Commutative algebra HW1

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Problem 1

Let $A = k[x, y]/(x^2y^4 - x^4y^2 + 1)$ where k is a field. Construct a k-algebra map as in Noether Normalization for A. Very briefly explain why it works.

Solution

As in the proof of Noether Normalization, we want to make a change of variables $(x, y) \rightarrow (z, y)$ such that y is integral over k[z]. We let $z = x - y^2$, and it turns out that the exponent 2 is large enough in order to make the resulting polynomial monic:

$$x^{2}y^{4} - x^{4}y^{2} + 1 \rightarrow 1 + y^{8} - y^{10} + 2y^{6}z - 4y^{8}z + y^{4}z^{2} - 6y^{6}z^{2} - 4y^{4}z^{3} - y^{2}z^{4}$$

Then by Lemma 1 proved in class $\phi: k[z] \to k[x,y]/(x^2y^4-x^4y^2+1)$ defined by $\phi(z)=x-y^2$ is a finite map.

Problem 2

Describe all prime ideals of $\mathbb{C}[x,y]/(xy)$ where \mathbb{C} is the field of complex numbers. Just list them in some way and explain briefly why they are primes and why you've got all of them.

Solution

As discussed in class, there is a 1-1 correspondence between ideals of $\mathbb{C}[x,y]/(xy)$ and ideals of $\mathbb{C}[x,y]$ that contain (xy). Moreover, if one of the latter is prime, then so is its correspondent. We will therefore consider prime ideals in $\mathbb{C}[x][y] = \mathbb{C}[x,y]$. Since $\mathbb{C}[x]$ is a PID, its prime ideals are of 3 types:

- 1) the 0 ideal, which does not contain (xy).
- 2) principal ideals (f(y)) generated by a polynomial f that is irreducible in $\mathbb{C}[x]$. The only such ideal that generates (xy) is (y).
- 3) (p, f(y)) where p is prime in $\mathbb{C}[x]$ and f is irreducible in $\mathbb{C}[x]/(p)$. The only such ideals that generate (xy) are $(x \lambda, y), (x, y \mu)$ and (x), for all $\lambda, \mu \in \mathbb{C}$.

It follows that all prime ideals of $\mathbb{C}[x,y]/(xy)$ are $(x+(xy)), (y+(xy)), (x-\lambda+(xy), y+(xy)), (x+(xy), y-\mu).$

Problem 3

Let k be a field. Prove that k[x, y] is not isomorphic to k[x, y, z].

Solution

Assume the two rings are isomorphic; then obviously the isomorphism is a finite map. Then there exist n generators that generate k[x,y,z] as a k[x,y] - module. This construction must span $z, z^2, ..., z^{n+1}$, all of which are linearly independent over k[x,y]. But this means that at least n+1 generators are required, which is a contradiction.

Problem 4

Let k be a field. Suppose A is a k-algebra and f is a nonzerodivisor of A such that k[x, y] is isomorphic to A_f as k-algebra. Show that A is isomorphic to k[x, y].

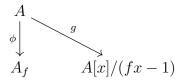
Solution

Since isomorphisms preserve units, f/1 and 1/f are units in k[x, y], therefore $f/1, 1/f \in k$. Since every map from a field to a ring is injective, the ring map $k \to A$ is an inclusion map, and therefore $f, f^{-1} \in A$. This means that localization does nothing to A, since we are inverting something which is already invertible. Thus $A \cong A_f \cong k[x, y]$.

Problem 5

Let A be a ring and let f be an element of A. Show that A_f is as an A-algebra isomorphic to A[x]/(fx-1).

Solution Consider the following maps:



 $\phi(a) = a/1$ and g(a) = a. The by the universality of localizations (see for example corollary 3.2 in Atiyah - Macdonald) there exists and isomorphism $h: A_f \to A[x]/(fx-1)$. The diagram above is easily shown to satisfy the requirements for the universality theorem:

- (a) $s \in \{f, f^2, ...\} \Rightarrow g(s)$ is a unit in A[x]/(fx-1)
- (b) $g(a) = 0 \Rightarrow as = 0$ for some $s \in \{f, f^2, ...\}$
- (c) every element of A[x]/(fx-1) is of the form $g(a)(g(s))^{-1}$

The first two are obvious, the third follows since $X = g(f)^{-1}$.