QFT Lecture 25

Matei Ionita

April 23, 2013

Look in chapter 38 for some useful identities about spinors. Also read chapter 40 for CPT. Today we talk about scattering, chapters 41-44 in Srednicki.

LSZ for spinors

reference: chapter 41

Dirac spinor:

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=\pm} b_s(k) u_s(k) e^{ikx} + d_s^\dagger(k) v_s(k) e^{-ikx}$$

$$\bar{\Psi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=\pm} b_s^\dagger(k) \bar{u}_s(k) e^{-ikx} + d_s(k) \bar{v}_s(k) e^{ikx}$$

LSZ formula for 2-2 scattering of "b" particles (electrons):

$$_{\text{OUT}}\langle k_1' s_1', k_2' s_2' | k_1 s_1, k_2 s_2 \rangle_{\text{IN}} - \text{"1"} =$$

$$= i^4 \int d^4x_1 d^4x_2 d^4x_1' d^4x_2' \quad e^{-ik_2'x_2'} \bar{u}_{s_2'}(k_2')(-i\partial\!\!\!/_{2'} + m)_{\alpha_2'} \quad e^{-ik_1'x_1'} \bar{u}_{s_1'}(k_1')(-i\partial\!\!\!/_{1'} + m)_{\alpha_1'}$$

$$\langle 0 | T \Psi_{\alpha_2'}(x_2') \Psi_{\alpha_2'}(x_1') \Psi_{\alpha_1}(x_1) \Psi_{\alpha_2}(x_2) | 0 \rangle \quad e^{ik_1x_1} \bar{u}_{s_1}(k_1)(i\partial\!\!\!/_1 + m)_{\alpha_1} \quad e^{ik_2x_2} \bar{u}_{s_2}(k_2)(i\partial\!\!\!/_2 + m)_{\alpha_2}$$

Once we know how the interaction looks (i.e. via photons) we can compute the 4-point function and plug it in here. Note that the amplitude we are computing is:

$$\langle 0 | b_{s_1} b_{s_2} b_{s_1'}^\dagger b_{s_1'}^\dagger | 0 \rangle$$

And order matters. If we want scattering of electrons and positrons, replace some b's by d's. The conditions we need to assume to derive this formula:

$$\langle 0 | \Psi(x) | 0 \rangle = 0$$

$$\langle k_b s | \Psi(x) | 0 \rangle = 0$$

$$\langle k_d s | \Psi(x) | 0 \rangle = u_s(k) e^{-ikx}$$

The second identity follows from the fact that Ψ contains b, but no $b^\dagger$. For a Majorana spinor we only have the first and third, since $b = d$.

2-pt function for free fermions (fermion free propagator)

Reference: chapter 42. Focus on Dirac spinors:

$$\langle 0|T\Psi_\alpha(x)\bar{\Psi}_\beta(y)|0\rangle = \langle 0|\theta(t_x - t_y)\Psi_\alpha(x)\bar{\Psi}_\beta(y) - \theta(t_y - t_x)\bar{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle = \frac{1}{i}S_{\alpha\beta}(x-y)$$

Let's first compute the first term. Notice that there's only one contributing term, since we must have a creation operator on the right and an annihilation operator on the left:

$$\langle 0|\Psi_\alpha(x)\bar{\Psi}_\beta(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\sum_{s,s'} e^{i(kx-k'y)}[u_s(k)]_\alpha[\bar{u}_s(k')]_\beta\langle 0|b_s(k)b^\dagger_{s'}(k')|0\rangle$$

We use the anticommutation relation for b's in order to exchange them; now the operators annihilate the vacuum and we are left with the anticommutator:

$$\langle 0|\Psi_\alpha(x)\bar{\Psi}_\beta(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_s e^{ik(x-y)}[u_s(k)]_\alpha[\bar{u}_s(k')]_\beta = \int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{ik(x-y)}(-\not{k}+m)_{\alpha\beta}$$

For the second term a similar computation gives:

$$\langle 0|\bar{\Psi}_\beta(y)\Psi_\alpha(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik(x-y)}(-\not{k}-m)_{\alpha\beta}$$

Putting these together, we find the free propagator:

$$\frac{1}{i}S_{\alpha\beta}(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\theta(t_x - t_y)e^{ik(x-y)}(-\not{k}+m)_{\alpha\beta} + \theta(t_y - t_x)e^{-ik(x-y)}(\not{k}m)_{\alpha\beta}\right]$$

Recall that in the scalar field case we can write the propagator as:

$$\frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}$$

Similarly, for spinors we can simplify the answer:

$$\frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{(-\not{k}+m)_{\alpha\beta}\; e^{ik(x-y)}}{k^2+m^2-i\epsilon}$$

In this form, it's easy to check that the propagator is the Green's function for the Dirac equation:

$$(-i\not{\partial}_x + m)_{\eta\alpha}S_{\alpha\beta}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{(\not{k}+m)_{\eta\alpha}(-\not{k}+m)_{\alpha\beta}}{k^2+m^2-i\epsilon}e^{ik(x-y)} = \delta^{(4)}(x-y)\delta_{\eta\beta}$$

Where we have used:

$$(\not{k}+m)_{\eta\alpha}(-\not{k}+m)_{\alpha\beta} = (k_\mu\gamma^\mu + m)(-k_\nu\gamma^\nu + m) = -k_\mu k_\nu\gamma^\mu\gamma^\nu + m^2 =$$

$$= -\frac{1}{2}k_\mu k_\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + m^2 = (k^2+m^2)\delta_{\eta\beta}$$

Note that $\langle 0|T\Psi_\alpha(x)\Psi_\beta(y)|0\rangle = 0$, unless we're talking about Majorana.

Fermionic path integral

Reference: chap. 43, 44. How do we integrate Grassman variables? Let's first think about integrating one variable:

$$\int da\ f(a)$$

Because a is anticommuting, the most general form of the function is $f(a) = c_0 + c_1 a$. We want the integral to be linear and invariant under shifts by a constant. The only form of the integral that satisfies these properties is:

$$\int da\ (c_0 + c_1 a) = c_1$$

Remarks: the integral is equal to the derivative; double integral or double derivative gives 0.