Riemann surfaces final exam

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December 7, 2013

Problem 1

a) We define $\theta_1 : \mathbb{C} \to \mathbb{C}$ by:

$$\theta_1(z|\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi i(n+1/2)^2\tau + 2\pi i(n+1/2)(z+1/2)]$$

The periodicity of this theta function is given by:

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau)$$

$$\theta_1(z+\tau|\tau) = -e^{-i\pi\tau-2\pi iz}\theta_1(z|\tau)$$

$$\theta_1(z+m+n\tau|\tau) = e^{i\pi(m+n)-i\pi n\tau-2\pi inz}\theta_1(z|\tau)$$

The sum used to define θ_1 is convergent everywhere, so θ_1 has no poles. Its zeros are all lattice points $z = m + n\tau$.

b) We show existence by explicit construction. dz is a holomorphic form on $\mathbb{C}$, which is also doubly periodic, so it descends to a holomorphic form on $\mathbb{C}/\Lambda$. Then we define:

$$\omega_{PQ}(z) = \left[\frac{\theta_1'(z-P)}{\theta_1(z-P)} - \frac{\theta_1'(z-Q)}{\theta_1(z-Q)}\right] dz$$

We first show that ω_{PQ} is doubly periodic. By the transofrmation law for θ_1:

$$\frac{\theta_1'(z-P+1)}{\theta_1(z-P+1)} = \frac{\theta_1'(z-P)}{\theta_1(z-P)}$$

$$\frac{\theta_1'(z-P+\tau)}{\theta_1(z-P+\tau)} = \frac{\theta_1'(z-P)}{\theta_1(z-P)} - 2\pi i$$

$$\Rightarrow \omega_{PQ}(z+1) = \omega_{PQ}(z+\tau) = \omega_{PQ}(z)$$

Therefore ω_{PQ} is well-defined on $\mathbb{C}/\Lambda$. We know from complex analysis that $\theta_1'(z-P)/\theta_1(z-P)$ has a simple pole with residue 1 whenever $\theta_1(z-P)$ has zeros, which happens for $z = P$. Since $\theta_1(z-P)$ has no poles, these are all the poles of $\theta_1'(z-P)/\theta_1(z-P)$. This shows that

ω_{PQ} has a simple pole with residue 1 at P, and a simple pole with residue -1 at Q.

c) Similarly, define:

$$\omega_P(z) = \left(\frac{\theta_1'(z-P)}{\theta_1(z-P)}\right)'$$

By the reasoning in part b), θ_1'/θ_1 is invariant under $z \to z+1$, and changes by a constant under $z \to z+\tau$. Then its derivative is doubly periodic. Moreover, θ_1'/θ_1 has a simple pole at P with residue 1. Therefore, in some small neighborhood of P, its Laurent expansion is:

$$\frac{\theta_1'(z-P)}{\theta_1(z-P)} = \frac{1}{z-P} + \text{ holomorphic}$$

And the expansion of its derivative is:

$$\left(\frac{\theta_1'(z-P)}{\theta_1(z-P)}\right)' = -\frac{1}{(z-P)^2} + \text{ holomorphic}$$

Which shows that $\omega_P(z)$ has a double pole at P.

Problem 2

a) Given a metric $h(z)$ on L, we define its curvature:

$$F_{\bar{z}z} = -\,\partial_z\,\partial_{\bar{z}} \log h$$

Then we define the first Chern class as:

$$c_1(L) = \frac{i}{2\pi}\int_X F_{\bar{z}z} dz \wedge d\bar{z}$$

b) In class we proved the following theorem. If ϕ is a meromorphic section of L which is not identically 0, then:

$$c_1(L) = \text{number of zeros of } \phi - \text{number of poles of } \phi$$

We see that, if $c_1(L) < 0$, then any meromorphic section must have at least a pole. Thus, no section is holomorphic.

c) We denote the vector space of holomorphic sections of L by $H^0(X, L)$. The Riemann-Roch theorem says that:

$$\dim H^0(X, L) - \dim H^0(X, K \otimes L^{-1}) = c_1(L) + \frac{1}{2}c_1(K^{-1})$$

We want to apply this to $L = K^n$. Note that, if h_1, h_2 are metrics on L_1, L_2, then $h_1 h_2$ is a metric on $L_1 \otimes L_2$. Then using the definition of curvature, which includes a logarithm, we see that the curvature is additive. Then c_1 must also be additive. In particular, $c_1(L^n) = nc_1(L)$ for all L. We obtain:

$$\dim H^0(X, K^n) - \dim H^0(X, K^{1-n}) = -nc_1(K^{-1}) + \frac{1}{2}c_1(K^{-1})$$

d) In general, we know that for $n = 0$ (holomorphic functions) the dimension is 1, and for $n = 1$ (holomorphic 1-forms) the dimension is g. For all other n, we split the computation into 3 cases:

