Commutative algebra HW8

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Problem 2

Let k be an algebraically closed field. Let $A = k[x, y]/(f)$ where f is an irreducible polynomial. Let K be the fraction field of A. Let $C = \{(s, t) \in k^2 | f(s, t) = 0\}$. Recall that the maximal ideals of A correspond 1-1 with points of the curve C.

(a) *Show that if every point of C is nonsingular, then the valuations of K centered on A are in 1-1 correspondence with points of C. (Hint: Above you showed that the local rings of A are regular at nonsingular points. You may use that a regular local ring of dimension 1 is a discrete valuation ring and hence gives rise to a discrete valuation, see for example Lemma Tag 00PD.)*

(b) *Give an example to show this is false when C is singular.*

Solution

(a) In HW4 we showed that, at nonsingular points $(s, t) \in C$, $A_{(x-s,y-t)}$ is a regular local ring. Also, $A_{(x-s,y-t)}$ has dimension 1 because it's the qoutient by a prime ideal of a dimension 2 ring. Therefore points on C are in 1-1 correspondence with regular local rings of dimension 1. By Lemma Tag 00PD, the latter are DVRs, and thus give rise to a discrete valuation. To see explicitly how this valuation is constructed, we follow the notes in http://www.math.nmsu.edu/∼pmorandi/math601f01/DiscreteValuationRings.pdf . We can first use Nakayama's lemma to show that the maximal ideal $\mathfrak{m} = (x - s, y - t)$ is principal in $A_\mathfrak{m}$, and then use this fact to show that any ideal of $A_\mathfrak{m}$ is equal to $\mathfrak{m}^n$ for some natural power n. This allows us to define a valuation v on $A_\mathfrak{m}$ by setting $v(a) = n$ iff $(a) = \mathfrak{m}^n$. v can be extended to $K = f.f.(A_\mathfrak{m}) = f.f.(A)$ by setting $v(a/b) = v(a) - v(b)$. Then it's easy to see that $\mathcal{O}_v = A_\mathfrak{m} \supset A$, and therefore v is centered on A. Moreover, v is the projection map from K^* to $K^*/A^*_\mathfrak{m}$, so it's the unique valuation on the DVR $A_\mathfrak{m}$.

b) Consider $f = y^2 - x^3 - x^2$. The curve C, depicted in the figure, is singular at $(0, 0)$. For all points of C except for $(0, 0)$, the argument of part (a) applies and we get 1-1 correspondence between these points and discrete valuations centered on A. However, we will see that at $(0, 0)$ we can define multiple valuations that are centered on A.

Consider the map $\phi : k[t] \to A = k[x,y]/(y^2 - x^3 - x^2)$ given by $\phi(t) = x + y$. We want to show that ϕ is an inclusion map, i.e. that it's injective. Assume there's some polynomial in t such that:

$$a_n t^n + \cdots + a_1 t + a_0 = g(x,y)(y^2 - x^3 - x^2)$$

Then we need to show that all $a_i = 0$. By factoring $y^2 - x^2$ we can rewrite the above as:

$$a_n t^n + \cdots + a_1 t + a_0 = g(x,y)(x+y)(x-y) - g(x,y)x^3$$

In particular, if we set $t = x + y = 0$, this reduces to:

$$a_0 = -g(x,-x)x^3$$

Which can only hold if $a_0 = g(x,-x) = 0$. Then we have $a_0 = 0$ and $g(x,y)$ is divisible by $x + y$, and then we can divide the relation by t and obtain:

$$a_n t^{n-1} + \cdots + a_1 = h(x,y)(y^2 - x^3 - x^2)$$

Proceeding by induction it follows that all $a_i = 0$ as desired. Then ϕ is a ring extension. Viewed as an extension of fraction fields, it has degree 3, because we can rewrite:

$$(x+y)(x-y) - x^3 = 0$$

$$\Leftrightarrow -(x+y)(x+y) + 2(x+y)x - x^3 = 0$$

Which is the degree 3 minimal polynomial for x over $k[t]$. Then we know that the valuation $\mathrm{ord}_{t=0}$ on $k[t]$ extends to a valuation v_1 on A such that $v_1(x+y) = e \cdot 1$, where $e \in \{1,2,3\}$.

Recall now that our goal is to construct 2 distinct valuations over the point $(x,y) = (0,0)$. The way we plan to do that is by constructing a similar map $\psi : k[s] \to A$, this time given by $\psi(s) = x - y$. This will produce a valuation v_2 such that $v_2(x-y) = e \cdot 1$. The key observation is that ϕ and ψ are related by the automorphism of A that takes $y \to -y$. Therefore $v_1(x+y) = v_2(x-y)$ and viceversa. Therefore it suffices to prove that $v_1(x+y) \neq v_1(x-y)$, as this implies that the two valuations are distinct. We do this by analyzing separately the cases $e = 1, 2, 3$. For each case we will make use of the properties:

(i) $v_1(x+y) + v_1(x-y) = 3v_1(x)$

(ii) $2x(x+y) + x^3 = (x+y)^2$

Case 1. e = 1. Then by (i) $v_1(x-y) \equiv 2 \bmod 3$. Then $v_1(x-y) \neq 1$.
Case 2. e = 2. Then by (i) $v_1(x-y) \equiv 1 \bmod 3$. Then $v_1(x-y) \neq 2$.
Case 3. e = 3. Assume first that $v_1(x) = 1$, then (ii) implies, by evaluating both sides, that 3=6, which is a contradiction. Then $v_1(x) \geq 2$, so $3 + v_1(x) < 3v_1(x)$. Then by (ii) $3 + v_1(x) = 6$, so $v_1(x) = 3$. Then by (i) we have $v_1(x-y) = 9 - 3 = 6$, so again it's not equal to $v(x+y)$. This finishes the proof.

Problem 3

Let $k = \mathbb{C}$ be the field of complex numbers. Let $f = 1 + x^n + y^n$ for some positive integer n. Let K be the fraction field of $A = k[x, y]/(f)$.

(a) *How many valuations of K/k are not centered on A?*

(b) *What would you guess is the number of "missing" valuations when you have a general irreducible $f \in k[x, y]$?*

(c) *Give an example to show that your guess is wrong!*

Solution

a) In problem 2a we proved that the valuations of K/k centered on A are in 1-1 correspondence with the points on the curve C. Therefore the only valuation left is the valuation at ∞, which is quite obviously not centered on A. For example, $\infty(x) = e \cdot \operatorname{ord}_\infty x = -e$ for some $e \geq 1$, therefore $\infty(x) < 0$.

b) Judging by problem 2b, we expect to get extra valuations centered on A when we have a singular point. In 2b, we had one singular point and that gave one extra valuation apart from the usual counting via the 1-1 correspondence. A fair guess then is that the valuations centered on A that we missed in the counting are in 1-1 correspondence with the number of singular points.

c) Consider $f = y^2 - x^3$, which gives rise to the cuspital curve. Here the point $(0, 0)$ is singular, so by our guess from part b we should get two distinct valuations centered on A corresponding to $(0, 0)$. However, it's easy to show, using the methods from HW7, problem 3, that the only valuation we get has ramification index 2 over $k[x]$ and is given by:

$$v(x) = 2 \qquad v(y) = 3$$

Therefore we have no missing valuations here, contrary to our guess in part b.