MATH564: MATHEMATICAL MODELING
Homework #2

Due on February 7 2020

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Problem 1

(a). Sample answer

Problem 2

We use *cobwebbing method* to visualize the approximate behavior of each equation, with an initial value slight off (add perturbation) the steady state, and guess from the plot whether the steady state is stable or not. To be more confident, the conclusions will be tested by calculation.

(a). The steady state $\bar{x}$ for equation $x_{n+1} = rx_n(1 - x_n)$ is $\bar{x} = 0$. Since there's one arbitrary constant r, whether the steady state is stable or not depends on the value of r.

We first try $r = 2$: It's clear in Figure 1 that when $r = 2$, $\bar{x} = 0$ is not a stable steady state.

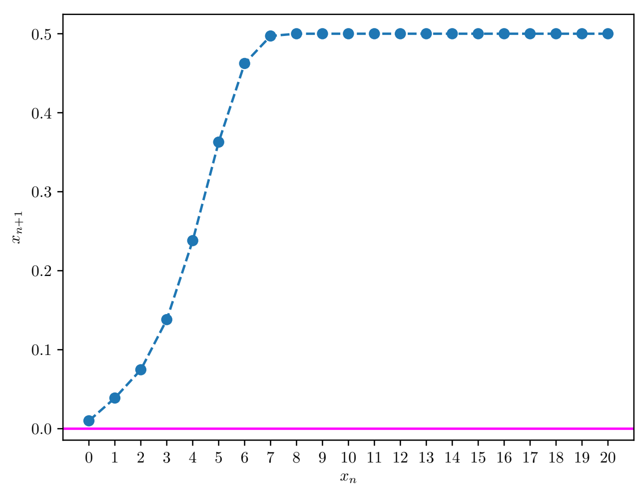

Figure 1: The behaviour of equation $x_{n+1} = 2x_n(1 - x_n)$. The steady state is marked by the horizontal magenta line.

The nexy try is $r = 0.5$: It seems that when $r = 0.5$, $\bar{x} = 0$ is a stable steady state. It's intuitive to

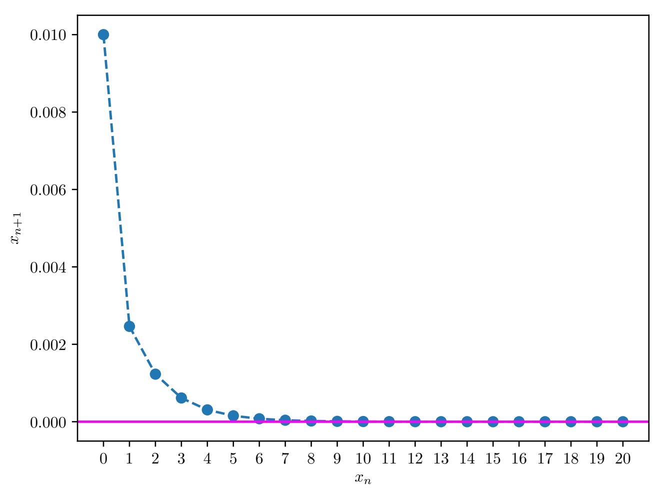

Figure 2: The behaviour of equation $x_{n+1} = \dfrac{x_n(1 - x_n)}{2}$, The steady state is marked by the horizontal magenta line.

guess that probably $r < 1$ will make $\bar{x} = 0$ be stable.

Problem 2 continued on next page...

Further exploration is shown in Figure 3 below:

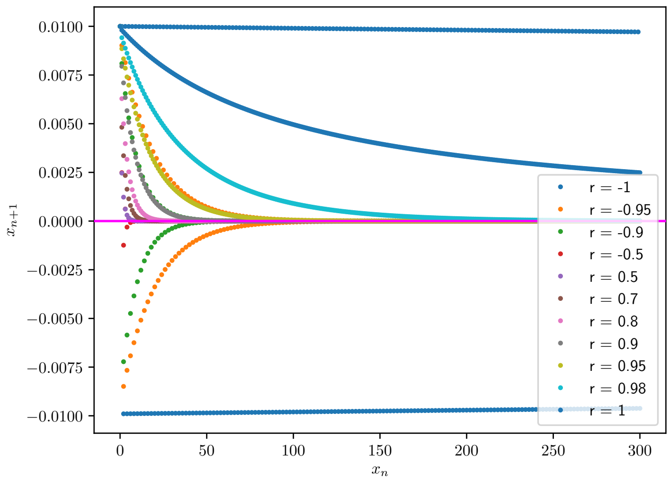

Figure 3: The behaviour of equations $x_{n+1} = rx_n(1 - x_n)$, with $r \in \{-1, -0.95, -0.5, 0.5, 0.7, 0.8, 0.9, 0.95, 0.98, 1\}$. It can be seen that the all equations except for the one with $r = -1$ converges to the steady state (marked by the magenta horizontal line).

