Commutative algebra HW6

Matei Ionita

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Problem 1

Prove that an axiomatic projective plane has the same number of points as lines. (You get extra points for noticing the missing axiom and fixing.)

Solution

This solution is heavily based on Rankeya Datta's proof, which is posted on the REU website. We begin by identifying the missing axiom: (3) There exist at least 4 distinct points such that no 3 of them are collinear. This gets rid of degenerate cases such as a single line incident to infinitely many collinear points.

Let $\mathbb{P}$ be the projective plane, $\mathcal{L}$ the set of lines and $\mathcal{P}$ the set of points. We first show that $\mathcal{L}$ infinite implies $\mathcal{P}$ infinite and viceversa. Assume that $\mathcal{L}$ is infinite, then by (3) and (1) we have 4 points with 4 lines, each line incident on exactly 2 points. If we then add the other lines in $\mathcal{L}$, we create an infinite number of intersection points. Conversely, assume that $\mathcal{P}$ is infinite. If $\mathcal{L}$ were finite, there would exist a line l_0 incident to infinitely many points. But by (3) there exist a point p_0 not incident to l_0. Then (1) tells us there must be infinitely many lines, each passing through p_0 and one of the points incident to l_0. This reduces the problem to 2 cases: both $\mathcal{L}$ and $\mathcal{P}$ are infinite, or both are finite.

We first prove the statement for both infinite. Let $\Delta_{\mathcal{L}}, \Delta_{\mathcal{P}}$ be the diagonals of $\mathcal{L} \times \mathcal{L}$ and $\mathcal{P} \times \mathcal{P}$ respectively. Then we have:

$$|\mathcal{L}| = |\mathcal{L} \times \mathcal{L}| = |\mathcal{L} \times \mathcal{L} - \Delta_{\mathcal{L}}|$$

$$|\mathcal{P}| = |\mathcal{P} \times \mathcal{P}| = |\mathcal{P} \times \mathcal{P} - \Delta_{\mathcal{P}}|$$

Axioms (1), (2) mean that there exist maps:

$$\pi_1 : \mathcal{L} \times \mathcal{L} - \Delta_{\mathcal{L}} \to \mathcal{P}$$

$$(l_1, l_2) \to l_1 \cap l_2$$

$$\pi_2 : \mathcal{P} \times \mathcal{P} - \Delta_{\mathcal{P}} \to \mathcal{L}$$

$$(p_1, p_2) \to \overline{p_1 p_2}$$

Where $\overline{p_1 p_2}$ denotes the line incident to p_1, p_2. If we can show that π_1, π_2 are surjective, we are done, because π_1 surjective implies $|\mathcal{P}| \leq |\mathcal{L}|$ and π_2 surjective implies $|\mathcal{L}| \leq |\mathcal{P}|$. To show that π_1 is surjective, take a point p_1. By (3) there exist points p_2, p_3 such that the three are not collinear. Then $\overline{p_1 p_2}$ and $\overline{p_1 p_3}$ are distinct, and $\overline{p_1 p_2} \cap \overline{p_1 p_2} = p_1$. To show that π_2 is surjective, take a line l. By (3), somewhere in $\mathbb{P}$ there exist 3 points which are noncolinear; then there exist three lines l_1, l_2, l_3 incident to each pair of points. If $l = l_1, l_2$ or l_3 we are done, otherwise (2) says that there exist p_1, p_2 such that $l_1 \cap l = p_1$ and $l_2 \cap l = p_2$. Then $\pi_2(p_1, p_2) = l$. This completes the proof for $\mathcal{P}, \mathcal{L}$ infinite.

