QFT Lecture 9

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Perturbation theory

$$S = \int d^4x(\mathcal{L}_0 + \mathcal{L}_{int})$$

$$\mathcal{L}_int = -\frac{1}{2}(Z_\phi - 1)(\partial\phi)^2 - \frac{1}{2}(Z_m - 1)m^2\phi^2 + \frac{1}{3!}Z_g g\phi^3 + Y\phi + \Lambda$$

$$Z_1[J] = \int D\phi e^{i\int d^4x\mathcal{L}_0 + 1/3! Z_g g\phi^3 + J\phi}$$

$$Z_1[J] = \sum_{V=0}^{\infty} \frac{1}{v!}\left(\frac{i}{3!}Z_g g\int d^3x(\delta/i\delta J_x)\right)^v \sum_{P=0}^{\infty}\frac{1}{P!}\left(\frac{1}{2}\int d^4yd^4z iJ_y iJ_z\Delta_{yz}/i\right)^p$$

$$Z_1[J] = e^{iW[J]}$$

Where $W[J]$ is the sum over all connected diagrams. For example:

$$\Theta = \frac{1}{12}(iZ^g g)^2\int d^4xd^4y\left(\frac{\Delta_{xy}}{i}\right)^3$$

$$dumbell = \frac{1}{8}(iZ^g g)^2\int d^4xd^4y\frac{\Delta_{xy}}{i}\left(\frac{\Delta(0)}{i}\right)^2$$

$$O - * = \frac{1}{2}(iZ_g g)\int d^4xd^4y(iJ_y)\frac{\Delta_{xy}}{i}\frac{\Delta(0)}{i}$$

$$* - * = \frac{1}{2}\int d^4xd^4y(iJ_x)(iJ_y)\frac{\Delta_{xy}}{i}$$

$$* - O - * = \frac{1}{4}(iZ_g g)^2\int d^4xd^4yd^4ad^4b(iJ_a)(iJ_b)\left(\frac{\Delta_{xy}}{i}\right)^2\frac{\Delta_{ax}}{i}\frac{\Delta_{yb}}{i}$$

Let's focus for a bit on diagrams with no J. For the free generating function:

$$Z_0[J = 0] = 1$$

With ϕ^3 interaction, this no longer holds:

$$Z_1[J=0] = e^{iW[J=0]}$$

i.e. the exponential of the sum of all bubble diagrams. This is not 1 in general. To correct this, we introduce the counterterm Λ. To get 1, we have to choose:

$$-i\int d^4x\Lambda = \Theta + dumbell + ...$$

$$\int d^4x\Lambda = -W(J=0)$$

It's not obvious why this works - why the sum over all bubbles is the volume integral of a constant. To see it, let's look at the Θ diagram:

$$\Theta = \frac{1}{12}(iZ^g g)^2 \int \frac{d^4x d^4y}{i^3}\left(\int \frac{d^4k_1}{2\pi^4}\frac{e^{ik_1(x-y)}}{k_1^2+m^2-i\epsilon}\right)(2)(3)$$

$$\Theta = \frac{1}{12}(iZ^g g)^2 \int \frac{d^4x}{i^3}\int \frac{d^4k_2 d^4k_3}{2\pi^8}\frac{1}{(-k_2-k_3)^2+m^2-i\epsilon}\frac{1}{k_1^2+m^2-i\epsilon}\frac{1}{k_1^2+m^2-i\epsilon}$$

The k integral goes like k^2 - diverges. Thus Λ is infinite. To go around this issue, we introduce the ULTRAVIOLET CUTOFF: integrate only up to $k = k_{UV}$. The infinite Λ problem is very bad because it appears with a - sign in $\mathcal{H}$. We get a $-\lambda$ energy density for vacuum. Intuitively, the bubble diagrams represent vacuum energy because only J terms are interactions.

Now we'll make the argument that we also want to cancel out $1J$ diagrams. Consider:

$$\langle 0|\hat{\phi}(x)|0\rangle = \left.\frac{\delta Z_1[J]}{i\delta J(x)}\right|_{J=0} = e^{iW[J]}\left.\frac{\delta iW(J)}{\delta iJ_x}\right|_{J=0}$$

The first factor is already 1 by the removal of bubbles. The second factor Gives the derivative of all diagrams with one J. Why do we want this to be 0? This is:

$$\langle 0|\hat{\phi}(x)|0\rangle = \langle 0|e^{iHt}\hat{\phi}(0,\mathbf{x})e^{-iHt}|0\rangle$$

We can make the exponentials act on the vacuum states, which they leave unchanged. Therefore the 1-point function should be time independent. Then we can take the time to be in the far future or far past, where the field behaves like a free field. Using the expansion of the free field in terms of a and $a^\dagger$, a annihilates the ket and $a^\dagger$ annihilates the bra. We get 0. An intuitive argument is that one J diagrams mean that a particle interacts with the vacuum; if we want a stable vacuum (one that doesn't randomly poop out particles), we make them 0. For this reason, we introduce the $Y\phi$ term. The vertices will have 1 leg sticking out, so there's only one possible diagram:

$$\mathrm{x} - * = (\mathrm{iY})\int \mathrm{d^4xd^4y(iJ_y)\Delta_{xy}/i}$$

We want this to cancel out the diagram:

$$O - * = \frac{1}{2}(iZ_g g) \int d^4x d^4y (iJ_y) \frac{\Delta_{xy}}{i} \frac{\Delta(0)}{i}$$

We make the choice:

$$Y = \frac{-Z_g g}{2} \frac{\Delta(0)}{i}$$

This is the $O(g)$ contribution to Y. Choose $O(g^3)$ contributions to Y to cancel out all the g^3 one J diagrams. Then we can do this order by order.