Commutative algebra HW12

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Problem 4

We follow the proof of proposition 1.18 in “The geometry of schemes” by Eisenbud and Harris. First note that, by the result of Problem 7 below, f_i generate the unit ideal iff the standard opens $D(f_i)$ cover $\operatorname{Spec} A$. Moreover, by the same problem we can assume that the f_i are finitely many.

Exactness at A means that any global section that restricts to 0 on each $D(f_i)$ must be 0 on A. We prove this as follows. If the global section g restricts to 0 on $D(f_i)$, then $g/1 = 0$ as an element of A_{f_i}. This means $gf_i^N = 0$ in A, for some power N. This holds for every f_i (which are finitely many), so let M be the maximal power among all these. Then the ideal generated by f_i^M annihilates g. We use now the fact that $1 = \sum e_i f_i$ for some $e_i \in A$, whence $1 = 1^P = \left(\sum e_i f_i\right)^P$ for every power P. If we let P be large enough, each term in the sum will contain some f_i to a power greater or equal to M. Therefore the ideal generated by f_i^M generates 1, and we see that $1 \cdot g = 0$. Therefore $g = 0$.

Exactness at $\prod_i A_{f_i}$ means that the existence of g_i on each $D(f_i)$ that agree on overlaps implies the existence of a global section g, such that $g|_{D(f_i)} = g_i$. To prove this, we want to construct a partition of unity that lets us glue together the local sections to form a global one. First note that $g_i \in A_{f_i}$ implies $f_i^N g_i \in A$ for large enough N. Since the f_i are finitely many, let M be the greatest such power, and define $h_i = f_i^M g_i \in A$. $g_i = g_j$ on $D(f_i, f_j)$, so on each overlap we have

$$f_j^M h_i = f_j^M f_i^M g_i = f_j^M f_i^M g_j = f_i^M h_j$$

We will use this identity shortly. We will also use the fact that $\{f_i\}$ generate 1, i.e.:

$$1 = \sum_i e_i f_i$$

Now define $g \in a$ as follows:

$$g = \sum_i e_i h_i$$

We show that indeed $g|_{D(f_j)} = g_j$. On each $D(f_j)$ we write:

$$f_j^M g = \sum_i f_j^M e_i h_i = \sum_i f_i^M e_i h_j = 1 \cdot h_j = f_j^M g_j$$

But on $D(f_j)$ f_j^M is a unit, so this implies $g = g_j$ as desired.

Problem 5

Let $Z = \operatorname{Spec}\mathbb{C}[x] = \{(0), (x-\lambda)\}$ and equip it with the structure sheaf. Consider the quotient $X = Z/\sim$, where $\sim$ identifies (x) and $(x-1)$. If $f : Z \to X$ is the natural projection, we can equip X with the direct image sheaf $\mathcal{O}(U) = \mathcal{O}_Z(f^{-1}(U))$. The stalk $\mathcal{O}_x$ will be the union $\bigcup_{\lambda\neq 0,1}\mathbb{C}[x]_{x-\lambda}$. In this union all monomials except for powers of x and $x-1$ have inverses. Therefore the stalk has two maximal ideals, (x) and $(x-1)$.

Problem 6

Take $X = \operatorname{Spec}\mathbb{Z}_{(p)}$, where p is a prime, and equip it with the constant sheaf determined by $\mathbb{Z}_{(p)}$. All opens in $\operatorname{Spec}\mathbb{Z}_{(p)}$ are connected, so the ring above each of them is $\mathbb{Z}_{(p)}$. Then the stalk at each point is $\mathbb{Z}_{(p)}$. Do likewise for $Y = \operatorname{Spec}\mathbb{C}[[t]]$. Then X and Y are locally ringed spaces. Define a morphism as the map on spectra induced by the inclusion $\mathbb{Z}_{(p)} \to \mathbb{C}[[t]]$. The induced map on stalks is the inclusion $\mathbb{Z}_{(p)} \to \mathbb{C}[[t]]$. It is not a local map, because it takes the maximal ideal (p) to constants in $\mathbb{C}[[t]]$.

Problem 7

We begin by showing that any collection of standard opens $\{D(f_i)\}$ covers $\operatorname{Spec}A$ iff $\{f_i\}$ generate 1. Note that $D(f_i)$ is the set of primes that don't contain f_i. These cover $\operatorname{Spec}A$ iff no prime $\mathfrak{p}\in\operatorname{Spec}A$ contains all f_i. Now if $\{f_i\}$ generate 1, any ideal that contains all of them contains 1, therefore no prime contains all f_i. Conversely, assume that $\{f_i\}$ don't generate 1, then the ideal they generate is contained in some maximal $\mathfrak{m}$. This $\mathfrak{m}$ is then a prime that contains all f_i.

Now, to prove that $\operatorname{Spec}A$ is quasicompact, take an open cover $\bigcup U_i$ of $\operatorname{Spec}A$. Since the standard opens form a basis for $\operatorname{Spec}A$, this cover has a refinement $\bigcup_i D(f_i)$. As proved above, this is equivalent to $\{f_i\}$ generating 1. Then $1 = \sum_i e_i f_i$ for some coefficients $e_i \in A$. But this linear combination must be finite, and we obtain that finitely many of the $\{f_i\}$ generate 1. Using the first paragraph again, this is equivalent to $\bigcup_{\text{finitely many } i} D(f_i)$ being a finite subcover. Thus $\operatorname{Spec}A$ is quasicompact.

Every standard open $D(f_i)$ is the set of primes that don't contain f_i, and therefore $D(f_i) = \operatorname{Spec}A_{f_i}$. Then applying the above to this spectrum shows that $D(f_i)$ is quasicompact.