# QM for Mathematicians HW7

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## Problem 1

Let's first find the commutation relations for $L_{jk}$. Note that we can express $L_{jk}$ as outer products of vectors $|e_j\rangle$ in the usual basis for $M(n)$:

$$L_{jk} = -|e_j\rangle\langle e_k| + |e_k\rangle\langle e_j|$$

Now the commutator is:

$$[L_{jk}, L_{mp}] = (-|e_j\rangle\langle e_k|+|e_k\rangle\langle e_j|)(-|e_m\rangle\langle e_p|+|e_p\rangle\langle e_m|)-(-|e_m\rangle\langle e_p|+|e_p\rangle\langle e_m|)(-|e_j\rangle\langle e_k|+|e_k\rangle\langle e_j|)$$

Using the fact that $\langle e_j|e_k\rangle = \delta_{jk}$ we get:

$$[L_{jk}, L_{mp}] = |e_j\rangle\langle e_p|\delta_{km} - |e_j\rangle\langle e_m|\delta_{kp} - |e_k\rangle\langle e_p|\delta_{jm} + |e_k\rangle\langle e_m|\delta_{jp}-$$
$$-|e_m\rangle\langle e_k|\delta_{jp} + |e_m\rangle\langle e_j|\delta_{kp} + |e_p\rangle\langle e_k|\delta_{jm} - |e_p\rangle\langle e_j|\delta_{km}$$
$$[L_{jk}, L_{mp}] = -L_{jp}\delta_{km} + L_{jm}\delta_{kp} + L_{kp}\delta_{jm} - L_{km}\delta_{jp}$$

Since $L_{jk} = -L_{kj}$, we can rewrite this for future convenience:

$$[L_{jk}, L_{mp}] = -L_{jp}\delta_{mk} - L_{mj}\delta_{kp} + L_{kp}\delta_{jm} + L_{mk}\delta_{jp}$$

For the generators of $\mathfrak{sp}$(n):

$$[1/2\gamma_j\gamma_k, 1/2\gamma_m\gamma_p] = \frac{1}{4}\gamma_j\gamma_k\gamma_m\gamma_p - \frac{1}{4}\gamma_m\gamma_p\gamma_j\gamma_k$$

Using $\gamma_j\gamma_k = -\gamma_k\gamma_j + 2\delta_{jk}$ twice in the second term, we bring $\gamma_m$ to the front. Then, using it two more times, we bring $\gamma_p$ to the second position.

$$[1/2\gamma_j\gamma_k, 1/2\gamma_m\gamma_p] = \frac{1}{4}\gamma_m\gamma_j\gamma_k\gamma_p - \frac{1}{4}\gamma_m\gamma_p\gamma_j\gamma_k - \frac{1}{2}\gamma_j\gamma_p\delta_{mk} + \frac{1}{2}\gamma_k\gamma_p\delta_{jm}$$

$$[1/2\gamma_j\gamma_k, 1/2\gamma_m\gamma_p] = \frac{1}{4}\gamma_m\gamma_p\gamma_j\gamma_k - \frac{1}{4}\gamma_m\gamma_p\gamma_j\gamma_k - \frac{1}{2}\gamma_j\gamma_p\delta_{mk} + \frac{1}{2}\gamma_k\gamma_p\delta_{jm} - \frac{1}{2}\gamma_m\gamma_j\delta_{kp} + \frac{1}{2}\gamma_m\gamma_k\delta_{jp}$$

The first two terms cancel out and we are left with an expression analogous with the one we got for the generators of $\mathfrak{so}$(n):

$$[1/2\gamma_j\gamma_k, 1/2\gamma_m\gamma_p] = -\frac{1}{2}\gamma_j\gamma_p\delta_{mk} - \frac{1}{2}\gamma_m\gamma_j\delta_{kp} + \frac{1}{2}\gamma_k\gamma_p\delta_{jm} + \frac{1}{2}\gamma_m\gamma_k\delta_{jp}$$

Comparing this with the above expression for the commutator of $L_{jk}$, we see that they are identical.

