Commutative algebra HW11

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Problem 1

Consider $F = X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2$. Dehomogenize on $\mathbb{P}^2 - V(X_0)$, to get:

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1^2x_2^2$$

Cancelling the first derivatives gives:

$$x_1(1 + x_2^2) = 0$$

$$x_2(1 + x_1^2) = 0$$

Together with $f = 0$, we get the unique solution $x_1 = x_2 = 0$, which corresponds to $[1 : 0 : 0] \in \mathbb{P}^2$. Since F is symmetric in X_0, X_1, X_2, we get a total of 3 singular points, $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$.

Problem 2

The genus as a function of degree for a nonsingular curve is given by $g = \frac{1}{2}(d-1)(d-2)$, which in this case is 3. We also know that each singular point decreases the genus by at least 1, so $g \leq 0$. Therefore the only possibility is $g = 0$.

Problem 3

a) The product of the degrees, $2 \cdot 3 = 6$, is an upper bound for the number of intersection points. To see that the bound is sharp, we show that the plane $X_3 = 0$ intersects D in exactly 6 points. Setting $X_3 = 0$ we get:

$$X_2^2 = -X_0^2 - X_1^2$$

$$X_2^3 = -X_0^3 - X_1^3$$

And so:

$$X_2 = \frac{X_0^3 + X_1^3}{X_0^2 + X_1^2}$$

$$(X_0^2 + X_1^2)^3 + (X_0^3 + X_1^3)^2 = 0$$

Using Mathematica, we see that the last equation has 6 different solutions in $\mathbb{P}^2$, as desired.

b) If $[X_0 : X_1 : X_2] \in D'$ has inverse image $[X_0 : X_1 : X_2 : X_3] \in D$, then we see similarly to part a) that:

$$X_3^2 = -X_0^2 - X_1^2 - X_2^2$$
$$X_3^3 = -X_0^3 - X_1^3 - X_2^3$$
$$F(X_0, X_1, X_2) = (X_0^2 + X_1^2)^3 + (X_0^3 + X_1^3)^2 = 0$$

We take D' to be defined by $F = 0$. Now, given X_0, X_1, X_2, we attempt to construct the preimage $[X_0 : X_1 : X_2 : X_3] \in D$ by:

$$X_3 = \frac{X_0^3 + X_1^3 + X_2^3}{X_0^2 + X_1^2 + X_2^2}$$

If this is well-defined, i.e. for $X_0^2 + X_1^2 + X_2^2 \neq 0$, it is the unique point in the preimage. If this is not defined, we have $X_0^2 + X_1^2 + X_2^2 = X_0^3 + X_1^3 + X_2^3 = 0$, and we are in the situation of part a), i.e. there are 6 points in the preimage.

c) The degree of D' is the degree of F, which is 6.

d)

$$\frac{\partial F}{\partial X_i} = 2(\sum X_j^3) 3X_i^2 + 3\left(\sum X_j^2\right)^2 2X_i = 0$$

Summing over all i we get:

$$6\left(\sum X_i^2\right)\left[\sum X_i^3 + \left(\sum X_i^2\right)\left(\sum X_i\right)\right] = 0$$

Using Mathematica, we see that the different ways in which this can be solved in $\mathbb{P}^2$ break down to the case $\sum X_i^2 = \sum X_i^3 = 0$, and by part a) there are 6 solutions to this. Therefore D' has 6 singular points.

e) We guess that each of the 6 singularities decreases the genus by exactly 1, and then the genus of D' is $g = 10 - 6 = 4$. Then, since D and D' are birational, we expect that the genus of D is also 4.