Lie groups HW1

Matei Ionita

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Problem 1

Prove that the matrix groups $SO(n)$ and $SU(n)$ are compact and connected.

Proof. Since any topological manifold is connected if and only if it is path connected, we can show that $SO(n)$ and $SU(n)$ are connected by finding a path from the identity to an arbitrary element. For the case of $SU(n)$, we use the fact that any unitary matrix has an eigenspace decomposition, and the eigenvalues are unit length complex numbers. Take $A \in SU(n)$, then:

$$A = U \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} U^\dagger$$

For some unitary U. Now consider the following family of matrices, for $0 \leq t \leq 1$:

$$A(t) = U \begin{pmatrix} e^{it\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{it\theta_n} \end{pmatrix} U^\dagger$$

Any $A(t)$ is unitary:

$$A^\dagger A = U \begin{pmatrix} e^{-i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-i\theta_n} \end{pmatrix} U^\dagger U \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} U^\dagger = 1$$

And has determinant 1, since $\det(A) = \det(U) e^{it\theta_1} ... e^{it\theta_n} \det(U^\dagger) = e^{it(\theta_1 + ... + \theta_n)}$. Since the determinant of the original A must be 1, we have $\theta_1 + ... + \theta_n = 0$, and therefore this determinant is also 1. Therefore, $A(t) \in SU(n)$. But $A(0) = 1$ and $A(t) = A$, so this is a path from 1 to A.

For $SO(n)$, we note that any matrix in $SO(n)$ is just a rotation in some appropriately chosen 2-plane. In other words, for every $A \in SO(n)$ there exists a choice of basis such that

A is of the form:

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) & \dots & 0 \\ -\sin(\theta) & \cos(\theta) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Then consider the family of $SO(n)$ matrices:

$$A(t) = \begin{pmatrix} \cos(t\theta) & \sin(t\theta) & \dots & 0 \\ -\sin(t\theta) & \cos(t\theta) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

For $t \in [0,1]$. We have $A(0) = 1$ and $A(t) = A$, therefore this is a path from 1 to A.

Now we turn to compactness. We proceed by showing that $O(n)$ is compact, and after that come back to $SO(n)$. We can regard $O(n)$ as a subset of $\mathbb{R}^{n^2}$, and therefore it's compact if it's closed and bounded in $\mathbb{R}^{n^2}$. The fact that $O(n)$ is bounded follows from the fact that all entries of $O(n)$ matrices are ≤ 1. To see that it's closed, take a sequence $\{A_n\} \subset O(n)$ that converges to some $A \in M(n)$. We have $A_n \to A$ and $A_n^T \to A^T$, therefore $A_n^T A_n \to A^T A$. But $A_n^T A_n = 1$ for all n, so $A^T A = 1$. Thus $A \in O(n)$. This completes the proof that $O(n)$ is compact. Now note that $SO(n)$ is closed in $O(n)$, because it's the inverse image of 1 by the continuous map $\det : O(n) \to \mathbb{R}$. Since closed subsets of compact sets are compact, this shows that $SO(n)$ is compact. The same argument works for $SU(n)$, which is closed in $U(n)$. □

Problem 2

Show that $SU(n)/SU(n-1) \cong S^{2n-1}$ and $SO(n)/SO(n-1) \cong S^{n-1}$.

Proof. We regard $S(n-1)$ as the set of unit length vectors in $\mathbb{R}^n$. Then, by the construction in problem 1, the action of $SO(n)$ on S^{n-1} is transitive. Take the point $(1, 0, ..., 0) \in S^{n-1}$. Its orbit is the entire S^{n-1}, by transitivity. Its stabilizer is the subgroup of $SO(n)$ which only performs rotations in the hyperplane orthogonal to $(1, 0, ..., 0)$, and since any lenght-preserving rotations in this plane are allowed, this subgroup is isomorphic to $SO(n-1)$. Then corollary 2.21 in Kirillov (a generalization for Lie groups of the orbit-stabilizer theorem) tells us that $SO(n)/SO(n-1) \cong S^{n-1}$.

Similarly, we regard S^{2n-1} as the set of unit length vectors in $\mathbb{C}^n$. Again, the action of $SU(n)$ on S^{2n-1} is transitive. If we take the point $(1, 0, ..., 0)$, its orbit will be S^{2n-1} and its stabilizer will be $SU(n-1)$. Then by the same theorem $SU(n)/SU(n-1) \cong S^{2n-1}$. □

Problem 3

Prove that the set of right-invariant vector fields forms a Lie algebra under the Lie bracket operation, and show that it is isomorphic to T_1G. Define the diffeomorphism

$$\phi : g \in G \to \phi(g) = g^{-g} \in G$$

Show that if X is a left-invariant vector field, then $d\phi(X)$ is a right-invariant vector field, whose value at 1 is the same as that of X. Show that

$$X \to d\phi(X)$$

gives an ismorphism of the Lie algebras of left and right invariant vector fields on G.

