Honors Complex Variables

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1 Power series

Theorem
Let f be analytic in an open set Ω and let $z_0 \in \Omega$. Choose $\rho > 0$ s.t. $\bar{D}(z_0, \rho) \subset \Omega$. Then f has a power series expansion at z_0, i.e. $\forall z \in D(z_0, \rho)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof
Let $\gamma = \partial D(z_0, \rho)$. $\forall z \in D(z_0, \rho)$, the Cauchy integral formula gives:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} dw$$

Since $|z - z_0| < \rho = |w - z_0|$,

$$\frac{1}{1 - \frac{z - z_0}{w - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

When plugging this back into the expression for $f(z)$, we use the fact that the sum converges uniformly in order to switch the order of summation and integration:

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw \right) (z - z_0)^n$$

By the Cauchy integral formulas,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw = \frac{f^{(n)}(z_0)}{n!}$$

Remark
The radius of convergence of the power series is the largest R s.t. f is analytic in $D(z_0, R)$.

Morera's theorem
Let f be a continuous function in an open disc D such that for any rectangle R in D we have $\int_R f(z)dz = 0$. Then f is analytic in D.

Proof
From the proof of the rectangle theorem we know that f has a primitive F in D. F is analytic in D since $F' = f$. By Cauchy's integral formulas, F is infinitely differentiable so, in particular, its second derivative exists. Therefore f is analytic.

Generalized Liouville theorem
Let f be entire. Suppose that there exist an integer $k \geq 0$ and positive constants A and B s.t. $\forall z$:

$$|f(z)| \leq A + B|z|^k$$

Then f is a polynomial of degree $\leq k$.

Proof
For any $z_0 \in \mathbb{C}$, let γ be the circle of radius R centered at z_0. Then by Cauchy's inequalities:

$$|f^{(m)}(z_0)| \leq \frac{m!}{R^m} \sup_{z\in\gamma} |f(z)| \leq \frac{m!}{R^m}(A + BR^k)$$

For $m > k$, letting $R \to \infty$ gives $f^{(m)}(z_0) = 0$. Then the power series expansion of f is:

$$f(z) = \sum_{n=0}^{k} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Uniqueness theorem
Let f be analytic in a region Ω. Let $\{w_k\}$ be a sequence of distinct points in Ω s.t. $f(w_k) = 0$ and $w_k \to z_0 \in \Omega$. Then f is identically 0 in Ω.

Proof
Since $z_0 \in \Omega$ open, then there exists $r > 0$ s.t. $D(z_0, r) \subset \Omega$. Let's first show that f is identically 0 in this disc. Suppose the contrary. Then, in the power series expansion of f about z_0, there is a smallest n such that $a_n \neq 0$. Since $f(z_0) = 0$ by continuity, $n \geq 1$. Then $f(z) = a_n(z - z_0)^n g(z)$, where

$$g(z) = 1 + \sum_{k=0}^{\infty} \frac{a_{n+k}}{a_n}(z - z_0)^k$$

When $z \to z_0$, $g(z) \to 1$. Therefore $g(w_k) \to 1$ when $k \to \infty$, so there exists some k_0 s.t. $|g(w_k)| \geq \frac{1}{2}$ when $k > k_0$. In this case,

$$|f(w_k)| = |a_n||w_k - z_0|^n |g(w_k)| \geq \frac{1}{2}|a_n||w_k - z_0|^n$$

But this contradicts $f(w_k) = 0$. So f must be identically 0 in $D(z_0, r)$.

Now let $W = \{z \in \Omega : f(z) = 0\}$ and let $U = \dot{W}$. We have just shown that $D(z_0, r) \subset U$, so $U \neq \emptyset$. By construction U is open. If we can show that U is closed, then $U = \Omega$, since the only clopen sets in a connected subspace of $\mathbb{C}$ are $\emptyset$ and the subspace itself. So let's show that U is closed.

We want to show that U contains all its limit points. Therefore consider a sequence $\{z_k\}$ in U s.t. $z_k \to z \in \Omega$. By continuity, $f(z) = 0$ so $z \in W$. Applying the first part of the proof to $\{z_k\}$, we obtain that there exists some disc $D(z, r)$ such that $D(z_0, r) \subset W$. Therefore $z \in U$. This shows that U is closed, so we can conclude that $U = \Omega$.

