# Commutative algebra HW1

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September 17, 2013

## Problem 1

Let $A = k[x,y]/(x^2y^4 - x^4y^2 + 1)$ where $k$ is a field. Construct a $k$-algebra map as in Noether Normalization for $A$. Very briefly explain why it works.

*Solution*
As in the proof of Noether Normalization, we want to make a change of variables $(x, y) \to (z, y)$ such that $y$ is integral over $k[z]$. We let $z = x - y^2$, and it turns out that the exponent 2 is large enough in order to make the resulting polynomial monic:

$$x^2y^4 - x^4y^2 + 1 \to 1 + y^8 - y^{10} + 2y^6z - 4y^8z + y^4z^2 - 6y^6z^2 - 4y^4z^3 - y^2z^4$$

Then by Lemma 1 proved in class $\phi : k[z] \to k[x,y]/(x^2y^4 - x^4y^2 + 1)$ defined by $\phi(z) = x - y^2$ is a finite map.

## Problem 2

Describe all prime ideals of $\mathbb{C}[x,y]/(xy)$ where $\mathbb{C}$ is the field of complex numbers. Just list them in some way and explain briefly why they are primes and why you've got all of them.

*Solution*
As discussed in class, there is a 1-1 correspondence between ideals of $\mathbb{C}[x,y]/(xy)$ and ideals of $\mathbb{C}[x,y]$ that contain $(xy)$. Moreover, if one of the latter is prime, then so is its correspondent. We will therefore consider prime ideals in $\mathbb{C}[x][y] = \mathbb{C}[x,y]$. Since $\mathbb{C}[x]$ is a PID, its prime ideals are of 3 types:

1) the 0 ideal, which does not contain $(xy)$.
2) principal ideals $\big(f(y)\big)$ generated by a polynomial $f$ that is irreducible in $\mathbb{C}[x]$. The only such ideal that generates $(xy)$ is $(y)$.
3) $\big(p, f(y)\big)$ where $p$ is prime in $\mathbb{C}[x]$ and $f$ is irreducible in $\mathbb{C}[x]/(p)$. The only such ideals that generate $(xy)$ are $(x - \lambda, y), (x, y - \mu)$ and $(x)$, for all $\lambda, \mu \in \mathbb{C}$.

It follows that all prime ideals of $\mathbb{C}[x,y]/(xy)$ are $\big(x+(xy)\big)$, $\big(y+(xy)\big)$, $\big(x-\lambda+(xy), y+(xy)\big)$, $\big(x + (xy), y - \mu\big)$.

## Problem 3

Let $k$ be a field. Prove that $k[x,y]$ is not isomorphic to $k[x,y,z]$.

*Solution*
Assume the two rings are isomorphic; then obviously the isomorphism is a finite map. Then there exist $n$ generators that generate $k[x,y,z]$ as a $k[x,y]$ - module. This construction must span $z, z^2, ..., z^{n+1}$, all of which are linearly independent over $k[x,y]$. But this means that at least $n+1$ generators are required, which is a contradiction.

## Problem 4

Let $k$ be a field. Suppose $A$ is a $k$-algebra and $f$ is a nonzerodivisor of $A$ such that $k[x,y]$ is isomorphic to $A_f$ as $k$-algebra. Show that $A$ is isomorphic to $k[x,y]$.

*Solution*
Since isomorphisms preserve units, $f/1$ and $1/f$ are units in $k[x,y]$, therefore $f/1, 1/f \in k$. Since every map from a field to a ring is injective, the ring map $k \to A$ is an inclusion map, and therefore $f, f^{-1} \in A$. This means that localization does nothing to $A$, since we are inverting something which is already invertible. Thus $A \cong A_f \cong k[x,y]$.

## Problem 5

Let $A$ be a ring and let $f$ be an element of $A$. Show that $A_f$ is as an $A$-algebra isomorphic to $A[x]/(fx-1)$.

*Solution* Consider the following maps:

$$\begin{tikzcd} A \arrow[d, "\phi"'] \arrow[dr, "g"] & \\ A_f & A[x]/(fx-1) \end{tikzcd}$$

$\phi(a) = a/1$ and $g(a) = a$. The by the universality of localizations (see for example corollary 3.2 in Atiyah - Macdonald) there exists and isomorphism $h : A_f \to A[x]/(fx-1)$. The diagram above is easily shown to satisfy the requirements for the universality theorem:
(a) $s \in \{f, f^2, ...\} \Rightarrow g(s)$ is a unit in $A[x]/(fx-1)$
(b) $g(a) = 0 \Rightarrow as = 0$ for some $s \in \{f, f^2, ...\}$
(c) every element of $A[x]/(fx-1)$ is of the form $g(a)\big(g(s)\big)^{-1}$
The first two are obvious, the third follows since $X = g(f)^{-1}$.