Lie groups HW6

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Problem 1 (Kirillov 6.5)

We introduce the following notation:

$$h_a = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \qquad H_{ab} = \begin{pmatrix} h_a & 0 \\ 0 & h_b \end{pmatrix} \qquad X = \begin{pmatrix} h_c & B \\ -B^T & h_d \end{pmatrix}$$

Note that X is the most general form for a $\mathfrak{so}(4)$ element. We want to show that $\mathfrak{h} = \{H_{ab}\}$ is its own centralizer. For this we compute:

$$[H_{ab}, X] = \begin{pmatrix} [h_a, h_c] & h_a B - B h_d \\ -h_a B^T + B^T h_b & [h_b, h_d] \end{pmatrix} = \begin{pmatrix} 0 & h_a B - B h_d \\ -h_a B^T + B^T h_b & 0 \end{pmatrix}$$

Therefore, if we seek $X \in C(\mathfrak{h})$, we must have $h_a B - B h_d = 0$ for all a, d. This happens iff $B = 0$, which shows that $C(\mathfrak{h}) = \mathfrak{h}$.

In order for $\mathfrak{h}$ to be a Cartan, the adjoint action of all its elements must be diagonalizable. We prove this by explicitly computing the eigenvectors, which will also give the root space decomposition. Define:

$$m = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad n = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad p = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \qquad q = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & m \\ -m^T & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & n \\ -n^T & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & p \\ -p^T & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & q \\ -q^T & 0 \end{pmatrix}$$

(Note that these have a more intuitive description in terms of sigma matrices: $m = \sigma_0 - \sigma_2, n = i\sigma_1 + \sigma_3, p = \sigma_0 + \sigma_2, q = i\sigma_1 - \sigma_3$.) Then M, N, P, Q are eigenvectors for the adjoint action of each H_{ab}:

$$[H_{ab}, M] = i(a-b)M$$
$$[H_{ab}, N] = i(a+b)N$$
$$[H_{ab}, P] = -i(a-b)P$$
$$[H_{ab}, Q] = -i(a+b)Q$$

These are 4 eigenvalues, and the eigenspaces for each have dimension 1. Together with the zero eigenvalue space $\mathfrak{h}$ of dimension 2, we get 6, which is the dimension of $\mathfrak{so}(4)$. This proves that all H_{ab} are diagonalizable. The root space decomposition is:

$$\mathfrak{so}(4) = \mathfrak{h} \oplus \mathbb{C}M \oplus \mathbb{C}N \oplus \mathbb{C}P \oplus \mathbb{C}Q$$

Problem 2 (Kirillov 7.2)

(1) The induced inner product on the dual space is:

$$(\alpha^\vee, \beta^\vee) = \frac{4}{(\beta,\beta)(\alpha,\alpha)}(\alpha,\beta)$$

We see that the induced inner product on E^* is just a rescaling of the one on E. In particular, if $R = \{\alpha_i\}$ span E, then $R^\vee = \{\alpha_i^\vee\}$ span E^*. Next we want to compute:

$$n_{\alpha^\vee\beta^\vee} = 2\frac{(\alpha^\vee,\beta^\vee)}{(\beta^\vee,\beta^\vee)} = 2\frac{4(\beta,\alpha)}{(\alpha,\alpha)(\beta,\beta)}\frac{(\beta,\beta)}{4} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$

Finally, we need to show that the reflection $s_{\alpha^\vee}(\beta^\vee) \in R^\vee$. We can actually show that $s_{\alpha^\vee}(\beta^\vee) = (s_\alpha(\beta))^\vee$. To see this, let $\gamma \in E$ be arbitrary, and compute:

$$\begin{aligned}\langle \gamma, s_{\alpha^\vee}(\beta^\vee)\rangle &= \langle \gamma, \beta^\vee\rangle - \frac{2(\alpha^\vee,\beta^\vee)}{(\alpha^\vee,\alpha^\vee)}\langle\gamma,\alpha^\vee\rangle \\ &= \frac{2(\gamma,\beta)}{(\beta,\beta)} - \frac{4(\alpha,\beta)(\gamma,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\end{aligned}$$

$$\begin{aligned}\langle \gamma, (s_\alpha(\beta))^\vee\rangle &= \frac{2(\gamma, s_\alpha(\beta))}{(s_\alpha(\beta), s_\alpha(\beta))} \\ &= \frac{2(\gamma,\beta)}{(\beta,\beta)} - \frac{4(\alpha,\beta)(\gamma,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\end{aligned}$$

