Representation theory HW1

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Problem 1 (8.7 in Kirillov)

Problem 2 (8.9 in Kirillov)

$V_n \otimes V_m$ obviously has a decomposition into irreducibles $\bigoplus V_k$; we just need to figure out what the k's are. Using the properties of characters, we get:

$$\operatorname{char}(V_n)\operatorname{char}(V_m) = \sum_k \operatorname{char}(V_k)$$

For irreducible reps of $\mathfrak{sl}(2,\mathbb{C})$, we know that $\operatorname{char}(V_k) = X^k + X^{k-2} + \dots + X^{-k}$, therefore:

$$(X^n + X^{n-2} + \dots + X^{-n})(X^m + X^{m-2} + \dots + X^{-m}) = \sum_k (X^k + X^{k-2} + \dots + X^{-k})$$

After distributing terms, and assuming WLOG that $n \geq m$, the LHS becomes:

$$\begin{aligned} \text{LHS} = X^{n+m} &+ 2X^{n+m-2} + 3X^{n+m-4} + \dots + (m+1)X^{n-m} + (m+1)X^{n-m-2} + \dots \\ &+ (m+1)X^{-n+m} + mX^{-n+m-2} + \dots + 2X^{-n-m+2} + X^{-n-m} \end{aligned}$$

Therefore we identify the k's as $n+m, n+m-2, n+m-4, \dots, n-m$. In other words, these are the integers that satisfy the Clebsch-Gordan relations. I am a little fatty.