Part I

Qualitative and Quantitative Features of Delay Differential Equations

Delay differential equations (DDEs) are also referred to as time-delay systems, systems with after-effect, memory, time-delay, hereditary systems, equations with deviating argument, or differential-difference equations. They belong to the class of functional differential equations that are infinite-dimensional, as opposed to ordinary differential equations (ODEs).

Recently, this class of differential equations has received considerable attention from researchers because the introduction of memory terms in a differential model significantly increases its complexity. Therefore, studying qualitative and quantitative behavior, numerical treatment of such models, parameter estimation, and sensitivity and stability analyses of delay integro-differential equations as well as stochastic delay differential equations (SDDEs) are essential. In this part (Chaps. 1–7), we will study the qualitative and quantitative features of DDEs, which have not been adequately investigated in the literature until now.

Chapter 1

Qualitative Features of Delay Differential Equations

1.1 Introduction

Ordinary and partial differential equations have long played an important role in bioscience, and they are considered to continue to serve as indispensable tools in future investigations as well. However, they frequently provide only a first approximation of the systems under consideration. More realistic models need to include some of the past states of these systems as well; that is, a real system needs to be modeled using differential equations with time-delays (or time-lags). Delay models formulated in mathematical biology include several types of functional differential equations, such as delay differential equations (DDEs), neutral delay differential equations (NDDEs), integro-differential equations, and retarded partial differential equations (RPDEs). Recently, stochastic delay differential equations (SDDEs) have attracted significant attention from researchers.

To create more realistic mathematical models for problems with time-lag or aftereffect, we need to consider using retarded functional differential equations (RFDEs) in place of ordinary differential equations (ODEs), such as

t, y(t), y(a(t, y(t))), (

t

y' (t) = f

K(t, s, y(t), y(s))ds
)
,
t t0 ,
(1.1)

where a(t, y(t)) t and y(t) = (t), t t 0 . Such retarded equations form a class of equations that is, in some sense, between ODEs and time-dependent partial differential equations (PDEs), and they generate infinite-dimensional dynamical systems. RFDEs (1.1), where the integral term is absent, are usually called delay differential equations (DDEs) and they assume forms such as

y' (t) = f (t, y(t), y(a(t, y(t)))) , a(t, y(t)) t.
(1.2)

Neutral delay differential equations (NDDEs) are defined by equations of the form

y' (t) = f (t, y(t), y(a(t, y(t))), y' ((t, y(t)))) ,
(1.3)

where a(t, y(t)), (t, y(t)) t. The introduction of the "lagging" or "retarded" argumentsa(t, y(t)), (t, y(t)) is to reect an "after-effect"; e.g., the gestation period in population modeling.

Mathematical modeling of several real-life phenomena in bioscience requires "differential equations" that depend partially on the past history rather than only the current state. Such examples occur in population dynamics (taking into account the gestation and the maturation time), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling, for example, hematopoiesis and respiration, where the delays are, respectively, due to cell maturation and blood transport between the lung and brain, etc.), chemical and enzyme kinetics (such as mixing reactants), biological immune response (in which the antibody production by the B-cell population depends on the antigenic stimulation at an earlier time), navigational control of ships and aircraft (with large and short lags, respectively), and more general control problems. An early use of DDEs was to describe technical devices such as control circuits. In that context, the delay is a measurable physical quantity; e.g., the time that the signal travels to the controlled object, the reaction time, and the time that the signal takes to return. Similarly, there are parallels in the reaction of the body to pain, for example. Refer to [1–11] for further examples of DDEs in biomathematics.

In many applications in the field of life sciences, a time-delay is introduced when there are certain hidden variables and processes that are not well understood but are known to cause a time-lag [12]. Thus, a delay may, in fact, represent a reaction chain or a transport process. We shall see later that the mathematical properties of DDEs justify such approximations. A well-known example is Cheyne-Stokes respiration (or periodic breathing), discovered in the nineteenth century, wherein some people show periodic oscillations of breathing frequency under constant conditions [13, 14]. This strange phenomenon is due to a delay caused by cardiac insufficiency in the physiological circuit controlling carbon dioxide levels in the blood.

