Algebraic topology HW2

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Problem 1

Let $f : X \to \{y\}$ be a homotopy equivalence, and $g : \{y\} \to X$ be the inverse equivalence. We construct a homotopy equivalence $F : X \times [0,1] \to [0,1]$ as follows:

$$F(x,t) = t \ , \forall x \in X$$

To see that this is indeed a homotopoy equivalence, consider the map $G : [0,1] \to X \times [0,1]$ given by:

$$G(t) = (g(y), t)$$

We see that $F \circ G = \mathrm{Id}_{[0,1]}$, and that $(G \circ F)(x,t) = (g(y), t)$. $G \circ F$ is the identity on the second component, and is equal to $g \times f$ on the first component. This shows that $G \circ F \simeq \mathrm{Id}_{X \times [0,1]}$, as desired.

Now note that F descends to a map on the suspension SX, since $F(x,0) = F(y,0)$ for all $x, y \in X$, and similarly $F(x,1) = F(y,1)$. Therefore, denoting by $\tilde{F}$ the induced map $SX \to [0,1]$, we see that $\tilde{F}$ is a homotopy equivalence. The fact that $[0,1]$ is contractible then implies that SX is contractible.

Problem 2

We view $\mathbb{RP}^n$ as the closed disk D^n with antipodal boundary points identified, and denote by ϕ the quotient map $\partial D^n \to \mathbb{RP}^{n-1}$. By translating the punctured point if necessary, we can assume that it lies in the origin of D^n. Then consider the family of maps $f_t : D^n - \{0\} \to \partial D^n$ given by:

$$f_t(\mathbf{x}) = \frac{\mathbf{x}}{1 - t + t|\mathbf{x}|}$$

As discussed in class, f_t is a deformation retraction, which shows that $D^n - \{0\}$ and ∂D^n are homotopy equivalent, and the retraction f_1 is a homotopy equivalence, whose inverse equivalence is the inclusion map $i : \partial D^n \to D^n - \{0\}$. Let $\tilde{f}_1, \tilde{i}$ be the maps on the quotients given by the following diagrams:

$$
\begin{tikzcd}
D^n - \{0\} \arrow[r, "f_1"] \arrow[d, "\phi"] & \partial D^n \arrow[d, "\phi"] \\
\mathbb{RP}^n - \{0\} \arrow[r, "\tilde{f}_1"] & \mathbb{RP}^{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
D^n - \{0\} \arrow[d, "\phi"] & \partial D^n \arrow[l, "i"'] \arrow[d, "\phi"] \\
\mathbb{RP}^n - \{0\} & \mathbb{RP}^{n-1} \arrow[l, "\tilde{i}"']
\end{tikzcd}
$$

Then $(\tilde{f}_1 \circ \tilde{i})(\tilde{x}) = (\phi \circ f_1 \circ i)(x)$, where $\tilde{x} \in \mathbb{RP}^{n-1}$ is an equivalence class, and x is any representative. Since $f_1 \circ i \simeq \mathrm{Id}_{\partial D^n}$, we obtain $\tilde{f}_1 \circ \tilde{i} \simeq \mathrm{Id}_{\mathbb{RP}^{n-1}}$. We proceed similarly for $\tilde{i} \circ \tilde{f}_1$, and we obtain that $\mathbb{RP}^n - \{0\} \simeq \mathbb{RP}^{n-1}$.

Problem 3

We prove the result by induction on n. The statement for $n = 0$ is just $S^2 \simeq S^2$, which is true. For the inductive step, we consider attaching $X_1 = D^2$ to $X_0 = \bigvee_n S^2$. We consider the attaching maps $f, g : \partial X_1 \to X_0$, where the image of f is a circle and the image of g is the center of that circle. f and g are clearly homotopic, via maps f_t which attach ∂X_1 to circles of decreasing radius. Then, by the second criterion for homotopy equivalence discussed in class, we obtain $X_1 \sqcup_f X_0 \simeq X_1 \sqcup_g X_0$. But $X_1 \sqcup_g X_0$ is homotopy equivalent to $\bigvee_{n+1} S^2$, by simply translating the point to which D^2 was attached. Therefore $X_1 \sqcup_f X_0 \simeq \bigvee_{n+1} S^2$, as desired.

Problem 4

Let D be the disc encolesd by the circle in which the bottle self-intersects. Since D is contractible, $X/D \simeq X$. Now consider the space Y pictured below; we have $Y \simeq X/D$, because X/D is the quotient of Y by a contractible line. Moreover, Y is homeomorphic to Z pictured to the right; the homeomorphism simply pushes the line a to the bottom of the sphere.

But Z is homotopy equivalent to $S^2 \vee S^1 \vee S^1$, by translating the point x_1 to x_2. Therefore $X \simeq S^2 \vee S^1 \vee S^1$.

Problem 5

Let X be a CW complex and assume that $X = X_1 \cup X_2$, with $X_1, X_2, X_1 \cap X_2$ contractible. Using the first criterion for homotopy equivalence discussed in class, we see that the quotient map:

$$X \to X/X_1$$

is a homotopy equivalence. When we contract X_1 to a point, a part of the remaining X_2 is also contracted, specifically $X_1 \cap X_2$. Then we have:

$$X \simeq X/X_1 = X_2/(X_1 \cap X_2)$$

We apply again the criterion for homotopy equivalence, to get:

$$X_2 \simeq X_2/(X_1 \cap X_2)$$

Putting these two relations together, we obtain $X \simeq X_2$, and therefore X is contractible.