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Logica Dinamica Propozitionala

Referat la Logica Matematica si Computationala

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1. Introducere

Logica Dinamica a fost conceputa de Vaughan Pratt in 1974 ca o incercare de a exprima formula $p\{\alpha\}q$ din Logica Hoare in felul urmator: $p \to [\alpha]q$. Pratt a continuat sa dezvolte aceasta interpretare, iar 1976 a publicat prima lucrare care descria Logica Dinamica drept un sistem logic de sine statator.

Asa cum logica propozitionala are drept scop formalizarea teoremelor, propozitiilor si demonstratiilor matematice, asa si logica dinamica isi propune formalizarea programelor, precum si descrierea altor aspecte legate de acestea, precum corectitudinea, finitudinea si echivalenta dintre programe. Logica dinamica poate fi perceputa ca o logica a actiunilor, a schimbarii. Acesta descriere informala reiese clar prin comparatie cu logica predicatelor. In aceasta din urma, adevarul este *static*: valoarea de adevar a unei formule ϕ este determinata de o evaluarea a variabilelor sale. In schimb, logica dinamica prezinta structuri numite *programe*, al caror rol este acela de a schimba valorile variabilelor in functie de pasul la care se afla programul, alterand, in acest mod, si valoarea de adevar. Spre exemplu, programul $x = x + 1$ schimba valoarea de adevar a propozitiei "x este par" la fiecare iteratie.

Ideea este aceea de a integra programele intr-un limbaj, transformandu-i in operatori logici. *logica dinamica propozitionala* are acelasi rol in logica dinamica pe care logica propozitionala il are in logica de ordin intai a predicatelor. Astfel, in logica dinamica propozitionala (prescurtata in aceasta lucrare LDP) se foloseste de doua structuri sintactice: formule si programe. In LDP nu exista insa notiunea de primire a unei valori de catre o variabila. In schimb, programele primitive sunt interpretate drept relatii binare intre mai multe stari ale unei multimi. In acelasi mod, formulele atomice sunt interpretate ca submultimi ale multimii de stari.

Spre exemplu, in formula LDP:

$$[\alpha](\phi \wedge \psi) \leftrightarrow [\alpha]\phi \wedge [\alpha]\psi$$

partea stanga exprima faptul ca $\phi \wedge \psi$ trebuie sa aiba loc dupa executia programului α, iar partea dreapta, ca ϕ are loc dupa executia lui α si ψ are loc dupa executia lui α.

2. Limbajul logicii si semantica

2.1. Limbajul logicii dinamice propozitionale

Limbajul LDP contine doua tipuri de expresii: $formule$ si $programe$. Programele atomice sunt notate cu a, b, c *etc.* si multimea tuturor programelor atomice este Π_0. Propozitiile atomice se noteaza $p, q, retc.$, iar multimea tuturor formulelor atomice este Φ_0. Multimea tutror programelor se noteaza Π, iar multimea tuturor formulelor, Φ. Programele si formulele sunt constuite inductiv pornind de la programele si formulele atomice utilizand urmatorii operatori:

Operatori propozitionali:

$$\rightarrow \quad implicatia$$
$$\neg \quad negatia$$

Operatorii programelor:

$$; \quad operatorul\ de\ compozitie$$
$$U \quad operatorul\ de\ alegere$$
$$* \quad operatorul\ de\ iterare$$

Operatori mixti:

$$[\,] \quad operatorul\ necesitate\ (se\ citeste\ "box")$$
$$? \quad operatorul\ test$$

Pentru a evita utilizarea excesiva a parantezelor, vom considera operatorii unari ([ ], ?) mai puternici decat cei binari(; , $\cup$).

Operatorii $\rightarrow$, 0, $\wedge$, $\wedge$ au aceleasi proprietati si caracteristici ca in logica propozitionala.

De asemenea, daca ϕ si ψ sunt formule, iar α si β sunt programe, atunci:

$$\phi \rightarrow \psi$$
$$0$$
$$[\alpha]\phi$$

sunt formule, iar:

$$\alpha; \beta$$
$$\alpha \cup \beta$$
$$\alpha^*$$
$$\phi?$$

sunt programe.

Obs La nivel intuitiv, putem intelege programele prezentate mai sus in felul urmator:

- $[\alpha]\psi$ Dupa executia lui α, ϕ este adevarat.
- $\alpha \cup \beta$ Executa α sau β.
- $\alpha; \beta$ Executa α, apoi β.
- α^* Executa α de un numar finit de ori.
- $\psi?$ Daca ψ este adevarat, continua. In caz contrar, intrerupe.

Multimile Π si Φ sunt construite ca fiind multimile cele mai mici cu urmatoarele proprietati:

- $\Phi_0 \subseteq \Phi$.
- $\Pi_0 \subseteq \Pi$.
- Daca $\phi, \psi \in \Phi$, atunci $\phi \to \psi \in \Phi$.
- $0 \in \Phi$.
- Daca $\alpha, \beta \in \Pi$, atunci $\alpha; \beta, \alpha \cup \beta$ si $\alpha^* \in \Pi$.
- Daca $\alpha \in \Pi$ si $\phi \in \Phi$, atunci $[\alpha]\phi \in \Phi$.
- Daca $\phi \in \Phi$, atunci $\phi? \in \varnothing$.