Time-delays occur naturally in biological systems, e.g., in a chemostat (a laboratory device for controlling the supply of nutrients to a growing cell population
[15]). The use of ODEs to model a chemostat carries the implication that changes occur instantaneously. This is a potential deficiency of the ODE model. There are two sources of delays in the chemostat model: (i) delays due to the possibility that the organism stores the nutrients (so that the "free" nutrient concentration does not reect the nutrients available for growth) and (ii) delays due to the cell cycle; see [16–18].

When delays are introduced in first-order non-linear differential equations, or in discrete difference equations, erratic solutions can appear (such a chaotic behavior is also observed in nature); see [19, 20] and Fig. 1.1.

Among the classical references for DDEs are the books by Bellman and Cooke
[21] and Elsgol'ts and Norkin [22]. These are rich sources for analytical techniques
and understanding the many interesting examples. Kolmanovskii et al. [23, 24] gave a rigorous treatment for a wide range of problems. The monographs of Hale [25] and Hale and Verduyn Lunel [26] are standard sources for understanding the theory of delay equations. Another important monograph is by Diekmann et al. [27]. Kuang
[28] and Banks [29] pay particular attention to problems in population dynamics, wherein the former looked at neutral equations. Gopalsamy [30] and Györi and Ladas
[31] addressed the question of oscillations in DDEs. Early books by Cushing [32], Driver [33], Halanay [34], MacDonald [18, 35], May [20], and Waltman [36] have been very stimulating for the development of the field.

Our concern in this chapter is with the qualitative features of DDEs. We show that DDEs have a richer mathematical framework for the analysis of biosystem dynamics compared with ODEs. First, we start from simple real-life problems and formulate them in terms of DDEs; see Sect. 1.2. We then briey study the stability of delay models described by linear and non-linear DDEs, and the conditions that ensure stable behavior; see Sects. 1.3, 1.4 and 1.5.

1.2 Delay Models in Population Dynamics

In this section, we briey discuss some simple mathematical models with time-delays of population dynamics. Naturally, the growth of a population subject to maturation delay is modeled by using either a discrete delay or a delay continuously distributed over the population. The use of a discrete delay might be seen as a rough approximation in modeling the delay distribution over a large population size. However, it is much more realistic to assume the delay being continuously distributed by a continuous distribution function, with a mean delay equal to the discrete delay.

1.2.1 Logistic Equation with Discrete Delay

Let y(t) be the population of a certain species that is independent of other species. The simple model of exponential growth is

y' (t) = y(t) (¿ 0).

(1.4)

From the hypothesis that the growth rate will decrease with increasing population y(t) due to lack of resources (food and space), one arrives instead at the deterministic model of Verhulst (1845)

y(t) y' (t) = ry(t) 1 , (K)

(1.5)

where in (1.4) is replaced by r

t .

If we now assume that the growth rate depends on the population of the preceding generation and take into account the hatching and maturation periods, then the above equation is replaced by a delay equation. Hutchinson (1948) [37] was one of the first to introduce a delay in a biological model. He modified the classical logistic equation (1.5) into the form

y(t) ' (t) = ry(t) 1 . (K)

(1.6)

Here,thederivativedependson y(t)andtheearlierstate y(t),wherethelag ¿ 0 represents the maturation time of individuals in the population. The non-negative parameters r and K are known as the intrinsic growth rate and the environmental carrying capacity, respectively.

Now, we illustrate how the presence of a delay in a differential equation can lead to a notable increase in the complexity of the observed behavior (stable steady states may be destabilized and consequently large amplitude oscillations can occur [38].) Consider a delayed logistic equation (1.6), which can be changed (by putting K y(t) = y(t), a = b) into the form

dy(t) = a y(t) [1 y(t 1)] . dt

(1.7)

It is observed that the qualitative picture (Fig. 1.1) of the solution set of Eq. (1.7) is significantly dependent upon the delay parameter and upon the initial function. For "large" values of , the equation possesses undamped oscillatory solutions; whereas for small values of , the equation behaves like an ODE. For 0 ¡ a ¡ /2, x = 1 is a stable steady state; but for a ¿ /2, chaotic behavior and periodic solution can

arise [39]. For a small a /2, Morris [40] proved that the period is approximately p 4 + 1a(/2) [(3 2] . We note from Fig. 1.1a that the stable periodic solution of (1.7) rapidly acquires a spiky form as increases; see Fowler [38]. The numerical solution at = 3.5 consists of a series of well-separated pulses. This simple example illustrates many of the complexities that arise with delays and has the advantage that results may be easily and explicitly worked out.