It seems that for $-1 < r < 1$, or $|r| < 1$, $\bar{x} = 0$ will be a stable steady state. That is to say, even with small perturbation (here is $x' = 0.01$) present, the equation still can eventually reach the steady state.

Test

Recall

$$\boxed{\bar{x} \text{ is a stable steady state} \iff \left| \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\bar{x}} \right| < 1}$$

We can derive that

$$\begin{aligned} &\left| \left.\frac{\mathrm{d}}{\mathrm{d}x}\left(rx(1-x)\right)\right|_{\bar{x}} \right| \\ &= \left| (r - 2rx)|_{\bar{x}} \right| \\ &= |r| < 1 \end{aligned}$$

It confirms with our conclusion that when $|r| < 1$, $\bar{x} = 0$ will be a stable steady state.

(b). Plotting the figure of equation $x_{n+1} = -x_n^2(1 - x_n)$ with initial condition $x_0 = 1.6$ slightly off the given steady state $\bar{x} = (1 + \sqrt{5})/2$ gives:

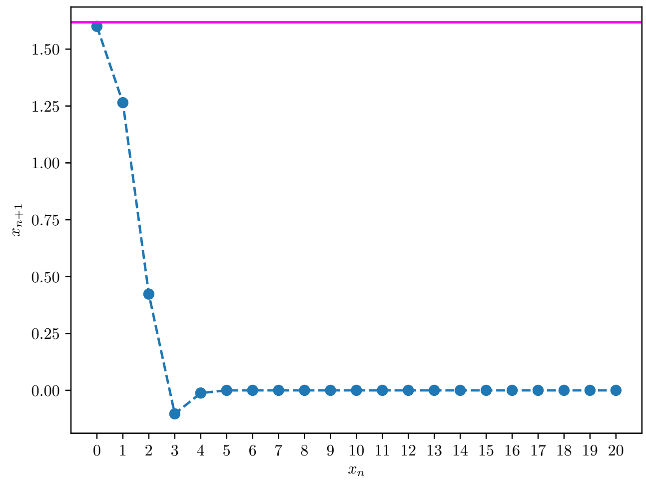

Figure 4: The behavior of equation $x_{n+1} = -x_n^2(1 - x_n)$

We can see that the steady state is unstable. The calculation of condition:

$$\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}x} \left(-x^2(1-x)\right)\Big|_{\bar{x}} \right| \\
&= \left| (-2x + 3x^2)\big|_{\bar{x}} \right| \\
&= \frac{7 + \sqrt{5}}{2} > 1
\end{aligned}$$

also confirms that $\bar{x} = (1 + \sqrt{5})/2$ is not stable.

(c). Plotting the figure of equation $x_{n+1} = 1/(2 + x_n)$ with initial condition $x_0 = 0.4$ slightly off the given steady state $\bar{x} = \sqrt{2} - 1$ gives:

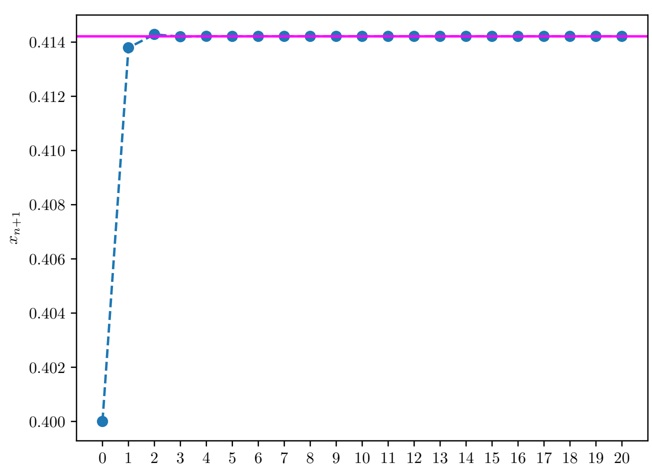

Figure 5: The behavior of equation $x_{n+1} = 1/(2 + x_n)$

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We can see that the steady state is stable. The calculation of condition:

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}x}\left(1/(2+x)\right)\Big|_{\bar{x}}\right|\\ &=\left|\left(\frac{-1}{(2+x)^2}\right)\Big|_{\bar{x}}\right|\\ &=3-2\sqrt{2}<1\end{aligned}$$

also confirms that $\bar{x}=\sqrt{2}-1$ is stable.

