

# **MATH564: MATHEMATICAL MODELING**

## **Homework #1**

Due on January 28 2020

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# MATH PART

## Problem 1

Consider the difference equation

$$x_{n+2} - 3x_{n+1} + 2x_n = 0.$$

(a). Show that the general solution to this equation is

$$x_n = A_1 + 2^n A_2$$

Now suppose that  $x_0 = 10$  and  $x_1 = 20$ . Then  $A_1$  and  $A_2$  must satisfy the system of equations

$$A_1 + 2^0 A_2 = x_0 = 10,$$

$$A_1 + 2^1 A_2 = x_1 = 20.$$

(b). Solve for  $A_1$  and  $A_2$  and find the solution to the above *initial value problem*.

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## Solution

(a). The characteristic solution is

$$\lambda^2 - 3\lambda + 2 = 0$$

The characteristic roots are

$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

By the principle of linear superposition, the general solution is indeed

$$x_n = A_1 + 2^n A_2$$

(b). Solving the system of equations

$$A_1 + 2^0 A_2 = x_0 = 10,$$

$$A_1 + 2^1 A_2 = x_1 = 20.$$

gives

$$A_1 = 0, \quad A_2 = 10.$$

Therefore, the solution should be

$$x_n = 10 \cdot 2^n$$

## Problem 2

Solve the following difference equations subject to the specified  $x$  values and sketch the solutions:

(a).  $x_n - 5x_{n-1} + 6x_{n-2} = 0$ ;  $x_0 = 2, x_1 = 5$ .

(c).  $x_n - x_{n-2} = 0$ ;  $x_1 = 3, x_2 = 5$ .

(e).  $x_{n+2} + x_{n+1} - 2x_n = 0$ ;  $x_0 = 6, x_1 = 3$ .

### Solution

(a). The characteristic function is

$$\lambda^2 - 5\lambda + 6 = 0$$

The characteristic roots are

$$\lambda_1 = 2, \quad \lambda_2 = 3.$$

The general solution should be

$$A_1 2^n + A_2 3^n = 0.$$

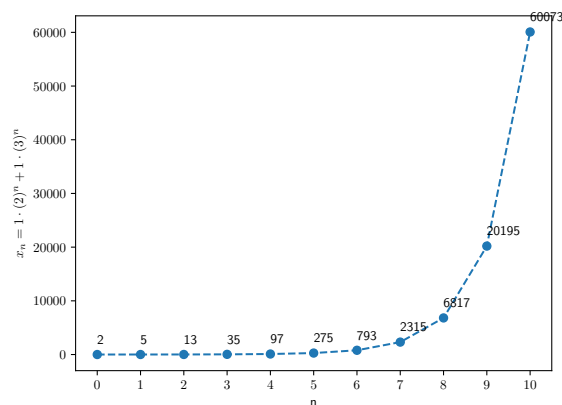
Plug in  $x_0 = 2$  and  $x_1 = 5$ , we have

$$\begin{cases} A_1 + A_2 = 2 \\ 2A_1 + 3A_2 = 5 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ A_2 = 1 \end{cases}$$

To sum up, the specific solution should be

$$x_n = 2^n + 3^n.$$

And the plot would look like



(c). The characteristic function is

$$\lambda^2 - 1 = 0$$

The characteristic roots are

$$\lambda_1 = -1, \quad \lambda_2 = 1.$$

The general solution should be

$$A_1(-1)^n + A_2 = 0$$

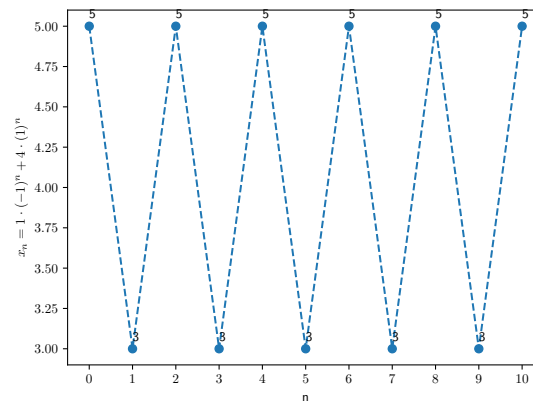
Plug in  $x_1 = 3$  and  $x_2 = 5$ , we have

$$\begin{cases} -A_1 + A_2 = 3 \\ A_1 + A_2 = 5 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ A_2 = 4 \end{cases}$$

To sum up, the specific solution should be

$$x_n = (-1)^n + 4.$$

And the plot would look like



(e). The characteristic function is

$$\lambda^2 + \lambda - 2 = 0$$

The characteristic roots are

$$\lambda_1 = 1, \quad \lambda_2 = -2.$$

The general solution should be

$$A_1 + A_2(-2)^n = 0$$

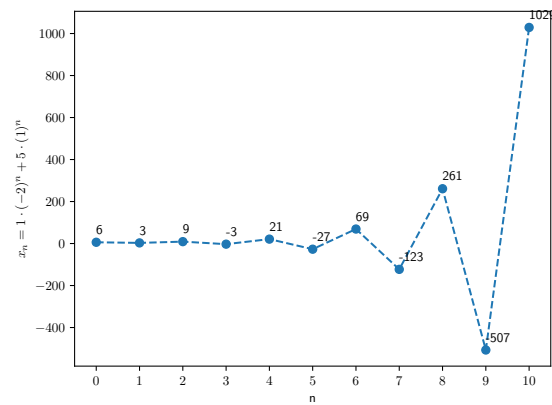
Plug in  $x_0 = 6$  and  $x_1 = 3$ , we have

$$\begin{cases} A_1 + A_2 = 6 \\ A_1 - 2A_2 = 3 \end{cases} \Rightarrow \begin{cases} A_1 = 5 \\ A_2 = 1 \end{cases}$$

