Middle Point Calculation on Hyperbolic Plane

A formulak

Az r1 es r2 a ket pont radialis koordinataja, ϕ az altaluk kozrezart szog, amely mindig 0 es π koze essen (tehat a szogkoordinatakbol igy szamoljuk ki). Az α szog legyen az a 0 es ϕ koze eso szog,
amely a felezopont szogkordinataja es az α szog egyik szara
az r2. A masik szara a felezoponthoz tartozo r hosszusagu
egyenes. Igy az r1 es r egyenesek altal bezart szog $\phi - \alpha$ lesz.

Az α - t kiszamito formula:

$$\alpha[r1,\ r2,\ \phi] := \mathrm{If}\left[\mathrm{ArcCot}\left[\frac{\mathrm{Sinh}[r2]}{\mathrm{Sinh}[r1]}\ \frac{1}{\mathrm{Sin}[\phi]} + \mathrm{Cot}[\phi]\right] < 0,\ \mathrm{ArcCot}\left[\frac{\mathrm{Sinh}[r2]}{\mathrm{Sinh}[r1]}\ \frac{1}{\mathrm{Sin}[\phi]} + \mathrm{Cot}[\phi]\right] + \pi,\ \mathrm{ArcCot}\left[\frac{\mathrm{Sinh}[r2]}{\mathrm{Sinh}[r1]}\ \frac{1}{\mathrm{Sin}[\phi]} + \mathrm{Cot}[\phi]\right]\right]$$

Tehat ha az α szogre a levezetett keplet alapjan negativ ertek jonne ki,
akkor hozza kell adni π - t, mivel α - nak mindenkeppen 0 es ϕ koze kell esnie.

Ha megvan az α, akkor az r - t az alabbi formula szerint szamolhatjuk:

$$r[r1,\ r2,\ \phi,\ \alpha] := \mathrm{Log}\left[\frac{\sqrt{-\mathrm{Cosh}[r1] + \mathrm{Cosh}[r2] - \mathrm{Cos}[\alpha - \phi]\ \mathrm{Sinh}[r1] + \mathrm{Cos}[\alpha]\ \mathrm{Sinh}[r2]}}{\sqrt{\mathrm{Cosh}[r1] - \mathrm{Cosh}[r2] - \mathrm{Cos}[\alpha - \phi]\ \mathrm{Sinh}[r1] + \mathrm{Cos}[\alpha]\ \mathrm{Sinh}[r2]}}\right]$$

Ha r - nek adodna kepzetes resze, akkor azt hagyjuk figyelmen kivul.

Ezek a formulak elvileg teljesen pontosak,
semmilyen kozelitest nem tartalmaznak.

A szamolasok

In[339]:= $$\mathbf{Solve}\left[\left\{\frac{\mathbf{Exp}[r2]}{\mathbf{Exp}[r1]} == \frac{\mathbf{Sin}[\phi - \alpha]}{\mathbf{Sin}[\alpha]}\right\},\ \alpha\right]$$

Out[339]= $\{\{\alpha \to \mathrm{ArcCot}[\mathrm{Csc}[\phi]\ (\mathrm{Cos}[\phi] + \mathrm{Cosh}[r1 - r2] - \mathrm{Sinh}[r1 - r2])]\}\}$

In[3]:= $$\mathbf{Sin}\left[\frac{\mathbf{ArcCot}\left[\frac{\mathbf{Exp}[r2-r1]}{\mathbf{Sin}[\phi]} + \mathbf{Cot}[\phi]\right]}{2}\right]^2$$

Out[3]= $$\mathrm{Sin}\left[\frac{1}{2}\ \mathrm{ArcCot}\left[\mathrm{Cot}[\phi] + e^{-r1+r2}\ \mathrm{Csc}[\phi]\right]\right]^2$$

In[4]:= **FullSimplify[%3]**

Out[4]= $$\mathrm{Sin}\left[\frac{1}{2}\ \mathrm{ArcCot}\left[\mathrm{Cot}[\phi] + e^{-r1+r2}\ \mathrm{Csc}[\phi]\right]\right]^2$$