This holds for all γ, which proves the desired relation.

(2) We first show that a choice of positive roots for R induces a choice of positive roots for $R^\vee$. Let $t \in E$ be such that $(t,\alpha) \neq 0$ for all $\alpha \in R$, then R_+ is defined as the subset of R for which $(t,\alpha) > 0$. In order to define positive roots for $R^\vee$, consider $t^\vee$. Then:

$$(t^\vee, \alpha^\vee) = \frac{4}{(\alpha,\alpha)(t,t)}(\alpha, t)$$

We see that $(t^\vee,\alpha^\vee) \neq 0$ iff $(\alpha,t) \neq 0$, and also $(t^\vee,\alpha^\vee) > 0$ iff $(\alpha,t) > 0$. Therefore $R_+^\vee = \{\alpha^\vee | \alpha \in R_+\}$ is a choice of positive roots.

Now we look for the simple roots associated to this $R_+^\vee$. By equation (7.17), the simple roots $\Pi^\vee$ are the $r = \dim E^* = \dim E$ positive roots such that:

$$C_+^\vee = \{\lambda^\vee \in E^* | (\lambda^\vee, \alpha^\vee) > 0,\ \forall \alpha^\vee \in R_+^\vee\} = \{\lambda^\vee \in E^* | (\lambda^\vee,\alpha^\vee) > 0,\ \forall\alpha^\vee \in \Pi^\vee\}$$

Working from the LHS we have:

$$\begin{aligned}C_+^\vee &= \{\lambda^\vee \in E^* | (\lambda,\alpha) > 0,\ \forall\alpha \in R_+\} \\ &= \{\lambda^\vee \in E^* | (\lambda,\alpha) > 0,\ \forall \alpha \in \Pi_+\} \\ &= \{\lambda^\vee \in E^* | (\lambda^\vee,\alpha^\vee) > 0,\ \forall\alpha \in \Pi\}\end{aligned}$$

We found $r = |\Pi|$ positive coroots that make the relation hold, and so these must be the simple coroots.

Problem 3 (Kirillov 7.11)

(1) Using Lemma 7.39, we see that the longest element w_0 is the unique one such that $w_0(C_+) = C_-$, and that $l(w_0) = |R_+|$. For $r = 2$, we have $|R_+| = m$, so $l(w_0) = m$. Since $s_i^2 = 1$, we can only have a sequence that alternates between the two simple roots s_1 and s_2. Therefore $w = s_1 s_2 s_1 \ldots$, with m terms.

Problem 4 (Kirillov 7.13)

(1) We know from theorem 7.52 that $e_1, \ldots, e_r$ generate $\mathfrak{n}_+$, and so it suffices to show that commutators of the form:

$$[...[[e_{i_1}, e_{i_2}], e_{i_3}]...e_{i_n}]$$

are zero for large enough n. We use the fact that $e_i \in \alpha_i$, and so:

$$[...[[e_{i_1}, e_{i_2}], e_{i_3}]...e_{i_n}] \in \mathfrak{g}_{\alpha_{i_1}+\cdots+\alpha_{i_n}}$$

We know that all the simple roots α_i are positive, and so their sum will also be positive. Moreover, the height of $\alpha_{i_1} + \cdots + \alpha_{i_n}$ is n. But $\mathfrak{g}$ is finite dimensional, so there is an upper bound m on the height of its positive roots. Then choose $n = m+1$, so that $\mathfrak{g}_{\alpha_{i_1}+\cdots+\alpha_{i_n}} = 0$, and the fact that $\mathfrak{n}_+$ is nilpotent follows.

