Commutative algebra HW13

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Problem 2

Consider the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$, which is a ring homomorphism. It induces a surjective map on spectra $\operatorname{Spec} \mathbb{C} \twoheadrightarrow \operatorname{Spec} \mathbb{R}$. $\operatorname{Spec} \mathbb{C}$ and $\operatorname{Spec} \mathbb{R}$, equipped with the structure sheaf, are affine schemes, and this makes the map on spectra into a morphism of schemes. More precisely, the morphism is:

$$f : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$$
$$f^{\#} : \mathcal{O}(\operatorname{Spec} \mathbb{R}) = \mathbb{R} \to \mathcal{O}(\operatorname{Spec} \mathbb{C}) = \mathbb{C}$$
$$\mathbb{R} \hookrightarrow \mathbb{C}$$

This morphism is surjective because the map f on underlying topological spaces is surjective. Even though f has a right inverse as a map on topological spaces, the morphism of schemes has no right inverse. That's because an inverse for $f^{\#}$ would be a surjective morphism from $\mathbb{C}$ to $\mathbb{R}$, which doesn't exist.

Problem 5

We want to show that the following functor is representable by an affine scheme:

$$F : \mathrm{Sch}^{\circ} \to \mathrm{Set}$$
$$T \mapsto \prod \operatorname{Mor}(T, X_i)$$

This means we are looking for an affine scheme Y such that:

$$\operatorname{Mor}(T, Y) = \prod \operatorname{Mor}(T, X_i)$$

Using the equivalence in Tag 01I1, this can be rewritten as:

$$\operatorname{Hom}(A, \mathcal{O}_T(T)) = \prod \operatorname{Mor}(A_i, \mathcal{O}_T(T))$$

Where A, A_i are the coordiante rings of the affine schemes Y, X_i respectively. But this is precisely the universal property of the coproduct $A = \coprod A_i$. For the case of commutative rings, elements of the coproduct A are linear combinations over $\mathbb{Z}$ of terms of the form $a_{i_1} a_{i_2} a_{i_3} \dots a_{i_n}$ where $a_{i_1} \in A_{i_1}$ etc. (Each term is a finite product, and if I'm not mistaken, this is called a free ring product.) With A defined this way, the affine scheme we are looking for is just $Y = \operatorname{Spec} A$.