In[5]:= $\text{FullSimplify}\left[\frac{\text{Exp}\left[\frac{r1-r2}{2}\right]\,\text{Sin}\left[\frac{\phi}{2}\right]}{\text{Sin}\left[\frac{\text{ArcCot}\left[\frac{\text{Exp}[r2-r1]}{\text{Sin}[\phi]}+\text{Cot}[\phi]\right]}{2}\right]^2}\right]$

Out[5]= $e^{\frac{r1-r2}{2}}\,\text{Csc}\left[\frac{1}{2}\,\text{ArcCot}\left[\text{Cot}[\phi]+e^{-r1+r2}\,\text{Csc}[\phi]\right]\right]^2\,\text{Sin}\left[\frac{\phi}{2}\right]$

(*Sajnos ugy nez ki nem nagyon lehet egyszerusiteni*)

In[132]:= $r[r1\_, r2\_, \phi\_] :=$
$\text{Log}\left[\text{Exp}\left[\frac{r1-r2}{2}\right]\,\text{Abs}\left[\text{Sin}\left[\frac{\phi}{2}\right]\right]\,\text{Csc}\left[\frac{1}{2}\,\text{ArcCot}\left[\text{Cot}[\phi]+\frac{\text{Sinh}[r2]}{\text{Sinh}[r1]}\,\text{Csc}[\phi]\right]\right]^2\right]$

In[78]:= $r\left[3, 5, \frac{\pi}{2.}\right]$

Out[78]= -0.509796

(*Nezzuk meg α szamitasat, hogy kb milyen.*)

In[300]:= $\alpha[r1\_, r2\_, \phi\_] := \text{If}\left[\text{ArcCot}\left[\frac{\text{Sinh}[r2]}{\text{Sinh}[r1]}\,\frac{1}{\text{Sin}[\phi]}+\text{Cot}[\phi]\right] < 0,\right.$
$\left.\text{ArcCot}\left[\frac{\text{Sinh}[r2]}{\text{Sinh}[r1]}\,\frac{1}{\text{Sin}[\phi]}+\text{Cot}[\phi]\right]+\pi,\ \text{ArcCot}\left[\frac{\text{Sinh}[r2]}{\text{Sinh}[r1]}\,\frac{1}{\text{Sin}[\phi]}+\text{Cot}[\phi]\right]\right]$

(*a fenti keplet jonak tunik, tudja azt,
hogy 0 es π kozeli szogek eseten 0 vagy -0 lesz az α,
ha r1 es r2 egyenlo, akkor a ϕ szoget megfelezi, tehat $\alpha=\frac{\phi}{2}$,
bar $\pi/2$ nel nagyobb szogek eseten $\frac{\phi}{2}-\pi$ t ad eredmenyul. *)

In[97]:= $\alpha[10., 10., 3.15]$

Out[97]= -1.56659

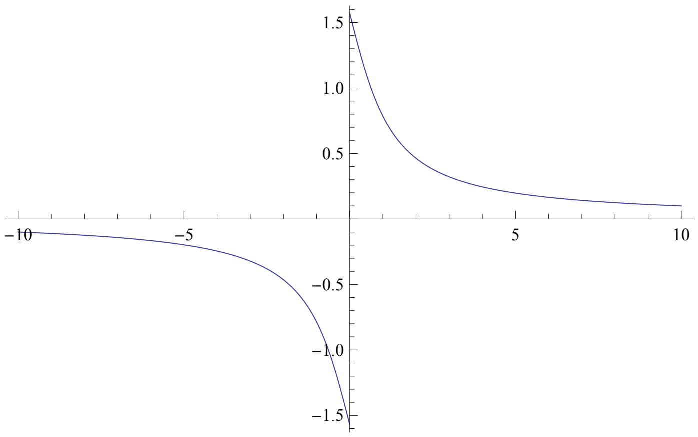
In[68]:= -0.6415926535897932\` + π

Out[68]= 2.5

In[59]:= π / 8.