1.2.2 Logistic Equation with Distributed Delay

Although Hutchinson's approach leading to Eq. (1.6) is quite useful to explain the appearance of sustained oscillations in a single-species population without any predatory interaction of other species, the underlying argument is somewhat questionable. We may ask: How can it be that the present change in population size depends exactly on the population size of time units earlier? The question has led people to consider integro-differential equations [41]

t 1 y' (t) = ry(t) 1 y(s)G(t s)ds , t t 0 . (K t)

(1.8)

Here, the derivative depends on y(t) and all the previous states after the initial moment t 0 . The delay is continuously distributed and the problem is said to have a fixed time-lag (or finite-memory) and a bounded retardation because the difference between t and t isfixed and bounded.

MacDonald [35] used the integro-differential equation

t y(t) y' (t) = ry(t) 1 y(s)G(t s)ds , 0

(1.9)

for parasite population growth that completes its life cycle within the same host and does not kill the host. (Immunological resistance by the host depends on exposure to the parasite population.) The delay here is continuously distributed and the problem is said to have an unbounded time-lag because the difference between 0 and t is unbounded. The initial time (t = 0) represents the start of the experiment or the time at which the naive host ingests the parasite. Here, it is possible to adopt the simple memory function G(t) = constant.

1.2.3 Delayed Lotka-Volterra System

Many mathematical studies using delay models to study ecology are built upon various generalizations of Volterra's integro-differential system with infinite delays, which are motivated by the characteristic nature of predator-prey dynamics, such as

x (t) = b 1 (y(s)k1 x(t) 1 c 11 x(t) c 12 (t s)ds) ,

(1.10)

t' y (t) = b 2 (+ x(s)k2 y(t) 1 c 21 (t s)ds) ,

where the variables x(t), y(t) represent the populations of the prey and the predator, and the parameters specifying the birth and interaction rates are non-negative 1 (see [32]).

In studying a similar interaction for predator-prey models, Wanggersky and Cunningham (1975) have used equations such as

x (t) = ax(t) mx(t) m bx(t)y(t), ( )

(1.11)

y (t) = cy(t) + dx(t)y(t).

More general delayed predator-prey models take the form

x (t) = x(t)F(t, x t , y t), y (t) = y(t)G(t, x t , y t),

(1.12)

where x t () = x(t +), y t () = y(t +) for 0, and F, G satisfy appropriate conditions (namely, F/x t 0, F/y t ¡ 0; G/x t ¿ 0, and G/y t 0), and (1.12) has positive solutions.

A question of great importance is how does the qualitative behavior depends on the form and magnitude of the delays? In other words, are discrete and continuous delays equivalent from the perspective of the qualitative dynamical properties of the model? The paper by [12] examines certain aspects of this question.

In the next two sections, we discuss the stability of different types of DDEs.

1.3 Stability of DDEs

Time-delay is, in many cases, a source of instability. However, for some systems, the presence of delay can have a stabilizing effect. In the well-known example

y (t) + y(t) y(t) = 0,

(1.13)

the system is unstable for = 1, but it is asymptotically stable when = 1. The approximation y (t) [y(t) y(t)] / explains the damping effect. The stability analysis and robust control of time-delay systems are, therefore, of theoretical and practical importance.

In the following subsections, we present a brief summary of some theories and analysis about the stability of linear and non-linear DDEs. We should first mention

the physical and mathematical interpretations of local and global stability. Local stability of an equilibrium point means that if you put the system somewhere near the point, then it will move itself to the equilibrium point in some time. However, global stability means that the system will come to the equilibrium point from any possible starting point (i.e., there is no "nearby" condition). Moreover, in local asymptotic stability, the solutions of the system must approach an equilibrium point under initial conditions close to the equilibrium point. Whereas in global asymptotic stability, the solutions must approach an equilibrium point under all initial conditions.

