Representation theory HW 1

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Problem 1

We begin by showing that the category of topological abelian groups with continuous homomorphisms between them is additive. First, for A, B abelian groups, we define addition on $\mathrm{Hom}(A, B)$ pointwise:

$$(f + g)(x) = f(x) + g(x)$$

This makes sense because elements of the abelian group B can be added. It's easy to see that addition makes $\mathrm{Hom}(A, B)$ into a group, with the zero morphism as identity; this group is abelian because the groups A and B are. The composition of two morphisms is bilinear, as follows immediately from homomorphism properties:

$$f(g + h)(x) = f(g(x) + h(x)) = f(g(x)) + f(h(x))$$

$$(f + g)(h)(x) = (f + g)(h(x)) = f(h(x)) + g(h(x))$$

Second, the trivial group 0 is the zero object in the category, satisfying $\mathrm{Hom}(0, 0) = \{0\}$. Third, given objects A, B, we define $A \oplus B = \{(a, b) | a \in A, b \in B\}$, with group operation $(a, b) + (c, d) = (a + c, b + d)$. With the product topology induced by the topologies of A and B, $A \oplus B$ is a topological abelian group. The group operation is continuous by componentwise continuity. To see that $A \oplus B$ is a product, take the projection morphisms to be $\pi_A : A \oplus B \to A$ to be $\pi_A(a, b) = a$, and similarly for π_B. Then, given an object C and morphisms $f_A : C \to A$, $f_B : C \to B$, these morphisms factor through $A \oplus B$, by the unique morhism $f : C \to A \oplus B$ given by $f(c) = (f_A(c), f_B(c))$. We can show analogously that $A \oplus B$ is a coproduct, with inclusion morphisms $\iota_A(a) = (a, 0)$ and $\iota_B(b) = (0, b)$.

Now we show that, because of the continuity requirement for morphisms, the category is not abelian. We construct a counterexample using the groups $(\mathbb{R}, +)$ with the Euclidean topology and $(\mathbb{Q}, +)$ with the subspace topology. Consider the inclusion $\iota : \mathbb{Q} \to \mathbb{R}$, which is a morphism. Suppose the cokernel of ι exists, i.e. there exist an object C and a morphism f that make the following diagram commute:

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\iota"] \arrow[dr, "0"'] & \mathbb{R} \arrow[d, "f"] \\ & C \end{tikzcd}$$

Then $f(x)$ must be 0 for any rational x. By continuity, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, this means that f must be identically 0. Since the cokernel is unique, it suffices to find one object C which satisfies the above, as well as the universality condition. Consider $C = \{0\}$, the zero object. Then for any other object C' and any map $g : \mathbb{R} \to C'$, the requirement that the diagram commutes implies again that $g = 0$. Then $g = 0_{C \to C'} \circ f$, so g factors through C as desired. This shows that the cokernel, if it exists, is 0. Then the image $\operatorname{Im} \iota = \ker f = \mathbb{R}$. But this is a contradiction, as the image of the inclusion is $\mathbb{Q}$.

Problem 2

We need to show that the functors Ind and Res are adjoint; this means that for all $A \in S - \text{Mod}, B \in R - \text{Mod}$ there is a natural bijection:

$$\tau_{AB} : \operatorname{Hom}_R(\operatorname{Ind} A, B) \to \operatorname{Hom}_S(A, \operatorname{Res} B)$$

We construct the bijection explicitly. Given $\psi \in \operatorname{Hom}_R(\operatorname{Ind} A, B)$, we define a morphism $\tilde{\psi} : A \to \operatorname{Res} B$ by $a \mapsto \psi(1 \otimes a)$. This morphism is S-invariant:

$$\tilde{\psi}(s \cdot a) = \psi(1 \otimes_S s \cdot a) = \psi(\phi(s) \otimes_S a) = \phi(s)\psi(1 \otimes_S a) = \phi(s)\tilde{\psi}(a)$$

Moreover, the association is injective. Indeed, if $\psi(1 \otimes_S a) = \psi'(1 \otimes_S a)$, then acting by $r \in R$ on this relation and using the R-invariance of the morphisms gives $\psi(r \otimes_S a) = \psi'(r \otimes_S a)$, so $\psi = \psi'$.