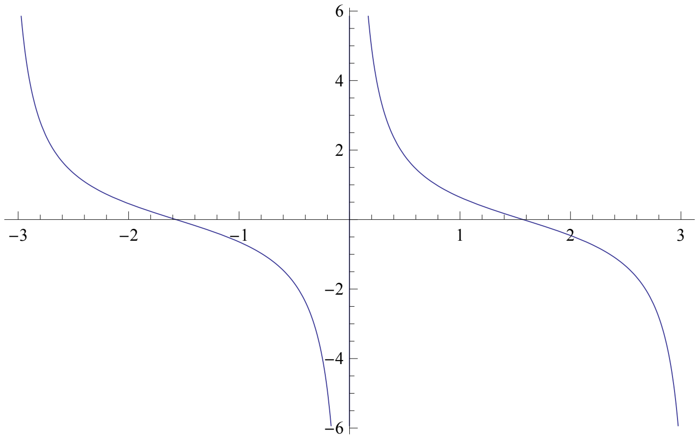
Out[59]= 0.392699

In[237]:= Plot[ArcCot[x], {x, -10, 10}]

Out[237]=


In[42]:= `Plot[Cot[x], {x, -3, 3}]`

Out[42]=

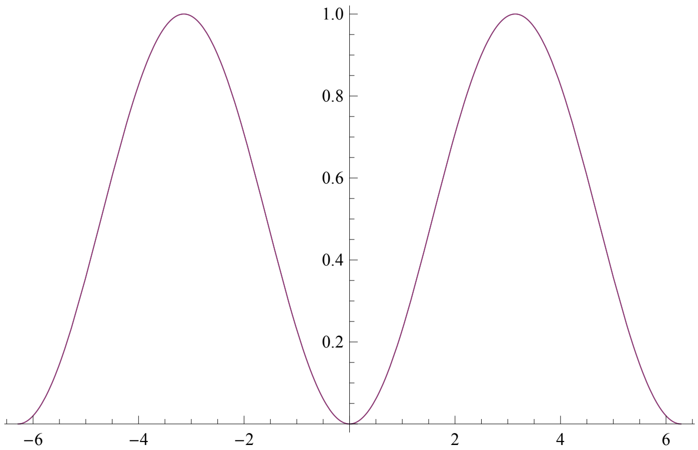


In[43]:= $\text{ArcCot}\left[\text{Cot}\left[\frac{\pi}{2} + .1\right]\right]$

Out[43]= -1.4708

In[51]:= $\text{Plot}\left[\left\{\frac{1 - \text{Cos}[\phi]}{2}, \text{Sin}\left[\frac{\phi}{2}\right]^2\right\}, \{\phi, -2\,\pi, 2\,\pi\}\right]$

Out[51]=



In[59]:= $\text{FullSimplify}\left[\frac{1 - \text{Cos}\left[\frac{\pi}{4}\right]}{2}\right]$

Out[59]= $\frac{1}{4}\left(2 - \sqrt{2}\right)$

In[56]:= $\text{Sin}\left[\frac{\pi}{8.}\right]^2$

Out[56]= 0.146447

In[60]:= $\frac{4}{\sqrt{2}\left(2 - \sqrt{2.}\right)}$

Out[60]= 4.82843

In[86]:= $\text{aux1}[\phi\_, \alpha\_] := \frac{\text{Sin}[\phi - \alpha]}{\text{Sin}[\alpha]} - \frac{\text{Sin}\left[\frac{\phi-\alpha}{2}\right]^2}{\text{Sin}\left[\frac{\alpha}{2}\right]^2}$

