Algebraic topology HW3

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Problem 1 (1.1.1 in Hatcher)

Since $g_0 \simeq g_1$ by some homotopy g_t, $\bar{g}_0 \simeq \bar{g}_1$ by the homotopy g_{1-t}. Let h_t be a homotopy between $f_0 \bullet g_0$ and $f_1 \bullet g_1$. Using the fact that path products preserve homotopy equivalence, we obtain a homotopy $h_t \bullet g_{1-t}$ between $f_0 \bullet g_0 \bullet \bar{g}_0$ and $f_1 \bullet g_1 \bullet \bar{g}_1$. Then we have:

$$f_0 \simeq f_0 \bullet g_0 \bullet \bar{g}_0 \simeq f_1 \bullet g_1 \bullet \bar{g}_1 \simeq f_1$$

Problem 2 (1.1.2 in Hatcher)

The change of basepoint homomorphism is defined by:

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\to \pi_1(X, x_0) \\ [f] &\mapsto [h \bullet f \bullet \bar{h}] \end{aligned}$$

Consider two homotopic paths $h_0 \simeq h_1$ from x_0 to x_1, and denote the homotopy by h_t. Since path products preserve homotopy equivalence, $h_t \bullet f \bullet \bar{h}_t$ is a homotopy between $h_0 \bullet f \bullet \bar{h}_0$ and $h_1 \bullet f \bullet \bar{h}_1$. Then:

$$\beta_{h_0}[f] = [h_0 \bullet f \bullet \bar{h}_0] = [h_1 \bullet f \bullet \bar{h}_1] = \beta_{h_1}[f]$$

Problem 3 (1.1.3 in Hatcher)

Assume first that $\pi_1(X)$ is abelian, i.e. $[f \bullet g] = [g \bullet f]$ for all f, g loops at some $x_1 \in X$. Consider two paths h_0, h_1 from x_0 to x_1. Then $\bar{h}_1 \bullet h_0$ is a loop at x_1. Take any other loop f at x_1, and by hypothesis we have:

$$\bar{h}_1 \bullet h_0 \bullet f \simeq f \bullet \bar{h}_1 \bullet h_0$$

By taking a product with h_1 on the left side and with $\bar{h}_0$ on the right side, this is equivalent to:

$$h_0 \bullet f \bullet \bar{h}_0 \simeq h_1 \bullet f \bullet \bar{h}_1$$

Which is to say $\beta_{h_0}[f] = \beta_{h_1}[f]$, for any paths h_0, h_1 with common endpoints.

Conversely, assume $\beta_{h_0}[f] = \beta_{h_1}[f]$. Take h_0 to be the constant loop at x_1, and h_1 an arbitrary loop at x_1. Then, given another arbitrary loop f at x_1, the hypothesis becomes:

$$f \simeq h_0 \bullet f \bullet \bar{h}_0 \simeq h_1 \bullet f \bullet \bar{h}_1$$

We take a product with h_1 on the right, and obtain:

$$f \bullet h_1 \simeq h_1 \bullet f$$

In other words, any two arbitrary loops at x_1 commute, so $\pi_1(X, x_1)$ is abelian. Since $\pi_1(X, x)$ are isomorphic for all $x \in X$, the fundamental group is abelian irrespective of basepoint.

Problem 4 (1.1.5 in Hatcher)

(a) $\Rightarrow$ (b) Consider $f : S^1 \to X$; by (a), there exists a homotopy f_t between f and x_0, where the latter is interpreted as the constant map at $x_0 \in X$. Define:

$$\begin{aligned} g : D^2 &\to X \\ (r, \theta) &\mapsto f_r(\theta) \end{aligned}$$

g is well-defined in the origin, because $f_0(\theta) = x_0$ for all θ. Moreover, it is continuous by definition of the homotopy f_t. $g|_{S^1} = g(1, \theta) = f_1(\theta) = f(\theta)$, so g indeed extends f.

(b) $\Rightarrow$ (c) Consider a loop $f : S^1 \to X$, which represents a homotopy class in $\pi_1(X, x_1)$ for $x_1 = f(0) = f(1)$. By (b), there exists a map $g : D^2 \to X$ such that $g|_{S^1} = f$, and this map satisfies $g(0, \theta) = x_0$ for all θ, as shown in the proof of the previous part. Define:

$$\begin{aligned} f_t : S^1 &\to X \\ \theta &\mapsto g(t, \theta) \end{aligned}$$

We see that $f_0 = x_0$, the constant map at x_0, and $f_1 = f$. Moreover, g is continuous by assumption, so the family f_t is continous. However, f_t is not a homotopy of paths, because its endpoints are not independent of t. This can be fixed by considering any family h_t of paths from $g(t, 0)$ to x_1, and constructing $\tilde{f}_t = \bar{h}_t \bullet f_t \bullet h_t$. Now $\tilde{f}_t$ is a homotopy of paths from the constant loop at x_1 to f. Indeed, the endpoints of each $\tilde{f}_t$ are at x_1 by construction, $\tilde{f}_1 \simeq f_1$ and $\tilde{f}_0 = \bar{h}_0 \bullet x_0 \bullet h_0 \simeq x_1$. Therefore $[x_1]$ is the unique homotopy class in $\pi_1(X, x_1)$, so $\pi_1(X, x_1) = 0$.

(c) $\Rightarrow$ (a) Consider a map $f : S^1 \to X$, then $[f] \in \pi_1(X, f(0))$. But $\pi_1(X, f(0)) = 0$ by hypothesis, so $f \simeq f(0)$, where the latter is the constant path at $f(0)$. Since any homotopy of paths is a homotopy of maps, (a) follows.

Note that (a) is equivalent to the statement that all maps $S^1 \to X$ are homotopic. For, if $f : S^1 \to X$ is homotopic to the constant map at x_0 and $g : S^1 \to X$ is homotopic to the constant map at x_1, the constant maps are homotopic by translation along a path from x_0 to x_1. This shows that $f \simeq g$. This fact, together with $(a) \Leftrightarrow (c)$, means that X is simply connected iff all maps $S^1 \to X$ are homotopic.

Problem 5 (1.1.7 in Hatcher)

First the easy part: we explicitly show a homotopy between f and the identity, that is stationary on $S^1 \times \{0\}$ but not on $S^1 \times \{1\}$:

$$\begin{aligned} f_t : S^1 \times I &\to S^1 \times I \\ (\theta, s) &\mapsto (\theta + 2\pi t s, s) \end{aligned}$$

Now assume that there exists a homotopy g_t that is stationary on both $S^1 \times \{0\}$ and $S^1 \times \{1\}$. For a fixed θ_0, this means that $g_t(\theta_0, 0) = g_t(\theta_0, 1) = \theta_0$. Let $P : S^1 \times I \to S^1$ denote the projection onto the first factor; then $P \circ g_t|_{\{\theta_0\} \times I}$ is a homotopy of paths between the two loops $P \circ f|_{\{\theta_0\} \times I}$ and θ_0. (The latter is the constant loop at θ_0.) But, using the notation from class, $P \circ f|_{\{\theta_0\} \times I} = \omega_1$ and $\theta_0 = \omega_0$. We showed that the map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ that takes n to $[\omega_n]$ is injective, therefore $\omega_0 \simeq \omega_1$ implies $0 = 1$, and we reach a contradiction.