Operatorul $< \alpha > \phi$

Prin prisma operatorului $[\alpha]\phi$ definim un nou operator:

$$< \alpha > \phi \; = \; \neg[\alpha]\neg\phi$$

care se citeste ”diamond”.

O observatie ce merita facuta este faptul ca operatorul $< >$ implica faptul ca programul α se termina, in timp ce operatorul $[\,]$ nu. Operatorii *box* si *diamond* pot fi intelesi mai bine in urmatorul mod:

$[\alpha]\phi$ **Orice** executie lui α va conduce la o stare in care ϕ este adevarata.

$< \alpha > \phi$ **Una** din executiile lui α va conduce la o stare in care ϕ este adevarata.

Acest lucru inseamna ca, in cazul $< \alpha > \phi$, **trebuie** sa existe o executie a lui α care conduce la o stare in care ϕ este adevarata. Pe de alta parte, $[\alpha]\phi$ ne spune practic ca **daca** exista o executie a lui α care conduce la o stare potrivita, atunci ϕ este adevarata, Insa nu se stie daca $[\alpha]\phi$ determina o astfel de stare.

Obs $[\alpha]0$ indica faptul ca α se termina, iar $[\alpha]1$ este mereu adevarat, pentru oricare $\alpha \in \Pi$.

Dam in continuare cateva exemple simple de programe:

- skip / continue $\Longleftrightarrow$ 1?
- fail / break $\Longleftrightarrow$ 0?
- if ϕ *then* α *else* $\beta \Longleftrightarrow (\phi?;\alpha) \cup (\neg\phi;\beta)$
- *while* ϕ *do* $\alpha \Longleftrightarrow (\phi?;\alpha)^*;\neg\phi?$

2.2. Semantica

Programele si formulele din LDP sunt interpretate prin un anumit tip de structuri, numite **structuri Kripke**. Aceasta este o pereche de forma:

$$k = (K, m_{\mathfrak{K}})$$

unde K este o multime de stari (notate $u, v, wetc.$), iar $m_{\mathfrak{K}}$ este o functie care asociaza o submultime a lui K fiecarei formule atomica, si o relatie binara pe K fiecarui program atomic:

$$m_{\mathfrak{K}}(p_0) \subseteq K, p_0 \in \Phi_0$$
$$m_{\mathfrak{K}}(a_0) \subseteq K \cdot K, a_0 \in \Pi_0$$

Aceasta definitie poate fi extinsa inductiv pentru toate programele si formulele:

$$m_{\mathfrak{K}}(p) \subseteq K, p \in \Phi$$
$$m_{\mathfrak{K}}(a) \subseteq K \cdot K, a \in \Pi$$

Obs: La nivel intuitiv, putem intelege $m_{\mathfrak{K}}(\phi)$ ca o multime de stari ce satisface formula ϕ in modelul k. De asemenea, relatia binara $m_{\mathfrak{K}}(\alpha)$ poate fi perceputa ca o multime de perechi de tip input / output de stari ale programului α.

Interpretarea programelor si formulelor compuse definite mai sus este:

$$m_{\mathfrak{K}}(\phi \to \psi) = (K - m_{\mathfrak{K}}(\phi)) \cup m_{\mathfrak{K}}(\psi)$$
$$m_{\mathfrak{K}}(0) = \varnothing$$
$$m_{\mathfrak{K}}([\alpha]\phi) = K - (m_{\mathfrak{K}}(\alpha) \circ (K - m_{\mathfrak{K}}(\phi)) = \{u \mid \forall v \in K, daca\ (u,v) \in m_{\mathfrak{K}}(\alpha)\ atunciv \in m_{\mathfrak{K}}(\phi)\}$$
$$m_{\mathfrak{K}}(\alpha;\beta) = m_{\mathfrak{K}}(\alpha) \circ m_{\mathfrak{K}}(\beta) = \{(u,v) | \exists w \in K\ cu\ (u,w) \in m_{\mathfrak{K}}(\alpha)\ si\ (w,v) \in m_{\mathfrak{K}}(\beta)\}$$
$$m_{\mathfrak{K}}(\alpha \cup \beta) = m_{\mathfrak{K}}(\alpha) \cup m_{\mathfrak{K}}(\beta)$$
$$m_{\mathfrak{K}}(\alpha^*) = m_{\mathfrak{K}}(\alpha)^* = \bigcup_{n \geqslant 0} m_{\mathfrak{K}}(\alpha)^n$$
$$m_{\mathfrak{K}}(\phi?) = \{(u,u) | u \in m_{\mathfrak{K}}(\phi)\}$$

,unde $\circ$ reprezinta operatorul de compunere, iar a doua utilizare a semnului $*$ in penultimul rand reprezinta inchiderea reflexiv-tranzitiva.