In[88]:=
```
aux1[.1, .01]
```
Out[88]= -71.958

In[80]:=
$$\text{aux2}[\phi\_, \alpha\_] := \frac{\text{Sin}\left[\frac{\phi-\alpha}{2}\right]^2}{\text{Sin}\left[\frac{\alpha}{2}\right]^2}$$

```
(*Egyelore ellentmondas van a szinusz
 tetel alkalmazasa es a koszinusz tetel kozott.*)
```

```
(*Csinalok egy numerikus peldat*)
```

```
(*r1=3, r2=5, ϕ=1.3*)
```

In[108]:=
```
{r1 = 3, r2 = 5, ϕ = 1.3}
```
Out[108]= {3, 5, 1.3}

In[92]:=
```
(*a tavolsag a ket pont kozott*)
ArcCosh[Cosh[r1] Cosh[r2] - Sinh[r1] Sinh[r2] Cos[ϕ]]
```
Out[92]= 6.99992

```
(*a levezetesem szerint a szog,
amely a felezoponton athalado egyeneshez tartozik*)
```

In[98]:=
```
α[r1, r2, ϕ]
```
Out[98]= 0.124898

In[134]:=
```
(*z altalam kiotolt szinusz tetel stimmel? Nagyjabol stimmel.*)
```
$$\frac{\text{Sinh}[5]}{\text{Sinh}[3]} - \text{Sin}[1.3 - 0.12489838117743689`] / \text{Sin}[0.12489838117743689`]$$

Out[134]= -8.88178×10^{-16}

In[112]:=
$$\text{FullSimplify}\left[\frac{\text{Sin}[\beta - \alpha]}{\text{Sin}[\alpha]}\right]$$

In[113]:=
```
Solve[Exp[x2 - x1] == -Cos[β] + Cot[α] Sin[β], α]
```
Out[113]=
$$\left\{\left\{\alpha \to \text{ConditionalExpression}\left[\text{ArcCot}\left[\text{Cot}[\beta] + e^{-x1+x2}\,\text{Csc}[\beta]\right] + \pi\,C[1],\ C[1] \in \text{Integers}\right]\right\}\right\}$$

In[135]:=
```
{r = 1.585,
 ArcCosh[Cosh[r] Cosh[r1] - Sinh[r] Sinh[r1] Cos[1.3 - 0.12489838117743689`]],
 ArcCosh[Cosh[r] Cosh[r2] - Sinh[r] Sinh[r2] Cos[0.12489838117743689`]]}
(*ez stimmel, visszaadja a ket tavolsag szamitas a felhosszakat*)
```
Out[135]= {1.585, 3.49974, 3.50018}

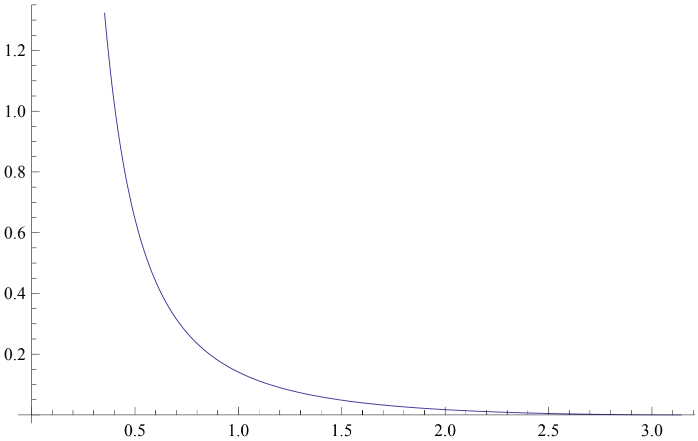
In[136]:=
$$\text{(*most az r alapjan*)} \left\{\text{Exp}[r + r1]\ \text{Sin}\left[\frac{1.3 - 0.12489838117743689`}{2}\right]^2, \text{Exp}[r + r2]\ \text{Sin}\left[\frac{0.12489838117743689`}{2}\right]^2\right\}$$

