Commutative algebra HW2

Matei Ionita

September 23, 2013

Problem 1

Let k be a field. Let $k[[t]]$ be the power series ring over k. Show that $k[[t]]$ is a local ring.

Solution
We can show that any power series with nonzero constant coefficient is invertible; thus the ideal (t) contains all non-units, and $\big(k[[t]], (t)\big)$ is a local ring. Consider:

$$F(t) = \left(a_0 + \sum_{n=1}^{\infty} a_n t^n\right)^{-1}$$

Then taking formal derivatives we can write $F(t) = \sum_{n=0}^{\infty} F^{(n)}(0)t^n/n!$, so $F(t) \in k[[t]]$, and we found an inverse for the original power series. This works as long as $F^{(n)}(0)$ doesn't have a zero denominator, which is always the case as long as $a_0 \neq 0$.

Problem 2

Give an example of (1) a local ring with 2 prime ideals and (2) a local ring with 3 prime ideals.

Solution
(1) Take $k[[t]]$ as in problem 1, its only primes are (0) and (t).
(2) Take $\mathbb{C}[x, y]/(xy)$. In HW1 we saw that its primes are $(x), (y), (x - \lambda, y), (x, y - \mu)$. If we localize this ring at (x, y), the only remaining primes will be the ones contained in (x, y), which are $(x), (y), (x, y)$.

Problem 3

Let $R = k[[t]]$ where k is a field. Give an example of a module M over R such that $M = tM$ (in other words, a module which contradicts the conclusion of Nakayama's lemma).

Solution
Consider $M = k[[t, t^{-1}]]$, which is an infinitely generated module over $k[[t]]$, with generators

t^{-n}. Any element of M looks like $a = \sum_{i=-\infty}^{\infty} k_i t^i$. Then $ta = \sum_{i=-\infty}^{\infty} k_{i-1} t^i$, so $tM \subset M$. Moreover, for all a there exists some $b \in M$ such that $a = bt$, namely $b = \sum_{i=-\infty}^{\infty} k_{i+1} t^i$. Therefore $M \subset tM$. So $M = tM$.

Problem 4

Let $R = \mathbb{C}[x]$ be the polynomial ring over the complex numbers. Let m_n, $n = 1, 2, 3, ...$ be an infinite sequence of pairwise distinct maximal ideals of R. Show that R does not surject onto the product of the rings R/m_n (contradicting the conclusion of the Chinese remainder theorem).

Solution
All maximal ideals of R are of the form $m_i = (z - \lambda_i)$, therefore $R/m_i = \mathbb{C}[x]/(z - \lambda_i) \cong \mathbb{C}$. Assume then that there exists a surjection

$$\phi : \mathbb{C}[x] \to \mathbb{C} \times \mathbb{C} \times \dots$$

Any homomorphism preserves $\mathbb{C} \subset \mathbb{C}[x]$, so we impose $\phi(\lambda) = (\lambda, \lambda, \dots)$ for all $\lambda \in \mathbb{C}$. Because ϕ is surjective, there exists $a \in \mathbb{C}[x]$ such that $\phi(a) = (1, 0, 0, \dots)$. But this means that $a \in m_i$ for all $i \geq 2$, so $a = f(x)(x - \lambda_2)(x - \lambda_3)\dots$ for some $f \in \mathbb{C}[x]$. a has an infinite number of roots, so $a = 0$. But this contradicts the fact that $\phi(0) = (0, 0, \dots)$. Therefore there exists no surjection $\mathbb{C}[x] \to \mathbb{C} \times \mathbb{C} \times \dots$

Problem 5

Let k be a field. Find the minimal prime ideals of $k[x, y, z]/(xy, xz, yz)$.

Solution
We are looking for primes of $k[x, y, z]$ that contain $(xy), (xz), (yz)$. The possible generators for primes in $k[x, y, z]$ are $x - \lambda, y - \lambda, z - \lambda$ as well as irreducible higher order polynomials f in x, y, z. But we do not care about the higher order polynomials, since these do not generate xy, xz or yz, and therefore any prime that contains f will contain another prime ideal which generates xy, xz, yz without the help of f. Therefore the primes which contain f will not be minimal. Similarly, we do not care about $x - \lambda, y - \lambda, z - \lambda$ for nonzero λ, since these do not help generate xy, xz, yz either, and they will prevent ideals from being minimal. All primes of interest are therefore generated by x, y, z; the possible ones are:

$$(x, y) \quad (x, z) \quad (y, z) \quad (x, y, z)$$

But (x, y, z) contains all other three primes, and thus the only minimal primes are $(x, y), (y, z), (x, z)$.