Lie groups HW3

Matei Ionita

October 24, 2013

Problem 1 (4.5 in Kirillov)

(1) Let V, W be irreducible representations of a Lie group G. Show that $(V \otimes W^*)^G = 0$ if V is non-isomorphic to W, and that $(V \otimes V^*)^G$ is canonically isomorphic to $\mathbb{C}$.
(2) Let V be an irreducible representation of a Lie algebra $\mathfrak{g}$. Show that V^* is also irreducible, and deduce from this that the space of $\mathfrak{g}$-invariant bilinear forms on V is either zero or 1-dimensional.

Proof. (1) $V \otimes W^*$ is canonically isomorphic to the space of linear maps $\phi : W \to V$, and the isomorphism is given by:

$$\phi \to \phi(w) \otimes w^*$$

This is an isomorphism because it is linear and it has an inverse given by:

$$(v \otimes w^*)(w) = v$$

Therefore the question reduces to Schur's lemma.

(2) Say we have a representation ρ of $\mathfrak{g}$ on V. We require any representation π of $\mathfrak{g}$ on V^* to preserve the action of V^* on V:

$$\big(\pi(g)v_1^*\big)\big(\rho(g)v_2\big) = v_1^*(v_2)$$

Take any subspace $W^* \subset V^*$ that is closed under the action of $\mathfrak{g}$. We want to show that $W^* = 0$ or $W^* = V^*$. For this, define:

$$W = \{w \in V : \exists w^* \in W^* \text{ such that } w^*(w) = 1\}$$

W and W^* are isomorphic by the map $w \to w^*$, therefore $\dim W = \dim W^*$. We show that W is $\mathfrak{g}$-invariant, and thus a subrepresentation of W:

$$\big(\pi(g)w^*\big)\big(\rho(g)w\big) = w^*(w) = 1$$

So we can exhibit $\pi(g)w^* \in W^*$ which maps $\rho(g)w$ to 1, which means $\rho(g)w \in W$. But V is irreducible, so $\dim W = 0$ or $\dim W = \dim V$. Then $\dim W^* = 0$ or $\dim W^* = \dim V^*$, which shows that V^* is irreducible.

Now we look at the space of $\mathfrak{g}$-invariant bilinear forms on V. By the same argument as in (1), $(V^* \otimes V^*)^{\mathfrak{g}}$ is canonically isomorphic to the space of $\mathfrak{g}$-invariant linear maps $\phi : V \to V^*$. But V, V^* are two irreducible representations of $\mathfrak{g}$, and thus all ϕ are intertwining operators for V and V^*. By Schur's lemma, either all ϕ are 0, in which case the space is 0 dimensional, or they are canonically isomorphic to $\mathbb{C}$, in which case it is 1 dimensional. (Note: Schur's lemma works the same way for Lie group and Lie algebra representations; it simply follows from the fact that the kernel and image of ϕ must be invariant subspaces of irreducible representations.) $\square$

Problem 2

(a) Show that

$$\pi : t \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

gives a representation of the group $\mathbb{R}$ on $\mathbb{C}^2$.
(b) Find all subrepresentations.
(c) Show this this representation is not unitary, that is is reducible, but not completely reducible.

Proof. (a) It is clear that $\pi(t)$ form a multiplicative group. We just need to show that π is a group homomorphism:

$$\pi(t)\pi(s) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t+s \\ 0 & 1 \end{pmatrix} = \pi(t+s)$$

(b) $\mathbb{C}^2$ is 2-dimensional, so any subrepresentation will be 1-dimensional. Let $a, b \in \mathbb{C}$, then:

$$\pi(t) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + tb \\ b \end{pmatrix}$$

We need $ab = b(a + tb)$ for all $t \in \mathbb{R}$, which only happens of $b = 0$. Therefore the only subrepresentation is the subspace spanned by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(c) Part (b) shows that $(\pi, \mathbb{C}^2)$ is reducible, because we exhibited a nontrivial subrepresentation of it. In fact we showed that it is the only nontrivial subrepresentation, so in particular its orthogonal complement is not a subrepresentation. Then $(\pi, \mathbb{C}^2)$ is not completely reducible. It is also not unitary, because not all $\pi(t)$ are unitary. Indeed, $[\pi(t)]^\dagger = \pi(t)$ implies $t = 0$. $\square$

Problem 3 (4.7 in Kirillov)

Let $\mathfrak{g}$ be a Lie algebra, and $(,)$ a symmetric ad-invariant bilinear form on $\mathfrak{g}$. Show that the element $\omega \in (\mathfrak{g}^*)^{\otimes 3}$ given by:

$$\omega(x, y, z) = ([x, y], z)$$

is skew-symmetric and ad-invariant.