Out[136]= {30.114, 2.82045}

In[137]:=
```
{ArcCosh[Cosh[r] Cosh[r1] - Sinh[r] Sinh[r1] Cos[1.3 - 0.12489838117743689`]],
 ArcCosh[Cosh[r] Cosh[r2] - Sinh[r] Sinh[r2] Cos[0.12489838117743689`]]}
```
Out[137]=
```
{3.49974, 3.50018}
```

```
{Exp[r] Exp[r1] (Sin[1/2 (1.3 - 0.12489838117743689`)]^2),
 Exp[r] Exp[r2] - Exp[r] Exp[r2] Cos[0.12489838117743689`]}
(*sajnos kis szogek eseten nagyon pontatlanna valik az expes kozelites*)
```
Out[164]=
```
{30.114, 5.6409}
```

In[178]:=
```
Plot[(Cosh[r] Cosh[r2] - Sinh[r] Sinh[r2] Cos[γ]) / (Exp[r + r2] Sin[γ/2]^2 / 2) - 1,
 {γ, 0, π}]
```
Out[178]=




```
(*a szogszamitas eleg pontosnak tunik,
abbol kellene valahogy az r/et szamitani*)
```

In[181]:=
```
Clear[r, r1, r2, ϕ]
```

In[190]:=
```
Solve[Cosh[r] Cosh[r1] - Sinh[r] Sinh[r1] Cos[ϕ - α] ==
  Cosh[r] Cosh[r2] - Sinh[r] Sinh[r2] Cos[α], r]
```
Out[190]=
```
{{r → ConditionalExpression[2 i π C[1] +
     Log[-(√(-Cosh[r1] + Cosh[r2] - Cos[α - ϕ] Sinh[r1] + Cos[α] Sinh[r2])) /
        (√(Cosh[r1] - Cosh[r2] - Cos[α - ϕ] Sinh[r1] + Cos[α] Sinh[r2]))],
    C[1] ∈ Integers]}, {r → ConditionalExpression[2 i π C[1] +
     Log[(√(-Cosh[r1] + Cosh[r2] - Cos[α - ϕ] Sinh[r1] + Cos[α] Sinh[r2])) /
       (√(Cosh[r1] - Cosh[r2] - Cos[α - ϕ] Sinh[r1] + Cos[α] Sinh[r2]))], C[
      1] ∈ Integers]}}
```

In[191]:=
```
r[r1_, r2_, ϕ_, α_] :=
 Log[(√(-Cosh[r1] + Cosh[r2] - Cos[α - ϕ] Sinh[r1] + Cos[α] Sinh[r2])) /
   (√(Cosh[r1] - Cosh[r2] - Cos[α - ϕ] Sinh[r1] + Cos[α] Sinh[r2]))]
```

In[192]:=
```
r[3, 5, 1.3, 0.1249]
```
Out[192]=
```
1.58527
```

In[305]:= α[3, 5, 0.0001]

Out[305]= 0.0000118947

In[306]:= r[3, 5, 0.0001, 0.000011894734977716024`]

Out[306]= 4.

In[307]:= α[3, 5, π - 0.0001]

Out[307]= 0.0000156077

In[308]:= r[3, 5, π - 0.0001, 0.000015607731580128384`]

Out[308]= 1.

In[309]:= {α[3, 5, 1.3], 1.3 - α[5, 3, 1.3]}

Out[309]= {0.124898, 0.124898}

In[304]:= α[3, 5, 2.] + α[5, 3, 2.]

Out[304]= 2.

In[320]:= α[5, 3, 2.] - π

Out[320]= -1.27093

In[321]:= r[5, 3, 2, -1.2709346124771161`]

Out[321]= -1.2144

In[322]:= α[5, 3, 2.]

Out[322]= 1.87066

In[323]:= r[5, 3, 2, 1.870658041112677`]

Out[323]= 1.2144 + 0. ⅈ

In[329]:= α[15, 12, π - .000000000000001]

Out[329]= 3.14159

In[330]:= r[15, 12, π - .000000000000001, 3.141592653589792`]

Out[330]= 1.5 + 0. ⅈ

(*spec eseteket eleg jol visszaadja*)