To sum up, the specific solution should be

$$x_n = (-2)^n + 5.$$

And the plot would look like



## Problem 3

- (a). In Section 1.3 it was shown that the general solution to equation (16a, b) is (22) provided  $\lambda_1 \neq \lambda_2$ . Show that if  $\lambda_1 = \lambda_2 = \lambda$  then the general solution is

$$A_1\lambda^n + A_2n\lambda^n.$$

- (b). Solve and graph the solutions to each of the following equations or systems

(ii)  $x_{n+2} - 2x_{n+1} + x_n = 0,$

(iii)  $x_{n+1} = -3x_n - 2y_n,$   
 $y_{n+1} = 2x_n + y_n.$

## Solution

- (a). *Proof* by Mathematical Induction: We know from (16a, b) that

$$x_{n+2} = B_1x_{n+1} + B_2x_n$$

is true for all  $n$ , where  $B_1$  and  $B_2$  are some constants. Suppose that  $\lambda_0$  is the repeated solution to the characteristic equation

$$\lambda^2 - B_1\lambda - B_2 = 0$$

By Vieta's formulas, we know that  $B_1 = 2\lambda_0$  and  $B_2 = -\lambda_0^2$ .

Suppose that the general solution formula is true for all integers from 0 through  $k$ , then we have

$$x_k = A_1\lambda_0^k + A_2k\lambda_0^k,$$

$$x_{k-1} = A_1\lambda_0^{k-1} + A_2(k-1)\lambda_0^{k-1}.$$

For  $x_{k+1}$ , we have

$$\begin{aligned} x_{k+1} &= B_1x_k + B_2x_{k-1} \\ &= B_1(A_1\lambda^k + A_2k\lambda^k) + B_2(A_1\lambda^{k-1} + A_2(k-1)\lambda^{k-1}) \\ &= A_1(B_1\lambda^k + B_2\lambda^{k-1}) + A_2(B_1k\lambda^k + B_2(k-1)\lambda^{k-1}) \\ &= A_1(2\lambda^{k+1} - \lambda^{k+1}) + A_2(2k\lambda^{k+1} + (1-k)\lambda^{k+1}) \\ &= A_1\lambda^{k+1} + A_2(k+1)\lambda^{k+1} \end{aligned}$$

The truth of  $x_0$  and  $x_1$  is automatic since  $A_1$  and  $A_2$  are numbers selected intentionally to make the following equations true:

$$x_0 = A_1, \quad x_1 = (A_1 + A_2)\lambda.$$

□

(b). Note: the constants  $A_1$  and  $A_2$  are set as 1 for both questions.

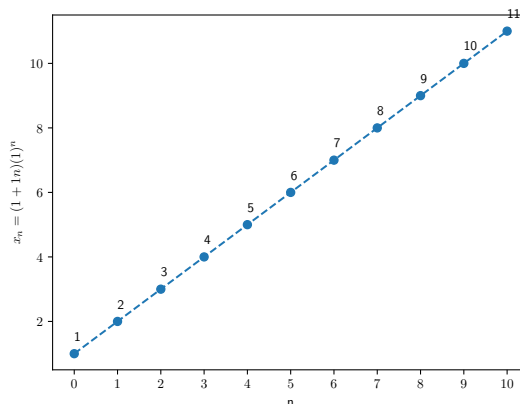
(ii) The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0.$$

So the roots should be  $\lambda_1 = \lambda_2 = \lambda = 1$ . The general solution would be

$$x_n = A_1 + A_2 n,$$

since the power of 1 is always 1.



(iii) The system of linear difference equation can be written in matrix form:

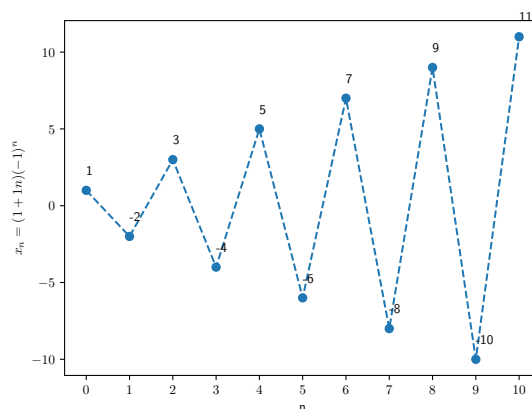
$$\begin{pmatrix} A_1 \lambda^{n+1} \\ A_2 \lambda^{n+1} \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} A_1 \lambda^n \\ A_2 \lambda^n \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} -3 - \lambda & -2 \\ 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Let determinant of the matrix of coefficients equal to 0, it leads to

$$\det \begin{pmatrix} -3 - \lambda & -2 \\ 2 & 1 - \lambda \end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 1 = 0$$

So we have two repeated eigenvalues  $\lambda_1 = \lambda_2 = \lambda = -1$ . And the general solution would be

$$x_n = (A_1 + A_2 n)(-1)^n,$$



## Problem 4

In Section 1.4 we determined that there are two values  $\lambda_1$  and  $\lambda_2$  and two vectors  $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$  and  $\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$  called *eigenvectors* that satisfy equation (29).