1.3.1 Stability of Linear Constant Coefficient DDEs

Consider a simple delay model of population growth given by the following linear DDE:

y (t) = y(t) + y(t), t t 0 , y(t 0) = (t), t t 0 .

(1.14)

One of the fundamental methods for finding the solution of (1.14) is to build up the solution as a sum of simple exponential terms. Assuming the solution to be of the form y(t) = ce st (where c, and s are constants), it will be a solution of (1.14) if and only if s is a zero of the transcendental function

h(s) = s e s .

(1.15)

(The equation h(s) = 0 is called the characteristic equation of (1.14), and s r is the characteristic root if it is a zero of this equation.) Bellman and Cooke [21] observed that the roots s r of (1.15) are infinite in number and complex conjugate and that all lie in the left half-plane Re(s) ¡ c, for some constant c.

Here, we summarize the necessary and sufficient conditions for the "asymptotical" stability of the linear DDEs (1.14). Driver [33], in the following theorem, provided the conditions for DDE (1.14) to be stable:

Theorem 1.1 A necessary and sufficient condition for all continuous solutions of (1.14) to approach zero as t is that all the characteristic roots have negative real parts.

The following results impose conditions on and in (1.15) for the roots of h(s) = 0 to have negative real parts (Re(s) ¡ 0):

• When and are complex. This case is also considered by Barwell [42] and he proved that: A sufficient condition that all the roots of (1.15) have negative real parts is

— — Re().

(1.16)

• When and are real, all roots of equation (1.15) have negative real parts if and only if (i) ¡ 1, (ii) ¡ 2 + 2 , where is the root of = tan() such that 0 ¡ ¡ (if = 0, take = 1 2 /); see Bellman and Cooke [21].

• When = 0 and is complex. This case has been considered by Barwell [42], and the result is: For = re i , a sufficient condition that all the roots of (1.15) have negative real parts is (i) Re() ¡ 0 (1 2 ¡ ¡ 3 2), (ii) 0 ¡ r ¡ min(3 2

, 1 2).

1.3.2 Asymptotical Stability Region for Linear DDEs

To find the asymptotical stability region [24] (which depends on the lag term), suppose, without any loss of generality, that = 1 in (1.14). We search for (,) values for which the first solution s crosses the imaginary axis (Re(s) = 0), i.e., s = i for real. If we insert this into (1.15), we obtain

= for = 0 (s real),

= i e i for = 0.

cos Byseparatingtherealandimaginaryparts,weget = , = validfor sin sin

all real and . Thus, the stability region of y (t) = y(t) + y(t 1) is bounded by = and the parametrized curve = cot(), = /sin(); see Fig. 1.2.

A smaller subset of the stability region, which has been classically considered in [42], is given by the set of pairs (,) such that the solution y(t) of (1.14) asymptotically vanishes independently of the lag (in the (,)-plane:

= (,) R 2 — + — — ¡ 0).

We next extend this analysis to linear neutral DDEs.

1.3.3 Stability of Linear NDDEs

Consider a linear neutral delay differential equation of the form

y (t) = y(t) + y(t) + y (t), t t 0 , y(t 0) = (t), t t 0 .
(1.17)

We summarize the necessary and sufficient conditions for the stability of linear NDDEs (1.17) as follows:

Theorem 1.2 Every solution (of the form y(t) = ce st) of (1.17) tends to zero as t if all roots of the characteristic equation

s = + e s + ses
(1.18)

have negative real parts and are bounded away from the imaginary axis.

Bellen et al. [43] gave a sufficient condition for the stability of the test equation (1.17) in the following theorem.

Theorem 1.3 A sufficient condition for all the roots of (1.18) to have negative real parts is

— — + — + — ¡ 2Re().

(, , and are complex parameters.)

