Commutative algebra HW9

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Problem 2

Show that every hyperelliptic curve is birational to a curve of the form $y^2 = f(x)$ where $f \in k[x]$ is a monic square free polynomial.

Solution
By definition, a hyperelliptic curve has function field:

$$f.f.\left[k(x)[y]/\big(g_2(x)y^2 + g_1(x)y + g_0(x)\big)\right]$$

We assume that g_2, g_1, g_0 have no common factors, otherwise we can just divide by them in the function field. By completing the square we obtain:

$$0 = g_2(x)y^2 + g_1(x)y + g_0(x) = g_2(x)\left[y + \frac{g_1(x)}{2g_2(x)}\right]^2 + g_0(x) - \frac{g_1(x)^2}{4g_2(x)}$$

$$0 = \left[2g_2(x)y + g_1(x)\right]^2 + 4g_2(x)g_0(x) - g_1(x)^2$$

Therefore we can use the rational map:

$$(x, y) \mapsto (x, 2g_2(x)y + g_1(x))$$

To map our curve into the one given by $y^2 + 4g_2(x)g_0(x) - g_1(x)^2 = 0$. The map is birational because its inverse is:

$$(x, y) \mapsto \left(x, \frac{y - g_1(x)}{2g_2(x)}\right)$$

Moreover, the polynomial $4g_2(x)g_0(x) - g_1(x)^2$ is squarefree by the assumption that g_2, g_1, g_0 have no common factors.

Problem 3

Conversely, show that every square free $f \in k[x]$ gives rise to a hyperelliptic curve in this way.

Solution
The curve given by $y^2 = f(x)$ has function field:

$$f.f\left[k(x)[y]/\big(y^2 - f(x)\big)\right]$$

Which is a degree 2 extension of the purely transcendental extension $k(x)$.

Problem 4

Give an example to show that two distinct monic square free $f \in k[x]$ can lead to isomorphic curves (for us this means that the function fields are isomorphic as extensions of k).

Solution
Just consider $x \mapsto x - 1$, whereby $f(x)$ becomes $f(x-1)$. This is obviously an isomoprhism and simply translates the curve by a unit.

Problem 5

Given a hyperelliptic curve $C : y^2 = f(x)$ as above let D be the zero divisor of x on C. The degree of D is 2. Show that $l(D) = 2$ if $g > 0$.

Solution
We note first that:

$$2 \leq l(D) \leq 3$$

The first inequality is true because $1, \frac{1}{x}$ are linearly independent functions in $L(D)$. The second one follows from Lemma 78 proved in class. Now we want to show that $l(D) = 3$ leads to $g = 0$, a contradiction. We use the fact that $D = P + Q$, where P, Q are two places, not necessarily distinct. Then $D \geq P$, so $l(P) \geq l(D) = 3$. This means that there exists some nonconstant function f such that $f \in L(P)$. Then, for $n > 0$, $\{1, f, \ldots, f^n\} \in L(nP)$, so $l(nP) \geq n + 1$. But applying Riemann-Roch to nP gives:

$$l(nP) - l(K - nP) = n - g + 1$$

By making n large enough, we can ensure that $l(K - nP) = 0$, since $L(K - nP)$ would require functions which have more zeros than poles. Then we have:

$$n + 1 \leq l(nP) = n - g + 1$$

So $g = 0$ as desired.

Problem 6

Show that a curve C which has a divisor D with deg$(D) = 2$ and $l(D) = 2$ is hyperelliptic (we may discuss this in class).

Solution
A divisor D with degree $d = 2$ gives a map:

$$\phi_D : C \to \mathbb{P}^{l(D)-1} = \mathbb{P}$$

That has degree $d = 2$. This means that the function field K of C is a degree 2 extension of the function field $k(t)$ of $\mathbb{P}$. Then C is a hyperelliptic curve.

Problem 7

Let $C : y^2 = f(x)$ as above. Consider the differential form $\omega = dx$. Compute its zeros and poles on C and as a consequence compute the genus of C. (The cases $\deg(f)$ *even or odd are slightly different. Just do one of the two cases.)*

Solution