QFT Lecture 11

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How do we compute cross-sections once we have the scattering amplitude?

$$i\mathcal{M} = (ig)^2 \left[\frac{1}{i} \frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{i} \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{i} \frac{1}{(k_1-k_2')^2+m^2} \right]$$

Reference: ch. 11 Srednicki.

Define Mandelstein Variables: $s = -(k_1+k_2)^2$, $t = -(k_1-k_1')^2$, $u = -(k_1-k_2')^2$. Properties:

$$s+t+u = m_1^2+m_2^2+m_3^2+m_4^2$$

In the center of momentum frame, s is the total energy:

$$s = (\omega_1+\omega_2)^2$$

Computing cross-sections

Go to the frame in which particle 1 is at rest. We can write the interaction rate as $n_2|\mathbf{v_1}-\mathbf{v_2}|\sigma$. Imagine putting the experiment in a box on finite volume V, and doing it an a finite time t. We will discover that the cross-section is independent of V and t, but these variables will be helpful along the way. We can look first at a differential cross-section:

$$d\sigma|\mathbf{v_1}-\mathbf{v_2}|\frac{1}{V} = \text{rateno.ofoutgoingstates}$$

Let's figure out the dimensions of $\mathcal{M}$. Action is dimensionless, mass is same dimension as inverse length. Looking at the expression for action, we see that Φ must have dimensions of mass. So g also has dimension of mass. So $\mathcal{M}$ is dimensionless! This only happens for 2-2 scattering. Note the way we normalized $\langle k|k'\rangle = \omega\delta^{(3)}$. So the scattering rate is:

$$\frac{1}{T} \frac{\left[(2\pi)^4\delta^{(4)}(k_1+k_2-k_1'-k_2')i\mathcal{M}\right]^2}{{}_{OUT}\langle k_1'k_2'|k_1'k_2'\rangle_{OUT}\ {}_{IN}\langle k_1k_2|k_1k_2\rangle_{IN}}$$

The number of outgoing states is:

$$\frac{d^3k_1'}{(2\pi)^3/V}\frac{d^3k_2'}{(2\pi)^3/V}$$

The result is (see the book for details):

$$d\sigma = \frac{1}{4\omega_1\omega_2|\mathbf{v_1} - \mathbf{v_2}|}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(k_1 + k_2 - k'_1 - k'_2)\frac{d^3k'_1}{(2\pi)^3 2\omega'_1}\frac{d^3k'_2}{(2\pi)^3 2\omega'_2}$$

This result works for any 2-2 scattering, not only ϕ^3. If we consider 2 particles colliding along the z axis, the prefactor is invariant under boosts in the z direction. The other stuff is invariant in general.

Look at a collision in the COM frame; we get:

$$d\sigma = \frac{1}{64\pi^2 s}\frac{|\mathbf{k'_1}|}{|\mathbf{k_1}|}|\mathcal{M}|^2 d\Omega$$

Scattering amplitude is, in general, angle-dependent.

$$\sigma_{TOTAL} = \frac{1}{2}\int d\sigma$$

The factor in front of the integral repsesints permutations of identical particles: $\Pi n_i!$.

In the COM frame:

$$s = (\omega_1 + \omega_2)^2 = 4(\mathbf{k}_1^2 + m^2) \geq 4m^2$$

$$t = -(k_1 - k_2)^2 = -|k_1|^2 - |k_2|^2 + 2\mathbf{k_1}\mathbf{k_2} = -2(\frac{s}{4} - m^2)(1 - \cos\theta) = -2|\mathbf{k_1}|^2(1 - \cos\theta)$$

In the limit $s >> m^2$, the scattering amplitude is:

$$i\mathcal{M} = \frac{(ig)^2}{i}\left[\frac{1}{-s} + \frac{1}{2s/4(1 - \cos\theta)} + \frac{1}{2s/4(1 + \cos\theta)}\right] = \frac{(ig)^2}{is}\frac{3 + \cos^2\theta}{\sin^2\theta}$$

Note that this is inversely proportional to center of mass energy, and that it favors head-on scattering. The decrease woth energy is related to the fact that the exponent of ϕ is smaller than the number of spacetime dimensions.

Higher (d) spacetime dimensions

reference: Srednicki chapter 12

$$S = \int d^dx\left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3\right)$$

Now ϕ has dimension mass$^{(d-2)/2)}$. g has dimension mass$^{(6-d)/2}$. Note that g is dimensionless if $d = 6$. We care about this because, since $\mathcal{M}$ is dimensionless, this allows it to not depend on the energy. We would like $\mathcal{M}$ to decrease as energy increases, so that it is renormalizable. So let's see how everything changes if we do it in dD. We always think about 1 time dimension and d-1 space, because otherwise it's just weird. Check George Sternham(?) paper on 2 time dimensions. We will always work with d, and not any number in particular, because we will do tricks with d in order to avoid divergence issues. We will work in a different d, and then analytically continue to the dimension of interest.

Propagator stuff

reference: Srednicki ch. 13

The full propagator for $x^0 > y^0$:

$$\frac{1}{i}\Delta(x-y) = \langle 0|\hat{\phi}_x\hat{\phi}_y|0\rangle = \sum_n \langle 0|\hat{\phi}_x|n\rangle\langle n|\hat{\phi}_y|0\rangle$$

We use the fact that:

$$\hat{\phi}(x) = e^{-iPx}\hat{\phi}(0)e^{iPx}$$

$$\frac{1}{i}\Delta(x-y) = \sum_n e^{ik_n(x-y)} \left|\langle 0|\hat{\phi}(0)|n\rangle\right|^2$$

We introduce a d-dimensional delta function:

$$\frac{1}{i}\Delta(x-y) = \int d^d p e^{iP(x-y)} \sum_n \left|\langle 0|\hat{\phi}(0)|n\rangle\right|^2 \delta^{(d)}(p-k_n)$$

Notation:

$$\frac{1}{(2\pi)^{d-1}}\theta(p^0>0)\rho_{tot}(-p^2) = \sum_n \left|\langle 0|\hat{\phi}(0)|n\rangle\right|^2 \delta^{(d)}(p-k_n)$$

$$\rho_{tot}(-p^2) = \int_0^\infty d\mu^2 \rho_{tot}(\mu^2)\delta(p^2+\mu^2)$$

So putting everything together:

$$\frac{1}{i}\Delta(x-y) = \int_0^\infty d\mu^2 \rho_{tot}(\mu^2)\int \frac{d^d p}{(2\pi)^{d-1}}\theta(p^0>0)e^{iP(x-y)}\delta(p^2+\mu^2)$$

$$\langle 0|T\hat{\phi}_x\hat{\phi}_y|0\rangle = \int_0^\infty d\mu^2 \rho_{tot}(\mu^2)\left[\theta(x^0>y^0)\int \frac{d^{d-1}p}{(2\pi)^{d-1}2\omega_p}e^{ip(x-y)} + \theta(x^0<y^0)...\right]$$

We recognize the stuff in the Bracket as the free Feynman propagator:

$$\langle 0|T\hat{\phi}_x\hat{\phi}_y|0\rangle = \int_0^\infty d\mu^2 \rho_{tot}(\mu^2)\frac{1}{i}\int \frac{d^d p}{(2\pi)^p}\frac{e^{ip(x-y)}}{p^2+\mu^2-i\epsilon}$$

In general, we expect for $\rho_{tot}(\mu^2)$ to have a delta function for m^2 , then the next nonzero value at $4m^2$, then a finite continuum that goes to 0 at ∞. If we have a theory capable of bound states, we can have some discrete nonzero values btn m^2 and $4m^2$.