3. Satisfiabilitate

Fie $k = (K, m_{\mathfrak{K}})$ o structura Kripke si ϕ o formula. Scriem $k,\ u \vDash \phi$ si spunem ca u satisface ϕ in k sau ca ϕ este satisfacuta in k. Daca ϕ este satisfacut pentru orice k, atunci spunem ca ϕ este satisfiabila. Semnul k poate fi omis in notatie, ramanand doar $u \nvDash \phi$.

Notatia $u \nvDash \phi$ denota ca u nu satisface ϕ.
Echivalent, putem scrie $u \in m_{\mathfrak{K}}(\phi)$ si spunem ca ϕ este adevarata pentru o stare u in k. $u \notin m_{\mathfrak{K}}(\phi)$ este o scriere echivalenta a $u \nvDash \phi$.

Putem rescrie definitiile date mai sus in felul urmator:

- $u \vDash \phi \leftrightarrow \psi \Longleftrightarrow u \vDash \phi \leftrightarrow u \vDash \psi$
- $u \vDash [\alpha]\phi \Longleftrightarrow (u, v) \in m_{\mathfrak{K}}(\alpha)\ si\ (v, w) \in m_{\mathfrak{K}}(\alpha), \forall v \in K$
- $(u, v) \in m_{\mathfrak{K}}(\alpha) \Longrightarrow \exists w\ a.i.\ (u, w) \in m_{\mathfrak{K}}(\alpha)\ si\ (w, v) \in m_{\mathfrak{K}}(\alpha)$
- $(u, v) \in m_{\mathfrak{K}}(\alpha \cup \beta) \Longrightarrow (u, v) \in m_{\mathfrak{K}}(\alpha)\ sau\ (u, v) \in m_{\mathfrak{K}}(\beta)$
- $(u, v) \in m_{\mathfrak{K}}(\alpha^*) \Longrightarrow \exists n \geqslant 0, \exists x_0, \cdots, x_n\ cu\ u\ =\ x_0\ si\ v\ =\ x_n\ cu\ proprietatea\ ca\ (x_i, x_{i+1}) \in m_{\mathfrak{K}}(\alpha), i\ = \overline{1, n}$
- $(u, v) \in m_{\mathfrak{K}}(\phi?) \Longrightarrow u\ =\ v\ si\ u \vDash \phi$

,iar operatorii definiti pana acum mostenesc, de asemenea, aceste proprietati:

$$\begin{aligned}
m_{\mathfrak{K}}(\phi \vee \psi) &= m_{\mathfrak{K}}(\phi) \cup m_{\mathfrak{K}}(\psi) \\
m_{\mathfrak{K}}(\phi \wedge \psi) &= m_{\mathfrak{K}}(\phi) \cap m_{\mathfrak{K}}(\psi) \\
m_{\mathfrak{K}}(\neg\phi) &= K - m_{\mathfrak{K}}(\phi) \\
m_{\mathfrak{K}}(< \alpha > \phi) &= \{u | \exists v \in K,\ (u, v) \in m_{\mathfrak{K}}(\alpha)\ si\ v \in m_{\mathfrak{K}}(\phi)\} = m_{\mathfrak{K}}(\alpha) \circ m_{\mathfrak{K}}(\phi) \\
m_{\mathfrak{K}}(1) &= K \\
m_{\mathfrak{K}}(skip) &= m_{\mathfrak{K}}(1?) \\
m_{\mathfrak{K}}(fail) &= m_{\mathfrak{K}}(0?) = \varnothing
\end{aligned}$$

3.1. Validitate

Daca $k, u \vDash \phi, \forall u \in K$,atunci scriem $k \vDash \phi$ si spunem ca ϕ este **valida in** k. De asemenea, daca $k \vDash \phi$ pentru orice k, atunci spunem ca ϕ este **valida**.

Fie Σ o multime de formule. Scriem $k \vDash \Sigma$ daca $k \vDash \phi, \forall \phi \in \Sigma$. O formula ψ se numeste *consecinta logica* a lui Σ daca $k \vDash \psi$ si $k \vDash \Sigma$. Scriem $\Sigma \vDash \psi$.

Obs 1: $\Sigma \vDash \psi$ nu este echivalent cu a scrie ca $k,\ u \vDash \psi$ si $k,\ u \vDash \Sigma$.

Obs 2: Regula de deductie

$$\frac{\phi_1, \cdots, \phi_n}{\phi}$$

este *valida* daca ϕ este o consecinta logica a multimii $\{\phi_1, \cdots, \phi_n\}$.

Obs 3: O formula ϕ este valida in k daca si numai daca $\neg\phi$ nu este satisfiabila in k.