## Problem 2

We use the fact, proved in the notes, that:

$$e^{\theta/2\gamma_j\gamma_k} = \cos(\theta/2) + \gamma_j\gamma_k \sin(\theta/2)$$

We have:

$$e^{-\theta/2\gamma_j\gamma_k}(v_j\gamma_j+v_k\gamma_k)e^{\theta/2\gamma_j\gamma_k} = [\cos(\theta/2) - \gamma_j\gamma_k\sin(\theta/2)]\,(v_j\gamma_j+v_k\gamma_k)\,[\cos(\theta/2) + \gamma_j\gamma_k\sin(\theta/2)] =$$

$$= \cos^2(\theta/2)v_j\gamma_j + \cos(\theta/2)\sin(\theta/2)v_j\gamma_j^2\gamma_k + \cos^2(\theta/2)v_k\gamma_k + \cos(\theta/2)\sin(\theta/2)v_k\gamma_k\gamma_j\gamma_k -$$

$$- \cos(\theta/2)\sin(\theta/2)v_j\gamma_j\gamma_k\gamma_j - \sin^2(\theta/2)v_j\gamma_j\gamma_k\gamma_j^2\gamma_k - \sin(\theta/2)\cos(\theta/2)v_k\gamma_j\gamma_k^2 - \sin^2(\theta/2)v_k\gamma_j\gamma_k^2\gamma_j\gamma_k$$

Now we can exploit the fact that $j \neq k$ to use $\gamma_j^2 = \gamma_k^2 = 1$ and $\gamma_j\gamma_k = -\gamma_k\gamma_j$, and get:

$$\cos^2(\theta/2)v_j\gamma_j + \cos(\theta/2)\sin(\theta/2)v_j\gamma_k + \cos^2(\theta/2)v_k\gamma_k - \cos(\theta/2)\sin(\theta/2)v_k\gamma_j +$$

$$+ \cos(\theta/2)\sin(\theta/2)v_j\gamma_k - \sin^2(\theta/2)v_j\gamma_j - \sin(\theta/2)\cos(\theta/2)v_k\gamma_j - \sin^2(\theta/2)v_k\gamma_k =$$

$$= \cos(\theta)v_j\gamma_j + \sin(\theta)v_j\gamma_k + \cos(\theta)v_k\gamma_k - \sin(\theta)v_k\gamma_j =$$

$$= [\cos(\theta)v_j - \sin(\theta)v_k]\gamma_j + [\sin(\theta)v_j + \cos(\theta)v_k]\gamma_k$$

So conjugation by $\exp[\theta/2\gamma_j\gamma_k]$ gives a rotation by angle $\theta$ in the j-k plane.

## Problem 3

Consider first the case $n = 4$, i.e. two fermionic variables. We have 4 basis elements for the spinors: $|0\rangle, a_1^\dagger|0\rangle, a_2^\dagger|0\rangle, a_1^\dagger a_2^\dagger|0\rangle$. Since all these are energy eigenstates, the action that the Hamiltonian generates on them is:

$$e^{iH\theta}|0\rangle = e^{iE_0\theta}|0\rangle = e^{-i\theta}|0\rangle$$

$$e^{iH\theta}a_1^\dagger|0\rangle = e^{iE_1\theta}a_1^\dagger|0\rangle = a_1^\dagger|0\rangle$$

$$e^{iH\theta}a_2^\dagger|0\rangle = e^{iE_2\theta}a_2^\dagger|0\rangle = a_2^\dagger|0\rangle$$

$$e^{iH\theta}a_1^\dagger a_2^\dagger|0\rangle = e^{iE_{12}\theta}a_1^\dagger a_2^\dagger|0\rangle = e^{i\theta}a_1^\dagger a_2^\dagger|0\rangle$$

Therefore, on a generic spinor the action is:

$$e^{iH\theta} \rightarrow \begin{pmatrix} e^{-i\theta} & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{i\theta} \end{pmatrix}$$

Which is a unitary representation. The Hamiltonian also generates an action on vectors $v_1\gamma_1 + v_2\gamma_2 + v_3\gamma_3 + v_4\gamma_4$ by conjugation. To see this, we first express H in terms of $\gamma_j$:

