QM for Mathematicians HW9

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Problem 1

Let $\Lambda = e^{tK_1}$; the commutator of Lie algebra elements is:

$$[K_1, P_m] = \frac{d}{dt}(\Lambda P_m \Lambda^{-1})|_{t=0}$$

For some translation $(a, 1)$, using the group law gives:

$$(0, \Lambda)(a, 1)(0, \Lambda^{-1}) = (\Lambda a, \Lambda)(0, \Lambda^{-1}) = (\Lambda a, 1)$$

Therefore:

$$[K_1, P] = \frac{d}{dt}(\Lambda P)|_{t=0} = \frac{d}{dt}\begin{pmatrix} \cosh t & \sinh t & & \\ \sinh t & \cosh t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

$$[K_1, P] = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore:

$$[K_1, P_0] = P_1 \quad [K_1, P_1] = P_0 \quad [K_1, P_2] = 0 \quad [K_1, P_3] = 0$$

In general:

$$[K_l, P_0] = P_l \qquad [K_l, P_m] = \delta_{lm} P_0 \text{ , for } m \neq 0$$

Problem 2

a)

$$\hat{P} = -i \int d^3x \hat{\Pi}(x)(-i\nabla)\hat{\phi}(x)$$

$$\hat{\Pi}(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2\omega_k)^{1/2}} i\omega_k \left(-a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)$$

$$\hat{P} = -\int \frac{d^3kd^3k'}{(2\pi)^3}\frac{1}{2}\int d^3x \left(a_k e^{-ikx} - a_k^\dagger e^{ikx}\right)\mathbf{k}'\left(a_{k'}e^{-ik'x} - a_{k'}^\dagger e^{ik'x}\right)$$

$$\hat{P} = \int \frac{d^3kd^3k'}{2(2\pi)^3}\mathbf{k}'\int d^3x \left[(a_k a_{k'}^\dagger + a_k^\dagger a_{k'})e^{i(k-k')x} - (a_k a_{k'} + a_k^\dagger a_{k'}^\dagger)e^{i(k+k')x}\right]$$

$$\hat{P} = \int \frac{d^3kd^3k'}{2(2\pi)^3}\mathbf{k}'\left[(a_k a_{k'}^\dagger + a_k^\dagger a_{k'})\delta(k-k') - (a_k a_{k'} + a_k^\dagger a_{k'}^\dagger)\delta(k+k')\right]$$

$$\hat{P} = \int \frac{d^3k}{2(2\pi)^3}\mathbf{k}(a_k a_k^\dagger + a_k^\dagger a_k) + \int \frac{d^3k}{2(2\pi)^3}\mathbf{k}(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)$$

Note that, since $[a_k, a_{-k}] = 0$ and $[a_k^\dagger, a_{-k}^\dagger] = 0$, the integrand in the second term is even, and thus the integral is 0. In the first term, we use the highly suspicious procedure of normal ordering to obtain:

$$\hat{P} = \int \frac{d^3k}{(2\pi)^3}\mathbf{k}\; a_k a_k^\dagger$$

b)

$$\hat{P} = -i\int d^3x \hat{\Pi}(x)(-i\nabla)\hat{\phi}(x) + \text{h. c.} = -i\int d^3x \left[\dot{\phi}^\dagger(x)(-i)\nabla\phi(x) + \nabla\phi^\dagger(x)(i)\dot{\phi}(x)\right]$$

We use:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2\omega_k)^{1/2}}\left(a_k e^{-ikx} + b_k^\dagger e^{ikx}\right)$$

$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2\omega_{k'})^{1/2}}\left(b_{k'}e^{-ik'x} + a_{k'}^\dagger e^{ik'x}\right)$$

