QFT Lecture 22

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Spinors

For left-handed spinors:

$$L_a^b(1+\delta\omega) = \delta_a^b + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a^b$$

Where $S_L^{ij} = \epsilon^{ijk}\frac{\sigma_k}{2}$ are the engular momenta and $S_L^{k0} = \frac{i}{2}\sigma_k$ are the boosts. The defining feature of the spinor representation is $K_k = iJ_k$. Important properties of the Pauli matrices:
a) $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma^k$
b) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$
c) $\sigma_k^T = -\epsilon\sigma_k\epsilon^{-1}$ where $\epsilon = \{(0,1),(-1,0)\}$

For right-handed spinors, the defining feature is $K_k = -iJ_k$. We have: $S_R^{ij} = \epsilon^{ijk}(-\sigma_k^*/2) = -(S_L^{ij})^*$ and $S_R^{k0} = i\sigma_k^*/2 = -(S_L^{k0})^*$.

Question: how does $\psi_a^\dagger$ transform? Like a right-handed spinor. Notation: $(\psi_a)^\dagger = \psi_{\dot{a}}^\dagger$

Now we want to write down an action for a free theory; it should be quadratic in the fields. For this, we need to form a scalar out of the spinors. Our scalar should be invariant under rotation and boosts: $\psi_2\chi_1 - \psi_1\chi_2$. In particular, for the same field, we get $\psi_2\psi_1 - \psi_1\psi_2$, which will not be 0, since these are anticommuting variables. To shorten notation, we use: $\epsilon^{ab}\psi_a\chi_b = \psi_2\chi_1 - \psi_1\chi_2$.

Let's prove that $\chi^T\epsilon\phi$ is a scalar. Let's see how it behaves under rotations:

$$\chi^T\epsilon\phi \rightarrow \left[(1+\frac{i}{2}\sigma_k\theta)\chi\right]^T\epsilon\left[(1+\frac{i}{2}\sigma_k\theta)\phi\right]$$

In order to have a real action, we need the mass term to be: $\phi\epsilon\phi + \phi^\dagger\epsilon\phi^{dagger}$.

How do we make vectors out of spinors? Use $\psi_{\dot{a}}^\dagger, \psi_b$. The vector is: $\psi^\dagger\bar{\sigma}^\mu\chi = \psi_{\dot{a}}^\dagger(\bar{\sigma}^\mu)^{\dot{a}b}\chi_b$. Here $\bar{\sigma}^\mu = (1, -\sigma_k)$. Similarly $\sigma^\mu = (1, \sigma_k)$. Note that this is a real vector if $\psi = \chi$. The Lagrangian for a single left-handed Weyl spinor is:

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\,\partial_\mu\,\psi - \frac{1}{2}m\psi\epsilon\psi - \frac{1}{2}m\psi^\dagger\epsilon\psi^\dagger$$

Note: the kinetic term looks weird. If we try to make it look like the free field: $\partial_\mu \psi \epsilon \partial^\mu \psi +$ h.c., we get particles with energy that is unbounded below. Let's check that the kinetic term is real:

$$(i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi)^\dagger = -i \partial_\mu \psi^\dagger \bar{\sigma}^\mu \psi$$

Now integrating by parts we get the original term. We obtained an action that is real, scalar and quadratic in the fields. The e.o.m:

$$0 = \frac{\delta S}{\delta \psi^\dagger} = i\bar{\sigma}^\mu \partial_\mu \psi - m\epsilon\psi^\dagger$$

$$0 = \frac{\delta S}{\delta \psi} = -i\bar{\sigma}^\mu \partial_\mu \psi^\dagger - m\epsilon^{-1}\psi$$

Multiplying through by ϵ in the last eq.:

$$0 = -i\bar{\sigma}^\mu \partial_\mu \psi^\dagger \epsilon - m\psi$$

Therefore the two equations are:

$$-i\bar{\sigma}^\mu \partial_\mu \psi + m\epsilon\psi^\dagger = 0$$

$$-i\sigma^\mu \partial_\mu (\epsilon\psi^\dagger) + m\psi = 0$$

Writing these in matrix form we get the Dirac equation:

$$\begin{pmatrix} m & -i\sigma^\mu \partial_\mu \\ -i\bar{\sigma}^\mu \partial_\mu & m \end{pmatrix} \begin{pmatrix} \psi \\ \epsilon\psi^\dagger \end{pmatrix} = 0$$

If we define:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

We get the Dirac equation in the form:

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0$$

The γ matrices have the property: $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$.