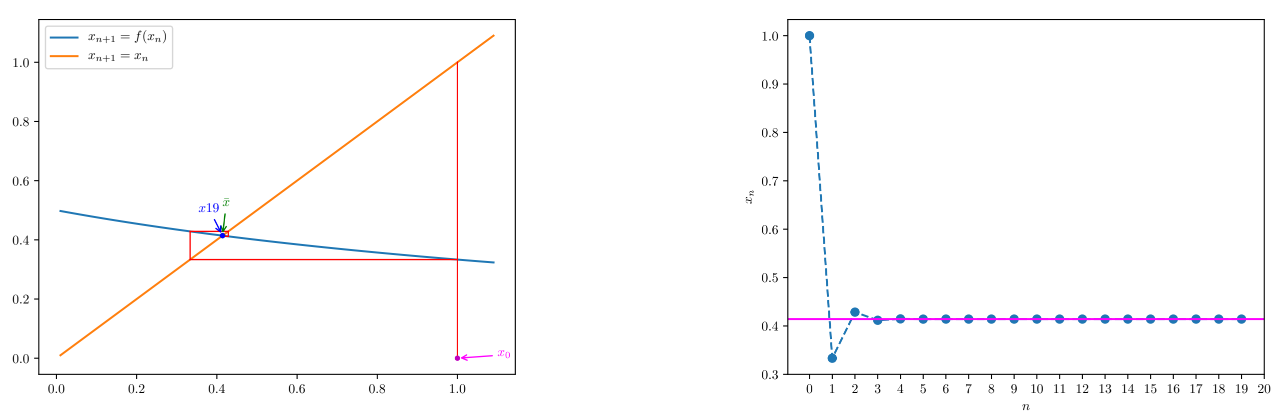
Figure 4: The behavior of equation $x_{n+1} = -x_n^2(1 - x_n)$

We can see that the steady state is unstable. The calculation of condition:

$$
\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(-x^2(1-x)\right)\Big|_{\bar{x}} \right| \\
&= \left| (-2x + 3x^2)|_{\bar{x}} \right| \\
&= \frac{7+\sqrt{5}}{2} > 1
\end{aligned}
$$

also confirms that $\bar{x} = (1+\sqrt{5})/2$ is not stable.

(c). Plotting the figure of equation $x_{n+1} = 1/(2 + x_n)$ with initial condition $x_0 = 0.4$ slightly off the given steady state $\bar{x} = \sqrt{2} - 1$ gives:



(a) x_{n+1} *v.s.* x_n

(b) x_{n+1} *v.s.* n

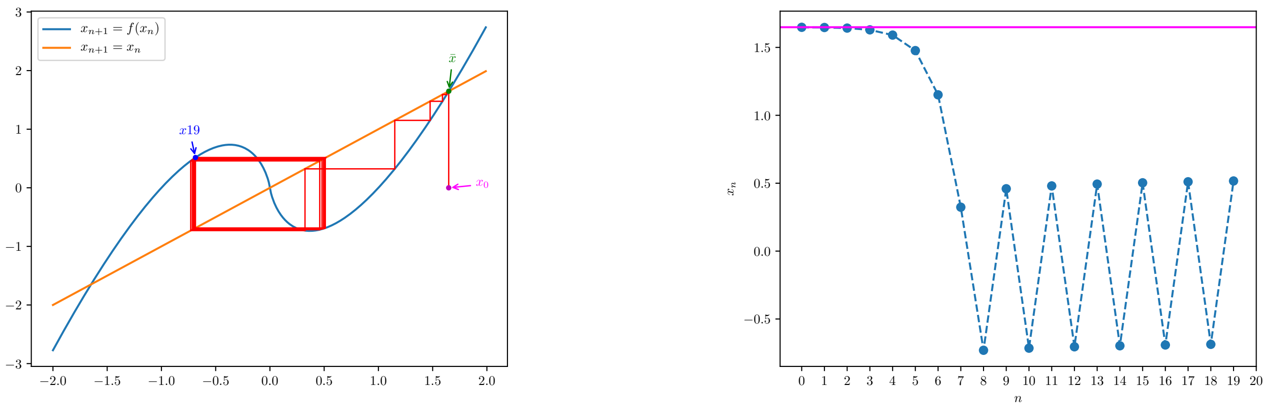
Figure 5: The behavior of equation $x_{n+1} = 1/(2 + x_n)$

We can see that the steady state is stable. The calculation of condition:

$$\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(1/(2+x)\right)\Big|_{\bar{x}} \right| \\
&= \left| \left(\frac{-1}{(2+x)^2} \right)\Big|_{\bar{x}} \right| \\
&= 3 - 2\sqrt{2} < 1
\end{aligned}$$

also confirms that $\bar{x} = \sqrt{2} - 1$ is stable.