First case: $c_1(K^{-1}) > 0$, which only happens when $g = 0$. This is equivalent to $c_1(K) < 0$, which also shows that $c_1(K^n) = nc_1(K) < 0$ for all $n > 0$. Using the result of part b), we see that $\dim H^0(X, K^n) = 0$ for all $n > 0$. In this case, part c) reduces to:

$$\dim H^0(X, K^{1-n}) = 2n - 1$$

For convenience, we make the substitution $m = 1 - n$, and we obtain that, for $m \leq 0$:

$$\dim H^0(X, K^m) = 1 - 2m$$

To sum up, the dimension of $H^0(X, K^n)$ is 0 for $n > 0$, and $1 - 2n$ otherwise.

Second case: $c_1(K^{-1}) = 0$, which only happens when $g = 1$. This implies that $c_1(K^n) = 0$ for all n. Using part b), we see that any meromorphic section of K^n has equal number of zeros and poles. In particular, holomorphic sections have no zeros. Now consider two nontrivial sections $\phi_1, \phi_2 \in \Gamma(X, K^n)$ and evaluate them at some point z. Let $w_1 = \phi_1(z)$ and $w_2 = \phi_2(z)$. We construct the linear combination:

$$\psi = w_1\phi_2 - w_2\phi_1 \in \Gamma(X, K^n)$$

Since $\psi(z) = 0$, ψ must be the trivial section. Therefore ϕ_1, ϕ_2 are linearly dependent. This shows that $\dim H^0(X, K^n) = 1$ for all n.

Third case: $c_1(K^{-1}) < 0$, which happens for $g \geq 2$. This implies that $c_1(K^{-n}) < 0$ for $n > 0$, therefore $\dim H^0(X, K^{1-n}) = 0$ for $n > 1$. In this case, part c) reduces to:

$$\dim H^0(X, K^n) = (2n - 1)(g - 1)$$

To sum up, the dimension of $H^0(X, K^n)$ is $(2n-1)(g-1)$ for $n > 1$, g for $n = 1$, 1 for $n = 0$, and 0 for $n < 0$.

e) We proved in class that the dimension of the moduli space of Riemann surfaces of genus g is equal to $\dim H^0(X, K^2)$. Using part d), we see that this is 0 for $g = 0$, 1 for $g = 1$ and $3(g - 1)$ for $g \geq 2$.

Problem 3

a) Let $\phi_1, \phi_2 \in \Gamma(X, L)$ and $\psi_1, \psi_2 \in \Gamma(X, L \otimes \bar{K})$. We define:

$$\langle \phi_1, \phi_2 \rangle = \int_X \phi_1 \bar{\phi}_2 h \; g_{\bar{z}z}$$

$$\langle \psi_1, \psi_2 \rangle = \int_X \psi_1 \bar{\psi}_2 h$$

To see that these definitions make sense, note that:

$$\phi_1 \bar{\phi}_2 h \; g_{\bar{z}z} \in \Gamma(X, L \otimes \bar{L} \otimes L^{-1} \otimes \bar{L}^{-1} \otimes K \otimes \bar{K}) = \Gamma(X, K \otimes \bar{K})$$

$$\psi_1 \psi_2 h \in \Gamma(X, L \otimes \bar{K} \otimes \bar{L} \otimes K \otimes L^{-1} \otimes \bar{L}^{-1}) = \Gamma(X, \bar{K} \otimes K)$$

Both expressions are 1-1 forms, so it makes sense to integrate them over X.

b) The formal adjoint $\bar{\partial}^\dagger$ is defined as:

$$\langle \bar{\partial}\phi, \psi \rangle = \langle \phi, \bar{\partial}^\dagger \psi \rangle \quad \forall \phi, \psi$$

Writing the inner products explicitly, this becomes:

$$\int_X (\bar{\partial}\phi)\bar{\psi} h = \int_X \phi \overline{(\bar{\partial}^\dagger \psi)} h \; g_{\bar{z}z}$$

After integrating by parts on the LHS:

$$\int_X \phi \; \bar{\partial}(\bar{\psi} h) = \int_X \phi \overline{(\bar{\partial}^\dagger \psi)} h \; g_{\bar{z}z}$$

