Algebraic topology HW4

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Problem 1 (1.1.10 in Hatcher)

We need to show that following a loop γ in $X \times \{y_0\}$ and then a loop δ in $\{x_0\} \times Y$ is the same as following δ, and then γ. That is, we need to find a homotopy between the following loops in $X \times Y$:

$$f_0(s) = \begin{cases} (\gamma(2s), y_0) & , 0 \leq s < 1/2 \\ (x_0, \delta(2s-1)) & , 1/2 \leq s \leq 1 \end{cases}$$

$$f_1(s) = \begin{cases} (x_0, \delta(2s)) & , 0 \leq s < 1/2 \\ (\gamma(2s-1), y_0) & , 1/2 \leq s \leq 1 \end{cases}$$

To construct the homotopy, we vary the point at which the loop γ is attached, continuously along δ. Specifically:

$$f_t(s) = \begin{cases} (x_0, \delta(2s)) & , 0 \leq s < t/2 \\ (\gamma(2s-t), \delta(t)) & , t/2 \leq s < (t+1)/2 \\ (x_0, \delta(2s-1)) & , (t+1)/2 \leq s \leq 1 \end{cases}$$

Problem 2 (1.1.12 in Hatcher)

Since $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$, we are looking at homomorphisms $f : \mathbb{Z} \to \mathbb{Z}$. Any such homomorphism is completely determined by $f(1)$, therefore $\mathrm{Hom}(\pi_1(S^1), \pi_1(S^1)) = \mathbb{Z}$ as sets. Thus the problem reduces to finding, for each $n \in \mathbb{Z}$, a map $\phi_n : S^1 \to S^1$ such that $[\phi_n \circ \omega_1] = [\omega_n]$. Regarding S^1 as the complex unit circle, the homotopy classes of S^1 are $\omega_n(s) = e^{2\pi i n s}$. Then take $\phi_n(z) = z^n$, which is continuous on $\mathbb{C}$, and therefore is continuous when restricted as a function $S^1 \to S^1$. We see that:

$$(\phi \circ \omega_1)(s) = (e^{2\pi i s})^n = e^{2\pi i n s} = \omega_n(s)$$

Then $\phi_*[\omega_1] = [\omega_n]$, as desired.

Problem 3 (1.1.16 in Hatcher)

Throughout we will be using proposition 1.17 in Hatcher: if a space X retracts onto a subspace A, then the homomorphism $i_* : \pi_1(A) \to \pi_1(X)$ induced by the inclusion $i : A \to X$ is injective.

a) $\pi_1(X) = 0$, and $\pi_1(A) = \mathbb{Z}$, since homeomorphic spaces have isomorphic fundamental groups. Since $\mathbb{Z}$ cannot be injected into 0, there cannot exist a retraction $X \to A$.

b) $\pi_1(X) = \pi_1(S^1) \times \pi_1(D^2) = \mathbb{Z} \times 0 = \mathbb{Z}$, and $\pi_1(A) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. If a retraction exists, we obtain an injective homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$. Such a homomorphism is determined by its values on the two generators of $\mathbb{Z}^2$:

$$\phi(1, 0) = m$$

$$\phi(0, 1) = n$$

But then we have $\phi(n, 0) = nm = \phi(0, m)$, which contradicts the injectivity of ϕ. Therefore no retraction exists.

c) The solid torus deformation retracts onto its central circle C, therefore it has the same homotopy type as S^1. Consider the inclusion map $i : A \to X$, and the induced homomorphism $i_* : \pi^1(A) \to \pi_1(X)$. Viewed as a loop in the solid torus, A is nullhomotopic, since it can be homotoped to a point as shown in the figure.