- (a). Show that this equation can be written in matrix form as

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M} \begin{pmatrix} A \\ B \end{pmatrix}$$

where  $\mathbf{M}$  is given by equation (27c).

- (b). Show that one way of expressing the eigenvectors in terms of  $a_{ij}$  is:

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix}$$

for  $a_{12} \neq 0$ .

- (c). Show that eigenvectors are defined only up to a multiplicative constant; i.e., if  $\mathbf{v}$  is an eigenvector corresponding to the eigenvalue  $\lambda$ , then  $\alpha \mathbf{v}$  is also an eigenvector corresponding to  $\lambda$  for all real numbers  $\alpha$ .

## Solution

- (a). Equation (29) in the book is

$$0 = A(a_{11} - \lambda) + B(a_{12})$$

$$0 = A(a_{21}) + B(a_{22} - \lambda)$$

Re-arranging the terms with  $\lambda$  to the left-hand side gives

$$\lambda A = a_{11}A + a_{12}B$$

$$\lambda B = a_{21}A + a_{22}B$$

which is equivalent of

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M} \begin{pmatrix} A \\ B \end{pmatrix}$$

- (b). *Proof.* Suppose the eigenvalue is  $\lambda_i$  and the eigenvector corresponding to it is  $\mathbf{v}_i = \begin{pmatrix} A_i \\ B_i \end{pmatrix}$ . Then we know by the definition of eigenvector that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

Suppose  $\mathbf{v}_i = \begin{pmatrix} 1 \\ \frac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix}$ , then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{v}_i = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix} = \begin{pmatrix} \lambda_i \\ a_{21} + \frac{a_{22}(\lambda_i - a_{11})}{a_{12}} \end{pmatrix}$$



And

$$\lambda_i \mathbf{v}_i = \begin{pmatrix} \lambda_i \\ \frac{\lambda_i^2 - a_{11}\lambda_i}{a_{12}} \end{pmatrix}$$

Suppose

$$\begin{pmatrix} \lambda_i \\ a_{21} + \frac{a_{22}(\lambda_i - a_{11})}{a_{12}} \end{pmatrix} = \begin{pmatrix} \lambda_i \\ \frac{\lambda_i^2 - a_{11}\lambda_i}{a_{12}} \end{pmatrix}$$

since we have  $a_{12} \neq 0$ , there will be

$$\lambda_i^2 - a_{11}\lambda_i = a_{21}a_{12} + a_{22}\lambda_i - a_{11}a_{12}$$

which gives

$$\lambda_i = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

where

$$\beta = a_{11} + a_{22}, \quad \gamma = (a_{11}a_{22} - a_{21}a_{12}).$$

And that's exactly the value of  $\lambda_i$ .

Thus we have shown that  $\mathbf{v}_i = \begin{pmatrix} 1 \\ \frac{\lambda_i - a_{11}}{a_{12}} \end{pmatrix}$  satisfies  $\mathbf{M}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ , i.e. it is indeed one way of expressing the eigenvector.

□

(c). *Proof.* Suppose that  $\mathbf{v}$  is the eigenvector corresponding to eigenvalue  $\lambda$ , then it is known that

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$$

For  $\alpha\mathbf{v}$ , we know from the properties of scalar multiplication

$$\mathbf{M}(\alpha\mathbf{v}) = \alpha(\mathbf{M}\mathbf{v})$$

and

$$\lambda(\alpha\mathbf{v}) = \alpha(\lambda\mathbf{v})$$

It's clear that the two quantities are equal. Thus, by the definition of eigenvectors,  $\alpha\mathbf{v}$  is also an eigenvector.

□

## Problem 5

The following complex numbers are expressed as  $\lambda = a + bi$ , where  $a$  is the real part and  $b$  is the imaginary part. Express the number in polar form  $\lambda = re^{i\theta}$ , and use your result to compute the indicated power  $\lambda^n$  of this complex number. Sketch  $\lambda^n$ , for  $n = 0, 1, 2, 3, 4$  as a function of  $n$ .