Using $\bar{h} = h, \bar{g}^{\bar{z}z} = g^{\bar{z}z}$ and $g^{\bar{z}z} g_{\bar{z}z} = 1$, we further rewrite the LHS:

$$\int_X \phi h \overline{g^{\bar{z}z} h^{-1} \, \partial(\psi h)} g_{\bar{z}z} = \int_X \phi \overline{(\bar{\partial}^\dagger \psi)} h \; g_{\bar{z}z}$$

Since this must hold for all ϕ, we obtain:

$$\bar{\partial}^\dagger \psi = g^{\bar{z}z} h^{-1} \, \partial(h\psi)$$

$$\bar{\partial}^\dagger \psi = g^{\bar{z}z} \nabla_z \psi$$

Where $\nabla_z : \Gamma(X, L \otimes \bar{K}) \to \Gamma(X, L \otimes \bar{K} \otimes K)$ is the covariant derivative on the bundle $L \otimes \bar{K}$.

c) We first show that $\operatorname{Ker} \Delta_+ = \operatorname{Ker} \bar{\partial}$, and the analogous statement will hold for Δ_-.

$$\operatorname{Ker} \Delta_+ = \{\phi \in \Gamma(X, L) | \bar{\partial}^\dagger \bar{\partial} \phi = 0\} \subset \{\phi | \langle \phi, \bar{\partial}^\dagger \bar{\partial} \phi \rangle = 0\}$$

$$= \{\phi | \; ||\bar{\partial}\phi||^2 = 0\} = \{\phi | \bar{\partial}\phi = 0\} = \operatorname{Ker} \bar{\partial}$$

But clearly $\operatorname{Ker} \bar{\partial} \subset \operatorname{Ker} \bar{\partial}^\dagger \bar{\partial} = \operatorname{Ker} \Delta_+$, so the two are equal. Therefore:

$$\dim \operatorname{Ker} \Delta_+ - \dim \operatorname{Ker} \Delta_- = \dim \operatorname{Ker} \bar{\partial} - \dim \operatorname{Ker} \bar{\partial}^\dagger$$

We can define the action of $e^{-t\Delta_\pm}$ on eigenfunctions $\phi_\pm^n$ as:

$$e^{-t\Delta_\pm}\phi_\pm^n = e^{-t\lambda_\pm^n}\phi_\pm^n$$

We consider only eigenfunctions that satisfy $||\phi_\pm^n|| = 1$. Assuming that the eigenvalues are discrete, we can define the trace of the exponential as:

$$\mathrm{Tr}\, e^{-t\Delta_\pm} = \sum_n \langle \phi_\pm^n, e^{-t\Delta_\pm}\phi_\pm^n\rangle = \sum_n e^{-t\lambda_\pm^n}$$

Now note that, if $\lambda \neq 0$ is an eigenvalue for Δ_+, it is also an eigenvalue for Δ_-. This is because:

$$\bar\partial^\dagger\bar\partial\phi = \lambda\phi \Rightarrow (\bar\partial\bar\partial^\dagger)(\bar\partial\phi) = \lambda(\bar\partial\phi)$$

The converse is proved analogously. We see that the nonzero eigenvalues of Δ_+ and Δ_- coincide, and therefore:

$$\mathrm{Tr}\, e^{-t\Delta_+} - \mathrm{Tr}\, e^{-t\Delta_-} = \sum_{\lambda_n=0} e^{-t\lambda_+^n} - \sum_{\lambda_n=0} e^{-t\lambda_-^n}$$

Recall that each n parametrizes a unit length eigenfunction, therefore each $\lambda_n = 0$ gives a one-dimensional subspace of the kernel. This becomes:

$$\mathrm{Tr}\, e^{-t\Delta_+} - \mathrm{Tr}\, e^{-t\Delta_-} = \dim \mathrm{Ker}\,\Delta_+ - \dim \mathrm{Ker}\,\Delta_-$$

And combining this with our previous result:

$$\mathrm{Tr}\, e^{-t\Delta_+} - \mathrm{Tr}\, e^{-t\Delta_-} = \dim \mathrm{Ker}\,\bar\partial - \dim \mathrm{Ker}\,\bar\partial^\dagger$$

d) The operator $\bar\partial$ is defined on $\Gamma(X, L)$ and:

$$\mathrm{Ker}\,\bar\partial = \{\phi \in \Gamma(X, L) | \bar\partial\phi = 0\} = H^0(X, L)$$

Moreover, in part b) we showed that:

$$\begin{aligned}\mathrm{Ker}\,\bar\partial^\dagger &= \{\psi \in \Gamma(X, L\otimes\bar K) |\, \partial_z(h\psi) = 0\}\\ &= \{\psi \in \Gamma(X, L\otimes\bar K) |\, \partial_{\bar z}(h\bar\psi) = 0\}\end{aligned}$$