(d). Plotting the figure of equation $x_{n+1} = x_n \ln x_n^2$ with initial condition $x_0 = 1.648$ slightly off the given steady state $\bar{x} = e^{1/2}$ gives:



(a) x_{n+1} *v.s.* x_n (b) x_{n+1} *v.s.* n

Figure 6: The behavior of equation $x_{n+1} = x_n \ln x_n^2$

We can see that the steady state is unstable. The calculation of condition:

$$\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(x \ln x^2\right)\Big|_{\bar{x}} \right| \\
&= 2.5 > 1
\end{aligned}$$

also confirms that $\bar{x} = e^{1/2}$ is not stable.

Problem 4

(a). The relationship between λ and K is depicted below in Figure 7. It's clear in Figure 7 that $\lambda > 1$ when

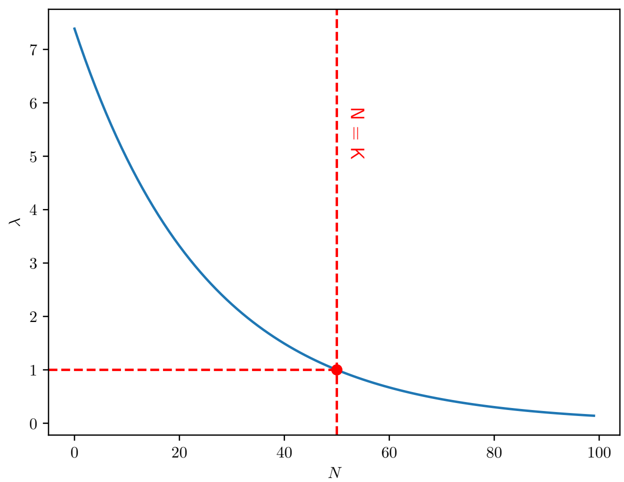


Figure 7: The relationship between $\lambda = \exp[r(1 - N_t/K)]$ and K. λ, the population's growth rate, is described as a function of K. Here K is selected to be 50 and r is selected to be 2.

$N < K$, i.e., the population can continue to grow and reproduce only if $N < K$.

(b). Plug in $\bar{N} = K$ into the equation, we'll have

$$N_t = \bar{N} = K,$$
$$N_{t+1} = N_t \exp[r(1 - N_t/K)] = K \exp[r(1 - K/K)] = K$$

Therefore, for $\bar{N}$ can make $N_{t+1} = N_t$ for two successive timepoints. It is indeed the steady state.

(c). The condition of stability for the steady state depends on:

$$\left| \frac{\mathrm{d}}{\mathrm{d}N} \left(N \exp[r(1 - N/K)]\right) \Big|_{\bar{N}} \right|$$
$$= \left| \left(\frac{N^2 - rkN + N^2}{N^2} \exp[r(1 - N/K)] \right) \Big|_{\bar{N}} \right|$$
$$= |(2 - r)e^r|$$

To make the steady state stable, we need

$$|(2 - r)e^r| < 1$$

Solving this inequity gives

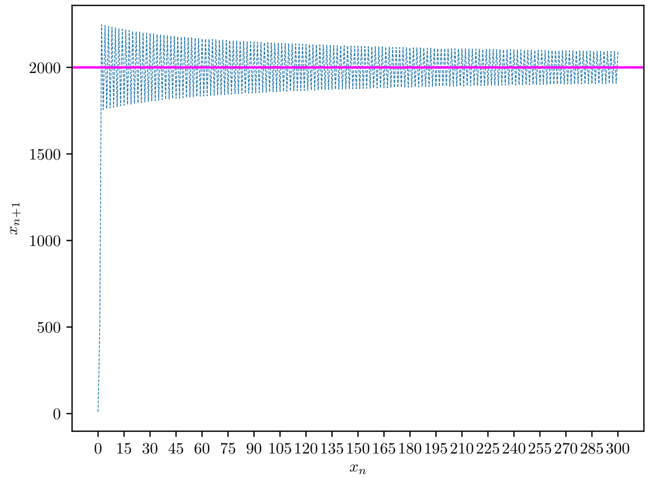
$$r < W_{-1}(-1/e^2) + 2$$
$$W(-1/e^2) + 2 < r < W(1/e^2) + 2$$