4. Sistemul deductiv

Folosim un sistem deductiv de tip Hilbert pentru Logica Dinamica Propozitionala:

Reamintim axiomele logicii propozitionale:

1. $(\phi \to (\psi \to \phi)$
2. $(\phi \to (\psi \to \chi) \to ((\phi \to \psi) \to (\phi \to \chi))$
3. $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$

Si introducem axiomele LDP:

1. $[\alpha](\phi \to \psi) \to \ ([\alpha]\phi \to [\alpha]\psi)$
2. $[\alpha](\phi \wedge \psi) \leftrightarrow \ ([\alpha]\phi \wedge [\alpha]\psi)$
3. $[\alpha](\phi \cup \psi) \leftrightarrow \ ([\alpha]\phi \wedge [\alpha]\psi)$
4. $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$
5. $[\psi?]\phi \leftrightarrow (\psi \to \phi)$
6. $\phi \wedge [\alpha][\alpha^*] \leftrightarrow \ [\alpha^*]\phi$
7. **Axioma de inductie:** $\phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$

   Obs: Axioma de inductie spune ca, daca ϕ este adevarata initial si daca, presupunand ca dupa un numar finit de rulari ale programului ea este adevarata, atunci ramane adevarata dupa inca o rulare a programului α. Atunci ϕ va fi adevarata dupa oricate iteratii ale lui α. Cu alte cuvinte, daca ϕ este initial adevarata, atunci valoarea lui ϕ se conservata prin rularea programului α de oricate ori.

Reguli de deductie:

Modus ponens:

$$\frac{\phi, \phi \to \psi}{\psi}$$

Generalizare:

$$\frac{\phi}{[\alpha]\phi}$$

5. Proprietati

5.1. Proprietati mostenite din Logica Modala

Urmatoarele proprietati nu sunt specifice LDP, ci mostenite din logica modala propozitionala:

Urmatoarele sunt reguli de deductie in LDP:

1. Generalizarea:
$$\frac{\phi}{[\alpha]\phi}$$

2. Monotonia lui $< \alpha >$:
$$\frac{\phi \to \psi}{< \alpha > \phi \to < \alpha > \psi}$$

3. Monotonia lui $[\alpha]$:
$$\frac{\phi \to \psi}{[\alpha]\phi \to [\alpha]\psi}$$

Din 2) avem ca $m_{\mathfrak{K}}(\alpha) \circ (K - m_{\mathfrak{K}}(\psi)) \subseteq m_{\mathfrak{K}}(\alpha) \circ (k_{m\mathfrak{K}}(\phi))$, de unde rezulta ca $K - (m_{\mathfrak{K}}(\alpha) \circ (K - m_{\mathfrak{K}}(\psi))) \subseteq K - (m_{\mathfrak{K}}(\alpha) \circ (K - m_{\mathfrak{K}}(\phi)))$.

Teorema 5.1.2.: Urmatoarele sunt formule ale LDP:

1. $< \alpha \cup \beta > \phi \leftrightarrow < \alpha > \phi \vee < \beta > \phi$
2. $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$

Demonstratie:
1) A arata ca $< \alpha \cup \beta > \phi \leftrightarrow < \alpha > \phi \vee < \beta > \phi$ este echivalent cu a demonstra ca $m_{\mathfrak{K}}(< \alpha \cup \beta > \phi) = m_{\mathfrak{K}}(< \alpha > \phi \vee < \beta > \phi)$.

Avem: $m_{\mathfrak{K}}(< \alpha \cup \beta > \phi) = (m_{\mathfrak{K}}(\alpha) \cup m_{\mathfrak{K}}(\beta)) \circ m_{\mathfrak{K}}(\phi)$ iar, $m_{\mathfrak{K}}(\alpha > \phi \vee < \beta > \phi) = (m_{\mathfrak{K}}(\alpha) \circ m_{\mathfrak{K}}(\phi)) \cup (m_{\mathfrak{K}}(\beta) \circ m_{\mathfrak{K}}(\phi))$.

Cele doua rezultate sunt egale, fapt ce reiese din distributivitatea $\circ$ fata de $\cup$.

2) A arata ca $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$ este echivalent cu a demonstra ca $m_{\mathfrak{K}}([\alpha \cup \beta]\phi) = m_{\mathfrak{K}}([\alpha]\phi \cap [\beta]\phi)$.

Avem: $m_{\mathfrak{K}}([\alpha \cup \beta]\phi) = K - (m_{\mathfrak{K}}([\alpha \cup \beta]) \circ (K - m_{\mathfrak{K}}(\phi))$ iar, $m_{\mathfrak{K}}([\alpha \cup \beta]\phi) = (K - (m_{\mathfrak{K}}([\alpha]) \circ (K - m_{\mathfrak{K}}(\phi))) \cap (K - (m_{\mathfrak{K}}([\beta]) \circ (K - m_{\mathfrak{K}}(\phi)))$.

Obs:

Punctul 1. al teoremei anterioare spune ca programul $\alpha \cup \beta$ se poate opri la un stadiu ce satisface ϕ daca si numai daca α sau β pot satisface ϕ.