Proof. Let $\mathcal{R}$ denote the set of right-invariant vector fields of G. Derivatives are linear maps, so:

$$DR_g(aX + bY) = aDR_g(X) + bDR_g(Y) = aX + bY$$

Therefore $\mathcal{R}$ is a vector space. To show it's a Lie algebra, we just need to show it's closed under Lie brackets. By the naturality of Lie brackets (see, for example, Lee 8.30 and 8.31):

$$DR_g[X, Y] = [DR_g(X), DR_g(Y)] = [X, Y]$$

Now define a map $\phi : T_1G \to \mathcal{R}$ by $\phi(X)|_g = (DR_g)|_1(X)$. To avoid confusion, note that X represents a *vector* in T_1G, and $\phi(X)$ represents a *vector field*. We first need to show that ϕ is well-defined, i.e. that $\phi(X)$ is smooth and right-invariant. To show smoothness, it suffices to show that $\phi(X)f$ is smooth whenever f is a smooth function. Following the proof of Lee 8.37, choose a smooth curve $\gamma : (-\delta, \delta) \to G$ such that $\gamma(0) = 1$ and $\gamma'(0) = X$. Then:

$$(\phi(X)f)(g) = \phi(X)|_g f = (DR_g)|_1(X)f = X(f \circ R_g) = \gamma'(0)(f \circ R_g) = \left.\frac{d}{dt}\right|_{t=0} (f \circ R_g \circ \gamma)(t)$$

Denote $f \circ R_g \circ \gamma(t)$ by $\psi(t, g)$. ψ is a composition of smooth maps, so it is smooth. Also, the computation above shows that $(\phi(X)f)(g) = \frac{\partial \psi}{\partial t}(0, g)$, which is smooth, since it's the partial derivative of a smooth map.

We now need to show that $\phi(X)$ is indeed right-invariant. This means $(DR_h)|_g\phi(X)|_g = \phi(X)_(gh)$. But by the composition law of right actions:

$$(DR_h)|_g\phi(X)|_g = (DR_h)|_g \circ (DR_g)|_1(X) = (DR_{gh})|_1(X) = \phi(X)|_{gh}$$

Now all that's left to show is that ϕ is bijective. If $\phi(X) = \phi(Y)$, then $\phi(X)(1) = \phi(Y)(1)$, so $X = Y$. Thus ϕ is injective. Now take some right-invariant vector field $\tilde{X}$ and let $X = \tilde{X}|_1$. Clearly $\tilde{X}|_g = DR_g(X) = \phi(X)|_g$, and thus ϕ is surjective. This completes the proof that $\mathcal{R} \cong T_1G$.

We look now at the map $\phi : G \to G$ given by $\phi(g) = g^{-1}$. In order to show that $D\phi(X)$ is right-invariant whenever X is left-invariant, we first prove that $\phi \circ L_{g^{-1}} = R_g \circ \phi$. Indeed, take some $h \in G$ and then:

$$\phi \circ L_{g^{-1}}(h) = \phi(g^{-1}h) = h^{-1}g$$

$$R_g \circ \phi(h) = R_g(h^{-1}) = h^{-1}g$$

Now if we differentiate the relation $\phi \circ L_{g^{-1}} = R_g \circ \phi$ and act it on X we obtain:

$$DR_g \circ D\phi(X) = D\phi \circ DL_{g^{-1}}(X) = D\phi(X)$$

Which proves that $D\phi(X)$ is right-invariant. We compute its value at the identity:

$$D\phi(X)|_1 = \frac{d}{dt}\bigg|_{t=0} \phi(e^{tX_1}) = \frac{d}{dt}\bigg|_{t=0} e^{-tX_1} = -X_1$$

$D\phi$ is a *vector space* isomorphism between left-invariant and right-invariant vector fields, since it's the derivative of a diffeomorphism. ($D\phi$ is actually its own inverse, as ϕ is its own inverse.) By the naturality of Lie brackets, $D\phi$ preserves the Lie bracket, so it's a Lie algebra isomorphism. □

Problem 4 (Kirillov 2.5)

Let $G(n,k)$ be the set of all dimension k subspaces in $\mathbb{R}^n$ (usually called the Grassmanian). Show that $G(n,k)$ is a homogeneous space for the group $O(n,R)$ and thus can be identied with coset space $O(n,R)/H$ for appropriate H. Use it to prove that $G(n,k)$ is a manifold and find its dimension.