Proof. The Lie bracket is skew-symmetric, so $\omega(x, y, z) = \omega(y, x, z)$ is immediate. The ad-invariance of $(\cdot, \cdot)$ means that:

$$([x, y], z) + (y, [x, z]) = 0$$

But then we can write:

$$\omega(x, y, z) = ([x, y], z) = -(y, [x, z]) = -([x, z], y) = -\omega(x, z, y)$$

And we can show similarly that $\omega(z, y, x) = -\omega(x, y, z)$, so ω is skew-symmetric. To show that it's ad-invariant we need to compute:

$$\begin{aligned}
&\omega([t, x], y, z) + \omega(x, [t, y], z) + \omega(x, y, [t, z]) \\
&= ([[t, x], y], z) + ([x, [t, y]], z) + ([x, y], [t, z]) \\
&= ([[t, x], y], z) + ([[y, t], x], z) + ([x, y], [t, z]) \\
&= -([[x, y], t], z) + ([x, y], [t, z]) \\
&= ([t, [x, y]], z) + ([x, y], [t, z]) \\
&= 0
\end{aligned}$$

Where we used the Jacobi identity for $[\cdot, \cdot]$ to obtain the fourth line, and ad-invariance for $(\cdot, \cdot)$ to get 0. $\square$

Problem 4 (4.10 in Kirillov)

Let $G = SU(2)$. Recall that we have a diffeomorphism $G \cong S^3$.

(1) Show that the left action of G on $G \cong S^3 \subset \mathbb{R}^4$ can be extended to an action of G by linear orthogonal transformations on $\mathbb{R}^4$.

(2) Let $\omega \in \Omega^3(G)$ be a left-invariant 3-form whose value at $1 \in G$ is defined by:

$$\omega(x_1, x_2, x_3) = \mathrm{Tr}([x_1, x_2]x_3), x_i \in \mathfrak{g}$$

Show that $\omega = \pm 4dV$ where dV is the volume form on S^3 induced by the standard metric in $\mathbb{R}^4$ (hint: let x_1, x_2, x_3 be some orthonormal basis in $\mathfrak{su}(2)$ with respect to $\frac{1}{2}\mathrm{Tr}(a\bar{b}^t)$). (Sign depends on the choice of orientation on S^3.)

(3) Show that $\left(\frac{1}{8\pi^2}\right)\omega$ is a bi-invariant form on G such that for appropriate choice of orientation on G, $\left(\frac{1}{8\pi^2}\right)\int_G = 1$.

Proof. (1) We treat $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ as quaternions $x_0 + x_1 i + x_2 j + x_3 k$. We also identify $SU(2)$ with the group of unit norm quaternions as follows:

$$x_0 + x_1 i + x_2 j + x_3 k \longleftrightarrow \begin{pmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{pmatrix}$$

Then the left action of $SU(2)$ on itself becomes just (left) quaternionic multiplication on unit quaternions. Now we can clearly extend multiplication by unit quaternions to all quaternions, i.e. all of $\mathbb{R}^4$. We just need to show that this action is orthogonal. Take two arbitrary quaternions q_1, q_2, then their inner product is $\mathrm{Re}(\bar{q}_1 q_2)$. Then left multiplication by unit quaternions preserves the inner product:

$$(\overline{qq_1})(qq_2) = \bar{q}_1 \bar{q} q q_2 = \bar{q}_1 |q|^2 q_2 = \bar{q}_1 q_2$$

(2) We know that ω is a left-invariant 3-form; we want to show that dV is also left-invariant. By definition a volume form is an element of $\Omega^3(G)$ that evaluates to ± 1 on any orthonormal basis in a tangent space, the sign depending on orientation of the basis. It suffices to check that:

$$dV(x_1, x_2, x_3) = (L_g^* dV)(x_1, x_2, x_3) = dV(L_{g*}x_1, L_{g*}x_2, L_{g*}x_3)$$

for some otrhonormal basis of $\mathfrak{su}(2)$, because the action of dV on all elements of $\mathfrak{su}(2)$ can then be generated by linear combinations. But L_{g*} is a vector space isomorphism, because L_g is a diffeomorphism. In particular, L_{g*} preserves the inner product of tangent spaces, so $\{x_i\}$ orthonormal implies $\{L_{g*}x_i\}$. Then:

$$dV(x_1, x_2, x_3) = dV(L_{g*}x_1, L_{g*}x_2, L_{g*}x_3) = \pm 1$$

Now we know that both ω and dV are left-invariant, so $\omega = cdV$, where c is a constant that can be determined from the relation between ω and dV at the identity. Take the orthonormal basis to be:

$$x_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

In order for dV to evaluate to 1 on these, we need to show that they are orthonormal with respect to the inner product inherited from $\mathbb{R}^4$. The easiest way to do this is to write them in quaternionic form:

$$x_1 = -i \qquad x_2 = -j \qquad x_3 = -k$$

In this form, the inner product inherited from $\mathbb{R}^4$ is $\langle x_i, x_j \rangle = \mathrm{Re}(\bar{x}_i x_j)$, and it's easy to check that this gives δ_{ij}. It follows that $dV(x_1, x_2, x_3) = \pm 1$. Then we have:

$$\omega(x_1, x_2, x_3) = \mathrm{Tr}([x_1, x_2]x_3) = \mathrm{Tr}(-2x_3 x_3) = \mathrm{Tr}(2\mathrm{Id}_{2,\mathbb{C}}) = 4$$

$$\omega(x_1, x_2, x_3) = cdV(x_1, x_2, x_3) = \pm c$$

This gives $c = \pm 4$ and so $\omega = \pm 4dV$.