Let's focus on the first term:

$$-\int \frac{d^3kd^3k'}{(2\pi)^3}\mathbf{k}'\int d^3x (b_{k'}e^{-ik'x} - a_{k'}^\dagger e^{ik'x})(a_k e^{-ikx} - b_k^\dagger e^{ikx})$$

$$-\int \frac{d^3kd^3k'}{(2\pi)^3}\mathbf{k}'\int d^3x \left[(b_{k'}a_k + a_{k'}^\dagger b_k^\dagger)e^{i(k+k')x} - (b_{k'}b_k^\dagger + a_{k'}^\dagger a_k)e^{i(k-k')x}\right]$$

$$\int d^3k\frac{1}{2}\mathbf{k}\left[-(b_{-k}a_k + a_{-k}^\dagger b_k^\dagger) + (b_k b_k^\dagger + a_k^\dagger a_k)\right]$$

Doing the same computation, we get the second term:

$$\int d^3k\frac{1}{2}\mathbf{k}\left[-(a_{-k}b_k + b_{-k}^\dagger a_k^\dagger) + (b_k^\dagger b_k + a_k a_k^\dagger)\right]$$

Adding them together gives:

$$\hat{P} = \int d^3k\frac{1}{2}\mathbf{k}\left[-(a_{-k}b_k + b_{-k}a_k + a_{-k}^\dagger b_k^\dagger + b_{-k}^\dagger a_k^\dagger) + (b_k b_k^\dagger + b_k^\dagger b_k + a_k^\dagger a_k + a_k a_k^\dagger)\right]$$

By the same argument as before, the first term is odd, so its integral is 0. In the second term we use normal ordering to get:

$$\hat{P} = \int d^3k\mathbf{k}(b_k b_k^\dagger + a_k a_k^\dagger)$$

Problem 3

The Lagrangian for a free theory with two complex fields is:

$$\mathcal{L} = -\,\partial^\mu\,\phi^\dagger\,\partial_\mu\,\phi - \partial^\mu\,\psi^\dagger\,\partial_\mu\,\psi - m^2\phi^\dagger\phi - m^2\psi^\dagger\psi$$

We need some linear transformation:

$$\begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = U \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Such that the Lagrangian is invariant, i.e. $\phi^\dagger\phi + \psi^\dagger\psi$ is unchanged. This can be written as:

$$\begin{pmatrix} \phi' & \psi' \end{pmatrix} \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = \begin{pmatrix} \phi & \psi \end{pmatrix} U^\dagger U \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi & \psi \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Therefore we need to impose $U^\dagger U = I_2$, so $U \in U(2)$. To find a basis for $\mathfrak{u}(2)$, note that this Lie algebra consists of skew-Hermitian matrices, so we impose:

$$\begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ ce^{i\gamma} & de^{i\delta} \end{pmatrix} = -\begin{pmatrix} ae^{-i\alpha} & ce^{-i\gamma} \\ be^{-i\beta} & de^{-i\delta} \end{pmatrix}$$

Where a, b, c, d are real. Thus, the most general form of a $\mathfrak{u}(2)$ matrix is:

$$\begin{pmatrix} ai & -ce^{-i\gamma} \\ ce^{i\gamma} & di \end{pmatrix} = a\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} + c\begin{pmatrix} 0 & -e^{-i\gamma} \\ e^{i\gamma} & 0 \end{pmatrix} =$$

$$= a\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} + c\cos\gamma\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + c\sin\gamma\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

We have found a basis for $\mathfrak{u}(2)$. Let's check what symmetries the four basis elements generate:

$$\text{Exp}\begin{pmatrix} ai & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{ia} & 0 \\ 0 & 0 \end{pmatrix}$$

This performs a $U(1)$ tranformation on ϕ. Similarly:

$$\text{Exp}\begin{pmatrix} 0 & 0 \\ 0 & di \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & e^{id} \end{pmatrix}$$

Which performs a $U(1)$ tranformation on ψ. Then:

$$\text{Exp}\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

Which performs an $SO(2)$ rotation of the two fields. Finally:

$$\mathrm{Exp}\begin{pmatrix} 0 & ci \\ ci & 0 \end{pmatrix} = \begin{pmatrix} \cos c & i\sin c \\ i\sin c & \cos c \end{pmatrix}$$

Therefore we can use this basis as our charge operators:

$$\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$