Proof. Take $V, W \in G(n,k)$; then V, W are k-dimensional vector spaces, and we can find orthonormal bases (v_i) and (w_i) for them. Transitivity of the $O(n)$ action then just means finding an $O(n)$ transformation that takes each $v_i \to w_i$. We can prove by induction on k that such a transformation exists. First, if $k = 1$ our claim reduces to taking a unit vector in $\mathbb{R}^n$ to another; by the transitivity of the $O(n)$ action on $\mathbb{R}^n$, this is always possible. For the inductive step, assume there exists $A \in O(n)$ that takes $(v_1, ..., v_{n-1}) \to (w_1, ..., w_{n-1})$. We now need a transformation B that takes $v_n \to w_n$ while leaving $(w_1, ..., w_{n-1})$ unchanged. This can always be found, by the transitivity of $O(n)$ on $\mathbb{R}^n$, as long as v_n is not in the space spanned by $(w_1, ..., w_{n-1})$. We can make sure this is the case by permuting the v_i until v_n is in the orthogonal complement of $(w_1, ..., w_{n-1})$. This finishes the proof that $O(n)$ acts transitively on $G(n,k)$.

We now want to find the stabilizer of some $V \in G(n,k)$. There exists an $O(k)$ subgroup of $O(n)$ that rotates the basis vectors of V inside the space; this clearly leaves V unchanged. There also exists an $O(n-k)$ subgroup that rotates vectors in the orthogonal complement of V but does nothing to the basis vectors of V; this also leaves V unchanged. We come to the conclusion that the stabilizer of V is $O(k) \times O(n-k)$. Since the orbit of V is equal to $G(n,k)$, corollary 2.21 in Kirillov gives $G(n,k) \cong O(n)/\big(O(k) \times O(n-k)\big)$. Also:

$$\begin{aligned}
\dim(G(n,k)) &= \dim(O(n)) - \dim\big(O(k) \times O(n-k)\big) \\
&= \frac{n(n-1)}{2} - \frac{k(k-1)}{2} - \frac{(n-k)(n-k-1)}{2} \\
&= k(n-k)
\end{aligned}$$

□

Problem 5 (Kirillov 2.8 - 2.10)

Define a basis in $\mathfrak{su}(2)$ by

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Show that the map

$$\begin{aligned}
\phi : SU(2) &\to GL(3,\mathbb{R}) \\
g &\to \text{matrix of Ad } g \text{ in the basis } i\sigma_1, i\sigma_2, i\sigma_3
\end{aligned}$$

Gives a morphism of Lie groups $SU(2) \to SO(3,\mathbb{R})$.

Proof. As a vector space, $\mathfrak{su}(2) \cong \mathbb{R}^3$, since any $X \in \mathfrak{su}(2)$ can be expressed as $X = x_1 i\sigma_1 + x_2 i\sigma_2 + x_3 i\sigma_3$ for $a, b, c \in \mathbb{R}$, i.e.:

$$X = \begin{pmatrix} ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & -ix_3 \end{pmatrix}$$

Notice that $\det(X) = x_1^2 + x_2^2 + x_3^2$, which is the length squared of an element of $\mathbb{R}^3$. The adjoint representation of $SU(2)$ is a map in $GL(\mathfrak{su}(2)) \cong GL(3,\mathbb{R})$:

$$(\mathrm{Ad}g)X = gXg^{-1}$$

This map preserves $\det(X)$, since $\det(gXg^{-1}) = \det(g)\det(X)\det(g^{-1}) = \det(X)$. Therefore it preserves the inner product of $\mathbb{R}^3$, and it's an $SO(3)$ map. To prove that it's a homomorphism, just note that:

$$\phi(gh)(X) = (gh)X(gh)^{-1} = g(hXh^{-1})g = \phi(g) \circ \phi(h)(X)$$

□

Let $\phi : SU(2) \to SO(3,\mathbb{R})$ be the morphism dened in the previous problem. Compute explicitly the map of tangent spaces $\phi_* : \mathfrak{su}(2) \to \mathfrak{so}(3,\mathbb{R})$ and show that ϕ_* is an isomorphism. Deduce from this that Kerϕ is a discrete normal subgroup in $SU(2)$, and that Imϕ is an open subgroup in $SO(3,\mathbb{R})$.