We now look at the case when $\mathcal{L}, \mathcal{P}$ are both finite. We first prove two claims, and then show how the proof follows from them. **Claim 1**: Let $\mathcal{L}_p$ denote the set of lines passing through p; then $|\mathcal{L}_p|$ is independent of p. To prove this, it suffices to show that $|\mathcal{L}_p| = |\mathcal{L}_q|$ for two distinct points p, q. By axiom (3) there exists a point $r \in \overline{pq}$ distinct from p, q. Let $l \in \mathcal{L}_p - \{\overline{pq}\}, m \in \mathcal{L}_r - \{\overline{pq}\}$. By (2), l, m are distinct. By (2) again, $l \cap m$ is a single point which is not on $\overline{pq}$. Now let $\mathcal{P}_{\mathbb{P}-\overline{pq}}$ denote the set of points of $\mathbb{P}$ which are not incident to $\overline{pq}$. We have a map:

$$\phi : (\mathcal{L}_p - \{\overline{pq}\}) \times (\mathcal{L}_r - \{\overline{pq}\}) \to \mathcal{P}_{\mathbb{P}-\overline{pq}}$$

$$(l, m) \to l \cap m$$

This is a bijection, because we can write down its inverse:

$$\phi^{-1} : \mathcal{P}_{\mathbb{P}-\overline{pq}} \to (\mathcal{L}_p - \{\overline{pq}\}) \times (\mathcal{L}_r - \{\overline{pq}\})$$

$$s \to (\overline{ps}, \overline{rs})$$

Therefore $(|\mathcal{L}_p| - 1)(|\mathcal{L}_r| - 1) = |\mathcal{P}_{\mathbb{P}-\overline{pq}}|$. Similarly one can show that $(|\mathcal{L}_q| - 1)(|\mathcal{L}_r| - 1) = |\mathcal{P}_{\mathbb{P}-\overline{pq}}|$. From these two equations we get $|\mathcal{L}_p| = |\mathcal{L}_q|$ as desired. Henceforth we denote $|\mathcal{L}_p|$ by c.

Claim 2: let $\mathcal{P}_l$ denote the set of points incident to l; then $|\mathcal{P}_l| = c$ for all l. To prove this, take a point p not incident to l. In particular $l \in \mathcal{L}_p$. Define a map:

$$\psi : \mathcal{L}_p \to \mathcal{P}_l$$

$$m \to l \cap m$$

This is a bijection, since we can write down its inverse:

$$\psi^{-1} : \mathcal{P}_l \to \mathcal{L}_p$$

$$s \to \overline{ps}$$

Therefore $|\mathcal{P}_l| = |\mathcal{L}_p| = c$, as desired.

Now we use these two claims to prove that $|\mathcal{L}| = c^2 - c + 1$ and $|\mathcal{P}| = c^2 - c + 1$. Let p, q be two distinct points, then:

$$\mathcal{P} = \mathcal{P}_{\overline{pq}} \sqcup \mathcal{P}_{\mathbb{P}-\overline{pq}}$$

In the proof of claim 1 we showed that $|\mathcal{P}_{\mathbb{P}-\overline{pq}}| = (|\mathcal{L}_p| - 1)^2 = (c-1)^2$. By claim 2 $|\mathcal{P}_{\overline{pq}}| = c$. Then $|\mathcal{P}| = (c-1)^2 + c = c^2 - c + 1$.

On the other hand, let $l \in \mathcal{L}$ and note that we can write:

$$\mathcal{L} = \left(\bigsqcup_{q \in \mathcal{P}_l} (\mathcal{L}_q - \{l\}) \right) \sqcup \{l\}$$

Because any line distinct from l intersects l in one point. By claim 2, $|\mathcal{P}_l| = c$ and by claim 1, $|\mathcal{L}_q - \{l\}| = c - 1$. This shows $|\mathcal{L}| = c(c-1) + 1$, and we are done.

Problem 3

Show that if P, Q, R are three pairwise distinct points on $\mathbb{P}^1$ then there exists a matrix A which determines a map $\mathbb{P}^1 \to \mathbb{P}^1$ mapping P, Q, R to $(1:0)$, $(0:1)$, and $(1:1)$.