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 - 1 = \frac{1}{4}(\gamma_1 + i\gamma_2)(\gamma_1 - i\gamma_2) + \frac{1}{4}(\gamma_3 + i\gamma_4)(\gamma_3 - i\gamma_4) - 1$$

$$H = \frac{i}{4}(\gamma_2\gamma_1 - \gamma_1\gamma_2 + \gamma_4\gamma_3 - \gamma_3\gamma_4) = i\left(\frac{\gamma_2\gamma_1}{2} + \frac{\gamma_4\gamma_3}{2}\right)$$

Then we can conjugate any vector by:

$$e^{-iH\theta}(v_1\gamma_1+v_2\gamma_2+v_3\gamma_3+v_4\gamma_4)e^{iH\theta} = e^{-\theta/2\gamma_1\gamma_2}e^{-\theta/2\gamma_3\gamma_4}(v_1\gamma_1+v_2\gamma_2+v_3\gamma_3+v_4\gamma_4)e^{\theta/2\gamma_1\gamma_2}e^{\theta/2\gamma_3\gamma_4}$$

We were able to factor the exponential because $[\gamma_2\gamma_1, \gamma_4\gamma_3] = 0$. Using the result of problem 2, note that this represents two rotations, one in the 1-2 plane and the other in the 3-4 plane:

$$[\cos\theta v_1 - \sin\theta v_2]\gamma_1 + [\sin\theta v_1 + \cos\theta v_2]\gamma_2 + [\cos\theta v_3 - \sin\theta v_4]\gamma_3 + [\sin\theta v_3 + \cos\theta v_4]\gamma_4$$

For a rotation in the j-k plane, problem 1 shows that we need to consider the element $\frac{1}{2}\gamma_j\gamma_k$. We will work with two cases. The first is when $\gamma_j$ and $\gamma_k$ correspond to the same copy of $\mathbb{C}$, i.e. they are the "coordinate" and "momentum" of the same fermionic variable. We choose $\gamma_1$ and $\gamma_2$ to illustrate this situation. The second case is when $\gamma_j$ and $\gamma_k$ correspond to different fermionic variables, and we will treat $\gamma_1$ and $\gamma_3$. Thus, for the first case:

$$e^{\theta/2\gamma_1\gamma_2} = \cos(\theta/2) + \sin(\theta/2)\gamma_1\gamma_2 = \cos(\theta/2) + \sin(\theta/2)i(a_1 + a_1^\dagger)(a_1 - a_1^\dagger)$$

$$e^{\theta/2\gamma_1\gamma_2} = \cos(\theta/2) + i\sin(\theta/2)(2N_1 - 1)$$

Where $N_1$ is the number of fermions of type 1. So the action on spinors is:

$$e^{\theta/2\gamma_1\gamma_2} \to \begin{pmatrix} e^{-\theta/2} & & & \\ & e^{-\theta/2} & & \\ & & e^{\theta/2} & \\ & & & e^{-\theta/2} \end{pmatrix}$$

For the second case:

$$e^{\theta/2\gamma_1\gamma_2} = \cos(\theta/2) + \sin(\theta/2)(a_1a_3 + a_1a_3^\dagger + a_1^\dagger a_3 + a_1^\dagger a_3^\dagger)$$

Thus the action is:

$$e^{\theta/2\gamma_1\gamma_3} \to \begin{pmatrix} \cos(\theta/2) & & & \sin(\theta/2) \\ & \cos(\theta/2) & \sin(\theta/2) & \\ & \sin(\theta/2) & \cos(\theta/2) & \\ \sin(\theta/2) & & & \cos(\theta/2) \end{pmatrix}$$

Which is not unitary.