(d). Plotting the figure of equation $x_{n+1}=x_n\ln x_n^2$ with initial condition $x_0=1.648$ slightly off the given steady state $\bar{x}=e^{1/2}$ gives:

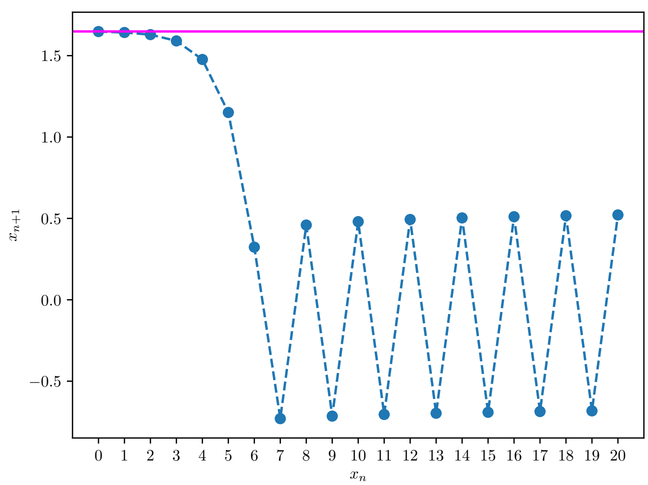

Figure 6: The behavior of equation $x_{n+1}=x_n\ln x_n^2$

We can see that the steady state is unstable. The calculation of condition:

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}x}\left(x\ln x^2\right)\Big|_{\bar{x}}\right|\\ &=2.5>1\end{aligned}$$

also confirms that $\bar{x}=e^{1/2}$ is not stable.

Problem 4

(a). The relationship between λ and K is depicted below in Figure 7. It's clear in Figure 7 that $\lambda > 1$ when

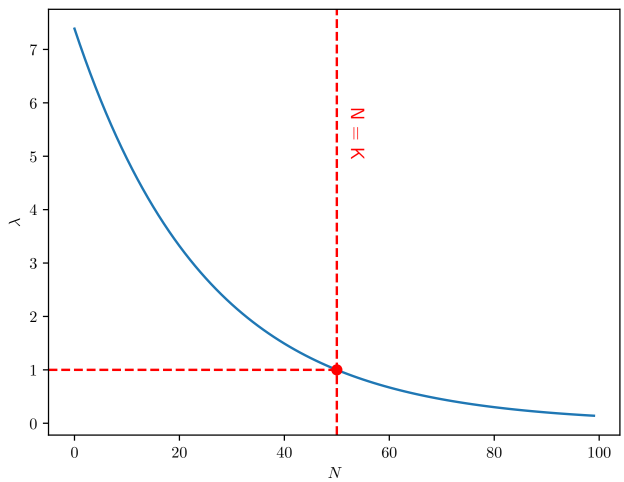

Figure 7: The relationship between $\lambda = \exp[r(1 - N_t/K)]$ and K. λ, the population's growth rate, is described as a fuction of K. Here K is selected to be 50 and r is selected to be 2.

$N < K$, i.e., the population can continue to grow and reproduce only if $N < K$.

(b). Plug in $\bar{N} = K$ into the equation, we'll have

$$N_t = \bar{N} = K,$$
$$N_{t+1} = N_t \exp[r(1 - N_t/K)] = K \exp[r(1 - K/K)] = K$$

Therefore, for $\bar{N}$ can make $N_{t+1} = N_t$ for two successive timepoints. It is indeed the steady state.

(c). The condition of stability for the steady state depends on:

$$\left| \frac{\mathrm{d}}{\mathrm{d}N} \left(N \exp[r(1 - N/K)] \right) \Big|_{\bar{N}} \right|$$
$$= \left| \left(\frac{N^2 - rkN + N^2}{N^2} \exp[r(1 - N/K)] \right) \Big|_{\bar{N}} \right|$$
$$= |(2 - r)e^r|$$

To make the steady state stable, we need

$$|(2 - r)e^r| < 1$$

Solving this inequity gives

$$r < W_{-1}(-1/e^2) + 2$$
$$W(-1/e^2) + 2 < r < W(1/e^2) + 2$$

The approximate values are

$$r < -1.15$$
$$1.84 < r < 2.12$$

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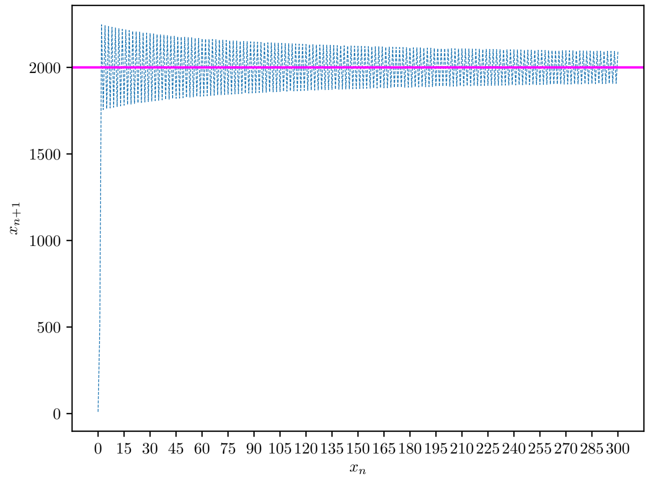
(d). $r = 2, K = 2000$


2000
1500
1000
500
0
x_{n+1}
0 15 30 45 60 75 90 105 120 135 150 165 180 195 210 225 240 255 270 285 300
x_n