# QFT Lecture 26

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## Perturbation theory

Srednicki ch. 43-48

Recall Gaussian integrals for the scalar field:

$$\int \frac{Dx}{\sqrt{(2\pi)^N \det M}} x_i x_j e^{-\frac{1}{2}x^T M x} = M^{-1}_{ij} = \langle x_i x_j \rangle_{\text{free}}$$

We want to do the same for Fermions. We want to show that:

$$\int D\Psi D\bar{\Psi}\ \Psi_\alpha(x)\Psi_\beta(y) e^{i\int d^4x\ i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi}$$

Gives what we obtained last lecture using operators:

$$\frac{1}{i}S_{\alpha\beta}(x_1 - x_2) = \frac{1}{i}\int \frac{d^4k}{(2\pi)^4} \frac{(-k\!\!\!/ + m)_{\alpha\beta}\ e^{ik(x_1-x_2)}}{k^2 + m^2 - i\epsilon}$$

Remember from last lecture the fermionic (Berezin) integral defined by $\int da\ 1 = 0$, $\int da\ a = 1$. Consider:

$$\int da_2 da_1\ e^{\frac{1}{2}a^T M a} = \int da_2 da_1 e^{a_1 M_{12} a_2} = \int da_2 da_1 (1 + a_1 M_{12} a_2 + ...) = M_{12}$$

Generalizing this we get:

$$\int da_n\ ...\ da_1\ e^{\frac{1}{2}a^T M a} = \sqrt{\det M}$$

$$\int da_n\ ...\ da_1\ a_i a_j e^{\frac{1}{2}a^T M a} = \sqrt{\det M}(-M^{-1}_{ij})$$

$$\int \frac{da_n d\bar{a}_n\ ...\ da_1 d\bar{a}_1}{\det M} a_i \bar{a}_j e^{\frac{1}{2}\bar{a}^T M a} = -M^{-1}_{ij} = \langle a_i \bar{a}_j \rangle$$

Let's apply this to compute the 2-point function for free theory:

$$\langle \Psi_\alpha(x_1)\bar{\Psi}_\beta(x_2)\rangle = -M^{-1}_{\alpha\beta}$$

Where:

$$M_{\gamma\alpha} = (id^4x_0)(i\partial\!\!\!/ - m)_{\gamma\alpha}d^4x_1\delta^{(4)}(x_0 - x_1)$$

We want to show that:

$$-\int d^4x_1\ M_{\gamma\alpha}(x_0 - x_1)\frac{1}{i}S_{\alpha\beta}(x_1 - x_2) = \delta(x_0 - x_2)\delta_{\gamma\beta}$$

The free 4-point function for a Dirac spinor:

$$\langle\Psi_{\alpha_1}(x_1)\bar{\Psi}_{\alpha_2}(x_2)\Psi_{\alpha_3}(x_3)\bar{\Psi}_{\alpha_4}(x_4)\rangle_{\text{free}} =$$

$$= \frac{1}{i}S_{\alpha_1\alpha_2}(x_1 - x_2)\frac{1}{i}S_{\alpha_3\alpha_4}(x_3 - x_4) - \frac{1}{i}S_{\alpha_1\alpha_4}(x_1 - x_4)\frac{1}{i}S_{\alpha_3\alpha_2}(x_3 - x_2)$$

## Interactions

Our toy theory will be:

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi - \frac{1}{2}(\partial\,\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + g\phi\bar{\Psi}\Psi$$

In fact this is not just a toy example; it models the interaction of fermions with the Higgs field. Consider $e^-e^- \to e^-e^-$ scattering:

$$\langle p_1's_1', p_2's_2'|p_1s_1, p_2s_2\rangle - "1" = (2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)i\mathcal{M} \sim$$

$$\sim \langle 0|b_{s_2'}(p_2')b_{s_1'}(p_1')b^\dagger_{s_1}(p_1)b^\dagger_{s_2}(p_2)|0\rangle$$

We are using:

$$\Psi(x) = \int\sum\left(b_s(k)u_s(k)e^{ikx} + d_s^\dagger(k)v_s(k)e^{-ikx}\right)$$

$$\bar{\Psi}(x) = \int\sum\left(b_s^\dagger(k)\bar{u}_s(k)e^{-ikx} + d_s(k)\bar{v}_s(k)e^{ikx}\right)$$

We are interested in:

$$\langle\Psi_{\alpha_{2'}}(x_2')\Psi_{\alpha_{1'}}(x_1')\bar{\Psi}_{\alpha_1}(x_1)\bar{\Psi}_{\alpha_2}(x_2)\rangle^C_{\text{full}} =$$

$$= \frac{(ig)^2}{2!}\int d^4yd^4z\langle\phi_y\bar{\Psi}_y\Psi_y\phi_z\bar{\Psi}_z\Psi_z\Psi_{\alpha_{2'}}(x_2')\Psi_{\alpha_{1'}}(x_1')\bar{\Psi}_{\alpha_1}(x_1)\bar{\Psi}_{\alpha_2}(x_2)\rangle^C_{\text{free}}$$

We need to connect these such that we have a $\phi - \phi$ propagator and four $\Psi - \bar{\Psi}$ propagators. One such option is:

$$\frac{(ig)^2}{2!}\int d^4yd^4z\ \frac{1}{i}\Delta(y - x)\left(\frac{1}{i}S(x_{1'} - y)\frac{1}{i}S(y - x_1)\right)_{\alpha_{1'}\alpha_1}\left(\frac{1}{i}S(x_{2'} - z)\frac{1}{i}S(z - x_2)\right)_{\alpha_{2'}\alpha_2}$$

We remove the 2! factor because of the diagram which swaps y and z. All in all, the 3 diagrams that we get are = | =. If we plug this into the LSZ formula, the Dirac operators kill the fermion propagators, and only the scalar propagator remains. We figure out the momentum of $\phi$ by conservation at each vertex. Thus:

$$i\mathcal{M} = (ig)^2\frac{1}{i}\frac{1}{(p_1 - p_{1'})^2 + m_\phi^2 - i\epsilon}\bar{u}_{s_1'}(p_1')\cdot u_{s_1}(p_1)\quad \bar{u}_{s_2'}(p_2')\cdot u_{s_2}(p_2)$$