Lecture 5

Matei Ionita

February 5, 2013

Path integrals

$$\langle q_f, t_f | q_i, t_i \rangle = \int Dq e^{iS[q]}$$

$$\langle q_f, t_f | \hat{q}(t_1) | q_i, t_i \rangle = \langle q_f, t_f | e^{iHt_1} \hat{q}_S e^{-iHt_1} | q_i, t_i \rangle =$$

$$= \int dq' dq'' \langle q_f, t_f | q'' \rangle \langle q'' | e^{iHt_1} \hat{q}_S e^{-iHt_1} | q' \rangle \langle q' | q_i, t_i \rangle =$$

$$= \int dq' \langle q_f, t_f | e^{-iH(t_F - t')} | q' \rangle q' \langle q' | e^{-iH(t' - t_i)} | q_i, t_i \rangle = \int Dq e^{iS[q]} q_S(t_1)$$

In general:

$$\langle q_f, t_f | T \hat{q}(t_1) \hat{q}(t_2) | q_i, t_i \rangle = \int Dq e^{iS[q]|_{t_i}^{t_f}} q(t_1) q(t_2)$$

$$\lim_{t \to \infty} | q_i, t_i \rangle = \lim_{t \to \infty} e^{-iHt_i} | q_i \rangle = \sum \lim_{t \to \infty} e^{-iE_n(1 - i\epsilon)t} | n \rangle \langle n | q_i \rangle = | 0 \rangle \langle 0 | q_i \rangle$$

Where $|0\rangle$ is the ground state. Thus, when $t_i \to -\infty$, $|q_i, t_i\rangle \sim |0\rangle$. When $t_f \to \infty$, $\langle q_f, t_f| \sim \langle 0|$. At the level of the path integral, in order to make it converge:

$$S = \int dt \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 = \int dt \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 (1 - i\epsilon) q^2$$

More rigorously, do Euclideization: $t \to it$. Claim: with this $i\epsilon$ prescription, we get:

$$\langle 0 | T \hat{q}(t_1) \hat{q}(t_2) | 0 \rangle = \int Dq e^{iS[q]|_{-\infty}^{\infty}} q(t_1) q(t_2)$$

We assume that:

$$1 = \langle 0 | 0 \rangle = \int Dq e^{iS[q]|_{-\infty}^{\infty}}$$

Gaussian integral:

$$\int d^n x e^{-\frac{1}{2} x^T M x} = \int d^n x e^{-\frac{1}{2} \sum x_i M_{ij} x_j} = (2\pi)^{\frac{n}{2}} |M|^{-\frac{1}{2}}$$

$$\int d^n x e^{-\frac{1}{2}x^T M x + J^T x} = (2\pi)^{\frac{n}{2}} |M|^{-\frac{1}{2}} e^{\frac{1}{2}J^T M^{-1} J}$$

$$\langle x_i x_j \rangle = M^{-1}_{ij}$$

$$\langle x^3 \rangle = \langle x^5 \rangle = 0 \quad \langle x^4 \rangle = \frac{3}{M^2} = 3\langle x^2 \rangle$$

Read n-point functions, Wick's theorem, connected part of n-point functions.

Path integral formulation of free scalar QFT

Read chapter 7 of Sredniki (PI treatment of SHO). Now we're doing chapter 8, PI for free scalar QFT. Zee I.3.

$$S_J = \int d^4 x \left[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + J\phi \right]$$

Notice that J acts like a source of some sort, by writing the e.o.m.:

$$\partial^2 \phi - m^2 \phi = -J$$

$$\langle 0|0 \rangle_J = \int D\phi \, e^{iS_J}$$

Usually, we normalize $D\phi$ such that $\langle 0|0 \rangle = 1$. Quantum version of causality: operators commute when evaluated at spacelike separations (i.e. the measurment of one does not affect the other).

$$\langle x_i x_j \rangle = \frac{\int d^n x x_i x_j e^{-\frac{1}{2}x^T M x}}{\int d^n x e^{-\frac{1}{2}x^T M x}} = M^{-1}_{ij}$$

The exponent is:

$$iS = i \int d^4 x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \right]$$
$$= -\frac{i}{2} \int d^4 x \phi(-\partial^2 - m^2)\phi$$

$$M_{ij} = (id^4 x)(-\partial^2 + m^2)\delta_{ij}$$
$$\sum M^{ij} M^{-1}_{ik} = \delta_{ik}$$
$$(id^4 x)(-\partial^2 + m^2) M^{-1}_{ik} = \delta_{ik}$$
$$i(-\partial_x^2 + m^2) M^{-1}_{xy} = \delta(x-y)$$

M^{-1} is the Green's function for the KG eq. Definition:

$$(-\partial_x^2 + m^2)\Delta(x-y) = \delta(x-y)$$