Punctul 2. al teoremei anterioare spune ca orice stadiu al programului $\alpha \cup \beta$ satisface ϕ daca si numai daca α si β satisfac ϕ concomitent.

Corolar: Daca $m_{\mathfrak{K}}(\alpha) \subseteq m_{\mathfrak{K}}(\beta)$, atunci, pentru oricare ϕ, au loc urmatoarele:

1. $k \vDash < \alpha > \phi \to < \beta > \phi$
2. $k \vDash [\alpha]\phi \to [\beta]\phi$

Teorema 5.1.3.: Urmatoarele sunt formule ale LDP:

1. $< \alpha; \beta > \phi \leftrightarrow < \alpha >< \beta > \phi$
2. $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$

Demonstratie:

1. $< \alpha; \beta > \phi \leftrightarrow < \alpha >< \beta > \phi \Leftrightarrow m_{\mathfrak{K}}(< \alpha; \beta > \phi) = m_{\mathfrak{K}}(< \alpha >< \beta > \phi) \Leftrightarrow (m_{\mathfrak{K}}(\alpha) \circ m_{\mathfrak{K}}(\beta)) \circ m_{\mathfrak{K}}(\phi) = m_{\mathfrak{K}}(\alpha) \circ (m_{\mathfrak{K}}(\beta) \circ m_{\mathfrak{K}}(\phi))$, fapt ce rezulta din asociativitatea operatiei "$\circ$".
2. $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi \Leftrightarrow m_{\mathfrak{K}}([\alpha; \beta]\phi) = m_{\mathfrak{K}}([\alpha][\beta]\phi)$. Notam $[\alpha; \beta]\phi := A$ si $[\alpha][\beta]\phi := B$. Avem:

   $$m_{\mathfrak{K}}([\alpha; \beta]\phi) = K - (m_{\mathfrak{K}}(\alpha; \beta) \circ (K - m_{\mathfrak{K}}(m_{\mathfrak{K}}(\phi))) = K - (m_{\mathfrak{K}}(\alpha) \circ m_{\mathfrak{K}}(\beta) \circ (K - m_{\mathfrak{K}}(\phi)))$$

   $$m_{\mathfrak{K}}([\alpha][\beta]\phi) = m_{\mathfrak{K}}(\alpha) \circ m_k(K - ([\beta]\phi)) = m_{\mathfrak{K}}(\alpha) \circ (K - (m_{\mathfrak{K}}(m_{\mathfrak{K}}(\beta) \circ (K - m_{\mathfrak{K}}(\phi))))$$

Teorema 5.1.4. Urmatoarele formule sunt formule *valide* ale LDP:

1. $< \phi? > \psi \leftrightarrow (\phi \wedge \psi)$
2. $[\phi?]\psi \leftrightarrow (\phi \to \psi)$

Demonstratie:

1. $< \phi? > \psi \leftrightarrow (\phi \wedge \psi) \Leftrightarrow m_{\mathfrak{K}}(< \phi? > \psi) = m_{\mathfrak{K}}(\phi \wedge \psi)$

   $m_{\mathfrak{K}}(< \phi? > \psi) = \{(u, u) | u \in m_{\mathfrak{K}}(\phi)\} \circ m_{\mathfrak{K}}(\psi) = \{u | u \in m_{\mathfrak{K}}(\psi)\} \cap m_{\mathfrak{K}}(\psi) = m_{\mathfrak{K}}(\phi) \cap_{\mathfrak{K}} (\psi) = m_{\mathfrak{K}}(\phi \wedge \psi)$
2. $[\phi?]\psi \leftrightarrow (\phi \to \psi) \Leftrightarrow m_{\mathfrak{K}}([\phi?]\psi) = m_{\mathfrak{K}}(\phi \to \psi)$

   $m_{\mathfrak{K}}([\phi?]\psi) = \{(u, u) | u \in m_{\mathfrak{K}}(\phi)\} \circ m_{\mathfrak{K}}(\psi) = \{u | u \in m_{\mathfrak{K}}(\psi)\} \cap m_{\mathfrak{K}}(\psi) = ... = (K - m_{\mathfrak{K}}(\phi)) \cup m_{\mathfrak{K}}(\psi)$

5.2. Operatorul de conversie α^-

Operatorul de conversie este un operator specific Logicii Dinamice. La nivel intuitiv, el ne lasa sa rulam un program de la final la inceput. Din punct de vedere semantic, relatia input / output a programului α este relatia output / input a programului α^-:

$$m_{\mathfrak{K}}(\alpha^-) = m_{\mathfrak{K}}(\alpha)^- = \{(v,u)|(u,v) \in m_{\mathfrak{K}}(\alpha)\}$$

Obs: Desi nu este intotdeauna utilizabil, operatorul de conversie ne ofera un mod de a formaliza metoda *backtracking* sau, mai important, pentru ne intoarce cu un pas inapoi in cadrul unui program.