Corollary (Analytic continuation)
Let f and g analytic in Ω and $f(z) = g(z)$ for all z in an open subset D of Ω. Then $f = g$ throughout Ω.

2 More applications of the Cauchy integral formulas

2.1 Mean Value Theorem

Mean Value Theorem
If f is analytic in an open set Ω and $z_0 \in \Omega$, then whenever $\bar{D}(z_0, r) \subset \Omega$:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

Proof
By the Cauchy Integral Formulas,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(z)}{z - z_0} dz$$

If we parametrize the boundary by $z = z_0 + re^{i\theta}$, the result follows.

Corollary
By taking the real part of both sides, we prove that $u = Re(f)$ also satisfies the mean-value property.

Theorem
Any real harmonic function is the real part of an analytic function.

Proof
Let $g = 2\frac{\partial u}{\partial z}$. Since u is harmonic, g has continuous partial derivatives. Furthermore, it satisfies the Cauchy-Riemann equation, since:

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(2\frac{\partial}{\partial z}u) = 2\frac{1}{4}\Delta u = 0$$

This shows that g is analytic. Then, by the integral theorem, there exists f analytic such that $f' = g$.
Now let $f = w + iv$. We want to show that $w = u$. A simple calculation gives $f' = 2\frac{\partial w}{\partial z}$. However, by construction $f' = g = 2\frac{\partial u}{\partial z}$. Thus, $\frac{\partial}{\partial z}(u - w) = 0$. This means that $u - w = a$, for some real a. Then $Re(f + a) = u$, so u is the real part of the analytic function $f + a$.

From the above, we can conclude that any real harmonic function satisfies the mean value property.

2.2 Maximum Modulus Theorem

Maximum Principle
If u is a real, non-constant harmonic function in a region Ω, then u can't attain a maximum value in Ω. (i.e. there is no $x_0 \in \Omega$ s.t. $u(x_0) \geq u(x), \forall x \in \Omega$)

Proof
Suppose there is an $x_0 \in \Omega$ s.t. $M = u(x_0) \geq u(x), \forall x \in \Omega$. Let:

$$A = \{x \in \Omega : u(x) < M\}$$

$$B = \{x \in \Omega : u(x) = M\}$$

Note that $A = u^{-1}(-\infty, M)$. Since the inverse image of an open set by a continuous map is open, A is open.
Now let's show that B is open. Let $z \in B$. $u(z) = M \geq u(x), \forall x \in \Omega$. Since $z \in \Omega$ and Ω is open, there is some r s.t. $\bar{D}(z, r) \subset \Omega$. Take any $s \in (0, r)$. Then by the Mean Value Theorem:

$$M = u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + se^{i\theta})d\theta$$

Let's prove by contradiction that $u(z + se^{i\theta}) = M, \forall \theta \in (0, 2\pi)$. Assume there is some θ_0 for which this does not hold. Then, since u is continuous, there is an arc of length t on the circle for which $u(z + se^{i\theta}) \leq M - \epsilon$. Then:

$$M = \frac{1}{2\pi}\int_0^t u(z+se^{i\theta})d\theta + \frac{1}{2\pi}\int_t^{2\pi} u(z+se^{i\theta})d\theta \leq \frac{1}{2\pi}(t(M-\epsilon)+(2\pi-\epsilon)M) = M - \frac{t\epsilon}{2\pi} < M$$

Which is a contradiction. Therefore $u(z) = M, \forall z$ on $\partial D(z, s)$. Since this holds for all $s \in (0, r)$, we have $u(z) = M, \forall z \in D(z, \frac{r}{2})$. This means $D(z, \frac{r}{2}) \subset B$, so B is open.
We now have two open subsets of D such that $A \cap B = \emptyset$ and $A \cup B = D$. Since D is connected and B is nonempty, $B = D$, so f is a constant.

Maximum Modulus Theorem
If f is a non-constant analytic function in a region Ω, then $|f|$ does not attain a maximum value in Ω.

Proof
Since f satisfies the Mean Value Theorem, $|f|$ also satisfies it. The proof of the maximum principle for harmonic functions only makes use of the MVT, so it can be applied to show that if $|f|$ has a maximum in Ω, then $|f|$ is constant in Ω. It is a simple exercise to show that an analytic function with constant modulus is a constant.