Remark 1.1 If , , and are real, then the condition — — + — + — ¡ 2Re() is equivalent to the condition — — ¡ and — — ¡ 1. If and are complex and = 0, then the hypothesis of Theorem 1.2 reduces to — — ¡ Re(), which gives a sufcient condition for the stability of the test equation (1.14).

1.3.4 Asymptotic Stability Region for Linear NDDEs

Suppose that = 1 in Eq. (1.17). We need to search for the stability regions in terms of parameters (,) for which the rst solution s of (1.18) crosses the imaginary axis (Re(s) = 0), i.e., s = i for real. By separating the real and imaginary parts, we obtain

= cot() for = 0 + ; sin()
(1.19)

= cos() sin() for = 0.
(1.20)

The stability regions for the NDDE (1.17) in the space of parameters (,) for = 0.9, 0.5, 0.5, 0.9 are shown in Fig. 1.3. Equation (1.17) is always unstable for — — ¿ 1; see [24].

1.4 Stability of Non-linear DDEs and Contractivity Conditions

Consider a more general, non-linear DDE with a xed time-lag

y (t) = f (t, y(t), y(t)), t t 0 , y(t) = (t), t t 0 ,
(1.21)

where y [t 0 ,] C n , f : [t 0 ,) C n C n C n and [t 0 , t 0] C n .

We wish to examine the effect that a small change in the initial conditions has on a solution. Thus, we consider another system, dened by the same function f (t, y, x) of (1.21) but with another initial condition:

z (t) = f (t, z(t), z(t)), t t 0 , z(t) = (t), t t 0 .

(1.22)

In the sense of Lyapunov [24], the stability of the solution of (1.21) is dened by the following denition:

Denition 1.1 If there exists a norm on C n such that for every t t 0 , the solution of (1.21) is said to be

(1) stable (with respect to perturbing the initial function), if for each ¿ 0 there exists = (, t 0) such that y(t) z(t) when (t) (t) ;

(2) asymptotically stable, if it is stable and y(t) z(t) 0 as t ;

(3) uniformly asymptotically stable, if under condition (ii) the number = () is independent of t 0 ;

(4) globally uniformly asymptotically stable, if can be an arbitrarily large, nite number;

(5) -exponentially stable, if it is asymptotically stable and, given t 0 , there exists a nite constant K such that y(t) z(t) Ke (tt 0) ,

where y(t) and z(t) are solutions of (1.21) and (1.22), respectively, and (t) and (t) are distinct and continuous functions.

Denition 1.2 The problem (1.21) is contractive (with respect to perturbing the initial function) if for every t t 0 :

y(t) z(t) max (t) (t) t t0

holds.

Corollary 1.1 The zero solution of (1.21) is stable if there exists a norm on C n such that for every t t 0 :

y(t) max (t) . t t0

The following theorem provides sufcient conditions for the contractivity of (1.21) (in the sense described above):

Theorem 1.4 (Contractivity Condition [44]) For a given inner product ., . in Cn and the corresponding norm . , let (t) and (t) be continuous functions such that

(t)

Re (f (t, y 1 , z) f (t, y 2 , z), y 1 y 2) sup z,y 1 ,y 2 C n y 1 y 2 2 y 1 =y 2

(1.23)

and

(t)