Teorema 5.2.1: Urmatoarele egalitati au loc pentru oricare $\alpha, \beta \in \Pi$:

1. $m_{\mathfrak{K}}((\alpha \cup \beta)^-) = m_{\mathfrak{K}}(\alpha^- \cup \beta^-)$
2. $m_{\mathfrak{K}}((\alpha; \beta)^-) = m_{\mathfrak{K}}(\alpha^-; \beta^-)$
3. $m_{\mathfrak{K}}(\phi?^-) = m_{\mathfrak{K}}(\phi?)$
4. $m_{\mathfrak{K}}(\alpha^{*-}) = m_{\mathfrak{K}}(\alpha^{-*})$
5. $m_{\mathfrak{K}}(\alpha^{--}) = m_{\mathfrak{K}}(\alpha)$

Demonstratie: 1. $m_{\mathfrak{K}}((\alpha \cup \beta)^-) = (m_{\mathfrak{K}}(\alpha) \cup m_{\mathfrak{K}}(\beta))^- = m_{\mathfrak{K}}(\alpha)^- \cup m_{\mathfrak{K}}(\beta)^- = m_{\mathfrak{K}}(\alpha^-) \cup m_{\mathfrak{K}}(\beta^-) = m_{\mathfrak{K}}(\alpha^-) \cup m_{\mathfrak{K}}(\beta^- 1) = m_{\mathfrak{K}}(\alpha^- \cup m_{\mathfrak{K}}(\beta^-))$

Teorema 5.2.2: Urmatoarele formule sunt formule *valide* ale LDP:

1. $\phi \to [\alpha] < \alpha^- > \phi$
2. $\phi \to [\alpha^-] < \alpha > \phi$
3. $< \alpha > [\alpha^-]\phi \to \phi$
4. $< \alpha^- > [\alpha]\phi \to \phi$

Teorema 5.2.3. Fie k o structura Kripke, A o multime (nu neaparat finita) de formule, si ϕ o formula. Stiind ca In LDP, functia $\phi \mapsto < \alpha > \phi$ este continua, atunci daca:

$$m_{\mathfrak{K}}(\phi) = \sup_{\psi \in A} m_{\mathfrak{K}}(\psi)$$

,atunci

$$\exists \sup_{\psi \in A} m_{\mathfrak{K}}(\psi) \ si \ sup_{\psi \in A} \, m_{\mathfrak{K}}(\psi) \;=\; m_{\mathfrak{K}}(< \alpha > \phi)$$

5.3. Operatorul de iteratie α^*

Operatorul de iteratie este un operator specific Logicii Dinamice si depaseste din punct de vedere al complexitatii toti ceilalti operatori. El poate fi interpretat drept inchiderea reflexiv tranzitiva pe o multime data.

$$m_{\mathfrak{K}}(\alpha^*) = m_{\mathfrak{K}}(\alpha)^* = \bigcup_{n<\epsilon} m_{\mathfrak{K}}(\alpha)^n$$

Obs: Din cauza operatorului de iteratie, LDP nu este compacta. Acest lucru reiese din faptul ca multimea $\{< \alpha^* > \phi\} \cup \{\neq \phi, \neg < \alpha > \phi \neg < \alpha^2 > \phi, \cdots\}$ este finit satisfiabil, insa nu satisfiabil.

Fie $\neg < \alpha^n > \phi$ submultime finita aleasa arbitrar, din extragem n-ul maxim, pe care il notam cu m. Dat fiind ca multimea din care am extras este infinita, atunci exista o structura Kripke cu $|K| = m + 2$ stari care satisface submultimea.

Teorema 5.3.1: Urmatoarele formule sunt *valide* in LDP:

1. $[\alpha^*]\phi \to \phi$
2. $\phi \to< \alpha^* > \phi$
3. $[\alpha^*]\phi \to \ [\alpha]\phi$
4. $< \alpha > \phi \to < \alpha^* > \phi$
5. $[\alpha^*]\phi \leftrightarrow \ [\alpha^*\alpha^*]\phi$
6. $< \alpha^* > \phi \leftrightarrow < \alpha^*\alpha^* > \phi$
7. $[\alpha^*]\phi \leftrightarrow \ [\alpha^* *]\phi$
8. $< \alpha^* > \phi \leftrightarrow \alpha^* * > \phi$
9. $[\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi$
10. $< \alpha^* > \phi \leftrightarrow \phi \wedge < \alpha >< \alpha^* > \phi$
11. $[\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha^*](\phi \to [\alpha]\phi)$
12. $< \alpha^* > \phi \leftrightarrow \phi \wedge < \alpha^* > (\phi \to< \alpha > \phi)$

6. Inductia

Este esential in a demonstra mai multe proprietati ale iterarii faptul ca α^* este inchiderea reflexiv-tranzitiva a lui α si nu orice relatie reflexiv-tranzitiva ce contine pe α. Acest lucru poate fi exprimat echivalent in urmatoarele moduri:

1. RTC (Reflexive Transitive Closure Rule)

$$\frac{(\phi \vee < \alpha > \psi) \to \psi}{< \alpha^* > \phi \to \psi}$$

2. LI (Loop Invariance)
$$\frac{\phi \to [\alpha]\phi}{\phi \to [\alpha^*]\phi}$$

3. Inductia:
   (a) Forma *box*:
   $$\phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$$
   (b) Forma *diamond*:
   $$< \alpha^* > \phi \to \phi \wedge < \alpha^* > (\phi \to < \alpha > \phi)$$

Obs:

1. Importanta RTC reiese cel mai bine din legatura ei cu formulele valide ale LDP. Spre exemplu, in punctul 10. al *Teoremei 3.5.1*, implicatia stanga-dreapta se obtine inlocuind $< \alpha^* > \phi$ cu N. Avem: $\phi \vee < \alpha > N \to N$, pe care o notam cu REL.
   Teorema 3.5.1. sugereaza faptul ca $< \alpha^* > \phi$ este o solutie pentru REL. Cu alte cuvinte, REl este valida cand N este substituit cu $< \alpha^* > \phi$. In aacest context, RTC ne asigura ca $< \alpha^* > \phi$ este cea mai ... solutie in raport cu implicatia. Altfel spus, $< \alpha^* > \phi$ este cea mai mica multime de stari care, substituit cu N, rezulta intr-o formula valida.

2. Despre axioma de inductie am discutat si in capitolul *4. Sistemul Deductiv*, insa putem oferi o explicatie a celor doua forme ale inductiei si la nivel intuitiv:

   Forma *box* ne spune ca, daca ϕ este initial adevarat, iar valoarea de adevar se parteaza dupa un numar oarecare de iteratii in programul α, atunci, dupa inca o iteratie, valoarea de adevar se va pastra. Asadar, ϕ va fi adevarata pentru oricate iteratii ale lui α.

   Forma *diamond* ne spune ca, daca este posibil ca o stare care satisface ϕ sa fie atinsa dupa un numar de iteratii ale lui α, atunci fie ϕ este adevarata in starea actuala, fie este posibil sa se ajunga la o stare in care ϕ e false, insa devine adevarata dupa inca o iteratie.

Teorema 6.1. RTC este valida.

Demonstratie: Vrem sa aratam ca in orice model k, daca $m_{\mathfrak{K}}(\phi) \subseteq m_{\mathfrak{K}}(\psi)$ si $m_{\mathfrak{K}}(< \alpha > \phi) \subseteq m_{\mathfrak{K}}(\psi)$, atunci $m_{(K)}(< \alpha^* > \phi) \subseteq m_{\mathfrak{K}}(\psi)$. Aratam prin inductie ca $m_{\mathfrak{K}}(\alpha^n) \subseteq m_{\mathfrak{K}}(\psi)$:

$m_{\mathfrak{K}}(\alpha^0)\phi = m_{\mathfrak{K}}(skip)\phi \subseteq m_{\mathfrak{K}}(\psi)$

Presupunem ca P(n): $m_{\mathfrak{K}}(< \alpha^n > \phi) \subseteq m_{\mathfrak{K}}(\psi)$ adevarat si demonstram P(n+1) adevarat. Avem:

$m_{\mathfrak{K}}(< \alpha^n > \phi) = m_{\mathfrak{K}}(< \alpha >< \alpha^n > \phi) \subseteq m_{\mathfrak{K}}(< \alpha > \psi \subseteq m_{\mathfrak{K}}(\psi))$, unde prima incluziune reiese din *monotonia diamond-ului*, iar a doua, din presupunerea facuta.

Prin urmare, $m_{\mathfrak{K}}(\alpha^n) \subseteq m_{\mathfrak{K}}(\psi)$ este adevarata pentru orice n, asadar $m_{\mathfrak{K}}(\alpha^n) \subseteq m_{\mathfrak{K}}(\psi)$ este valida.

Lema 6.1. Monotonia box-ului si a diamond-ului pot fi deduse si fara utilizarea axiomei de inductie.

Teorema 6.2. In LDP urmatoarele trei axiome si reguli pot fi deduse una din cealalta doua cate doua:

1. Axioma de inductie;
2. LI;
3. RTC.

Demonstratie:

1. Axioma de inductie $\Longrightarrow$ LI

   Presupunem ca $\phi \to [\alpha]\phi$. Rezulta din regula de generalizare $\frac{\phi}{[\alpha]\phi}$ ca $[\alpha^*](\phi \to [\alpha]\phi)$ e valida. Prin inductie avem ca $\Longrightarrow \phi \to \phi \wedge [\wedge^*](\phi \to [\alpha]\phi)$. Asadar, avem ca $\phi \to [\alpha^*]\phi$ este valida.