Hence $i_*[1_A] = [0_X]$; the generator of $\pi_1(A)$ is mapped to the zero element in $\pi_1(X)$. Now assume that there exists a retraction $r : X \to A$. By definition, $r \circ i = \mathrm{Id}_A$, which means that $r_* \circ i_* = \mathrm{Id}_{\pi_1(A)}$. In particular, $r_* \circ i_*[1_A] = [1_A]$. However, using the fact that $i_*[1_A] = [0_X]$, we obtain $r_*[0_X] = [1_A]$. This is a contradiction, since any group homomorphism must map the zero element to zero.

d) We claim that $\pi_1(X) = 0$. First, $\pi_1(D^2) = 0$ for each copy of D^2, therefore any loop that lies entirely in one of the copies is nullhomotopic. Consider now a loop f passing through both copies. We can regard it as the product $f_1 \bullet f_2 \bullet \dots$, where each f_i is contained in one copy of D^2. Then each f_i is nullhomotopic, so their product f is also nullhomotopic. Therefore $\pi_1(X) = 0$. On the other hand,$\pi_1(A) \neq 0$; in fact, $\pi_1(A)$ contains a $\mathbb{Z}^2$ subgroup given by all loops that go around one copy of S^1 or the other, but not both. This means that $\pi_1(A)$ cannot be injected into $\pi_1(X)$.

e) A disk with two boundary points identified deformation retracts onto one of its boundary circles, so it has the same homotopy type as S^1. Therefore $\pi_1(X) = \mathbb{Z}$. But $\pi_1(A)$ contains a $\mathbb{Z}^2$ subgroup, and since there is no injective homomorphism from $\mathbb{Z}^2$ to $\mathbb{Z}$, there cannot exist one from $\pi_1(A)$ to $\pi_1(X)$.

f) The Mobius band deformation retracts onto its central circle C, therefore it has the same homotopy type as S^1. Consider the inclusion map $i : A \to X$, and the induced homomorphism $i_* : \pi^1(A) \to \pi^1(X)$. Travelling once around 1_A, the generator of $\pi^1(A)$, takes us twice around 1_X, therefore $i_*[1_A] = [1_X]^2$. Now assume there exists a retraction $r : X \to A$. By definition, $r \circ i = \mathrm{Id}_A$, which means that $r_* \circ i_* = \mathrm{Id}_{\pi_1(A)}$. In particular, $r_* \circ i_*[1_A] = [1_A]$. However, using the fact that $i_*[1_A] = [1_X]^2$, we obtain $(r_*[1_X])^2 = [1_A]$. Since r_* is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, we have $r_*[1_X] = [1_A]^n$ for some $n \in \mathbb{Z}$. Therefore $2n = 1$, which is a contradiction.

Problem 4 (1.2.4 in Hatcher)

X deformation retracts onto Y, a 2-sphere missing $2n$ points x_i. Choose a basepoint $y \in Y$, and for each $1 \leq i \leq 2n-1$ construct loops f_i based at y, such that f_i encloses x_i and none of the other missing points. The interior of each f_i is homeomorphic to $D^2 - \{0\}$, which deformation retracts onto its boundary. Therefore the interior of each f_i deformation retracts onto f_i. After this has been done, what's left of Y is the exterior of all the loops, which contains the missing point x_{2n}. This is again homeomorphic to $D^2 - \{0\}$, so it deformation retracts onto its boundary $f_1 \bullet \cdots \bullet f_{2n-1}$.

This shows how Y deformation retracts to $\bigvee_{2n-1} S^1$, where the copies of S^1 are precisely the loops f_i. By the van Kampen theorem, $\pi_1(Y) = *_{2n-1}\mathbb{Z}$.

Problem 5 (1.2.8 in Hatcher)

We begin from the cell decomposition of X. The 1-skeleton is a wedge of the three circles a, b, c, therefore its fundamental group is the free group on 3 generators. There are two 2-cells attached, and we use proposition 1.26 in Hatcher to analyze their influence on $\pi_1(X)$.

The 2-cells are attached to the loops $aba^{-1}b^{-1}$ and $aca^{-1}c^{-1}$, therefore $\pi_1(X)$ has the presentation:

$$\langle a, b, c \,|[a, b], [a, c]\rangle$$

Where $[a, b] = aba^{-1}b^{-1}$ denotes the commutator of a and b. We see that a commutes with both b and c, while between b and c there exists no relation. Therefore $\pi_1(X) = \mathbb{Z} \times \mathbb{Z} * \mathbb{Z}$, where the first (commuting) factor corresponds to a.