For the inverse association, given a morphism $\tilde{\psi} : A \to \operatorname{Res} B$, we define $\psi : \operatorname{Ind} A \to B$ by $\psi(r \otimes_S a) = r \cdot \tilde{\psi}(a)$. This is R-invariant because [...]. This association is also injective, because if $r \cdot \tilde{\psi}(a) = r \cdot \tilde{\psi}'(a)$ for all r, we take $r = 1$ and get $\tilde{\psi} = \tilde{\psi}'$. Therefore we have obtained a bijective association τ_{AB}.

For this bijection to be natural, given any $f : A \to A'$ and $g : B \to B'$, the following diagrams need to commute:

$$\begin{tikzcd}
\operatorname{Hom}_R(\operatorname{Ind} A', B) \arrow[r, "\circ \operatorname{Ind} f"] \arrow[d, "\tau_{A'B}"] & \operatorname{Hom}_R(\operatorname{Ind} A, B) \arrow[d, "\tau_{AB}"] \\
\operatorname{Hom}_S(A', \operatorname{Res} B) \arrow[r, "\circ f"] & \operatorname{Hom}_S(A, \operatorname{Res} B)
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Hom}_R(\operatorname{Ind} A, B) \arrow[r, "g\circ"] \arrow[d, "\tau_{AB}^{-1}"] & \operatorname{Hom}_R(\operatorname{Ind} A, B') \arrow[d, "\tau_{AB'}^{-1}"] \\
\operatorname{Hom}_S(A, \operatorname{Res} B) \arrow[r, "\operatorname{Res} g\circ"] & \operatorname{Hom}_S(A, \operatorname{Res} B')
\end{tikzcd}$$

But this is pretty obvious. For the first diagram, given $\psi : \operatorname{Ind} A' \to B$, we need to show that

$$\tilde{\psi} \circ f = (\psi \circ \widetilde{\operatorname{Ind} f})$$

It's easy to see that, for a given a, both maps return $\psi(1 \otimes f(a))$. For the second diagram we see similarly that both maps are $r \otimes a \mapsto r \cdot g(\tilde{\psi}(a))$.

Problem 3

Take some $M \in \mathcal{O}$. By the first axiom of $\mathcal{O}$, M is finitely generated as an $\mathcal{U}\mathfrak{g}$-module. Denote the generators by v_i; by the second axiom of $\mathcal{O}$, we can choose each v_i to be in some M_{λ_i}. We view $\mathcal{U}\mathfrak{g}$ as $\mathcal{U}\mathfrak{n}_- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}_+$, and use the third axiom of $\mathcal{O}$, which says that $\mathcal{U}\mathfrak{n}_+ v_i$ is finite for each v_i. This means $\mathcal{U}\mathfrak{n}_+ v_i = \{v_{ij}\}$ for finitely many j. Then v_{ij} are finitely many, and they generate M as a $\mathcal{U}\mathfrak{n}_- \otimes \mathcal{U}\mathfrak{h}$-module. However, $\mathcal{U}\mathfrak{h}$ acts on v_{ij} just by rescaling, therefore v_{ij} generate M as a $\mathcal{U}\mathfrak{n}_-$-module.

