Commutative algebra notes

Matei Ionita

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Lecture 3

Lemma 10

Let $A \overset{\phi}{\to} B$ be a finite ring map. Then:
a) for $I \subset A$ ideal, the ring map $A/I \to B/I$ is finite.
b) for $S \subset A$ multiplicative subset, $S^{-1}A \to S^{-1}B$ is finite.
c) for $A \to A'$ ring map, $A' \to B \otimes_A A'$ is finite.

Proof. □

Lemma 11

Suppose k is a field, A is a domain and $k \to A$ a finite ring map. Then A is a field.

Proof. Since A is an algebra, multiplication by an element $a \in A$ defines a map $A \to A$. A is a k-module, so this map is k-linear. The map is also injective: $\mathrm{Ker}(a) = \{a' \in A | aa' = 0\} = \{0\}$, because A has no zero divisors. But, since $\dim_k(A)$ is finite, injectivity implies surjectivity. Then there exists a'' such that $aa'' = 1$, so a is a unit. □

Lemma 12

Let k be a field and $k \to A$ a finite ring map. Then:
a) $\mathrm{Spec}(A)$ is finite.
b) there are no inclusions among prime ideals of A.
[In other words, $\mathrm{Spec}(A)$ is a finite discrete topological space WRT the Zariski topology.]

Proof. By lemma 11, all primes of A are maximal (???), which proves claim b. Moreover, by the Chinese remainder theorem: □

Lemma 13 (Chinese remainder theorem)

Let A be a ring, and $I_1, ..., I_n$ ideals of A such that $I_i + I_j = A, \forall i \neq j$. Then there exists a ring map $A \to A/I_1 \times ... \times A/I_n$ with kernel $I_1 \cap ... \cap I_n = I_1 ... I_n$.

Lemma 14

Let $A \to B$ be a finite ring map. The fibers of $\mathrm{Spec}(\phi)$ are finite.

Proof. $\square$

Lemma 15

Suppose that $A \subset B$ is a finite extension (i.e. there exists a finite injective map $A \to B$). Then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Proof. We want to reduce the problem to the case where A is a local ring. For this, let $p \subset A$ be a prime. By part b of lemma 10, the map $A_p \to B_p$ is finite. By lemma 8, the same map is injective. Then we can replace A and B in the statement of the lemma by A_p and B_p.

Now, assuming that A is local, p is the maximal ideal of A, and we denote it by m in what follows. The following statements are equivalent:

$$\begin{aligned} \exists q \subset B \text{ lying over } m &\Leftrightarrow \exists q \subset B \text{ such that } mB \subset q \\ &\Leftrightarrow B/mB \neq 0 \end{aligned}$$

But the last statement is always true, since Nakayama's lemma (see below) says that $mB = B$ implies $B = 0$. $\square$

Lemma 16 (Nakayama's lemma)

Let A be a local ring with maximal ideal m, and let M be a finite A-module such that $M = mM$. Then $M = 0$.

Proof. Let $x_1, ..., x_r \in M$ be generators of M. Since $M = mM$ we can write $x_i = \sum_{j=1}^{r} a_{ij} x_j$, for some $a_{ij} \in m$. Then define the $r \times r$ matrix $B = 1_{r\times r} - (a_{ij})$. The above relation for the generators translates into:

$$B \begin{pmatrix} x_1 \\ ... \\ x_r \end{pmatrix} = 0$$

Now consider B^{ad}, the matrix such that $B^{\mathrm{ad}} B = \det(B) 1_{r\times r}$. Multiplying the above equation on the left by B^{ad} we obtain:

$$\det(B) \begin{pmatrix} x_1 \\ ... \\ x_r \end{pmatrix} = 0$$

Thus $\det(B) x_i = 0$ for all i. If we assume that the generators of M are nonzero, the fact that $\det(B)$ annihilates all generators implies that it is equal to 0. But, by expanding out the determinant of $B = 1_{r\times r} - (a_{ij})$, we see that it is of the form $1 + a$ for some $a \in m$. Since (A, m) is a local ring, this implies that $\det(a)$ is a unit. A unit cannot be zero in (A, m), so this is a contradiction. Thus all generators of M are zero, and $M = 0$. $\square$

Lemma 17 (Going up for finite ring maps)

Let $A \to B$ be a finite ring map, p a prime ideal in A and q a prime ideal in B which belongs to the fiber of p. If there exists a prime p' such that $p \subset p' \subset A$, then there exists a prime q' such that $q \subset q' \subset B$ and q' belongs to the fiber of p'.

Proof. Consider $A/p \to B/q$. This is injective since $p = A \cup q$ and finite by lemma 3 (???). p'/p is a prime ideal in A/p, and by lemma 15 its preimage is nonempty. Thus there exists a prime q'/q in A/p which maps to p'/p, and this corresponds to a prime q' in B that contains q. $\square$