Algebraic topology HW1

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Problem 1

If X is contractible, there exist a point y and a homotopy equivalence $f : X \to \{y\}$. Let $g : \{y\} \to X$ be a map such that $f \circ g = \mathrm{Id}_{\{y\}}$ and $g \circ f \simeq \mathrm{Id}_X$. Furthermore, let $g(y) = x_0$. It suffices to prove that there is a path from every $x \in X$ to x_0, for then one can get a path between $x_1, x_2 \in X$ by concatenating the path from x_1 to x_0 with the one from x_0 to x_2.

Now, since $g \circ f \simeq \mathrm{Id}_X$, there exists a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{Id}_X$ and $f_1 = g \circ f$. We construct the desired path as follows:

$$\begin{aligned} \gamma : X \times I &\to X \\ (x, t) &\mapsto f_t(x) \end{aligned}$$

By definition of a homotopy, γ is continuous, and we have $\gamma(x, 0) = x$, $\gamma(x, 1) = x_0$ for all $x \in X$. Therefore γ is a path from x to x_0.

Problem 2

By hypothesis there exists a homotopy $f_t : X \to Y$ from f_0 to f_1 and a homotopy $g_t : Y \to Z$ from g_0 to g_1. We claim that $h_t = g_t \circ f_t$ is a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$. Indeed, by construction we have $h_0 = g_0 \circ f_0$ and $h_1 = g_1 \circ f_1$. Moreover, the map $H(x, t) = h_t(x)$ satisfies $H(x, t) = (G \circ F)(x, t)$, where G, F are defined similarly. By the definition of a homotopy we have that G, F are continuous, so H, a composition of continuous maps, is also continuous. This shows that h_t is a homotopy.

Problem 3

Consider the family of maps:

$$\begin{aligned} f_t : \mathbb{R}^n - \{0\} &\to \mathbb{R}^n - \{0\} \\ \mathbf{x} &\mapsto \frac{\mathbf{x}}{1 - t + t|\mathbf{x}|} \end{aligned}$$

Clearly $f_0 = \mathrm{Id}_{\mathbb{R}^n - \{0\}}$. Also $f_1(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$, which surjects onto S^{n-1}. If we restrict f_t to S^{n-1}, we have $|\mathbf{x}| = 1$, and so $f_t|_{S^{n-1}}(\mathbf{x}) = \mathbf{x}$. Finally, the maps f_t are continuous, because they are compositions of rational and square root functions of the coordinates $x_1, \ldots, x_n, t$ in the Euclidean space $(\mathbb{R}^n - \{0\}) \times I$.

Problem 4

a) Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be homotopy equivalences; then there exist maps $g_1 : Y \to X$ and $g_2 : Z \to Y$ such that $f_1 \circ g_1 \simeq \mathrm{Id}_Y$, $g_1 \circ f_1 \simeq \mathrm{Id}_X$, $f_2 \circ g_2 \simeq \mathrm{Id}_Z$, $g_2 \circ f_2 \simeq \mathrm{Id}_Y$. Then $f_2 \circ f_1 : X \to Z$ is a homotopy equivalence, because:

$$(g_1 \circ g_2) \circ (f_2 \circ f_1) \simeq g_1 \circ \mathrm{Id}_Y \circ f_1 = g_1 \circ f_1 \simeq \mathrm{Id}_X$$

$$(f_2 \circ f_1) \circ (g_1 \circ g_2) \simeq f_2 \circ \mathrm{Id}_Y \circ g_2 = f_2 \circ g_2 \simeq \mathrm{Id}_Z$$

This proves that homotopy equivalence is transitive. It's clear that it's also reflexive (using Id_X as homotopy equivalence from X to itself) and symmetric (exchanging the roles of f and g). Therefore it's an equivalence relation.

b) To show that the relation is reflexive, we construct a homotopy from any map $f : X \to Y$ to itself by $f_t(x) = f(x)$. To show that it's symmetric, given a homotopy f_t from $f_0 : X \to Y$ to $f_1 : X \to Y$, we construct a homotopy from f_1 to f_0 by $g_t(x) = f_{1-t}(x)$. Finally, if f_t is a homotopy from f_0 to f_1, and g_t is a homotopy from f_1 to f_2, then we get a homotopy from f_0 to f_2 by defining piecewise:

$$h_t(x) = \begin{cases} f_{2t} \text{ for } t \leq 1/2 \\ g_{2t-1} \text{ for } t > 1/2 \end{cases}$$

c) Let $f : X \to Y$ be a homotopy equivalence, $g : Y \to X$ be the associated homotopy equivalence, and $f' \simeq f$. Since $g \simeq g$, we use the result of problem 2 to get:

$$f' \circ g \simeq f \circ g \simeq \mathrm{Id}_Y$$

$$g \circ f' \simeq g \circ f \simeq \mathrm{Id}_X$$

Therefore f' is also a homotopy equivalence.

Problem 5

If $r : X \to A$ is a retract, and $i : A \to X$ is the inclusion map, then $r \circ i = \mathrm{Id}_A$ and $i \circ r \simeq \mathrm{Id}_X$ (with the homotopy given by the deformation retraction). This means that $X \simeq A$. By definition of contractibility, we also have $X \simeq \{y\}$ for some point y. But homotopy equivalence is an equivalence relation, so these imply $A \simeq \{y\}$.