QFT Lecture 23

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More on the Dirac eq.

Reference: ch. 36

Recall the Lagrangian for a left-handed Weyl spinor:

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + \frac{1}{2} m\psi\epsilon\psi - \frac{1}{2} m\psi^\dagger \epsilon\psi^\dagger$$

To get to Srendicki's notation, in which a sign differs:

$$\frac{1}{2} m\psi\epsilon\psi - \frac{1}{2} m\psi^\dagger \epsilon\psi^\dagger =$$

$$= \frac{1}{2} m\psi_b \epsilon^{ba} \psi_a - \frac{1}{2} m\psi^\dagger_{\dot{a}} \epsilon^{\dot{a}\dot{b}} \psi^\dagger_{\dot{b}} =$$

$$= -\frac{1}{2} m\epsilon^{ab} \psi_b \psi_a - \frac{1}{2} m\psi^\dagger_{\dot{a}} \epsilon^{\dot{a}\dot{b}} \psi^\dagger_{\dot{b}} =$$

$$= -\frac{1}{2} m\psi^a \psi_a - \frac{1}{2} m\psi^\dagger_{\dot{a}} \psi^{\dagger\dot{a}} =$$

$$= -\frac{1}{2} m\psi\psi - \frac{1}{2} m\psi^\dagger \psi^\dagger$$

Also recall the useful relations:

$$\bar{\sigma}^\mu = (1, -\sigma^\mu)$$

$$\epsilon\sigma_i\epsilon = \sigma_i^T$$

$$\epsilon\sigma^\mu\epsilon = -\bar{\sigma}^\mu$$

For the e.o.m:

$$\frac{\delta S}{\delta \psi^\dagger} = 0 \Rightarrow i\bar{\sigma}^\mu \partial_\mu \psi - m\epsilon\psi^\dagger = 0$$

Let's explicitly compute a term:

$$\frac{\delta(\psi^\dagger_{\dot{a}} \epsilon^{\dot{a}\dot{b}} \psi^\dagger_{\dot{b}})}{\delta \psi^\dagger_{\dot{c}}} = \epsilon^{\dot{c}\dot{b}} \psi^\dagger_{\dot{b}} - \psi^\dagger_{\dot{b}} \epsilon^{\dot{b}\dot{c}} = 2\epsilon^{\dot{c}\dot{b}} \psi^\dagger_{\dot{b}}$$

The two e.o.m. can be combined into the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

Where the dirac spinor and gamma matrices are:

$$\Psi = \begin{pmatrix} \psi \\ \epsilon\psi^\dagger \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Useful to know that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

We used Wigner's trick to obtain spinors. Another route, which also works for dimensions other than 3+1, is based on the porperties of the Clifford algebra. We can form a representation of the Lorentz group by setting:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Then these obey the correct relations for the Lorentz Lie algebra representation:

$$[S^{\mu\nu}, S^{\rho\sigma}] = -i(\eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho})$$

As of now, ψ is its own antiparticle. To describe something like electron-positron, we consider two spinors ψ_1, ψ_2. We should be careful not to confuse these indices with the components of the spinor. Now we write a Lagrangian for the system as the sum of Lagrangians for the two particles:

$$\mathcal{L} = \sum_{i=1}^{2} i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i + \frac{1}{2}m\psi_i\epsilon\psi_i - \frac{1}{2}m\psi_i^\dagger\epsilon\psi_i^\dagger$$

Note that the Lagrangian has an $SO(2)$ symmetry. Rewrite it in terms of $\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$, $\xi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$:

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger\bar{\sigma}^\mu \partial_\mu \xi + m\chi\epsilon\xi - m\xi^\dagger\epsilon\chi^\dagger$$

Two of the e.o.m. are:

$$\frac{\delta S}{\delta\chi^\dagger} \Rightarrow i\bar{\sigma}^\mu \partial_\mu \chi - m\epsilon\xi^\dagger = 0$$

$$\frac{\delta S}{\delta\xi} \Rightarrow i\sigma^\mu \partial_\mu \epsilon\xi^\dagger - m\chi = 0$$

Combining these gives a Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \qquad \Psi = \begin{pmatrix} \chi \\ \epsilon\xi^\dagger \end{pmatrix}$$

The e.o.m. looks nicer and simpler in terms of the Dirac spinor. Similarly, we can also write a simpler Lagrangian in terms of the Dirac spinor:

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \qquad \bar{\Psi} = \Psi^\dagger\gamma^0$$

In this form, the Lagrangian has a $U(1)$ symmetry: $\Psi \to e^{-i\theta}\Psi$. The Noether current:

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi$$

In terms of the two initial fields:

$$j^\mu = \chi^\dagger\bar{\sigma}^\mu\chi - \xi^\dagger\bar{\sigma}^\mu\xi$$

Next semester, we will couple this current to the electromagnetic field:

$$\mathcal{L}_{\text{interaction}} = eA_\mu j^\mu$$

A special case of the Dirac spinor is the Majorana spinor, where $\chi = \xi$:

$$\Psi = \begin{pmatrix} \chi \\ \epsilon\chi^\dagger \end{pmatrix}$$

This can be used to model a Majorana neutrino. Note that the conserved current is 0. Define charge conjugation as the operation that flips χ and ξ:

$$\Psi = \begin{pmatrix} \xi \\ \epsilon\chi^\dagger \end{pmatrix}$$

I.e. a Majorana spinor satisfies $\Psi = \Psi^C$. We will see later that this is the analog of a real scalar field, and not $\Psi = \Psi^\dagger$. We define the charge conjugation operator as:

$$\Psi^C = \mathbb{C}\bar{\Psi}^T$$

Check:

$$\mathbb{C}\bar{\Psi}^T = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} (\Psi^T\gamma^0)^T =$$

$$= \begin{pmatrix} \xi\epsilon^T & \chi^\dagger \end{pmatrix}^T = \begin{pmatrix} \xi \\ \epsilon\chi^\dagger \end{pmatrix}$$

For the Majorana spinor, better to work with the Lagrangian in a form which already incorporates the Majorana condition:

$$\mathcal{L} = \frac{i}{2}\psi^T\mathbb{C}\gamma^\mu\,\partial_\mu\,\Psi - \frac{1}{2}m\Psi^T\mathbb{C}\Psi$$

Solving Dirac eq.

Reference: ch. 37

Try:

$$\Psi(x) = u(k)e^{ikx}$$

We know that k is on-shell, because Ψ also obeys the KG eq. Plug it into the Dirac eq. to get:

$$(-\gamma^\mu k_\mu - m)u(k) = 0$$

$$(\not{k} + m)u(k) = 0$$

Let's go to rest frame $k^\mu = (m, 0, 0, 0)$. Then $(\gamma^0 k_0 + m)u = 0$, so $\gamma^0 u = u$. Two solutions are:

$$u_+ = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_- = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

If, instead, we look for $\Psi(x) = u(k)e^{-ikx}$, we will get $\gamma^0 v = -v$. Two solutions are:

$$v_+ = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad v_- = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

For the most general solution, we make a superposition of u_+, u_-, v_+, v_- and boost it in any direction.