(a).  $1 + i$

(d).  $-1 + \sqrt{3}i$

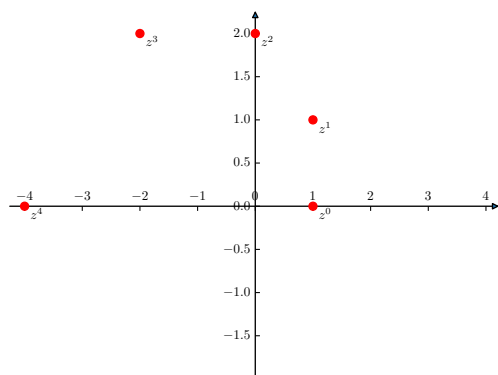
(b).  $1 - i$

(e).  $-\frac{1}{2} - \frac{i}{2}$

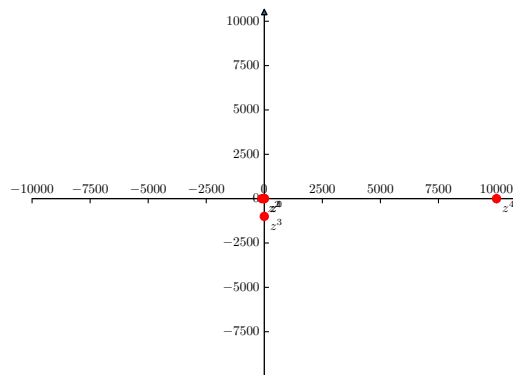
(c).  $10i$

## Solution

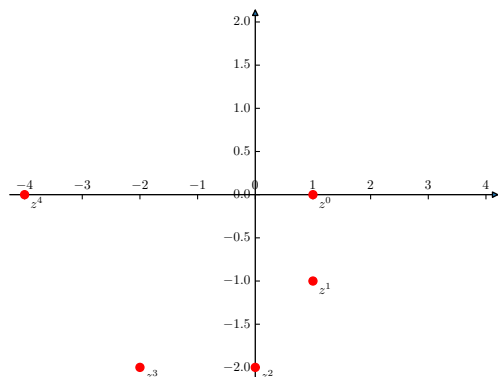
(a). **Polar form:**  $z^n = (\sqrt{1^2 + 1^2}e^{i \tan^{-1}(1)})^n$   
 $= (\sqrt{2})^n e^{i\pi n/4}$ .



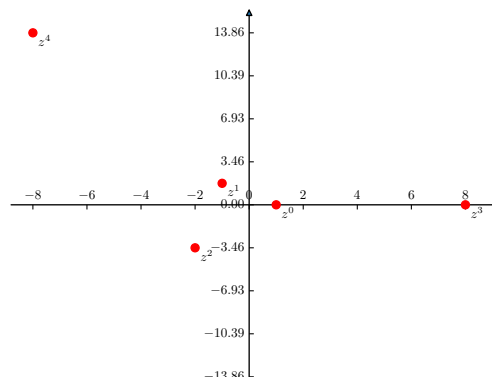
(c). **Polar form:**  $z^n = (\sqrt{0^2 + 10^2}e^{i \tan^{-1}(10/0)})^n$   
 $= 10^n e^{i\pi n/2}$ .



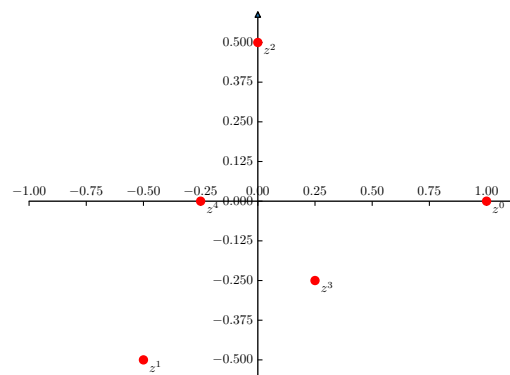
(b). **Polar form:**  $z^n = (\sqrt{1^2 + (-1)^2}e^{i \tan^{-1}(-1)})^n$   
 $= (\sqrt{2})^n e^{-i\pi n/4} (\cos > 0 \& \sin < 0)$



(d). **Polar form:**  $z^n = (\sqrt{(-1)^2 + (\sqrt{3})^2}e^{i \tan^{-1}(-\sqrt{3})})^n$   
 $= 2^n e^{2i\pi n/3}$



(e). **Polar form:**  $z^n = (\sqrt{(-1/2)^2 + (-1/2)^2}) e^{i \tan^{-1}(1)}$   
 $= (\frac{1}{\sqrt{2}})^n e^{i 5\pi n/4} (\sin \& \cos < 0)$



## Problem 6

*Complex eigenvalues.* Solve and graph the solutions to the following difference equations.

(a).  $x_{n+2} + x_n = 0$ ,

(b).  $x_{n+2} - x_{n+1} + x_n = 0$ ,

## Solution

(a). The characteristic equation is

$$\lambda^2 + \lambda = 0,$$

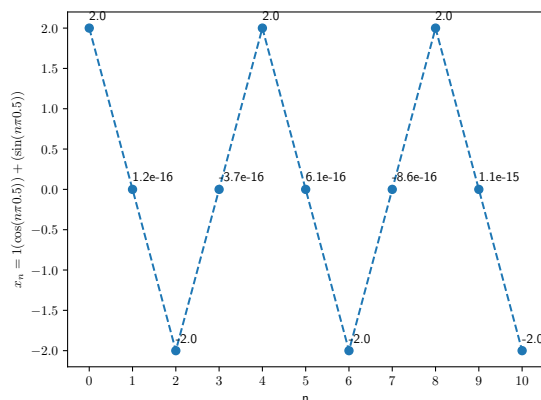
with the complex conjugate roots  $\lambda = 0 \pm i$ . Thus  $a = 0$  and  $b = 1$ , so that

