QFT Lecture 21

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Spinor fields

References: Srednicki 33-35, Zee II.3, appendix E. We'll only talk about spinors in 4 dimensions. Also look at Polchinski II. Appendix about how to construct spinors in arbitrary dimensions.

Let's remember what scalar means. $x \to \Lambda x$ leads to $\phi(x) \to \phi(\Lambda^{-1}x)$. (I.e. ϕ at some point P has the same value in the Lorentz-transformed coordiantes). For a vector field, we would have $A^\mu(x) \to \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$. We can wonder if there exists some object in between scalar and vector. Look at an object ψ_a with arbitrary number of components. We make it transform like: $\psi_a(x) \to L_a^b(\Lambda)\psi_b(\Lambda^{-1}x)$. In operator language: $U(\Lambda)^{-1}\hat{\psi}_a U(\Lambda) = L_a^b(\Lambda)\hat{\psi}_b(\Lambda^{-1}x)$. Some properties we would like L_a^b to have are:

1.
$$L_a^b(\Lambda = 1) = \delta_a^b$$

2.
$$U(\Lambda_2)^{-1}\left[U(\Lambda_1)^{-1}\hat{\psi}_a U(\Lambda_1)\right]U(\Lambda_2) = L_a^b(\Lambda_1)L_b^c(\Lambda_2)\hat{\psi}_c(\Lambda^{-1}x)$$
$$U(\Lambda_1\Lambda_2)^{-1}\hat{\psi}_a U(\Lambda_1\Lambda_2) = L_a^b(\Lambda_1)L_b^c(\Lambda_2)\hat{\psi}_c(\Lambda^{-1}x)$$

Thus: $L_a^c(\Lambda_1\Lambda_2) = L_a^b(\Lambda_1)L_b^c(\Lambda_2)$, so the set of L matrices has the same group structure as Lorentz transformations. Therefore the L matrices form a representation of the Lorentz group. By finding all representations of $SO(1,3)$, we get all particles.

Example: trivial representation: $\forall\Lambda, L_a^b(\Lambda) = \delta_a^b$. Vector representation: $L_a^b(\Lambda) = \Lambda_a^b$.

Expand L_a^b:
$$L_a^b(\Lambda) = \delta_a^b + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_a^b$$

We will recover $K^i = S^{i0}$ and $J^i = \frac{1}{2}\epsilon_{ijk}S^{jk}$. Earlier in the semester we computed the generators for the Lorentz group: $[J^i, J^j] = i\epsilon_{ijk}J^k$, $[J^i, K^j] = i\epsilon_{ijk}K^k$, $[K^i, K^j] = i\epsilon_{ijk}J^k$.

Representations of J are labeled by spin. In the spin 1/2 representation, they are the Pauli matrices. How do we find all representations of the Lorentz group? Wigner defined $N_i = \frac{1}{2}(J_i - iK_i)$ and $N_i^\dagger = \frac{1}{2}(J_i + iK_i)$. With these definitions, we get the Lie algebra: $[N_i, N_j] = i\epsilon_{ijk}N_k$, $[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger$, $[N_i, N_j^\dagger] = 0$. We thus have two copies of $SU(2)$; label them by j_L, j_R. For $j_L = j_R = 0$ we get the 1-1 repsesentation, i.e. a scalar.

Let's look at $j_L = 1/2, j_R = 0$, called a 2-1 representation. If we rotate along N, the object rotates, but about $N^\dagger$ it behaves like a scalar. In other words, $N^\dagger = 0$ for this representation. We need one index, so: $\psi_a \to L_a^b \psi_b$. $N^\dagger = 0 \Rightarrow K_i = iJ_i$. Therefore $J_i = \sigma_i/2$ and $K_i = i\sigma_i/2$.