(3) Part (2) shows that $\omega = \pm 4dV$. We proved that dV is left invariant, and we can prove analogously that it is right-invariant (R_g is also a diffeomorphism). Then ω is bi-invariant. We choose the basis $\{x_i\}$ defined above to be positively oriented, and then:

$$\frac{1}{8\pi^2}\int_G \omega = \frac{1}{2\pi^2}\int_G dV = \frac{V_{S^3}}{2\pi^2}$$

It remains to show that the volume of the unit 3-sphere is $2\pi^2$. (Since we are talking about the 3-sphere and not the 3-ball, "volume" actually means surface area.) To compute the surface area, we write the equation of the sphere as $x^2 + y^2 + z^2 + t^2 = 1$, and let $t = \sin\theta$. For a fixed value of θ we get a 2-sphere $x^2 + y^2 + z^2 = \cos^2\theta$ of radius $\cos\theta$; the surface area of this 2-sphere is $4\pi\cos^2\theta$. We get the area of the 3-sphere by integrating this from $\theta = -\pi/2$, which corresponds to $t = -1$, to $\theta = \pi/2$, which corresponds to $t = 1$.

$$V_{S^3} = \int_{-\pi/2}^{\pi/2} 4\pi\cos^2\theta d\theta = 2\pi^2$$

$\square$

Problem 5

Prove that the Frobenius-Schur indicator

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2)$$

for a complex irreducible representation takes 3 possible values: -1, 0, 1. For real irreducible representations, what are the possibilities for

$$\mathrm{Hom}_G(V; V)$$

Which ones does one get when restricting the three possible sorts of complex irreducible representations given above?

Proof. We prove that the Frobenius-Schur indicator takes the 3 possible values by relating it to the dimensions of spaces of forms on V. Concretely, note that $\chi_{V^*} = \bar{\chi}_V$, and then by the properties of characters (see Fulton & Harris, 2.1):

$$\chi_{\Lambda^2 V^*}(g) = \frac{\chi_{V^*}(g)^2 - \chi_{V^*}(g^2)}{2} = \frac{\overline{\chi_V(g)^2} - \overline{\chi_V(g^2)}}{2}$$

$$\chi_{\mathrm{Sym}^2 V^*}(g) = \frac{\chi_{V^*}(g)^2 + \chi_{V^*}(g^2)}{2} = \frac{\overline{\chi_V(g)^2} + \overline{\chi_V(g^2)}}{2}$$

Substracting these two equations and conjugating the result gives:

$$\chi_V(g^2) = \overline{\chi_{\mathrm{Sym}^2 V^*}(g)} - \overline{\chi_{\Lambda^2 V^*}(g)}$$

Which we can then average over G:

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\mathrm{Sym}^2 V^*}(g)} - \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\Lambda^2 V^*}(g)}$$

Using the discussion in Fulton & Harris, 2.4 we see that the terms on the RHS are projections onto the spaces of G-invariant symmetric and alternating bilinear forms. Then:

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \dim(\mathrm{Sym}^2 V^*)^G - \dim(\Lambda^2 V^*)^G$$

We now quote the result of theorem 3.37 in Fulton & Harris:

An irreducible representation V is one and only one of the following:

(i) Complex: χ_V is not real-valued; V does not have a G-invariant nondegenerate bilinear form;

(ii) Real: $V = V_0 \otimes \mathbb{C}$, a real representation; V has a G-invariant symmetric nondegenerate bilinear form;

(iii) Quaternionic: χ_V is real, but V is not real; V has a G-invariant skew-symmetric nondegenerate bilinear form.

Using this information about the spaces of nondegenerate bilinear forms, we conclude that the Frobenius indicator gives 1 iff V is real, 0 iff V is complex and -1 iff V is quaternionic.

In order to determine $\mathrm{Hom}_G(V, V)$ for V a real representation, we modify the proof of Schur's lemma accordingly. It's still true that, for $F \in \mathrm{Hom}_G(V, V)$, $\mathrm{Ker}\, F$ and $\mathrm{Im} F$ are G-invariant subspaces of V, so we still get that any nontrivial F must be an isomorphism. Therefore $\mathrm{Hom}_G(V, V)$ is a division algebra over $\mathbb{R}$, and there are three such: $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

$\square$