Commutative algebra HW4

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Problem 5

What is the dimension of the local ring of $k[x,y,z]/(x^2y^2z^2, x^3y^2z)$ at the maximal ideal (x,y,z)?

Solution
The dimension of $k[x,y,z]_{(x,y,z)}$ is 3. This ring is a domain, so taking a quotient by the nonzero element $x^2y^2z^2$ will decrease dimension by 1. Therefore $\dim\left(k[x,y,z]/(x^2y^2z^2)_{(x,y,z)}\right) = 2$. If we quotient again by the zero divisor x^3y^2z, we can't tell immediately if the dimension stays 2 or drops to 1. But we can construct a chain of 3 primes, which proves that the dimension is 2. To see this, consider the following chain in $k[x,y,z]/(x^2y^2z^2)_{(x,y,z)}$:

$$(x) \subsetneq (x,y) \subsetneq (x,y,z)$$

All three primes generate the element x^3y^2z, so they are also primes in $k[x,y,z]/(x^2y^2z^2, x^3y^2z)_{(x,y,z)}$. Furthermore, the inclusions in the chain remain proper when we pass to the quotient, because the equation $x^3y^2z = 0$ provides no way of solving for one of the generators x, y, z in terms of the others. Thus the dimension is 2.

Problem 6

What is the dimension of the local ring of $A = k[x,y,z]/(x^3-y^2, x^5-z^2, y^5-z^3)$ at the maximal ideal (x,y,z)?

Solution
$k[x,y,z]$ is a domain, therefore modding out by (x^3-y^2) will decrease its dimension from 3 to 2. Furthermore, (x^3-y^2) is a prime in $k[x,y,z]$, and so $k[x,y,z]/(x^3-y^2)$ is a domain. Then modding out by (x^5-z^2), a nonzerodivisor, again drops the dimension from 2 to 1. Therefore the dimension of A is at most 1. To show it is at least 1, we construct the following

map:

$$\begin{aligned}\phi : A &\to k[t]\\ a &\to a \text{ for } a \in k\\ x &\to t^2\\ y &\to t^3\\ z &\to t^5\end{aligned}$$

And the rest is defined by homomorphism properties. Let's first check that this map is well-defined, in the sense that it takes the same value for all representatives of an equivalence class in A. It suffices to check this for the class of 0, as homomorphism porperties take care of the rest:

$$\phi(x^3 - y^2) = t^6 - t^6 = 0$$
$$\phi(x^5 - z^2) = t^{10} - t^{10} = 0$$
$$\phi(y^5 - z^3) = t^{15} - t^{15} = 0$$

Thus ϕ is well-defined. We claim now that $k[t]$ is integral over A. Any element in $k \subset k[t]$ is also an element of A. Then, because 2 and 3 are relatively prime, $t^2 = \phi(x)$ and $t^3 = \phi(y)$ generate all powers of t apart from 1. Finally, t satisfies the monic polynomial:

$$t^2 - \phi(x) = 0$$

And so $k[t]$ is integral over A. But then, by lemma 10.106.3 in the Stacks project, $\dim A \geq \dim k[t] = 1$. Then $\dim A = 1$.

Problem 7

Let k be a field. Let $f \in k[x, y]$ be a polynomial. Let $a, b \in k$ be elements such that $f(a, b) = 0$. Let $m = (x - a, y - b)$ be the corresponding maximal ideal in the ring $A = k[x, y]/(f)$. Prove that A_m is a regular local ring if and only if one of df/dx, df/dy doesn't vanish at (a, b).

