Commutative algebra notes

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Lemma 1. *Given an A-module M, the rule $\mathcal{B} \to A$-modules, $U = D(f) \mapsto M_f = M \otimes_A A_f$ is a sheaf (of A-modules) on $\mathcal{B}$.*

Proof. Lemma 1 shows this is well defined ang gives the restriction mappings. It's on HW to prove the sheaf condition. $\square$

Definition 1. The structure sheaf of $Spec(A)$ is the sheaf of maps $\mathcal{O}_{Spec(A)}$ which corresponds to the rule:

$$D(f) \mapsto A_f$$

on the basis $\mathcal{B}$ of standard opens.

Remark. Similarly we have the sheaf $\tilde{M}$ corresponding to $D(f) \mapsto M_f$. Observe that $\tilde{M}$ is a sheaf of $\mathcal{O}_{Spec(A)}$-modules.

Stalk of $\mathfrak{p}$. Since $\mathcal{B}$ is a bassi for the topological space, to compute the stalk we need only consider pairs $(D(f), s)$ where $\mathfrak{p} \in D(f)$ and $s \in A_f$. , i.e.:

$$f \in A - \mathfrak{p}, s = \frac{a}{f^n}$$

Then 2 pairs $(D(f), a/f^n)$ and $(D(g), b/g^n)$ give the same element of the stalk iff there exists $h \in A - \mathfrak{p}$ such that $D(h) \subset D(f), D(h) \subset D(g)$ and $1/f^n$ and $1/g^n$ map to the same element of A_h. Contemplate the diagram. We conclude that we get a well-defined, injective and surjective map. In particular, $\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} = A_{\mathfrak{p}}$.

Lemma 2. *The stalk of $\mathcal{O}_{\mathrm{Spec}(A)}$ at $\mathfrak{p}$ is $A_{\mathfrak{p}}$. The stalk of $\tilde{M}$ at $\mathfrak{p}$ is $M_{\mathfrak{p}}$.*

Remark. If $(X, \mathcal{O}_X)$ is a locally ringed space and $U \in X$ is open, then $(U, \mathcal{O}_X|_U)$ is a locally ringed space. Moreover there is an inclusion morphism:

$$j : (U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$$

of locally ringed spaces.

$$V \subset X \quad j^{\#} : \mathcal{O}_X(V) \overset{\rho^V_{U \cap V}}{\to} \mathcal{O}_X|_U(j^{-1}V) = \mathcal{O}_X(U \subset V)$$

Remark. Open subspaces of schemes are, again, schemes. To see this, it's enough to show that $(D(f), \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)}) = (\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)})$.

Ring maps and morphisms

Let $A \xrightarrow{\phi} B$ be a ring map. Then:

$$\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is a continuous map of top spaces. Moreover, if $f \in A$ then:

$$\mathrm{Spec}(\phi)^{-1}(D(f)) = D(\phi(f))$$

Lemma. *Let $f : X \to Y$ be a continuous map of top spaces. Let $\mathcal{B}, \mathcal{C}$ be a basis for the top on X, Y respectively, both closed under intersections. Assume $f^{-1}v \in \mathcal{B}$ for all $V \in \mathcal{C}$. Then given sheaves $\mathcal{F}, \mathcal{G}$ on X, Y respectively. To give a collection of maps:*

$$\phi(V) : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$$

for all V open in Y compatible with the restriction maps, is the same as giving a collection:

$$\phi(V) : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$$

for all $V \in \mathcal{C}$ compatible with the reztriction maps.

Remark. Such a collection of maps is called an f-map from $\mathcal{G}$ to $\mathcal{F}$.

Proof. Given $\phi(V)$ defined for $V \in \mathcal{C}$ and $W \subset Y$ open. Choose an open covering $W = \bigcup V_i$, $V_i \in \mathcal{C}$ and then define $\phi(W)$ by:

$$\mathcal{G}(W) = \{(s_i) \in \prod \mathcal{G}(V_i) | \rho^{V_i}_{V_i \cap V_j}(s_i) = \rho^{V_j}_{V_i \cap V_j}(s_j)\}$$

$$\mathcal{F}(f^{-1}W) = \{(t_i) \in \prod \mathcal{F}(f^{-1}V_i) | \dots\}$$

□

Going back to $A \xrightarrow{\phi} B$ we let:

$$\mathrm{Spec}(\phi) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

Defined by rules:

$$\mathrm{Spec}(\phi)(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \quad , \mathfrak{q} \in \mathrm{Spec}(B)$$

$$A_f = \mathcal{O}_{\mathrm{Spec}(A)}(D(f)) \to \mathcal{O}_{\mathrm{Spec}(B)}(D(\phi(f))) = B_{\phi(f)}$$

$$\frac{a}{f^n} \mapsto \frac{\phi(a)}{\phi(f)^n}$$

Morphism of ringed spaces. To check it's a morphism of schemes we need to check the induced maps:

$$\mathcal{O}_{\mathrm{Spec}(A)} \dots$$

is a local homo of local rings. This is OK as it's the map defined by ϕ.

Remark. $X \xrightarrow{f} Y, \phi : \mathcal{G} \to \mathcal{F}$ and f-map. Need to get:

$$\phi_x : \mathcal{G}_{f(x)} \to \mathcal{F}_x$$

$$(V, t) \mapsto (f^{-1}V, \phi(V)(t))$$

Lemma 3. *Let A be a ring and $f \in A$. The ring map $A \to A_f$ induces an isom:*

$$(\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)}) \xrightarrow{\cong} (D(f), \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)})$$

Lemma 4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Then f is an iso iff:*

(a) f is a homeo

(b) f induces isos on stalks.

Proof. Obvious by Lemma 6. □

Lemma 5. *Let $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ be a map of sheaves on a top space X. Then α is an iso iff $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is an iso for all $x \in X$.*

Proof. We will make $\beta : \mathcal{G} \to \mathcal{F}$ which is inverse of α. To do this it's enough if $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is a bijection for all U. We first show it's injective. Suppose $\alpha(s) = \alpha(s')$ for some $s, s' \in \mathcal{F}(U)$. Then $(U, \alpha(s)), (U, \alpha(s'))$ define the same element of the stalk $\mathcal{G}_x$ for all $x \in U$. By assumption this means that $(U, s), (U, s')$ define the same element in $\mathcal{F}_x$ for all $x \in U$. By definition this means that for all $x \in U$ there exists $x \in U_x \subset U$ open such that $s|_{U_x} = s'|_{U_x}$. Then $U = \bigcup_{x \in U} U_x$ is an open covering and the sheaf condition for $\mathcal{F}$ shows $s = s'$. Surjectivity is similar. □