A positive formulation of the maximum modulus theorem
If f is continuous on a compact set E and analytic in the interior $\dot{E}$ of E, then the maximum of $|f|$ is attained on ∂E.

Proof
f is continuous on the compact set E, so by the Weirstrass extreme value theorem $|f|$ has a maximum at some $z_0 \in E$. If $z_0 \in \partial E$, we are done. Otherwise $z_0 \in \dot{E}$. Then, since $\dot{E}$ is open, there exists some r s.t. $D(z_0, r) \subset \dot{E}$. But this means that $|f|$ attains its maximum inside the connected component D of $\dot{E}$ that contains $D(z_0, r)$, so by the maximum modulus theorem f is constant on D. So there exists $z_1 \in \partial D$ s.t. $|f(z_1)| = |f(z_0)| \geq |f(z)|, \forall z \in E$. Since $\partial D \subset \partial E$, this proves the desired result.

Minimum modulus theorem
If f is a nonconstant analytic function in a region Ω, then then no point $z_0 \in \Omega$ can be a minimum for $|f|$ unless $f(z_0) = 0$.

Proof
Suppose the contrary. This implies that f is nonzero throughout Ω. Define $g = \frac{1}{f}$; g is nonconstant and analytic in Ω. By construction $|g|$ attains its maximum at z_0, which contradicts the maximum modulus theorem.

Application of the maximum and minimum modulus theorems
Let f be entire and non-constant and $|f(z)| = 1$ when $|z| = 1$. Then $f(z) = cz^n$ for some $n \geq 1$ and $|c| = 1$.

Proof
If f has an infinite number of zeros in the unit disc D, then these zeroes accumulate in $\bar{D}$

and f is identically 0, by the uniqueness theorem. This contradicts the hypothesis. So f has finitely many zeros in D. Call them a_j, $j \in \{1...n\}$. Let

$$g(z) = \frac{f(z)}{\Pi_{j=1}^{n} \frac{z-a_j}{1-\bar{a}_j z}}$$

$|g(z)| = 1$ when $|z| = 1$, since Blaschke factors have unit modulus when $|z| = 1$ and $|a_j| \leq 1$. Furthermore, $g(z) \neq 0, \forall z \in D$. Assume g is not constant in D. Then by the maximum modulus theorem $|g(z)| < 1$ when $|z| < 1$. Then by continuity $|g|$ has a nonzero minimum inside D, which contradicts the minimum modulus theorem. So g must be a constant in D. Thus:

$$f(z) = c\Pi_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}$$

holds in D. By analytic continuation, it holds for all $z \in \mathbb{C}$. But, since f is entire, it can have no singularities. This forces $a_j = 0, \forall j$. Then $f(z) = cz^n$.

2.3 Schwarz lemma

Schwarz lemma
If f is analytic in the unit disc D and satisfies:
a) $|f(z)| \leq 1, \forall z \in D$
b) $f(0) = 0$,
Then:
i) $|f(z)| \leq |z|, \forall z \in D$.
ii) $|f'(0)| \leq 1$.
iii) If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, or $|f'(0)| = 1$, then $f(z) = cz$, where c is a constant of unit modulus.

Proof
Expand f in a power series about 0:

$$f(z) = a_0 + a_1 z + a_2 z^2 + ...$$

Since $f(0) = 0$, $a_0 = 0$. So $f(z) = zg(z)$, where $g(z)$ is analytic and its power series expansion is:

$$g(z) = a_1 + a_2 z + a_3 z^2 + ...$$

Note that $f'(0) = a_1 = g(0)$. Pick some $r < 1$ and let $|z| = r$. Then:

$$g(z) = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

By the maximum modulus theorem, $|g(z)| \leq \frac{1}{r}$ on the compact set $\{|z| \leq r\}$. By letting $r \to 1$, we obtain $|g(z)| \leq 1$ on D. Then $|f(z)| = |z||g(z)| \leq |z|$, which proves i).

$|f'(0)| = |g(0)| \leq 1$, which proves ii). If $|f'(0)| = 1$, $|g(0)| = 1$. If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$. In either case, $|g|$ attains its maximum value inside D. By the maximum modulus theorem, $|g|$ is a constant c. Hence $f(z) = cz$, which proves iii).