f (t, y, z 1) f (t, y, z 2) sup y,z 1 ,z 2 C n z 1 z 2 z 1 =z 2
.
(1.24)
If
(t) + (t) 0, for every t t 0 ,
(1.25)
then it holds that
y(t) z(t)   max (x) (x) , t t 0 . x to
(1.26)
Corollary 1.2 Suppose that f (t, y(t), y(t)) = y(t) + y(t), as in Eq. (1.14). Then, (t) = Re() and (t) = — — . In this case, if Re()   — — , using theorem (1.4) we get — y(t) — max — (t) — for every t t 0 .
t t0
To prove Theorem 1.4, the following theorems are needed:
Theorem 1.5 Consider the initial value problems of the form
y (t) = (t)y(t) + g(t), t t 0 , y(t 0) = y 0 ,
(1.27)
with y, , g : [t 0 , +) C and Re((t)) ¡ 0 for every t t 0 . Then, the solution y(t) of the initial value problem (1.27) is such that:
— y(t) — max y 0 —; t 0 max x t — g(x)/(Re((x))) — . —
t0 Proof Dene A(t) := t (x)dx; we note that Re(A(t)) ¡ 0 for every t t 0 . The solution of (1.27) is
t y(t) = y 0 e A(t) + e A(t) e A(x) g(x)dx. t0
We have that
t t e Re(A(x)) g(x)dx = Re(A(x)) g(x)/(Re((x))) — — Re((x))e dx — [ ] — t 0 t 0
t max — g(x)/(Re((x))) — — Re((x))e Re(A(x)) dx — , t 0 x t t 0
and
t Re((x))e ReA(x)) dx = e Re(A(t)) 1. t0
Therefore,
— e Re(A(x)) g(x)dx — t 0 max x t g(x)/(Re((x))) — e Re(A(t)) 1 — . t 0
Hence:
— y(t) — e Re(A(t)) — y 0 — + (1 e Re(A(t))) t 0 max x t — g(x)/(Re((x)) — ,
and so, for every t t 0 :
— y(t) — max y 0 —; t 0 max x t — g(x)/(Re((x))) — . —
Theorem 1.6 Consider, two initial value problems,

y (t) = f (t, y(t), u(t)), t t 0 , y(t 0) = y 0 ,

(1.28)

and

z (t) = f (t, z(t), v(t)), t t 0 , z(t 0) = z 0 ,

(1.29)

with f : [t 0 , +) C n C n C n and y, z, u, v: [t 0 , +) C n , and y 0 = z 0 . Assume there exists an inner product ., . on C n such that (1.25) holds (x = x, x for every x C n). Then, for every t t 0 :

y(t) z(t)   max y 0 z 0 ; t 0 max x t   (x) u(x) v(x) /((x))) .

Proof We have

1 d y(t) z(t) = Re y (t) z (t), y(t) z(t)   2 2 dt

= Re f (t, y(t), u(t)) f (t, z(t), v(t)), y(t) z(t)

= Re f (t, y(t), u(t)) f (t, y(t), v(t)), y(t) z(t) +

Re f (t, y(t), v(t)) f (t, z(t), v(t)), y(t) z(t) .

It follows from the denitions of (t) and (t) and from Schwartz inequality that

1 d y(t) z(t)   f (t, y(t), u(t)) f (t, y(t), v(t)) y(t) z(t) + (t) y(t) z(t) 2 2 2 dt

(t) u(t) v(t) y(t) z(t) + (t) y(t) z(t) 2 .

Dene

Y(t) := y(t) z(t) .

Note that Y(t) ¿ 0 for every t ¿ t 0 because we assume that the function f is such that (1.28) has a unique solution y(t) for every initial condition y(t 0) = y 0 .

Then,

1 d d = y(t) z(t) (t), y(t) z(t) 2 y(t) z(t) = Y(t)Y 2 dt dt

so we have

Y(t)Y (t) (t)Y 2 (t) + (t) u(t) v(t) Y(t),

and hence

Y (t) (t)Y(t) + (t) u(t) v(t) .

Dene g(t) := (t) u(t) v(t) ; Therefore,

Y (t) (t)Y(t) + g(t),

and, by Theorem 1.5, for t t 0 :

Y(t) max Y 0 ; max g(x)/((x)) , t 0 x t

i.e.,

y(t) z(t)   max y 0 z 0 ; t 0 max x t   (x) u(x) v(x) /((x))) .

Proof Theorem 1.4. From Theorem 1.6 we know that, for every t t 0 , the solutions y(t) and z(t) of (1.21) and (1.22), respectively, are such that y(t) z(t) max (t 0) (t 0) ; max t 0 x t (x) y(x) z(x) /((x)) .

Assume that (t) (t) and ¿ 0 for every t t 0 ; therefore:

y(t) z(t) max (t 0) (t 0) ; t 0 max x t y(x) z(x) ,

i.e.,

y(t) z(t) max (t) (t) . t t0

Therefore, the DDE (1.21) is stable if conditions (1.23)–(1.25) are satised.

Next, we will study global stability using Lyapunov functionals.

