Commutative algebra HW3

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Problem 1

Find a ring A and an ideal I such that I is generated by countably many elements $f_1, f_2, f_3, \ldots$ such that $f_i^2 = 0$ but such that I is not a nilpotent ideal (in other words for all $n > 0$ the ideal I^n is not zero).

Solution
Take $A = \mathbb{C}[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$, and $I = (x_1, x_2, \ldots)$. Then obviously each of the x_i's is nilpotent, but I^n will contain the element $x_1 \ldots x_n \neq 0$.

Problem 2

Let $A \subset B$ be an extension of domains. Let K be the fraction field of A and L be the fraction field of B, so that we have an extension of fields $K \subset L$. Show that if (a) B is a finite type A-algebra and (b) L is a finite extension of K, then the image of $Spec(B) \to Spec(A)$ contains a nonempty open subset of $Spec(A)$.

Solution

$$\begin{tikzcd} B \arrow[r, hook] & L \\ A \arrow[u, hook, "\phi"] \arrow[r, hook] & K \arrow[u, hook] \end{tikzcd}$$

Let $b \in B$, then $b \in L$, and since L is a finite extension of K we have $k_n b^n + \cdots + k_0 = 0$ for some n. By cancelling denominators, we can get this to the form $a_n b^n + \cdots + a_0 = 0$ for $a_i \in A$, but we do not know that a_n has an inverse, so this polynomial may not be monic. But we can localize a_n, in order to make b integral. Therefore, if $(x_1, \ldots, x_n)$ generate B as an A-algebra, let c_i denote the leading coefficient in the polynomial for x_i, and we localize A and B at the multiplicative subset $S = \{c_i\}$. Then $x_i \in S^{-1}B$ are integral over $S^{-1}A$, and this implies $S^{-1}\phi : S^{-1}A \to S^{-1}B$ is a finite map. The localization functor is exact, so $S^{-1}\phi$ is also injective. By Lemma 15 proved in class, $\text{Spec}(S^{-1}\phi)$ is surjective. Then:

$$\text{Spec}(S^{-1}A) = \text{Im Spec}(S^{-1}\phi) \subset \text{Im Spec}(\phi) \qquad (*)$$

But $\mathrm{Spec}(S^{-1}A)$ contains all primes in $\mathrm{Spec}(A)$ that avoid $S = \{c_i\}$, so:

$$\mathrm{Spec}(S^{-1}A) = \mathrm{D}(c_1 \dots c_n) \qquad (**)$$

By (*) and (**), $\mathrm{D}(c_1 \dots c_n) \subset \mathrm{Im}\ \mathrm{Spec}(\phi)$.

Problem 3

Let k be a field. Let $f, g \in k[t]$ be two polynomials in a variable t with coefficients in k. Show that there exists a nonzero two variable polynomial $P \in k[x, y]$ such that $P(f, g) = 0$ in $k[t]$.

Solution
Consider the map:

$$\begin{aligned} \phi : k[x, y] &\to k[t] \\ P(x, y) &\to P\big(f(t), g(t)\big) \end{aligned}$$

This is a ring homomorphism. Assume there is no nonzero P such that $P\big(f(t), g(t)\big) = 0$, then ϕ is injective. We want to show that ϕ is also finite. We have that $k[t]$ is a finitely generated $k[x, y]$-algebra, with generator t. By Lemma 1 proved in class, if t satisfies a monic equation of the form

$$t^n + \phi(a_1)t^{n-1} + \dots + \phi(a_n) = 0$$

With $a_i \in k[x, y]$, then ϕ is finite. But we see that $\phi(x) = f(t) = \sum_{i=0}^{n} b_i t^i$, and therefore:

$$t^n + \sum_{i=0}^{n-1} b_i/b_n t^i + \phi(x)/b_n = 0$$

So ϕ is a finite injective map. By Lemma 15, $\mathrm{Spec}(\phi)$ is surjective, and then by Lemma 29 $\dim(k[x, y])$=$\dim(k[t])$. But this is a contradiction, since $\dim(k[x, y]) = 2$ and $\dim(k[t]) = 1$.

Problem 4

Give an example of an Artinian ring which is not an algebra of finite type over a field.

Solution
$\mathbb{Z}_4$ is Artinian because it has a finite number of ideals. It is not a finite-type algebra over a field because there exists no ring homomorphism from a field other than $\mathbb{Z}_4$ to $\mathbb{Z}_4$.