Solution
We first look for a matrix A such that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \end{pmatrix}$$

For some nonzero μ, λ. Solving for the coefficients we get:

$$A = \frac{1}{Q_1 P_0 - Q_o P_1} \begin{pmatrix} -Q_1 \lambda & Q_0 \lambda \\ P_1 \mu & -P_0 \mu \end{pmatrix}$$

Since the points are distinct, the denominator is not 0. Now we impose:

$$\frac{1}{Q_1 P_0 - Q_o P_1} \begin{pmatrix} -Q_1 \lambda & Q_0 \lambda \\ P_1 \mu & -P_0 \mu \end{pmatrix} \begin{pmatrix} R_0 \\ R_1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma \end{pmatrix}$$

Solving for λ, μ in terms of γ we get:

$$A = \gamma \begin{pmatrix} \frac{-Q_1}{Q_0 R_1 - Q_1 R_0} & \frac{Q_0}{Q_0 R_1 - Q_1 R_0} \\ \frac{P_1}{P_0 R_1 - P_1 R_0} & \frac{-P_0}{Q_0 R_1 - Q_1 R_0} \end{pmatrix}$$

As expected, all multiples of the solution are also solutions.

Problem 4

Find a field K and a conic as defined above without any points.

Solution
Let $K = \mathbb{R}$ and:

$$F = (X_0 - 2X_1)^2 + (X_1 - 2X_2)^2 + (X_2 - 2X_0)^2$$

Over $\mathbb{R}$, $F = 0$ iff each term is 0, and this gives us $X_0 = 2X_1 = 4X_2 = 8X_0$, and similarly $X_1 = 8X_1$, $X_2 = 8X_2$. Therefore the only solution is $(0 : 0 : 0)$, which does not belong to $\mathbb{P}$.

Problem 5

Prove that a degree two morphism $P^1 \to P^2$ maps onto either a line or a conic.

Solution
Since the morphisms are degree 2, they are nonconstant, and therefore it suffices to show that they map *into* a conic or line. Also, a line squares to a (reducible) conic, so it suffices to show that morphisms map into an arbitrary conic. For this, we write the morphism as:

$$(a : b) \to (G_1, G_2, G_3)$$

$$G_i(ab) = c_{i1}a^2 + c_{i2}ab + c_{i3}b^2$$

We need to show there exist 6 coefficients α_{ij} such that:

$$\sum_{i \leq j} \alpha_{ij} G_i G_j = 0$$

Writing G_i, G_j explicitly, this condition becomes:

$$f_1(\alpha_{ij}, c_{ij})a^4 + f_2(\alpha_{ij}, c_{ij})a^3 b + f_3(\alpha_{ij}, c_{ij})a^2 b^2 + f_4(\alpha_{ij}, c_{ij})ab^3 + f_5(\alpha_{ij}, c_{ij})b^4 = 0$$

Where the functions f_k are linear in α_{ij}. But a, b are arbitrary and $a^4, a^3 b, \ldots$ are linearly independent, so this is equivalent to $f_k = 0$ for all k. This is then a system of 5 linear equations in 6 variables α_{ij}. Since all constant terms are 0, it is consistent, so it admits a solution. (In fact, we expect it to admit infinitely many, since rescaling all α_{ij} by any nonzero constant produces another solution.)

Problem 6

Let k be an algebraically closed field. Let $k(t)$ be the field of rational functions over k. Let $k(t) \subset K$ be a finite extension. Prove or look up the proof of the following statements: (a) the integral closure of $k[t]$ in K is finite over $k[t]$, (b) for every discrete valuation v on $k(t)$ there are finitely many discrete valuations w_i on K whose restriction to $k(t)$ is $e_i v$ for some integer e_i, and (c) we have $\sum_i e_i = [K : k(t)]$.

Solution

a) $k[t]$ is a UFD, so it is integrally closed over $k(t)$. We need only examine $x \in K - k(t)$ which are integral over $k[t]$. If x satisfies some monic polynomial $f \in k[t][x]$, we can regard $f \in k(t)[x]$, which proves that x is algebraic over $k(t)$. But we know that $K/k(t)$ is finite, so algebraic elements over $k(t)$ form an n-dimensional vector space over $k(t)$, where $n = [K : k(t)]$. By cancelling denominators, the same elements form an n-dimensional module over $k[t]$. Since all x belong to this module, the integral closure is finite over $k[t]$.

b) This is the statement of Lemma 73 proved in class.