$$r = \sqrt{a^2 + b^2} = 1,$$

$$\theta = \tan^{-1}(1/0) = \pi/2.$$

Thus the real-valued solution is

$$x_n = C_1 \cos(n\pi/2) + C_2 \sin(n\pi/2)$$



(b). The characteristic equation is

$$\lambda^2 - \lambda + 1 = 0,$$

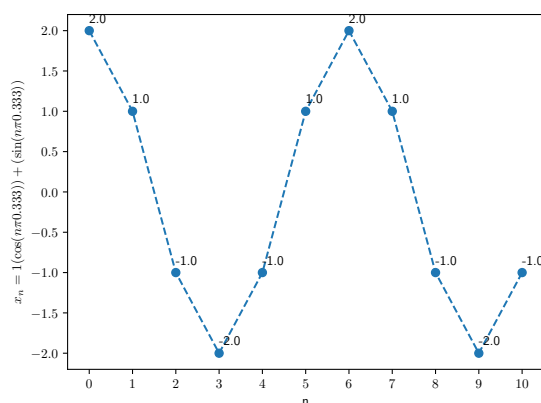
with the complex conjugate roots  $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ . Thus  $a = \frac{1}{2}$  and  $b = \frac{\sqrt{3}}{2}$ , so that

$$r = \sqrt{a^2 + b^2} = 1,$$

$$\theta = \tan^{-1}(b/a) = \pi/3.$$

Thus the real-valued solution is

$$x_n = C_1 \cos(n\pi/3) + C_2 \sin(n\pi/3)$$



## Problem 7

- Consider the growth of an aphid population described in Section 1.1. If the fractional mortality of aphids is 80% and the sex ratio (ratio of females to the total number of aphids) is 50%, what minimum fecundity  $f$  is required to prevent extinction.
- Establish a general condition on the fecundity of aphids to guarantee population growth given a fixed survivorship and a known sex ratio.

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## Solution

- Extinction will be prevented if the growth rate  $\lambda = rf(1 - m) > 1$ . Solving the inequity by plugging in given values gives  $f = 10$ , i.e. a fecundity of 10 progeny per female.
- In general, we need  $f \geq \frac{1}{r(1 - m)}$

## BIO PART

### Problem 16: *Red blood cell production*

(a). We have

$$R_{n+1} = (1 - f)R_n + M_n, \quad (1)$$

$$M_{n+1} = \gamma f R_n \quad (2)$$

Equation (2) can be re-written into

$$M_n = \gamma f R_{n-1} \quad (3)$$

Substitute (3) into (1), we'll get

$$R_{n+1} = (1 - f)R_n + \gamma f R_{n-1}$$

(b). From (a), it's easy to see that the characteristic function is

$$\lambda^2 - (1 - f)\lambda - \gamma f$$

Hence, the eigenvalues are indeed given by

$$\lambda_{1,2} = \frac{(1 - f) \pm \sqrt{(1 - f)^2 + 4\gamma f}}{2}$$

Since  $f$  is definitely a fraction value between 0 and 1 (it's horrible to think that your spleen removes 100% of your RBCs in circulation), and  $\gamma$  is larger than 0, so that there will be RBC produced.

Observe the boundary stated above, which makes the model biologically reasonable, we know

$$\sqrt{(1 - f)^2 + 4\gamma f} > \sqrt{(1 - f)^2} = (1 - f)$$

Therefore, we should have a positive and a negative eigenvalue, i.e.,  $\lambda_1 > 0$  and  $\lambda_2 < 0$ .

(c). Suppose the positive eigenvalue  $\lambda_1 = 1$ , then

$$\begin{aligned} \frac{(1 - f) + \sqrt{(1 - f)^2 + 4\gamma f}}{2} &= 1 \\ \sqrt{(1 - f)^2 + 4\gamma f} &= 1 + f \\ (1 - f)^2 + 4\gamma f &= (1 + f)^2 \\ 4\gamma f &= 4f \\ \gamma &= 1 \end{aligned}$$

(d). From (c) we know  $\gamma = 1$ , so

$$\sqrt{(1 - f)^2 + 4\gamma f} = 1 + f$$

Then

$$\lambda_2 = -2f/2 = -f$$

And the solution

$$R_n = A\lambda_1^n + B\lambda_2^n = A + B(-f)^n$$

The solution will oscillate around constant  $A$ , and since  $f$  is a fraction the oscillation amplitude will decrease as  $n$  increases. This means that the RBC number in circulation will eventually reach an equilibrium.

**Problem 17:** *Annual plant propagation*

- (a). The model for annual plants was condensed into a single equation (15) for  $p_n$ , the number of plants. Show that it can also be written as a single equation in  $S_n^1$ .

*Solution:* We have

$$p_n = \alpha s_n^1 + \beta s_n^2 \quad (4)$$

$$\overline{s_n^1} = (1 - \alpha)s_n^1 \quad (5)$$

$$\overline{s_n^2} = (1 - \alpha)s_n^2 \quad (6)$$

$$s_n^0 = \gamma p_n \quad (7)$$

$$s_{n+1}^1 = \sigma s_n^0 \quad (8)$$

$$s_{n+1}^2 = \sigma \overline{s_n^1} \quad (9)$$

Make several substitutions:

$$(7) \rightarrow (8) \quad S_{n+1}^1 = \sigma \gamma p_n \quad (10)$$

$$(5) \rightarrow (9) \quad S_n^2 = \sigma(1 - \alpha)S_{n-1}^1 \quad (11)$$

$$(4) \rightarrow (10) \quad S_{n+1}^1 = \sigma \gamma (\alpha s_n^1 + \beta s_n^2) \quad (12)$$

$$(11) \rightarrow (12) \quad S_{n+1}^1 = \sigma \gamma (\alpha s_n^1 + \beta \sigma(1 - \alpha)S_{n-1}^1) \quad (13)$$

And equation (13) should be the answer.