This gives an isomorphism:

$$\psi \in \mathrm{Ker}\,\bar\partial^\dagger \longleftrightarrow h\bar\psi \in \mathrm{Ker}\,\bar\partial|_{\Gamma(X, K\otimes L^{-1})}$$

Which shows that $\dim \mathrm{Ker}\,\bar\partial^\dagger = \dim H^0(X, K\otimes L^{-1})$. Together with the result of part c), we get:

$$\mathrm{Tr}\, e^{-t\Delta_+} - \mathrm{Tr}\, e^{-t\Delta_-} = \dim H^0(X, L) - \dim H^0(X, K\otimes L^{-1})$$

Problem 4

a) On $X_\mu \cap X_\nu$, ϕ^α satisfy the glueing condition:

$$\phi^\alpha_\mu(z_\mu) = t_{\mu\nu}{}^\alpha{}_\beta(z)\phi^\beta_\nu(z_\nu)$$

The transition matrix is holomorphic, $\partial_{\bar{j}}\, t = 0$, therefore:

$$\frac{\partial}{\partial\, \bar{z}^j_\mu}\phi^\alpha_\mu(z_\mu) = t_{\mu\nu}{}^\alpha{}_\beta(z)\frac{\partial}{\partial\, \bar{z}^j_\mu}\phi^\beta_\nu(z_\nu) = t_{\mu\nu}{}^\alpha{}_\beta(z)\frac{\partial\, \bar{z}^k_\nu}{\partial\, \bar{z}^j_\mu}\frac{\partial}{\partial\, \bar{z}^k_\nu}\phi^\beta_\nu(z_\nu)$$

Which shows that $\partial_{\bar{j}}\, \phi^\alpha \in \Gamma(X, E \otimes \Lambda^{0,1})$. In the case of $\nabla_j\phi$, we have $H_{\bar{\beta}\gamma} \in \Gamma(X, \bar{E}^* \otimes E^*)$, so $H_{\bar{\beta}\gamma}\phi^\gamma \in \Gamma(X, \bar{E}^*)$. This is an antiholomorphic bundle, so the same reasoning as for $\partial_{\bar{j}}$ above shows that $\partial_j(H_{\bar{\beta}\gamma}\phi^\gamma) \in \Gamma(X, \bar{E}^* \otimes \Lambda^{1,0})$ is well-defined. Finally, since $H^{\alpha\bar{\beta}} \in \Gamma(X, E \otimes \bar{E})$, we see that $\nabla_j\phi \in \Gamma(X, E \otimes \Lambda^{1,0})$.

b)

$$\nabla_j\phi^\alpha = H^{\alpha\bar{\beta}}H_{\bar{\beta}\gamma}\, \partial_j\, \phi^\gamma + H^{\alpha\bar{\beta}}\, \partial_j\, H_{\bar{\beta}\gamma}\phi^\gamma = \delta^\alpha_\gamma\, \partial_j\, \phi^\gamma + (H^{\alpha\bar{\beta}}\, \partial_j\, H_{\bar{\beta}\gamma})\phi^\gamma$$

Therefore $A^\alpha_{j\gamma} = H^{\alpha\bar{\beta}}\, \partial_j\, H_{\bar{\beta}\gamma}$. Now we write the commutator:

$$\begin{aligned}
[\nabla_j, \nabla_{\bar{k}}]\phi^\alpha &= [\partial_j, \partial_{\bar{k}}]\phi^\alpha + A^\alpha_{j\gamma}(\partial_{\bar{k}}\, \phi^\gamma) - \partial_{\bar{k}}(A^\alpha_{j\gamma}\phi^\gamma) \\
&= -(\partial_{\bar{k}}\, A^\alpha_{j\gamma})\phi^\gamma
\end{aligned}$$

Therefore $F_{\bar{k}j}{}^\alpha{}_\gamma = -\,\partial_{\bar{k}}\, A^\alpha_{j\gamma} = -\,\partial_{\bar{k}}(H^{\alpha\bar{\beta}}\, \partial_j\, H_{\bar{\beta}\gamma})$.

c) We begin by computing dA:

$$\begin{aligned}
dA = d(A_j dz^j) &= (\partial_k\, A_j dz^k + \partial_{\bar{k}}\, A_j d\bar{z}^k) \wedge dz^j \\
&= \frac{1}{2}(\partial_k\, A_j - \partial_j\, A_k)dz^k \wedge dz^j + F_{\bar{k}j}d\bar{z}^k \wedge dz^j \\
&= \frac{1}{2}[\partial_k(H^{-1}\, \partial_j\, H) - \partial_j(H^{-1}\, \partial_k\, H)]dz^k \wedge dz^j + F \\
&= \frac{1}{2}[-H^{-1}(\partial_k\, H)H^{-1}(\partial_j\, H) - (j \leftrightarrow k)]dz^k \wedge dz^j + F \\
&= \frac{1}{2}(A_jA_k - A_kA_j)dz^k \wedge dz^j + F \\
&= -A \wedge A + F
\end{aligned}$$

