# Commutative algebra HW7

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## Problem 2

*Let $k$ be an algebraically closed field. Let $K = k(t)$. Denote $v_c = \mathrm{ord}_{t=c}$ the valuation corresponding to $c$ in $k$. Denote $\infty$ the valution corresponding to the point at infinity. With this notation:*

1. *Give a basis for $L(D)$ when $D = 2v_0 + 3v_1$.*
2. *Give a basis for $L(D)$ when $D = 2v_0 + 2\infty$.*

*Solution*
When $D = 2v_0 + 3v_1$ we allow poles of order at most 2 at 0 and at most 3 at 1. However, we don't allow any poles at $\infty$, so the degree of the functions in $L(D)$ has to be at most 0. Then a basis for $L(D)$ is:

$$\left\{1, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x-1}, \frac{1}{(x-1)^2}, \frac{1}{(x-1)^3}\right\}$$

When $D = 2v_0 + 2\infty$, we allow poles of order at most 2 at 0, and we allow the degree of functions in $L(D)$ to be at most 2. Then a basis is:

$$\left\{1, \frac{1}{x}, \frac{1}{x^2}, x, x^2\right\}$$

## Problem 3

*Assume $k$ does not have characteristic 2. Let $K$ be the degree 2 extension of $k(t)$ defined by $y^2 = f(t)$ for some cubic squarefree polynomial $f$. Find all the discrete valuations on $K/k$. In other words, analyze the structure of these discrete valuations as we did in the class for the field $k(t)$, but try to use as much as you can the lemmas from the lectures. (If you like you can pick a specific $f$ and a specific $k$.)*

*Solution*
In order to simplify matters we study the case $k = \mathbb{C}$. In this case the polynomial $f(t)$

factors into $(t-\lambda_1)(t-\lambda_2)(t-\lambda_3)$. We will use throughout the lemma proved in the last homework, i.e. that any valuation $w_i$ on $K$ restricts to $e_i v$ for $e_i$ natural and $v$ a valuation on $k(t)$, where we have $\sum_i e_i = [K : k(t)] = 2$. This means that, for every $\lambda \in k(t)$, we have two possibilities for the valuations lying over $\lambda$. Either there is only one with ramification index 2, or there are two of them, each with ramification index 1. We also use the fact that the valuations on $k(t)$ are $\operatorname{ord}_{t=\lambda}$ and $\operatorname{ord}_\infty$.

We start by analyzing the way valuations act on $(t-\lambda_i)$, for $i = 1, 2, 3$. This will give us all information about the valuations over $\lambda_i$ and over $\infty$. Assume that there exists some valuation $w$ such that $w(t-\lambda_i) > 0$. Then $w$ restricts on $k(t)$ to $m \operatorname{ord}_{t=\lambda_i}$ for some natural $m$. Using properties of valuations, we compute:

$$\begin{aligned}
2w(y) &= w(y^2) \\
&= w(t-\lambda_1) + w(t-\lambda_2) + w(t-\lambda_3) \\
&= m \operatorname{ord}_{t=\lambda_i}(t-\lambda_1) + m \operatorname{ord}_{t=\lambda_i}(t-\lambda_2) + m \operatorname{ord}_{t=\lambda_i}(t-\lambda_3) \\
&= m
\end{aligned}$$

We see that $m$ is a multiple of 2, but since $m$ is at most 2 we get $m = 2$. Therefore there is only one valuation over each $\lambda_i$; it is ramified and has $w(y) = 1$. Now assume there exists some valuation $w$ such that $w(\lambda_i) < 0$ for $i = 1, 2$ or 3. Then $w$ restricts to $n \operatorname{ord}_\infty$, and we compute:

$$\begin{aligned}
2w(y) &= w(y^2) \\
&= w(t-\lambda_1) + w(t-\lambda_2) + w(t-\lambda_3) \\
&= m \operatorname{ord}_\infty(t-\lambda_1) + m \operatorname{ord}_\infty(t-\lambda_2) + m \operatorname{ord}_\infty(t-\lambda_3) \\
&= -3m
\end{aligned}$$

Which can only be fulfilled if $m = 2$ and $w(y) = -3$. This means that there is only one valuation above $\infty$, which is also ramified.

We have already found 4 ramification points for the curve $y^2 = (t-\lambda_1)(t-\lambda_2)(t-\lambda_3)$. This curve defines a torus, so its genus is 1. But there cannot be more than $2(g+1)$ ramification points on the curve, or else these would determine an additional cut, which would increase its genus. Therefore the 4 ramification points we found are the only ones, and we can conclude that above any other $\lambda$ there are two valuations, each with ramification index 1.