1.5 Stability of DDEs in Lyapunov Method

Lyapunov functions are an essential tool in the stability analysis of dynamical systems,bothintheoryandapplications.Asinsystemswithoutdelay,anefcientmethod for stability analysis of DDEs is the Lyapunov method. For DDEs, there exist two main Lyapunov methods: the Krasovskii method of Lyapunov functionals [45] and the Razumikhin method of Lyapunov functions [46, 47]. The two Lyapunov methods for linear DDEs result in linear matrix inequalities (LMIs) conditions. The LMI approach to analysis and design of DDEs provides constructive nite-dimensional conditions, despite signicant model uncertainties [48].

Consider a simple DDE of the form

y (t) = f (t, y(t)), t t 0 ,

(1.30)

where f : R C [, 0] R n is continuous in both arguments and is locally Lipschitz continuous in the second argument. We assume that f (t, 0) = 0, which guarantees that (1.30) possesses a trivial solution y(t) = 0. The system is uniformly asymptotically stable if its trivial solution is uniformly asymptotically stable.

The core concept of Lyapunov stability theory is to construct a functional V (y(t)) (total energy stored in a system) to be dened and its derivative along the trajectories of the system.

Denition 1.3 Let V : R n R be a Lyapunov function if

(i) V (y(t)) 0 with equality if and only if y = 0, and

(ii) dt d V (y(t)) 0.

Theorem 1.7 (Lyapunov's Second Theorem on R) If there exists a Lyapunov function V , then y = 0 is Lyapunov stable. Furthermore, if V (y(t)) ¡ 0, then equilibrium y = 0 is asymptotically stable.

Given a DDE of the form:

y (t) = f (y(t), y(t)), f (0, 0) = 0,

where f (., .) is locally Lipschitz in its arguments. Let us assume that V (t) = y 2 (t), which is a typical Lyapunov function for n = 1. Then, we have along the system:

V (t) = 2y(t)y (t) = 2y(t) f (y(t), y(t)).

For the feasibility of inequality V (t) 0, we need to ensure that y(t) f (y(t), y(t)) 0 for all sufciently small — y(t) — and — y(t) — . This essentially restricts the class of equations considered. For example, y (t) = y(t)y 2 (t) is stable based on the above arguments.

1.5.1 Lyapunov-Krasovskii Sense

Let V : R C [, 0] R be a continuous functional, and let y s (t,) be the solution of (1.30) at time s t with the initial condition y t = (t). We dene the right ˙ upper derivative V (t,) along (1.30) as follows:

1 V ˙ (t,) = lim sup (t,) V (t,)] . V (t t, yt + t + [t0 + t

˙ Intuitively, a non-positive (t,) indicates that yt the system under consideration is stable.

does not grow with t, meaning that

Theorem 1.8 (Lyapunov-Krasovskii Theorem, Gu et al. [49]) Suppose that f : R C [, 0] R n maps R (bounded sets) in C [, 0] into bounded sets of R n and that u ; v ; w : R + R + are continuous nondecreasing functions, u(s) and v(s) are positive for s ¿ 0, and u(0) = v(0) = 0. The trivial solution of (1.30) is uniformly stable if there exists a continuous functional V : R C [, 0] R + , which is positive-denite, i.e.,

u(— (0) —) V (t,) v(— (0) —),

(1.31)

and such that its derivative along (1.30) is non-positive in the sense that

V ˙ (t,) w(C).

(1.32)

If w(s) ¿ 0 for s ¿ 0, then the trivial solution is uniformly asymptotically stable. If in addition lim u(s) = , then it is globally uniformly asymptotically stable.

s

1.5.2 Lyapunov-Razumikhin Sense

In Razumikhin approach, the derivative V along the solution y(t) of (1.30) of a differentiable function V : R R n R + is dened as follows:

d V (t, y(t) V (t, y(t)) V ˙ (t, y(t)) = + f (t, y t). V (t, y(t)) = dt t y

(1.33)

Theorem 1.9 (Lyapunov-Razumikhin Theorem, Gu et al. [49]) Suppose that f : R C [, 0] R n maps R (bounded sets) in C [, 0] into bounded sets of R n and that u ; v ; w : R + R + are continuous nondecreasing functions, u(s) and v(s) are positive for s ¿ 0, and u(0) = v(0) = 0, v is strictly increasing. The trivial solution of (1.30) is uniformly stable if there exists a continuous functional V : R C [, 0] R + , which is positive-denite, i.e.,

u(— y —) V (t, y) v(— y —),

(1.34)

and the derivative along (1.30) satises

V ˙ (t, y(t)) w(— y(t) —), i f V (t + , y(t +) ¡ V (t, y(t)), f or [0,] .