The approximate values are

$$r < -1.15$$
$$1.84 < r < 2.12$$

Problem 4 continued on next page. . .

(d). $r = 2, K = 2000$


2000
1500
1000
500
0
x_{n+1}
0 15 30 45 60 75 90 105 120 135 150 165 180 195 210 225 240 255 270 285 300
x_n

Problem 17: *Blood CO_2 and Ventilation*

(a). The old model supposes that the loss of CO_2 in only related to ventilation. However, CO_2 can also diffuse spontaneously from high concentration region to low concentration region. The new model includes this factor.

(b). The system of equations for C_n and V_n is

$$C_{n+1} = C_n - \beta V_n C_n + m \tag{1}$$
$$V_{n+1} = \alpha C_n. \tag{2}$$

Writing it as a single equation that only depends on C_n:

$$C_{n+1} = C_n - \alpha\beta C_n C_{n-1} + m \tag{3}$$

(c). Suppose $\bar{C} = C_{n+1} = C_n = C_{n-1}$, substituting $\bar{C}$ into (3) gives:

$$\begin{aligned}\bar{C} &= \bar{C} - \alpha\beta\bar{C}^2 + m \\ \bar{C}^2 &= m/\alpha\beta \\ \bar{C} &= \sqrt{m/\alpha\beta} = \bar{V}/\alpha\end{aligned}$$

To determine whether it's stable or not, we need to make the absolute value of the roots of the characteristic function smaller than 1:

$$\lambda^2 - b\lambda + c = 0$$

where

$$\begin{aligned}b &= \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} + \left.\frac{\partial f}{\partial V}\right|_{\bar{C},\bar{V}} \\ c &= \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial V}\right|_{\bar{C},\bar{V}} - \left.\frac{\partial f}{\partial V}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial C}\right|_{\bar{C},\bar{V}}\end{aligned}$$

Here

$$\begin{aligned}f(C,V) &= C - \beta CV + m \\ g(C,V) &= \alpha C\end{aligned}$$

So

$$\begin{aligned}b &= (1 - \beta\bar{V}) = 1 - \sqrt{\alpha\beta m} \\ c &= -\alpha(-\beta\bar{C}) = \sqrt{\alpha\beta m}\end{aligned}$$

Recall that we derived in the book that if both eigenvalues have magnitude less than 1, then

$$\begin{cases} 2 > 1 + c > |b|, & b^2 - 4c > 0 \\ 1 > c > |b/2|^2, & b^2 - 4c < 0 \end{cases}$$

Either way, when $c = \sqrt{\alpha\beta m} < 1$, both inequities hold. So if we want the steady state to be stable, we need to have

$$m\alpha\beta < 1$$

(d). We can have oscillation when the characteristic equation has complex eigenvalues. To make this happen, we need to have

$$b^2 - 4c = (1 - \sqrt{m\alpha\beta})^2 - 4\sqrt{m\alpha\beta} < 0$$

Substitute $x = \sqrt{m\alpha\beta}$, then we should have

$$x^2 - 6x + 1 < 0$$

Problem 17 continued on next page. . .

Solving this inequity gives

$$3 - 2\sqrt{2} < x < 3 + 2\sqrt{2}$$

i.e., when we have $|x-3| < 2\sqrt{2}$, where $x = \sqrt{m\alpha\beta}$, there will be oscillation in V_n and C_n.