Application of the Schwarz lemma
If $f : D \to D$ is bijective and analytic, then there exist $\theta \in \mathbb{R}$ and $\alpha \in D$ s.t.:

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}$$

Schwarz reflection principle
Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real axis (i.e. $z \in \Omega \Rightarrow \bar{z} \in \Omega$). Let Ω^+, Ω^- denote the parts of Ω that lie in the upper and lower half plane. Let $\Omega \cap \mathbb{R} = I$. Consider an analytic function f in Ω^+ that extends continnuously to I and is real valued on I. Then there exists a function F analytic on Ω s.t. $F = f$ on Ω^+.

Proof
For $z \in \Omega^-$, let $F(z) = \overline{f(\bar{z})}$. Then:

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0} \right)} = \lim_{\bar{z} \to \bar{z}_0} \overline{\left(\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0} \right)} = \overline{f'(\bar{z})}$$

So F is analytic in Ω^-. Since f is real valued on I, $f(z) = \overline{f(\bar{z})}$, so F is contiuous on Ω. Then the symmetry principle, stated and proved below, tells us that F is analytic on Ω.

Schwarz symmetry principle
If f^+ and f^- are analytic in Ω^+ and Ω^-, extend continuously to I and $f^+(x) = f^-(x), \forall x \in I$, then $f(z)$ defined by $f(z) = f^+(z)$ for $z \in \Omega^+$ and $f(z) = f^-(z)$ for $z \in \Omega^-$ is analytic in Ω.

Proof
Morera's theorem tells us that f is analytic in Ω if its integral over any triangle contained in Ω is 0. We consider 3 cases:

i) A triangle T fully contained in Ω^+. Since f is analytic in Ω, the integral of f over T is 0 by Goursat's theorem.

ii) A triangle T fully contained in Ω^+, with the exception of an edge or a vertex that touches I. In this case we look at T_ϵ, the triangle obtained by translating T upwards by an amount ϵ. By Goursat's theorem the integral of f over T_ϵ is 0. Letting $\epsilon \to 0$, the continuity of f tells us that the integral of f over T will also be 0.

iii) A triangle T whose interior intersects I. T can be split into smaller triangles that fall into the categories i) and ii). The integral of f over T is 0 since it is the sum of the integrals of f over these smaller triangles.

2.4 Sequences of analytic functions

Theorem
Let $\{f_n\}$ be a sequence of functions analytic in a region Ω that converges uniformly to a function f in every compact subset of Ω. Then f is analytic in Ω.

Proof
Let R be any rectangle in Ω and K some compact subset of Ω that contains R. By Goursat's theorem,

$$\int_{\partial R} f_n(z)dz = 0$$

The limit $n \to \infty$ of this integral will also be 0. But $f_n \to f$ uniformly in K, so we can exchange the limit and the integral to obtain:

$$\int_{\partial R} f(z)dz = 0$$

Then, by Morera's theorem, f is analytic.

Remark This result does not hold in real analysis. If we let $f : [0,1] \to \mathbb{R}$ be continuous, the Weirstrass approximation theorem tells us that there exists a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on $[0,1]$. However, f is not necessarily differentiable.

Theorem Under the conditions of the previous theorem, $f_n' \to f'$ on every compact subset K of Ω.

Proof
For any $\delta > 0$ construct the compact sets K_δ and K_δ':

$$K_\delta = \{z \in \Omega : \overline{D}(z,\delta) \subset \Omega\}$$

$$K_\delta' = \{z \in \Omega : \overline{D}(z,\delta) \subset K\}$$

Let F be analytic in K. $\forall z \in K'$ let $\gamma = \partial D(z,\delta)$. By Cauchy's integral formulas:

$$|F'(z)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}dz\right| \leq \frac{1}{2\pi}\frac{\sup_{w\in\gamma}|F(w)|}{\delta^2}2\pi\delta \leq \frac{1}{\delta}\sup_{w\in K}|F(w)|$$

Applying this to $F = f_n - f$:

$$\sup_{z\in K'}|f_n'(z) - f'(z)| \leq \frac{1}{\delta}\sup_{w\in K}|f_n(w) - f(w)|$$

Since δ is fixed by the choice of K' and $f_n \to f$ uniformly, this shows $f_n' \to f'$.

