Lecture 6

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Two-point function for free theory

we showed that the two-point function can be computed as:

$$\langle \phi_1 \phi_2 \rangle = \langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle = \frac{\int D\phi \ e^{iS} \phi(x_1) \phi(x_2)}{\int D\phi \ e^{iS}}$$

We looked at S with no J, i.e. no source. If we want a source, we add a term $J\phi$ and call the action S_J. Often we choose the implicit normalization of the measure such that the denominator is 1. For free theory, we worked this out to be:

$$e^{iS} = e^{-\frac{1}{2} \int d^4 x i \phi (-\partial^2 + m^2) \phi}$$

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{i} \Delta(x_1 - x_2)$$

Where Δ_{12} is the Green's function for the Klein Gordon eq.:

$$(-\partial_1^2 + m^2) \Delta_{12} = \delta^{(4)}(x_1 - x_2)$$

Comment:

$$\int D\phi e^{iS} = (2\pi)^{\frac{n}{2}} \left| \frac{\Delta}{i} \right|^{\frac{1}{2}}$$

Using the Fourier space to find the Green's function:

$$\Delta_{12} = \int \frac{d^4 k}{(2\pi)^4} f(k) e^{ik(x_1 - x_2)}$$

$$\int \frac{d^4 k}{(2\pi)^4} (k^2 + m^2 - i\epsilon) f(k) e^{ik(x_1 - x_2)} = \delta(x_1 - x_2)$$

$$\Delta(x_1 - x_2) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik(x_1 - x_2)}}{k^2 + m^2 - i\epsilon}$$

We integrate over dk^0, so k is not (yet) on-shell. The denominator is:

$$-(k^0 - (-\omega_k + i\epsilon))(k^0 - (\omega_k - i\epsilon))$$

We use $-i\epsilon$ and NOT $i\epsilon$ because this is what makes the path integral converge (see argument with the Hamiltonian from last time).

$$\Delta(x_1 - x_2) = i \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x_1}-\mathbf{x_2})} \left(\theta(t_1 - t_2) \frac{e^{-i\omega_k(t_1-t_2)}}{2\omega_k} + \theta(t_2 - t_1) \frac{e^{i\omega_k(t_1-t_2)}}{2\omega_k} \right)$$

Changing variables from k to -k in the second integral:

$$\langle \phi_1 \phi_2 \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(\theta(t_1 - t_2) e^{i\mathbf{k}(\mathbf{x_1}-\mathbf{x_2})} e^{-i\omega_k(t_1-t_2)} + \theta(t_2 - t_1) e^{-i\mathbf{k}(\mathbf{x_1}-\mathbf{x_2})} e^{i\omega_k(t_1-t_2)} \right)$$

If we let $k^0 = \omega_k$:

$$\langle 0 | T \hat{\phi}_1 \hat{\phi}_2 | 0 \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(\theta(t_1 - t_2) e^{ik(x_1-x_2)} + \theta(t_2 - t_1) e^{-ik(x_1-x_2)} \right)$$

We derived this result using path integrals. In the HW, we will check it using canonical quantization.

Classically we would have:

$$\phi(x) = \int d^4x' \Delta_{xx'} J(x')$$

Where Δ is a retarded Green function. But ours is half retarded and half advanced. This is called Feynman Green's function. Check causality: $[\hat{\phi}_1, \hat{\phi}_2] = 0$ when 1 and 2 are spacelike separated.

For free theory, $\langle \phi_1 ... \phi_m \rangle =$ Wick pairs, for example:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle + \langle \phi_1 \phi_3 \langle \phi_2 \phi_4 \rangle + \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle$$

For even number, the n-point function is 0.

n-point function for J

$$Z[J] = \int D\phi e^{iS_J}$$

Is called the generating function of the n-point function because:

$$\frac{\delta Z[J]}{\delta i J(x_1)} = \int D\phi \phi(x_1) e^{iS_J}$$

Where, implicitly, $\frac{\delta J(x)}{\delta J(x_1)} = \delta(x - x_1)$. Therefore we get:

$$\left. \frac{\delta Z[J]}{\delta i J(x_1)} \right|_{J=0} = \langle \phi(x_1) \rangle \qquad \left. \frac{\delta^n Z[J]}{\delta i J(x_1) ... \delta i J(x_n)} \right|_{J=0} = \langle \phi(x_1) ... \phi(x_n) \rangle$$

Static source

$J = J(x)$. Let's solve the classical problem:

$$(-\partial_x^2 + m^2)\phi_x = J_x$$

$$\phi_x = \int d^4x' J(\mathbf{x}') \int \frac{d^4k}{(2\pi)^2} \frac{e^{ik(x-x')}}{k^2+m^2}$$

$$\phi_x = \int d^3x' J(\mathbf{x}') \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2+m^2} = \frac{1}{4\pi} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

This is the Yukawa potential. Behaves like $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ when $|\mathbf{x}-\mathbf{x}'| << \frac{1}{m}$, and is exponentially supressed when $|\mathbf{x}-\mathbf{x}'| >> \frac{1}{m}$. We will see later why we call this 'potential'. Now let's compute the energy of our $\phi(x)$ configuration:

$$\mathcal{H} = \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - J\phi$$