- (b). Seeds produced this year (year  $n$ ) which survived the winter and will germinate next year (year  $(n+1)$ )
- (c). We know from the book that

$$\lambda_{1,2} = \frac{\sigma \gamma \alpha}{2} (1 \pm \sqrt{1 + \delta})$$

where

$$\delta = \frac{4\beta}{\gamma\alpha} \left( \frac{1}{\alpha} - 1 \right)$$

We know that  $\delta$  is a positive quantity since  $\alpha < 1$ . So we have a positive eigenvalue and a negative eigenvalue.

If we want the population to increase in size, we should have the positive eigenvalue  $\lambda_1 > 1$ .

Plug in  $\alpha = \beta = 0.001$  and  $\sigma = 1$ , we'll have

$$\begin{aligned} \delta &= \frac{4}{\gamma} \left( \frac{1}{0.001} - 1 \right) \approx \frac{4000}{\gamma} \\ \lambda_1 &= \frac{0.001\gamma}{2} (1 + \sqrt{1 + \delta}) \end{aligned} \quad \rightarrow \quad \lambda_1 = \frac{\gamma}{2000} \left( 1 + \sqrt{1 + \frac{4000}{\gamma}} \right) > 1$$

Solving the inequity gives

$$\gamma > 500$$

- (d). In case (1), the parameters are:

$$\alpha = 0.5, \quad \beta = 0.25, \quad \gamma = 2.0, \quad \sigma = 0.8,$$

we can compute that

$$\begin{aligned} \delta &= \frac{4\beta}{\gamma\alpha} \left( \frac{1}{\alpha} - 1 \right) = 1 \\ \lambda_1 &= \frac{\sigma\gamma\alpha}{2} (1 + \sqrt{1 + \delta}) = 0.4(1 + \sqrt{2}) \\ &\approx 0.97 < 1 \end{aligned}$$

So the population decreases.

An in case (2), the parameters are:

$$\alpha = 0.6, \quad \beta = 0.3, \quad \gamma = 2.0, \quad \sigma = 0.8,$$

we can compute that

$$\begin{aligned} \delta &= \frac{4}{\gamma} \frac{\beta}{\alpha} \left( \frac{1}{\alpha} - 1 \right) = \frac{2}{3} \\ \lambda_1 &= \frac{\sigma\gamma\alpha}{2} (1 + \sqrt{1 + \delta}) = 0.48(1 + \sqrt{5/3}) \\ &\approx 1.10 > 1 \end{aligned}$$

So the population increases.

(e). Consider the positive eigenvalue as

$$\lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2}$$

where

$$a = \alpha\sigma\gamma, \quad b = \beta\sigma^2(1 - \alpha)\gamma$$

If we want  $\lambda_1 > 1$ , then

$$\begin{aligned} \frac{a + \sqrt{a^2 + 4b}}{2} &> 1 \\ \sqrt{a^2 + 4b} &> 2 - a \\ a^2 + 4b &> a^2 - 4a + 4 \\ a + b &> 1 \end{aligned}$$

(Note: If  $(2 - a) < 0$  then the inequity will always be trivially true, thus it's not of too much interest. We only consider  $(2 - a) > 0$  here.) Substitute back  $a = \alpha\sigma\gamma$  and  $b = \beta\sigma^2(1 - \alpha)\gamma$ , we'll have

$$\gamma > \frac{1}{\alpha\sigma + \beta\sigma^2(1 - \alpha)}$$

**Problem 18:** *Blood  $CO_2$  and ventilation*

(a). We know from the book that

$$C_{n+1} = C_n - \mathcal{L}(V_n, C_n) + m, \quad (14)$$

$$V_{n+1} = \mathcal{S}(C_n). \quad (15)$$

From the question description we know that

$$\mathcal{L}(V_n, C_n) = \beta V_n, \quad V_{n+1} = \alpha C_n$$

Make some simple substitution, we'll have

$$C_{n+1} = C_n - \alpha\beta C_n + m$$

i.e.

$$C_{n+1} - C_n + \alpha\beta C_n = m$$

(b). (1) Suppose  $C_n = m/\alpha\beta$ , then

$$C_{n+1} - C_n + \alpha\beta C_n = m/\alpha\beta - m/\alpha\beta + \alpha\beta \cdot \frac{m}{\alpha\beta} = m$$

Hence,  $C_n = m/\alpha\beta$  is a particular solution.

(2) Let  $m = 0$ , then it'll turn into a homogeneous problem. The characteristic equation would be

$$\lambda^2 - \lambda + \alpha\beta = 0$$

with eigenvalues

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4\alpha\beta}}{2},$$

which is the *complementary/homogeneous solution*.

Combined the complementary solution with the particular solution, the general solution would be

$$C_n = \frac{m}{\alpha\beta} + C_1 \left( \frac{1 + \sqrt{1 - 4\alpha\beta}}{2} \right)^n + C_2 \left( \frac{1 - \sqrt{1 - 4\alpha\beta}}{2} \right)^n$$

(c). (1) Assume that  $4\alpha\beta < 1$ , then  $\alpha\beta < 1/4$ . This implies

$$\mathcal{L}(V_n, C_n) = \beta V_n = \alpha\beta C_{n-1} < \frac{C_{n-1}}{4}$$

i.e., the amount of  $CO_2$  loss / breathed out will not be larger than  $\frac{1}{4}$  of the blood  $CO_2$  amount.