(1.35)

If, in addition, w(s) ¿ 0 for s ¿ 0, and there exists a continuous nondecreasing function (s) ¿ 0, for s ¿ 0, such that condition (1.35) is strengthened to

V ˙ (t, y(t)) w(— y(t) —), i f V (t + , y(t +) ¡ (V (t, y(t))), f or [0,] , (1.36)

then the trivial solution is uniformly asymptotically stable. If in addition lim u(s) = s , then it is globally uniformly asymptotically stable.

1.5.3 Stability of Linear Systems with Discrete Delays

Given the linearized system

y˙ (t) = Ay(t) + By(t (t)), y(t) = (t), t [M , 0] ,

(1.37)

(t) [0,] is abounded. A simple Lyapunov-Krasovskii functional for the above system has the form

t V (t, y(t)) = y T (t)Py(t) + y(s)Qy(s)ds, t(t)

where P ¿ 0 and Q ¿ 0 are n n matrices. Clearly V satises the positivity condition V (t, y(t)) — y(t) 2 , for ¿ 0. Then, differentiating V along the system, we have

V ˙ (t, y(t)) = 2y T (t)P ˙x(t) + y T (t)Qy(t) (1 ˙)y T (t)Qy(t).

If we further substitute y˙ (t), the right-hand side of the DDEs system, with ˙ d 1, we arrive at

y(t) V yT ˙ (t, (y(t)) (t)y T (t)W 2 , for ¿ 0, if — y(t) — [y(t)]

A T P + PA + Q PB ¡ 0. [B T A (1 d)Q]

(1.38)

The linear matrix inequality (LMI) (1.38) does not depend on and it is, therefore, delay-independent (but delay-derivative dependent). The feasibility of LMI (1.38) is a sufcient condition for the delay-independent asymptotic stability of systems with slowly varying delays; see [49].

However, delay-independent conditions cannot be applied for the stabilization of unstable plants through a feedback with delay. For such systems, delay-dependent conditions are then needed. Now, we derive stability conditions by applying Razumikhin's approach and using the Lyapunov function:

V (t, y(t)) = y T (t)Py(t)

with P ¿ 0 that satises the positivity condition (1.34). Consider the derivative of V along(1.37).WewillapplytheLyapunov-Razumikhintheoremwith(s) = .s ¿ 1, where the constant ¿ 1. Whenever Razumikhin's condition:

y T (t)Py(t) y T (t (t))Py(t (t)) ¿ 0

holds for + 1, with ¿ 0. We then conclude that, for any q ¿ 0, there exists ¿ 0 such that

V ˙ (t, y(t)) =2y T (t)P [Ay(t) + A 1 y(t (T))]

(1.39)

2y T (t)P [Ay(t) + A 1 y(t)] +

q [y T (t)Py(t) y T (t (t))Py(t (t))] — y(t) — 2

if

A T P + PA + qP PB ¡ 0. [B T P q P]

(1.40)

The MLI (1.40) does not depend on . Therefore, the feasibility of (1.40) is sufcient for delay-independent uniform asymptotic stability for systems with fast-varying delays (without any constraints on the delay-derivatives); see [49].

1.6 Concluding Remarks

In this chapter, we have provided a general introduction on DDEs and examined the stability of delay models described by linear and non-linear DDEs along with conditions that ensure local and global asymptotic stable behavior. Next, we will study approximation solutions and numerical schemes of DDEs. We will also discuss how the Runge-Kutta methods, which are so popular for ODEs, can be extended to DDEs.

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