Proof. We want to compute the derivative of the map:

$$\begin{aligned} \phi : SU(2) &\to SO(3) \\ g &\to \text{matrix of Ad } g \end{aligned}$$

We begin by computing $\mathrm{ad}Y(X)$, the derivative of $\mathrm{Ad}g(X) = gXg^{-1}$. For this, take a curve $\gamma : (-\delta, \delta) \to SU(2)$ such that $\gamma(0) = 1$ and $\gamma'(0) = Y$. Then:

$$\begin{aligned} \mathrm{ad}Y(X) &= \left.\frac{d}{dt}\right|_{t=0} Ad(\gamma(t))(X) \\ &= \left.\frac{d}{dt}\right|_{t=0} \gamma(t)X\gamma^{-1}(t) \\ &= \gamma(0)X(\gamma^{-1})'(0)\gamma(0)X\gamma(0) \\ &= \gamma'(0)X\gamma(0) - \gamma(0)X\gamma^{-1}(0)\gamma'(0)\gamma^{-1}(0) \\ &= YX - XY \\ &= [Y, X] \end{aligned}$$

It suffices to compute $\mathrm{ad}Y(X)$ in the case when $Y = i\sigma_j$, one of the three generators of $\mathfrak{su}(2)$. Take $Y = i\sigma_1$ for example. Let X be arbitrary, i.e. $X = x_1 i\sigma_1 + x_2 i\sigma_2 + x_3 i\sigma_3$. Then:

$$[Y, X] = -x_2[\sigma_1, \sigma_2] - x_3[\sigma_1, \sigma_3] = -2x_2 i\sigma_3 + 2x_3 i\sigma_2$$

We have found that the map $\mathrm{ad}(i\sigma_1)$ takes $(x_1, x_2, x_3) \to (0, 2x_3, -2x_2)$. Therefore:

$$\phi_*(i\sigma_1) = \begin{pmatrix} & & \\ & & 2 \\ & -2 & \end{pmatrix} = 2l_1$$

Similarly we find that:

$$\phi_*(i\sigma_2) = \begin{pmatrix} & & -2 \\ & & \\ 2 & & \end{pmatrix} = 2l_2 \quad \phi_*(i\sigma_3) = \begin{pmatrix} & 2 & \\ -2 & & \\ & & \end{pmatrix} = 2l_3$$

Where l_j are the generators of $\mathfrak{so}(3)$. This proves that ϕ_* is an isomorphism of $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ *as vector spaces*. In order to show that these are also isomorphic *as Lie algebras*, we show that ϕ_* preserves Lie brackets:

$$\phi_*([i\sigma_1, i\sigma_2]) = \phi_*(-2i\sigma_3) = -4l_3 = 4[l_1, l_2] = [\phi_*(i\sigma_1), \phi_*(i\sigma_2)]$$

And similarly for the other 2 brackets. This concludes the proof the $\phi_* : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is a Lie algebra isomorphism.

The inverse function theorem for manifolds now tells us that, since ϕ_* is an isomorphism, ϕ

is a local diffeomorphism. Take some $g \in \mathrm{Ker}(\phi)$, and there exists a nighborhood of g that is mapped diffeomorphically onto a neighborood of 0. Then in this neighborhood there is no other $g' \in \mathrm{Ker}(\phi)$. This shows that $\mathrm{Ker}(\phi)$ is discrete in $SU(2)$. By the first isomorphism theorem, $\mathrm{Ker}(\phi)$ is also a normal subgroup of $SU(2)$, and $SU(2)/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$. Moreover, since ϕ_* is surjective ϕ is a submersion, and since all submersions are open maps, $\mathrm{Im}(\phi)$ is an open subset of $SO(3)$. $\square$

Prove that the map ϕ used in two previous exercises establishes an isomorphism $SU(2)/\mathbb{Z}_2 \to SO(3,\mathbb{R})$ and thus, since $SU(2) \cong S^3$, $SO(3,\mathbb{R}) \cong \mathbb{RP}^3$.

Proof. Since $\mathrm{Im}(\phi)$ is an open subset of $SO(3)$, Corollary 2.10 in Kirillov tells us that $\mathrm{Im}(\phi) = SO(3)$. Then ϕ is a covering map, and we know from algebraic topology that covering maps are classified by subroups of the fundamental group of the target space. In this case, $\pi_1(SO(3)) = \mathbb{Z}_2$, so the only possibility is $\mathrm{Ker}(\phi) = \mathbb{Z}_2$. Therefore $SO(3) \cong SU(2)/\mathbb{Z}_2 \cong S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$. $\square$