3 Conformal maps

Lemma
If $f : U \to V$ is analytic and injective, then:
a) $f'(z) \neq 0, \forall z \in U$
b) $f^{-1} : f(U) \to U$ is analytic
c) f preserves angles

Proof
a) Suppose $f'(z_0) = 0$ for some $z_0 \in U$. Then, for some $k \geq 2$:

$$f(z) - f(z_0) = a_k(z - z_0)^k + (z - z_0)^{k+1}h(z) = a_k(z - z_0)^k + H(z)$$

For any $r > 0$, there is some M s.t. $h(z) \leq M$ on $D(z_0, 2r)$. Choose $t \leq r$, then on $\partial D(z_0, t)$ we have:

$$|H(z)| \leq Mt^{k+1}$$

Define $F(z) = a_k(z - z_0)^k - w$ for some $|w| \leq \frac{1}{2}|a_k|t^k$ (I'll buy a beer to anyone who can explain the purpose of substracting w). On $\partial D(z_0, t)$, we have:

$$|F(z)| \geq |a_k|t^k - |w| \geq \frac{1}{2}|a_k|t^k$$

In order to use Rouche's theorem, we would like to have $|F(z)| > |H(z)|$ on $\partial D(z_0, t)$. This happens when $t < \frac{a_k}{2M}$, which is allright, since we are free to choose t. Rouche's theorem says that $F(z)$ and $K(z) = F(z) + H(z)$ have the same number of zeros inside $D(z_0, 2r)$. Since $|F(z)|$ has k such zeros, K must have the same. But $K(z) = f(z) - f(z_0) - w$, so $K'(z) = f'(z) \neq 0$. Therefore all the zeros of k have multiplicity 1, so they have k distinct values. f maps these k distinct values to $f(z_0)+w$, which contradicts the fact that f is injective.

b) Let $g = f^{-1} : f(U) \to U$. Let $w = f(z)$ and $w_0 = f(z_0)$. Since f and g are continuous, $z \to z_0 \Leftrightarrow w \to w_0$, so

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}$$

Since f' is defined and nonzero on U, the limit exists, so g is analytic.

c) Let $\gamma(t)$ be a smooth curve with $\gamma(t_0) = z_0$. The angle between the tangent line to γ at z_0 is $\arg(\gamma'(z_0))$. Let $\Gamma = f(\gamma)$. The angle between the tangent line to Γ at $f(z_0)$ and the positive real axis is:

$$\arg\left(f'(\gamma(t_0))\right) = \arg(f'(z_0)\gamma'(t_0)) = \arg(f'(z_0)) + \arg(\gamma'(t_0))$$

Therefore, if we have two curves γ_1 and γ_2, the angle between them is the same as the angle between their images:

$$\arg\left(f'(\gamma_1(t_0))\right) - \arg\left(f'(\gamma_2(t_0))\right) = \arg(\gamma_1'(t_0)) - \arg(\gamma_2'(t_0))$$

Definition
A bijective analytic function $f : U \to V$, with U and V open, is called a conformal map. U and V are said to be conformally equivalent or biholomorphic.

Given two open sets U and V, we can ask whether there exists a conformal map from U to V. Let's look at two cases:

1) $U = \mathbb{C}$. A conformal map exists $\Leftrightarrow V = \mathbb{C}$. To show this, first note that if $V = \mathbb{C}$, then $f(z) = z$ is a conformal map. Conversely, suppose $f : \mathbb{C} \to V$ conformal. f is entire and injective, so by HW9 it is a linear map (the proof considers singularities at ∞ and shows that these singularities must be poles of order 1). But the image of a linear map is $\mathbb{C}$.

2) $V = D$, the unit disc. A conformal map exists $\Leftrightarrow U \neq \mathbb{C}$ and is simply connected. The Riemann mapping theorem proves the existence of a conformal map if U satisfies the given conditions. The proof of that theorem is not included here. Conversely, suppose a conformal map f exists. The proof of 1) above shows that $U \neq \mathbb{C}$. To see that U must be simply connected, take a closed curve $\gamma : [0,1] \to U$. Then $f \circ \gamma : [0,1] \to D$ ia a curve in D. Since D is simply connected, we can deform $f \circ \gamma$ to a point z by some analytic map F. By the lemma, f^{-1} is analytic, so $F \circ f^{-1}$ is analytic. Then γ can be deformed to $f^{-1}(z)$ by $F \circ f^{-1}$, so U is simply connected.

Automorphisms of the unit disc
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