And, since  $4\alpha\beta < 1$ , then  $0 < \sqrt{1 - 4\alpha\beta} < 1$ . Let  $\sqrt{1 - 4\alpha\beta} = \delta$ , then

$$\lambda_1 = \frac{1 + \delta}{2} \in \left( \frac{1}{2}, 1 \right)$$

$$\lambda_2 = \frac{1 - \delta}{2} \in \left( 0, \frac{1}{2} \right)$$

Therefore, the absolute value of each eigenvalue  $|\lambda_i| < 1$ , implying

$$\lim_{n \rightarrow \infty} (C_n) = \frac{m}{\alpha\beta}$$

since the power of a fraction will diminish towards 0. Under this scenario, a steady state will eventually be established regardless of the initial conditions. And the steady ventilation rate would be

$$V_n = \alpha \frac{m}{\alpha\beta} = \frac{m}{\beta}$$



- (2) Assume that  $4\alpha\beta > 1$ , then  $1 - 4\alpha\beta < 0$ , which means that the characteristic equation would have complex eigenvalues. The conjugated complex eigenvalues  $\lambda = a \pm bi$  have

$$a = \frac{1}{2}, \quad b = \frac{\sqrt{4\alpha\beta - 1}}{2}$$

i.e.

$$r = \sqrt{a^2 + b^2} = \sqrt{\alpha\beta}, \quad \theta = \tan^{-1}(b/a) = \tan^{-1}(\sqrt{4\alpha\beta - 1})$$

Then

$$C_n = r^n (C_1 \cos(n\theta) + C_2 \sin(n\theta)) = (\sqrt{\alpha\beta})^n \sin(n\theta + \phi)$$

where  $\tan \phi = C_1/C_2$ . It is clear that the solution will oscillate at frequency  $f = \theta/2\pi$

If  $\alpha\beta \geq 1$ , then the oscillation will increase in magnitude. When  $\alpha\beta = 1$ , the oscillation frequency would be  $f = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ .

This result could correspond to Cheyne-Stokes respiration, which is a disorder characterized by recurrent oscillation between apnea and hyperpnea (May, 1978 and Naughton, 1998).

- (d). For now the ventilation rate is proportional to blood  $CO_2$  concentration. In real biological systems, body will likely adjust the rate of breath according to  $C_n$ , just like the relationship between air friction and velocity. It's suitable to suppose that

$$\mathcal{S}(C_n) = \alpha C_n^2$$

Then the equation will turn into

$$C_{n+1} = C_n - \alpha\beta C_{n-1}^2 + m \quad C_{n+1} - C_n + \alpha\beta C_{n-1}^2 = m$$

Suppose  $C_{n+1} = C_n$ , then we can derive a steady-state where

$$C_n = \sqrt{\frac{m}{\alpha\beta}}$$

It would also be interesting to think about the equilibrium shifting between bicarbonates and  $CO_2$  and the formation of carbamino hemoglobin ( $HbCO_2$ ). But deriving a nice math equation with respect to these factors is out of our ability.

## Problem 20

Take  $p_n^1$  as the number of newborns, then it should be compute by

$$\begin{aligned} p_{n+1}^1 &= \sum_{i=1}^m \left( \begin{array}{c} \text{number of births} \\ \text{in age class } i \\ \text{in last year } (n) \end{array} \right) \left( \begin{array}{c} \text{number of individuals} \\ \text{in age class } i \\ \text{in last year } (n) \end{array} \right) \\ &= \sum_{i=1}^m \alpha_i p_n^i \end{aligned}$$

And for each age class  $1 < i < m$ , it can be computed by

$$p_{n+1}^i = \sigma_{i-1} p_n^{i-1}$$

Since  $m$  is the oldest age class,  $p_n^m$  will not survive to next year. The system of equations can be written in the form of

$$\begin{aligned} p_{n+1}^1 &= \alpha_1 p_n^1 + \alpha_2 p_n^2 + \alpha_3 p_n^3 + \cdots + \alpha_{m-1} p_n^{m-1} + \alpha_m p_n^m \\ p_{n+1}^2 &= \sigma_1 p_n^1 \\ p_{n+1}^3 &= \sigma_2 p_n^2 \\ &\vdots \\ p_{n+1}^m &= \sigma_{m-1} p_n^{m-1} \end{aligned}$$

It's clear that it can be written as the matrix form:

$$\mathbf{P}_{n+1} = \mathbf{A} \mathbf{P}_n$$

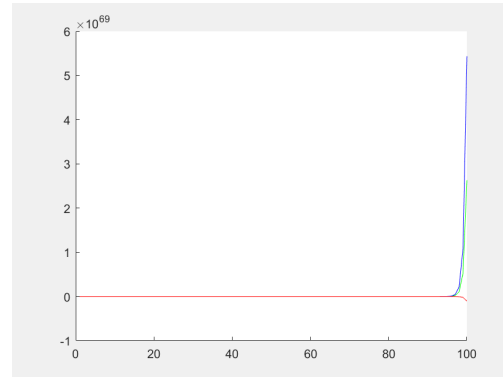
where

$$\mathbf{P}_k = \begin{pmatrix} p_k^1 \\ p_k^2 \\ \vdots \\ p_k^m \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} & \alpha_m \\ \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{m-1} & 0 \end{pmatrix}$$