Problem 4

We first show that $V \otimes W$ satisfies axioms 2 and 3 of category $\mathcal{O}$, irrespective of whether the factors are finite dimensional or not. For axiom 2, we have:

$$V \otimes W = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \otimes \bigoplus_{\mu \in \mathfrak{h}^*} W_\mu = \bigoplus_{\lambda, \mu \in \mathfrak{h}^*} V_\lambda \otimes W_\mu$$

Consider $v \in V_\lambda, w \in W_\mu$; for $h \in \mathfrak{h}$, we have:

$$h \cdot (v \otimes w) = (h \cdot v) \otimes w + v \otimes (h \cdot w) = (\lambda(h) + \mu(h))(v \otimes w)$$

Therefore $V_\lambda \otimes W_\mu$ is a weight $\lambda + \mu$ space. Then we can regroup the direct sum as:

$$V \otimes W = \bigoplus_{\lambda+\mu} (V \otimes W)_{\lambda+\mu}$$

For axiom 3, take $\sum_i v_i \otimes w_i \in V \otimes W$. The action of $e \in \mathfrak{n}_+$ is:

$$e \cdot \sum_i v_i \otimes w_i = \sum_i (e \cdot v_i) \otimes w_i + v_i \otimes (e \cdot w_i)$$

Repeated action by elements of $\mathfrak{n}_+$ therefore reduces to a sum of terms of the form $(e_1 \cdot v_i) \otimes (e_2 \cdot w_i)$ By axiom 3 applied to V and W, there are only finitely many $e_1 \cdot v$ and $e_2 \cdot w$ for each v and w, therefore there are finitely many $e \cdot \sum_i v_i \otimes w_i$.

Now we show that, if one of V or W is finite (say W is), then axiom 1 holds for $V \otimes W$. Let v_i be a set of generators for V as a $\mathcal{U}\mathfrak{g}$-module, and let w_j be a set of elements that span W as a vector space. Denote by M the module that $v_i \otimes w_j$, which are finitely many, generate; we want to show that this is exactly $V \otimes W$. First, it's clear that $v_i \otimes w \in M$, for any $w \in W$, since w is just a linear combination of the w_j's. Then we can use, for any $X \in \mathfrak{g}$:

$$X \cdot (v_i \otimes w) = (X \cdot v_i) \otimes w + v_i \otimes (X \cdot w)$$

Note that the LHS is in M, and the second term on the RHS is in M. This means that $(X \cdot v_i) \otimes w \in M$. But by repeated applications of elemens of $\mathfrak{g}$ on the v_i's we can generate all of V, therefore $v \otimes w \in M$, for all $v \in V, w \in W$. Since $v \otimes w$ span $V \otimes W$, we see that $M = V \otimes W$.

Finally, we show that, if both V and W are infinite dimensional, then axiom 1 doesn't have to hold for $V \otimes W$. To keep things simple, we use $\mathfrak{g} = \mathfrak{sl}_2$ to construct a counterexample. Let $V = M_\lambda, W = M_\mu$ be Verma modules; their elements have weights $\lambda - 2i$ and $\mu - 2j$ respectively, for $i, j \in \mathbb{Z}$. Assume that $\{v_k\}$ is a finite set of generators for $M_\lambda \otimes M_\mu$. By axiom 2, each $v_k = \sum_j v_{kj}$, where the sum is finite and each v_{kj} has weight a_{kj}. Then we might as well take the finite set of all $\{v_{kj}\}$ as generators, and after relabelling, we end up with a set of generators $\{v_k\}$ where each has weight a_k. Let a_i be the lowest of the weights a_k (since the weights are integers, ordering them makes sense). Now, elements of $\mathfrak{g}$ act by the Leibniz rule, creating two terms of the form $v \otimes w$ for each one term that they act on. Therefore for any $X \in \mathcal{U}\mathfrak{g}$, $X \cdot v_k$ will have an even number of terms of weight smaller than a_k. In particular, all elements of the module generated by $\{v_k\}$ will have an even number of terms of weight smaller than a_i. This means that $v_a \otimes w_b$, for $a + b < a_i$, is not contained in this module, since it consists of only one term. However, such an element $v_a \otimes w_b$ always exists in $M_\lambda \otimes M_\mu$, the weight lattices of both Verma modules are infinite in the negative direction.