# Lie groups HW5

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## Problem 1 (Kirillov 5.2)

We plan to use the result of exercise 4.5 in Kirillov, namely that the space of $\mathfrak{g}$ invariant bilinear forms on an irrep of $\mathfrak{g}$ is 1-dimensional. We begin, then, by showing that the adjoint representation of $\mathfrak{sl}(n, \mathbb{C})$ is irreducible. Assume the contrary, i.e. that there exists a subspace of $\mathfrak{sl}(n, \mathbb{C})$ that is invariant under the adjoint action of $\mathfrak{sl}(n, \mathbb{C})$. Then this subspace is an ideal. But $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra, so the ideal must be trivial. Therefore the adjoint representation is irreducible, and we can use exercise 4.5 to obtain that:

$$K(x, y) = c\,\mathrm{Tr}(xy)$$

Where $c \in \mathbb{C}$. To determine the constant, we take $x = y = h_1$ in the equation above, where $h_1$ is the basis element of $\mathfrak{sl}(n, \mathbb{C})$ having 1 on the $(1, 1)$ position and $-1$ on the $(2, 2)$ position. We obtain $K(h_1, h_1) = 2c$. Now we need to compute the matrix $\mathrm{ad}\, h_1$. First note that $h_1$ commutes with all other diagonal elements $h_i$. We need only consider its action on linear combinations on $e_i, f_i$. For this, write such a linear combination in the defining representation of $\mathfrak{sl}(2, \mathbb{C})$ in terms of $\mathbb{C}$ coefficients $x_i, y_j$:

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ y_1 & & x_4 & x_5 \\ y_2 & y_3 & & x_6 \\ y_4 & y_5 & y_6 & \end{pmatrix}$$

(We use $n = 4$ in order to simplify notation, but it will be obvious that the case for any $n$ is analogous.) Then the adjoint action is:

$$\begin{pmatrix} & x_1 & x_2 & x_3 \\ y_1 & & x_4 & x_5 \\ y_2 & y_3 & & x_6 \\ y_4 & y_5 & y_6 & \end{pmatrix} \mapsto \begin{pmatrix} & x_1 & x_2 & x_3 \\ -y_1 & & -x_4 & -x_5 \\ & & & \\ & & & \end{pmatrix} - \begin{pmatrix} & -x_1 & & \\ y_1 & & & \\ y_2 & -y_3 & & \\ y_4 & -y_5 & & \end{pmatrix} = \begin{pmatrix} & 2x_1 & x_2 & x_3 \\ -2y_1 & & -x_4 & -x_5 \\ -y_2 & y_3 & & \\ -y_4 & y_5 & & \end{pmatrix}$$

This shows that $\mathrm{ad}\, h_1$ is diagonal and acts as:

$$e_1 \mapsto 2e_1 \quad e_2 \mapsto e_2 \quad e_3 \mapsto e_3 \quad e_4 \mapsto -e_4 \quad e_5 \mapsto -e_5 \quad e_6 \mapsto 0$$

$$f_1 \mapsto -2f_1 \quad f_2 \mapsto -f_2 \quad f_3 \mapsto f_3 \quad f_4 \mapsto -f_4 \quad f_5 \mapsto f_5 \quad f_6 \mapsto 0$$

Generalizing this to arbitrary $n$, it's clear that only the first two rows and first two columns of the commutator will have nonzero elements. Upon squaring, $\operatorname{ad} h_1$, all these elements will be positive. Therefore we have:

$$\operatorname{Tr}(\operatorname{ad} h_1, \operatorname{ad} h_1) = 2(2 + 2(n-1)) = 4n$$

$$4n = 2c \Rightarrow c = 2n$$

$$K(x, y) = 2n \operatorname{Tr}(xy)$$

## Problem 2 (Kirillov 5.3)

1) We need to show that $\mathfrak{g}$ is closed under commutators. Using multiplication rules for block matrices:

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} - \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} =$$

$$\begin{pmatrix} AA' & AB' + BD' \\ 0 & DD' \end{pmatrix} - \begin{pmatrix} A'A & A'B + B'D \\ 0 & D'D \end{pmatrix} =$$

$$\begin{pmatrix} [A, A'] & AB' + BD' - A'B - B'D \\ 0 & [D, D'] \end{pmatrix} \in \mathfrak{g}$$

2) First note that the given subspace of $\mathfrak{g}$, which we denote by $J$, is an ideal, since:

$$\begin{pmatrix} \lambda I & B \\ 0 & \mu I \end{pmatrix}\begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} - \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix}\begin{pmatrix} \lambda I & B \\ 0 & \mu I \end{pmatrix} = \begin{pmatrix} 0 & (\lambda - \mu)B' + BD' - A'B \\ 0 & 0 \end{pmatrix} \in J$$