Thus $F = dA + A \wedge A$. We take another exterior derivative of this equation and use the fact that $d^2 = 0$:

$$\begin{aligned}
dF = d(A \wedge A) &= dA \wedge A - A \wedge dA \\
&= (-A \wedge A + F) \wedge A - A \wedge (-A \wedge A + F) \\
&= F \wedge A - A \wedge F
\end{aligned}$$

d) If such ∇_j exists, it has to satisfy:

$$\begin{aligned}
\phi^\alpha(\nabla_j\psi_\alpha) &= \partial_j(\phi^\alpha\psi_\alpha) - (\nabla_j\phi^\alpha)\psi_\alpha \\
&= (\partial_j\,\phi^\alpha)\psi_\alpha + \phi^\alpha(\partial_j\,\psi_\alpha) - (\partial_j\,\phi^\alpha)\psi_\alpha - A^\alpha_{j\beta}\phi^\beta\psi_\alpha \\
&= \phi^\alpha(\partial_j\,\psi^\alpha) - A^\beta_{j\alpha}\phi^\alpha\psi_\beta \\
&= \phi^\alpha(\partial_j\,\psi^\alpha) - \phi^\alpha A^\beta_{j\alpha}\psi_\beta
\end{aligned}$$

On the third line we simply relabeled the dummy indices α and β. We see that the following definition does the job:

$$\begin{aligned}
\nabla_j\psi_\alpha &= \partial_j\,\psi_\alpha - \psi_\beta A^\beta_{j\alpha} \\
\nabla_j\psi &= \partial_j\,\psi - \psi A_j
\end{aligned}$$

For $T \in \Gamma(X, \mathrm{End}(E))$, we proceed similarly:

$$\begin{aligned}
(\nabla_j T^\alpha{}_\beta)\phi^\beta &= \nabla_j(T^\alpha{}_\beta\phi^\beta) - T^\alpha{}_\beta(\nabla_j\phi)^\beta \\
&= \partial_j(T^\alpha{}_\beta\phi^\beta) + A^\alpha_{j\gamma}T^\gamma{}_\beta\phi^\beta - T^\alpha{}_\beta\,\partial_j\,\phi^\beta - T^\alpha{}_\beta A^\beta_{j\gamma}\phi^\gamma \\
&= \partial_j\,T^\alpha{}_\beta\phi^\beta + A^\alpha_{j\gamma}T^\gamma{}_\beta\phi^\beta - T^\alpha{}_\gamma A^\gamma_{j\beta}\phi^\beta \\
\nabla_j T^\alpha{}_\beta &= \partial_j\,T^\alpha{}_\beta + A^\alpha_{j\gamma}T^\gamma{}_\beta - T^\alpha{}_\gamma A^\gamma_{j\beta} \\
\nabla_j T &= \partial_j\,T + [A_j, T]
\end{aligned}$$

e) The usual exterior derivative d on scalar valued forms is defined as:

$$d(\omega_{\bar{k}j}dz^j \wedge d\bar{z}^k) = (\partial_l\,\omega_{\bar{k}j})dz^l \wedge dz^j \wedge d\bar{z}^k + (\partial_{\bar{l}}\,\omega_{\bar{k}j})d\bar{z}^l \wedge dz^j \wedge d\bar{z}^k$$

We emulate this behavior and define:

$$\begin{aligned}
d_A(F_{\bar{k}j}dz^j \wedge d\bar{z}^k) &= (\nabla_l F_{\bar{k}j})dz^l \wedge dz^j \wedge d\bar{z}^k + (\nabla_{\bar{l}} F_{\bar{k}j})d\bar{z}^l \wedge dz^j \wedge d\bar{z}^k \\
&= (\partial_l\,F_{\bar{k}j}dz^l + \partial_{\bar{l}}\,F_{\bar{k}j}d\bar{z}^l) \wedge dz^j \wedge d\bar{z}^k + (A_l F_{\bar{k}j} - F_{\bar{k}j}A_l)dz^l \wedge dz^j \wedge d\bar{z}^k \\
d_A F &= dF + A \wedge F - F \wedge A
\end{aligned}$$

Together with the result of part c), this shows $d_A F = 0$.