The statement for $\mathfrak{n}_-$ is proved analogously; the only difference is that commutators of f_i will live in root spaces which are linear combinations of α_i with coefficients -1 instead of 1.

(2) The Serre relations give $[h_i, e_j] \in \mathfrak{n}_+$. Together with $[e_i, e_j] \in \mathfrak{n}_+$ and $[h_i, h_j] = 0$, this shows that the commutator of any elements in $\mathfrak{b}$ is in $\mathfrak{n}_+$. This reduces commutators of the form:

$$[[[b_1, b_2], [b_3, b_4]], ...]$$

to commutators in $\mathfrak{n}_+$. But $\mathfrak{n}_+$ is nilpotent, so taking enough commutators give 0. Therefore $\mathfrak{b}$ is solvable.

Problem 5 (Kirillov 16)

(1) We start from:

$$h_\alpha = i_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And the definition of the adjoint action:

$$\begin{aligned}
\operatorname{Ad} S_\alpha(h_\alpha) &= \frac{d}{dt}\bigg|_{t=0} \left(S_\alpha e^{th_\alpha} S_\alpha^{-1}\right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left[i_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \exp\left(t i_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) i_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\
&= i_\alpha \frac{d}{dt}\bigg|_{t=0} \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\
&= i_\alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= -h_\alpha
\end{aligned}$$

Now if we have some h such that $\langle h, \alpha \rangle = 0$, we show that h commutes with $\frac{\pi}{2}(f_\alpha - e_\alpha)$. Using the Serre relations:

$$\begin{aligned}
[h, \frac{\pi}{2}(f_\alpha - e_\alpha)] &= \frac{\pi}{2}[h, f_\alpha] - \frac{\pi}{2}[h, e_\alpha] \\
&= \frac{\pi}{2}\langle h, \alpha \rangle (-f_\alpha - e_\alpha) \\
&= 0
\end{aligned}$$

From this we deduce that $S_\alpha e^{th} = e^{th} S_\alpha$, which immediately implies that $\operatorname{Ad} S_\alpha(h) = h$.

We define the action of S_α on $\mathfrak{g}^*$ in the obvious way:

$$\langle \operatorname{Ad} S_\alpha(\lambda), \operatorname{Ad} S_\alpha(X) \rangle = \langle \lambda, X \rangle$$

We have seen above that $\operatorname{Ad} S_\alpha(h_\beta) \in \mathfrak{h}$, therefore $\operatorname{Ad} S_\alpha(\beta) \in \mathfrak{h}^*$. Moreover, $\operatorname{Ad} S_\alpha(h_\alpha) = -h_\alpha$ implies $\operatorname{Ad} S_\alpha(\alpha) = -\alpha$, and $\operatorname{Ad} S_\alpha(h_\beta) = h_\beta$ implies $\operatorname{Ad} S_\alpha(\beta) = \beta$. This shows that $\operatorname{Ad} S_\alpha|_{\mathfrak{h}^*} = s_\alpha$.

(2) We saw in part 1 that $\operatorname{Ad} S_\alpha|_{\mathfrak{h}^*} = s_\alpha$. Now we have to prove a similar statement for arbitrary w in the Weyl group. We prove it for $w = s_\alpha s_\beta$, and the proof for longer elements follows analogously. It's easy to see that:

$$(\operatorname{Ad} S_\alpha \operatorname{Ad} S_\alpha)|_{\mathfrak{h}^*} = s_\alpha s_\beta$$

But $G \overset{\operatorname{Ad}}{\to} Gl(\mathfrak{g})$ is a homomorphism, so $\operatorname{Ad} S_\alpha \operatorname{Ad} S_\alpha = \operatorname{Ad}(S_\alpha S_\beta)$.