Solution

$k[x, y]$ is a domain, so f is not a zero divisor. Therefore taking a qoutient by (f) reduces the dimension of $(k[x, y])_m$ from 2 to 1. In order for A_m to be regular, we must have $\dim_{A/(f)} m/m^2 = 1$. Let's analyze m and m^2. First, we know that $f(x, y)$ is a polynomial that vanishes at (a, b). Therefore if we Taylor expand it around (a, b) there will be no constant term:

$$f(x, y) = \sum_{i+j \geq 1}^{\infty} c_{ij}(x - a)^i (y - b)^j \quad , c_{ij} \in k$$

Note that this Taylor series must be finite, since we are dealing with a polynomial, and the degree of the LHS and RHS must be equal. By the same argument, we can write the ideal

m as:

$$m = \left\{ \sum_{i+j \geq 1} d_{ij}(x-a)^i(y-b)^j | d_{ij} \in k \right\}$$

Then:

$$m^2 = \left\{ \sum_{i+j \geq 2} d_{ij}(x-a)^i(y-b)^j | d_{ij} \in k \right\}$$

$$m/m^2 = \{d_{10}(x-a) + d_{01}(y-b)\} \cong k^2$$

But remember that we are looking at this module in $k[x,y]/(f)$, so we must mod by (f) in the above. But $(f)/m^2 = c_{10}(x-a) + c_{01}(y-b)$. If $c_{10} = f_x$ and $c_{01} = f_y$ are both 0, then the expression above is 0, so quotienting by it leaves $m/m^2 \cong k^2$, and therefore its dimension over k is 2. However, if not both derivatives are 0, then c_{10}, c_{01} span a line in k^2, and qoutienting by it leaves $m/m^2 \cong k$. In this case the dimension is 1, and we obtain the desired result.

Problem 9

Let $k = \mathbb{C}$ be the field of complex numbers. What are the singular points of the curve C defined by $f = x^n + y^n + 1$, $f = xy^2 + x^2y$, $f = x^2 - 2x + y^3 - 3y^2 + 3y$?

Solution
For $f(x,y) = x^n + y^n + 1$, singular points (x,y) satisfy $x^{n-1} = y^{n-1} = 0$, so $(x,y) = (0,0)$. But this point does not belong to the curve, so we have no singular points on C.

For $f(x,y) = xy^2 + x^2y$, $df/dx = df/dy = 0$ gives the unique solution $x = y = 0$. This singular point clearly belongs to the curve.

For $f(x,y) = x^2 - 2x + y^3 - 3y^2 + 3y$, $df/dx = df/dy = 0$ gives the unique solution $x = y = 1$. This singular point clearly belongs to the curve.

Problem 10

Let k be an algebraically closed field. Let $f \in k[x,y]$ be a squarefree polynomial of degree $\leq d$. What is the maximum number of singular points the associated curve C can have? Start with $d = 1, 2, 3, \ldots$ and make a guess for the general answer. To prove it in general is too hard right now.

Solution
<u>d=1</u>: $f(x,y) = ax + by + c$, and for a singular point we need $\frac{\partial f}{\partial x} = a = 0$ and $\frac{\partial f}{\partial y} = b = 0$. Therefore unless $f = 0$ there will be no singular points.
<u>d=2</u>: $f(x,y) = ax^2 + by^2 + cxy + dx + ey + g$. For a singular point we need $2ax + cy + d = 0$ and $2by + cx + e = 0$. This system has an infinite number of solutions if the two lines coincide, i.e.

$2a/c = c/2b = d/e$. However, by plugging these into the expression for $f(x, y)$ and choosing g such that $f = 0$ on this line, we see that f becomes a square in this case, which is not allowed. Then there is a unique solution, and by choosing the parameter f appropriately in the expression for $f(x, y)$ we can make this point lie on the curve. Therefore there is one singular point.

d=3: Here the system of equations $df/dx = 0, df/dy = 0$ consists of two quadratic equations. After splitting the system we get a quadratic in x and a quadratic in y, which can be solved to give 2 solutions for each. Therefore we get 4 singular points. We hope that we can somehow make all these lie on the curve.

d arbitrary: We expect $(d-1)^2$ singular points, because the system $df/dx = 0, df/dy = 0$ will give equations of order $d-1$ for each variable. Even if it turns out that we can't make all these lie on the curve, $(d-1)^2$ is a decent upper bound.