# CODE PART

## Problem 6

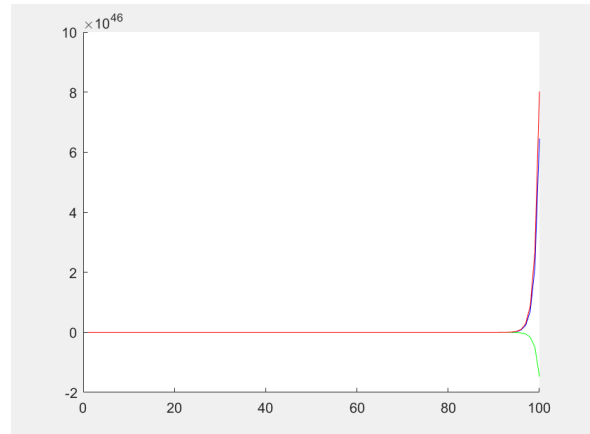
- (a). As seen in this graph of 6a, each initial condition produces a unique exponential mapping, however, all the mappings follow the same pattern and are unstable.



$$x_{n+2} - 7x_{n+1} + 10x_n = 0$$

```
clear all
eig1=5; eig2=2;
x(1)=0;
x(2)=1;
x(3)=0.5;
x(4)=-0.1;
x(5)=2;
x(6)=3;
n=linspace(1,500);
for k=1:2:6
    syms A B
    [solA,solB] = solve(
        A*eig1^x(k) + B*eig2^x(k) == x(k),
        A*eig1^x(k+1) + B*eig2^x(k+1) == x(k+1));
    n=[1:1:100];
    if k == 1
        hold on
        g=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,g,'g')
    elseif k == 3
        hold on
        b=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,b,'b')
    else
        hold on
        r=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,solA.*eig1.^n + solB.*eig2.^n,'r')
    end
end
end
```

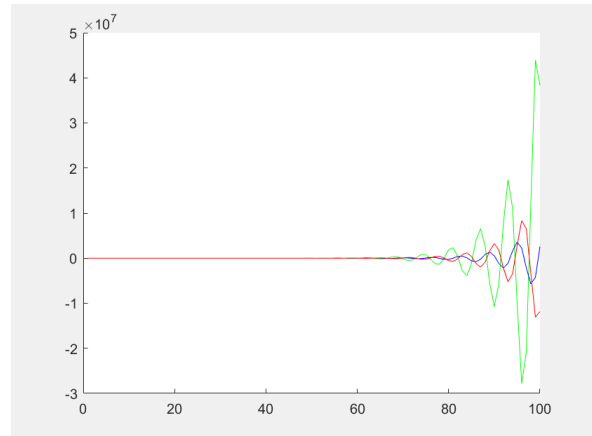
(c). As seen in this graph of 6c, each initial condition produces a unique exponential mapping, however, all the mappings follow the same pattern and are unstable.



$$x_{n+2} - \sigma_1 x_{n+1} + \sigma_2 x_n = 0$$

```
clear all
s1= -1;
s2= -2;
beta= 3;
eig1=(1/2)*(-s1+(sqrt(s1^2 -4*s2*beta)))
eig2=(1/2)*(-s1-(sqrt(s1^2 -4*s2*beta)))
x(1)=7;
x(2)=1;
x(3)=0.5;
x(4)=-0.1;
x(5)=2;
x(6)=3;
n=linspace(1,500);
for k=1:2:6
    syms A B
    [solA,solB] = solve(
        A*eig1^x(k) + B*eig2^x(k) == x(k),
        A*eig1^x(k+1) + B*eig2^x(k+1) == x(k+1));
    n=[1:1:100];
    if k == 1
        hold on
        g=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,g,'g')
    elseif k == 3
        hold on
        b=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,b,'b')
    else
        hold on
        r=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,solA.*eig1.^n + solB.*eig2.^n,'r')
    end
end
end
```

- (f). As seen in this graph of 6f, each initial condition produces a unique oscillating mapping, however, all the mappings follow the same pattern and are unstable.



$$x_{n+2} - \frac{5}{4}x_{n+1} + \frac{11}{8}x_n = 0$$

```
clear all
eig1=(1/2)*(1.25+sqrt(1.5625-5.5))
eig2=(1/2)*(1.25-sqrt(1.5625-5.5))
x(1)=7;
x(2)=1;
x(3)=0.5;
x(4)=-0.1;
x(5)=2;
x(6)=3;
n=linspace(1,500);

for k=1:2:6
    syms A B
    [solA,solB] = solve(
        A*eig1^x(k) + B*eig2^x(k) == x(k),
        A*eig1^x(k+1) + B*eig2^x(k+1) == x(k+1));
    n=[1:1:100];
    if k == 1
        hold on
        g=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,g,'g')
    elseif k == 3
        hold on
        b=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,b,'b')
    else
        hold on
        r=solA.*eig1.^n + solB.*eig2.^n;
        plot(n,solA.*eig1.^n + solB.*eig2.^n,'r')
    end
end
end
```