Now let's do the same calculation for $n = 6$ (3 fermionic variables). The action that the

Hamiltonian generates on the 8 basis elements is:

$$e^{iH\theta} \to \begin{pmatrix} e^{-3\theta/2} & & & & & & & \\ & e^{-\theta/2} & & & & & & \\ & & e^{-\theta/2} & & & & & \\ & & & e^{-\theta/2} & & & & \\ & & & & e^{\theta/2} & & & \\ & & & & & e^{\theta/2} & & \\ & & & & & & e^{\theta/2} & \\ & & & & & & & e^{3\theta/2} \end{pmatrix}$$

To see the action by conjugation of H on vectors, we write it in terms of $\gamma_j$:

$$H = i\left(\frac{\gamma_2\gamma_1}{2} + \frac{\gamma_4\gamma_3}{2} + \frac{\gamma_6\gamma_5}{2}\right)$$

$$e^{-iH\theta}(v_1\gamma_1 + v_2\gamma_2 + v_3\gamma_3 + v_4\gamma_4 + v_5\gamma_5 + v_6\gamma_6)e^{iH\theta} =$$
$$= [\cos\theta v_1 - \sin\theta v_2]\gamma_1 + [\sin\theta v_1 + \cos\theta v_2]\gamma_2 + [\cos\theta v_3 - \sin\theta v_4]\gamma_3 + [\sin\theta v_3 + \cos\theta v_4]\gamma_4 +$$
$$+[\cos\theta v_5 - \sin\theta v_6]\gamma_5 + [\sin\theta v_5 + \cos\theta v_6]\gamma_6$$

For rotations in the j-k plane, we work once again with two cases. For $\gamma_1$ and $\gamma_2$ we have:

$$e^{\theta/2\gamma_1\gamma_2} = \cos(\theta/2) + i\sin(\theta/2)(2N_1 - 1)$$

$$e^{\theta/2\gamma_1\gamma_2} \to \begin{pmatrix} e^{-\theta/2} & & & & & & & \\ & e^{\theta/2} & & & & & & \\ & & e^{-\theta/2} & & & & & \\ & & & e^{\theta/2} & & & & \\ & & & & e^{-\theta/2} & & & \\ & & & & & e^{\theta/2} & & \\ & & & & & & e^{-\theta/2} & \\ & & & & & & & e^{\theta/2} \end{pmatrix}$$

For the $\gamma_1$ and $\gamma_3$ case:

$$e^{\theta/2\gamma_1\gamma_2} = \cos(\theta/2) + \sin(\theta/2)(a_1a_3 + a_1a_3^\dagger + a_1^\dagger a_3 + a_1^\dagger a_3^\dagger)$$

The action on each basis element is:

$$e^{\theta/2\gamma_1\gamma_3}|0\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)a_1^\dagger a_3^\dagger|0\rangle$$

$$e^{\theta/2\gamma_1\gamma_3}a_1^\dagger|0\rangle = \cos(\theta/2)a_1^\dagger|0\rangle + \sin(\theta/2)a_3^\dagger|0\rangle$$
$$e^{\theta/2\gamma_1\gamma_3}a_2^\dagger|0\rangle = \cos(\theta/2)a_2^\dagger|0\rangle + \sin(\theta/2)a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle$$

$$
\begin{aligned}
e^{\theta/2\gamma_1\gamma_3}a_3^\dagger|0\rangle &= \cos(\theta/2)a_3^\dagger|0\rangle + \sin(\theta/2)a_1^\dagger|0\rangle \\
e^{\theta/2\gamma_1\gamma_3}a_1^\dagger a_2^\dagger|0\rangle &= \cos(\theta/2)a_1^\dagger a_2^\dagger|0\rangle + \sin(\theta/2)a_2^\dagger a_3^\dagger|0\rangle \\
e^{\theta/2\gamma_1\gamma_3}a_1^\dagger a_3^\dagger|0\rangle &= \cos(\theta/2)a_1^\dagger a_3^\dagger|0\rangle + \sin(\theta/2)|0\rangle \\
e^{\theta/2\gamma_1\gamma_3}a_2^\dagger a_3^\dagger|0\rangle &= \cos(\theta/2)a_2^\dagger a_3^\dagger|0\rangle + \sin(\theta/2)a_1^\dagger a_2^\dagger|0\rangle \\
e^{\theta/2\gamma_1\gamma_3}a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle &= \cos(\theta/2)a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle + \sin(\theta/2)a_2^\dagger|0\rangle
\end{aligned}
$$