(e). The system of equations will look like

$$C_{n+1} = C_n - \beta V_n C_n + m \tag{4}$$

$$V_{n+1} = \frac{V_{\max} C_n^{\ell}}{K^{\ell} + C_n^{\ell}} \tag{5}$$

It's trivial to show that if we write (5) as

$$V_n = \frac{V_{\max} C_{n-1}^{\ell}}{K^{\ell} + C_{n-1}^{\ell}} \tag{6}$$

and plug (6) to (4), then we'll have

$$C_{n+1} = C_n - \beta \frac{V_{\max} C_{n-1}^{\ell} C_n}{K^{\ell} + C_{n-1}^{\ell}} + m \tag{7}$$

(f). The relationship between $\mathcal{S}(C)$ and ℓ is shown in Figure 8. K is chosen to be 50.

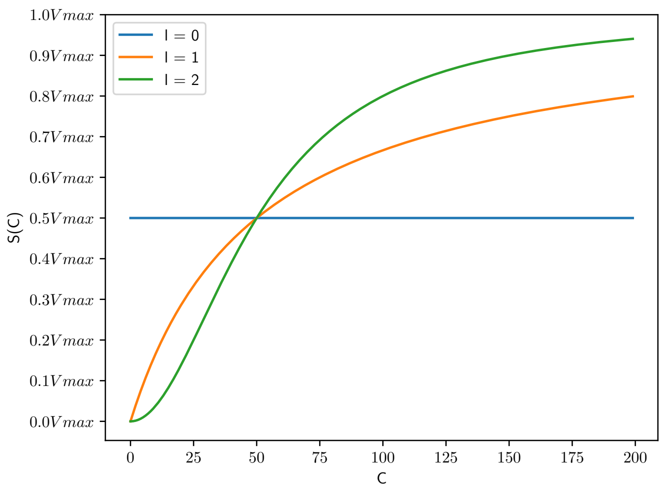


Figure 8: The relationship between $\mathcal{S}(C)$ and ℓ.

(g). {UNSOLVED}

(h). {UNSOLVED}

When $\ell = 1$,

$$C_{n+1} = C_n - \beta V_n C_n + m$$

$$V_{n+1} = \frac{V_{\max} C_n}{K + C_n}$$

and

$$C_{n+1} = C_n - \beta \frac{V_{\max} C_{n-1} C_n}{K + C_{n-1}} + m \tag{8}$$

Suppose $\bar{C} = C_{n+1} = C_n = C_{n-1}$ and plug it into (8),

$$\begin{aligned}
\bar{C} - \beta\frac{V_{\max}\bar{C}^2}{K+\bar{C}} + m &= \bar{C} \\
\frac{\bar{C}^2}{K+\bar{C}} &= \frac{m}{\beta V_{\max}} \\
\bar{C}^2 - \frac{m}{\beta V_{\max}}\bar{C} - \frac{m}{\beta V_{\max}}K &= 0
\end{aligned}$$

Let $\delta = \dfrac{m}{\beta V_{\max}}$, then

$$\begin{aligned}
\bar{C} &= \frac{1}{2}\left(\delta \pm \sqrt{\delta + 4K\delta}\right) \\
\bar{V} &= \frac{V_{\max}\bar{C}}{K+\bar{C}}
\end{aligned}$$

We need to make the roots for the characteristic equation have a magnitude < 1 again:

$$\lambda^2 - b\lambda + c = 0$$

where

$$\begin{aligned}
b &= \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} + \left.\frac{\partial f}{\partial V}\right|_{\bar{C},\bar{V}} \\
c &= \left.\frac{\partial f}{\partial C}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial V}\right|_{\bar{C},\bar{V}} - \left.\frac{\partial f}{\partial V}\right|_{\bar{C},\bar{V}} \left.\frac{\partial g}{\partial C}\right|_{\bar{C},\bar{V}}
\end{aligned}$$

Here

$$\begin{aligned}
f(C,V) &= C - \beta CV + m \\
g(C,V) &= \frac{V_{\max}C}{K+C}
\end{aligned}$$

So

$$\begin{aligned}
b &= (1-\beta\bar{V}) = 1 - \frac{\beta V_{\max}\bar{C}}{K+\bar{C}} \\
c &= \frac{\beta V_{\max}K\bar{C}}{(K+\bar{C})^2}
\end{aligned}$$

Recall again that we derived in the book that if both eigenvalues have magnitude less than 1, then

$$\begin{cases} 2 > 1 + c > |b|, & b^2 - 4c > 0 \\ 1 > c > |b/2|^2, & b^2 - 4c < 0 \end{cases}$$

(i). {UNSOLVED}