MATH564: MATHEMATICAL MODELING
Homework #3

Due on February 18 2020

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Problem 18

Recall the denotation in this problem:

$$\begin{aligned} p &= \text{frequency of allele } A \\ q &= \text{frequency of allele } a \\ u &= \text{frequency of } AA \text{ genotype} \\ v &= \text{frequency of } Aa \text{ genotype} \\ w &= \text{frequency of } aa \text{ genotype} \end{aligned}$$

(a). From the definition of u, v and w, it's clear that $u + v + w = 1$. The frequency of allele $A = u + \frac{1}{2}v$ and the frequency of allele $a = \frac{1}{2}v + w$. It's easy to see that $p + q = u + \frac{1}{2}v + \frac{1}{2}v + w = 1$.

(b). Table 3.1 is filled up and presented below as Table (there should be a ref). The new values are marked in red.

	Genotype	Frequency %	Fathers		
			AA	Aa	aa
			u	v	w
Mothers	AA	u	u^2	uv	uw
	Aa	v	uv	v^2	vw
	aa	w	uw	vw	w^2

(c). {UNSOLVED}

(d). Table 3.2 is filled up and presented below as Table (there should be a ref). The new values are marked in red.

Type of Parents	Genotype	Offspring Genotype Frequencies		
		AA	Aa	aa
$AA \times AA$	u^2	u^2	0	0
$AA \times Aa$	$2uv$	uv	uv	0
$AA \times aa$	$2uw$	0	$2uw$	0
$Aa \times Aa$	v^2	$v^2/4$	$v^2/2$	$v^2/4$
$Aa \times aa$	$2vw$	0	vw	vw
$aa \times aa$	w^2	0	0	w^2
	Total	$(u^2 + uv + v^2/4)$	$(uv + 2uw + vw + v^2/2)$	$(w^2 + vw + v^2/4)$

(e). It can be seen clearly in the table above that the frequencies are governed by

$$u_{n+1} = u_n^2 + u_n v_n + \frac{1}{4}v_n^2, \tag{1}$$

$$v_{n+1} = u_n v_n + 2u_n w_n + \frac{1}{2}v_n^2 + v_n w_n, \tag{2}$$

$$w_{n+1} = \frac{1}{4}v_n^2 + v_n w_n + w_n^2. \tag{3}$$

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(f).

$$
\begin{aligned}
u_{n+1} + v_{n+1} + w_{n+1} &= (u_n^2 + u_n v_n + \frac{1}{4}v_n^2) + (u_n v_n + 2u_n w_n + \frac{1}{2}v_n^2 + v_n w_n) + (\frac{1}{4}v_n^2 + v_n w_n + w_n^2) \\
&= u_n^2 + 2u_n v_n + v_n^2 + 2v_n w_n + 2u_n w_n + w_n^2 \\
&= (u_n + v_n)^2 + 2(u_n + v_n)w_n + w_n^2 \\
&= (u_n + v_n + w_n)^2
\end{aligned}
$$

According to the original setting we know that $u_n + v_n + w_n = 1$. Henceforth, we've proven that $u_{n+1} + v_{n+1} + w_{n+1} = 1$.

(g). At steady states $(\bar{u}, \bar{v}, \bar{w})$, we have

$$
\begin{aligned}
\bar{u} &= \bar{u}^2 + \bar{u}\bar{v} + \frac{1}{4}\bar{v}^2, && (4) \\
\bar{v} &= \bar{u}\bar{v} + 2\bar{u}\bar{w} + \frac{1}{2}\bar{v}^2 + \bar{v}\bar{w}, && (5) \\
\bar{w} &= \frac{1}{4}\bar{v}^2 + \bar{v}\bar{w} + \bar{w}^2. && (6)
\end{aligned}
$$

Divided $\bar{u}$ on both sides of (4) (since $\bar{u}$ is not very likely if not impossible to be 0 this is doable), we can get

$$1 = \bar{u} + \bar{v} + \frac{\bar{v}^2}{\bar{u}}$$

From (f) we know that $\bar{u} + \bar{v} + \bar{w} = 1$, so

$$
\begin{aligned}
\bar{u} + \bar{v} + \bar{w} &= \bar{u} + \bar{v} + \frac{\bar{v}^2}{\bar{u}} \\
\bar{w} &= \frac{\bar{v}^2}{\bar{u}} \\
\bar{u} &= \bar{v}^2/\bar{w}
\end{aligned}
$$

□

(h). From (f) we know that $w_{n+1} = 1 - u_{n+1} - v_{n+1}$. It can also be written as $w_n = 1 - u_n - v_n$. Substituting this back into (1) and (2) gives:

$$u_{n+1} = u_n^2 + u_n v_n + \frac{1}{4}v_n^2, \tag{7}$$

$$
\begin{aligned}
v_{n+1} &= u_n v_n + 2u_n(1 - u_n - v_n) + \frac{1}{2}v_n^2 + v_n(1 - u_n - v_n) \\
&= -2u_n^2 - 2u_n v_n - \frac{v_n^2}{2} + v_n + 2u_n
\end{aligned} \tag{8}
$$

(i). From (7) it's trivial to see that

$$u_{n+1} = \left(u_n + \frac{v_n}{2}\right)^2$$

And we have

$$
\begin{aligned}
v_{n+1} &= (u_n + \frac{1}{2}v_n)[2 - 2(u_n + \frac{1}{2}v_n)] \\
&= 2u_n + v_n - 2(u_n + \frac{1}{2}v_n)^2 \\
&= 2u_n + v_n - 2u_n^2 - 2u_n v_n - \frac{v_n^2}{2},
\end{aligned}
$$

which is exactly (8).

(j). Let X represents $u_n + \frac{v_n}{2}$, then:

$$\begin{aligned} u_{n+1} + \frac{v_{n+1}}{2} &= X^2 + X(1-X) \\ &= X \end{aligned} \qquad i.e., \left(u_{n+1} + \frac{v_{n+1}}{2}\right) = \left(u_n + \frac{v_n}{2}\right)$$

(k). Recall that

$$\begin{aligned} p_n &= u_n + \frac{v_n}{2} \\ q_n &= \frac{v_n}{2} + w_n \end{aligned}$$

So,

$$\begin{aligned} p_{n+1} &= u_{n+1} + \frac{v_{n+1}}{2} \\ q_{n+1} &= \frac{v_{n+1}}{2} + w_{n+1} \end{aligned}$$

We've known from (j) that

$$\left(u_{n+1} + \frac{v_{n+1}}{2}\right) = \left(u_n + \frac{v_n}{2}\right)$$

which means $p_{n+1} = p_n$. From (a) we know that $p+q=1$. Since p doesn't change accross generations, q also trivially won't change. I don't understand why we need to use $p^2 + 2pq + q^2 = 1$ to prove this.