Moreover, $J$ is solvable, since by computation above taking one commutator only leaves the top right block, which will be killed by taking a second commutator. We know then that $J \subset \operatorname{rad} \mathfrak{g}$. Now let's examine the commutation law proved in 1), in order to see if $\operatorname{rad} \mathfrak{g}$ can be any bigger than $J$. We see that, in order to eventually have 0 on the diagonal, we need commutators of the type $[[[A_1, A_2], [A_3, A_4]], \ldots]$ and $[[[D_1, D_2], [D_3, D_4]], \ldots]$ to eventually vanish. This means that $A, D$ must belong to the radical of $\mathfrak{gl}(k), \mathfrak{gl}(n-k)$ respectively. This means $A = \lambda I$ and $D = \mu I$, because $\mathfrak{gl}(k), \mathfrak{gl}(n-k)$ are reductive. Therefore $J = \operatorname{rad} \mathfrak{g}$.

Two elements of $\mathfrak{g}$ are equivalent in $\mathfrak{g}/\operatorname{rad} \mathfrak{g}$ if:

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} - \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} = \begin{pmatrix} \lambda I & C \\ 0 & \mu I \end{pmatrix}$$

In particular, all $B$ are equivalent to 0, and $A \sim A'$ if they differ by $\lambda I$. This means that each equivalence class contains exactly one matrix with trace 0, and therefore the set of equivalence classes of $A$ is isomorphic to $\mathfrak{sl}(k, \mathbb{C})$. Similarly, the set of equivalence classes of $D$ is isomorphic to $\mathfrak{sl}(n-k, \mathbb{C})$. Then we have:

$$\mathfrak{g}/\operatorname{rad} \mathfrak{g} = \mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{sl}(n-k, \mathbb{C})$$

## Problem 3 (Kirillov 5.4)

We need to show that for all nonzero $x \in \mathfrak{sp}(n, \mathbb{K})$, there exists some $y \in \mathfrak{sp}(n, \mathbb{K})$ such that $\mathrm{Tr}(xy) \neq 0$. For this, we first show that $x \in \mathfrak{sp}(n, \mathbb{K})$ implies $x^\dagger \in \mathfrak{sp}(n, \mathbb{K})$. We take the adjoint of the equation:

$$x + J^{-1} x^T J = 0$$
$$x^\dagger + J^\dagger \bar{x} J^{-1\dagger} = 0$$

Note that $\bar{x} = x^{\dagger T}$ and $J$ satisfies $J^\dagger = J^{-1}$. Therefore:

$$x^\dagger + J^{-1} x^{\dagger T} J = 0$$

Which shows $x^\dagger \in \mathfrak{sp}(n, \mathbb{K})$. Now we can compute:

$$\mathrm{Tr}(xx^\dagger) = \sum_{i,j} x_{ij} \bar{x}^T_{ji} = \sum_{i,j} x_{ij} \bar{x}_{ij} = \sum_{i,j} |x_{ij}|^2$$

Thus, $\mathrm{Tr}(xx^\dagger) = 0$ gives $x = 0$.

## Problem 4 (Kirillov 5.5)

Because of the ad-invariance of the Killing form, we have $\operatorname{ad} X \in \mathfrak{so}(\mathfrak{g})$, for all $X \in \mathfrak{g}$. This means that $\operatorname{ad} X = -(\operatorname{ad} X)^T$, so we have:

$$K(X, X) = \mathrm{Tr}(\operatorname{ad} X, -(\operatorname{ad} X)^T) = -\sum_{i,j} (\operatorname{ad} X)_{ij} (\operatorname{ad} X)^T_{ji} = -\sum_{i,j} [(\operatorname{ad} X)_{ij}]^2$$

$\mathfrak{g}$ is a real Lie algebra, so $\operatorname{ad} X \in \mathrm{End}(\mathfrak{g})$ has real entries. This shows that $K$ is negative definite. But, by hypothesis, $K$ is positive definite, so we must have $\mathfrak{g} = 0$.

## Problem 5 (Kirillov 6.1)

Consider the basis $J_x, J_y, J_z$ for $\mathfrak{so}(3)$. We want to compute the dual basis with respect to the Killing form. Using the commutation relations:

$$[J_i, J_j] = \epsilon_{ij}{}^k J_k$$

We can compute the matrix forms of $\operatorname{ad} J_i$:

$$\operatorname{ad} J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \operatorname{ad} J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \operatorname{ad} J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And we see that:

$$(\operatorname{ad} J_x)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (\operatorname{ad} J_y)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (\operatorname{ad} J_z)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore $K(J_x, J_x) = \mathrm{Tr}(\mathrm{ad}\, J_x)^2 = -2$, and thus $J_x^* = -\frac{1}{2}J_x$. Similarly, $J_y^* = -\frac{1}{2}J_y$ and $J_z^* = -\frac{1}{2}J_z$. Then we have:

$$C = \sum_i J_i^* J_i = -\frac{1}{2}(J_x^2 + J_y^2 + J_z^2)$$