$$
e^{\theta/2\gamma_1\gamma_3} \to \begin{pmatrix}
\cos(\theta/2) & & & & & \sin(\theta/2) & & \\
& \cos(\theta/2) & & \sin(\theta/2) & & & & \\
& & \cos(\theta/2) & & & & & \sin(\theta/2) \\
& \sin(\theta/2) & & \cos(\theta/2) & & & & \\
& & & & \cos(\theta/2) & & \sin(\theta/2) & \\
& & & & \sin(\theta/2) & \cos(\theta/2) & & \\
\sin(\theta/2) & & & & & & \cos(\theta/2) & \\
& & \sin(\theta/2) & & & & & \cos(\theta/2)
\end{pmatrix}
$$

Which is not only nonunitary, but also quite ugly.

## Problem 4

In class, starting from the Grassman algebra we define "complex" variables $\psi_j = \theta_1 + i\theta_2$. Then the operators $\psi_j$ and $\frac{\partial}{\partial \psi_j}$, acting on polynomials of $\psi_1, ..., \psi_m$ satisfy:

$$\left\{ \frac{\partial}{\partial \psi_i}, \psi_j \right\}_+ = \delta_{ij}$$

$$\{\psi_i, \psi_j\}_+ = 0 = \left\{ \frac{\partial}{\partial \psi_i}, \frac{\partial}{\partial \psi_j} \right\}_+$$

So they are good classical analogs of $a_F^\dagger$ and $a_F$. The classical spinors that they act on are all polynomials in $\psi_j$, which are spanned by the basis:

$$1, \psi_i, \psi_i\psi_j, ..., \psi_1\psi_2...\psi_m$$

The analog of the Bargmann-Fock inner product we can define for these polynomials is:

$$\langle f_1|f_2\rangle = \int \overline{f_1(\psi)} f_2(\psi) e^{\overline{\psi}\psi} d\psi d\overline{\psi}$$

Where the integral used is the Berezin integral. We can extend this to m variables as:

$$\langle f_1|f_2\rangle = \int \overline{f_1(\psi_1, ..., \psi_m)} f_2(\psi_1, ..., \psi_m) e^{\sum \overline{\psi}_j \psi_j} \Pi d\psi_j d\overline{\psi}_j$$

In order to see that $\psi_j$ and $\frac{\partial}{\partial \psi_j}$ are adjoints with respect to this inner product, we have to show that:

$$\langle f_1|\psi_j|f_2\rangle = \langle f_2|\partial/\partial\psi_j|f_1\rangle^*$$

Let's proceed from the RHS:

$$\langle f_2|\partial/\partial\psi_i|f_1\rangle^* = \left(\int \overline{f_2(\psi_1,...,\psi_m)}\frac{\partial}{\partial\psi_i}f_1(\psi_1,...,\psi_m)e^{\Sigma\overline{\psi}_j\psi_j}\Pi d\psi_j d\overline{\psi}_j\right)^* =$$

$$= \int f_2(\psi_1,...,\psi_m)\frac{\partial}{\partial\overline{\psi}_i}\overline{f_1(\psi_1,...,\psi_m)}e^{\Sigma\overline{\psi}_j\psi_j}\Pi d\psi_j d\overline{\psi}_j$$

We don't have to worry about the ordering of $\psi_j$ and $\overline{\psi}_j$ after we take a complex conjugate, because they commute. Now we can integrate by parts, which in the Berezin integral does not bring a minus sign:

$$\langle f_2|\partial/\partial\psi_i|f_1\rangle^* = \int f_2(\psi_1,...,\psi_m)\overline{f_1(\psi_1,...,\psi_m)}\frac{\partial}{\partial\overline{\psi}_i}e^{\Sigma\overline{\psi}_j\psi_j}\Pi d\psi_j d\overline{\psi}_j =$$

$$= \int \overline{f_1(\psi_1,...,\psi_m)}f_2(\psi_1,...,\psi_m)\psi_j e^{\Sigma\overline{\psi}_i\psi_j}\Pi d\psi_j d\overline{\psi}_j = \langle f_1|\psi_i|f_2\rangle$$

Which is the desired result.