2. LI $\Longrightarrow$ RTC

   Stim ca: $\alpha > p \leftrightarrow \neg[\alpha]\neg p$. Substituind in RTC, avem:

   $$\frac{(\phi \vee \neg[\alpha]\neg\psi) \to \psi}{\neg[\alpha^*]\neg\phi \to \psi} \iff \frac{\neg\psi \to (\neg\phi \wedge [\alpha]\neg\psi)}{\neg\psi \to [\alpha^*]\neg\phi} \iff \frac{\psi \to (\phi \wedge [\alpha]\psi)}{\psi \to [\alpha^*]\phi}$$

   Daca substituim ψ cu ϕ, rezulta LI:

   $$\frac{\psi \to [\alpha]\psi}{\psi \to [\alpha^*]\psi} \;,$$

   stiind ca $\psi \to (\psi \wedge [\alpha]\psi) \equiv \psi \to [\alpha]\psi$.
   Presupunem ca $\psi \to \phi$ si $\psi \to [\alpha]\psi$. Din LI aplicata celei de-a doua peesupuneri avem ca $\psi \to [\alpha^*]\psi$. Din $\psi \to \phi$ si monotonia lui [ ] avem ca $\psi \to [\alpha^*]\phi$.

3. RTC $\Longrightarrow$ Axioma de inductie

   Din Teorema 5.3.1. punctele 3.$[\alpha^*]\phi \to\ [\alpha]\phi$ si 7. $[\alpha^*]\phi \leftrightarrow\ [\alpha^**]\phi$ avem ca $\phi \wedge [\alpha^*](\phi \to [\alpha]\phi)$.
   $\phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \Longrightarrow \phi \wedge (\phi \to [\alpha]\phi) \wedge [\alpha][alpha^*](\phi \to [\alpha]\phi)) \Longrightarrow \phi \wedge [alpha]\phi \wedge [\alpha][\alpha^*](\phi \to$
   $alpha]\phi) \Longrightarrow \phi \wedge [\alpha](\phi \wedge [\alpha^*](\phi \to [\alpha]\phi))$.

   $(\phi \wedge [\alpha^*](\phi \to [\alpha]\phi)) \Rightarrow \phi \wedge [\alpha](\phi \wedge [\alpha^*](\phi \to [\alpha]\phi))$. Daca aplicam primul rezultat de la punctul anterior, avem ca:

   $\phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$.

7. Teorema de completitudine

7.1. Filtrare

7.1.1. Inchiderea Fischer - Ladner

Desi multe sisteme bazate pe logica modala utilizazea inductia pentru a demonstra relatii intre formule, in Logica Dinamica Propozitionala acest lucru este foarte complicat, in mare parte din cauza operatorului de iteratie "*".

Cu toate acestea, putem folosi relatii bine definitie intre subexpresii in inductie. Consideram expresie ca fiind un program sau o formula. Fiecare poate fi o subexpresie a celeilalte, datorita operatorilor [ ] < > si ?.

Definim inductiv functiile:

$$FL\ :\ \Phi \to 2^{\Phi}$$
$$FL^{\Box}:\ \{[\alpha]\phi|\alpha \in \Pi, \phi \in \Phi\} \to 2^{\Phi}$$

1. $FL(p) = \{p\}, p \in \Phi_0$
2. $FL(\phi \to \psi) = \{\phi \to \psi\} \cup FL(\phi) \cup FL(\psi)$
3. $FL(\mathbf{0}) = \{\mathbf{0}\}$
4. $FL([\alpha]\phi) = FL^{\Box}([\alpha]\phi) \cup FL(\phi) = \{[\alpha]\phi\} \cup \{\phi\}$
5. $FL^{\Box}([a]\phi) = \{[a]\phi\}, a \in \Pi_0$
6. $FL^{\Box}([\alpha \cup \beta]\phi) = \{[\alpha \cup \beta]\phi\} \cup FL^{\Box}([\alpha]\phi \cup FL^{\Box}([\beta]\phi)$
7. $FL^{\Box}([\alpha;\beta]\phi) = \{[\alpha;\beta]\phi\} \cup FL^{\Box}([\alpha \cup \beta]\phi) \cup FL([\beta]\phi)$
8. $FL^{\Box}([\alpha^*]\phi) = \{[\alpha^*]\phi\} \cup FL^{\Box}([\alpha][\alpha^*]\phi)$
9. $FL^{\Box}([\psi?]\phi) = \{[\psi?]\phi\} \cup FL(\psi)$

Obs: Din cauza punctului $8.FL^{\Box}([\alpha^*]\phi) = \{[\alpha^*]\phi\} \cup FL^{\Box}([\alpha][\alpha^*]\phi)$, aceasta definitie pare a fi una circulara. Scopul functei auxiliare $FL^{\Box}$ este acela de a evita acest caracter circular. La nivel intuitiv, atunci cand este aplicata formulelor de forma $[\alpha]\phi$, produce elementele lui $FL([\alpha]\phi)$ prin descompunerea lui α in subprograme si ignorand ϕ.

Lema 7.1.1.:

1. Daca $\sigma \in FL(\phi)$, atunci $FL(\sigma) \subseteq FL(\phi)$.
2. Daca $\sigma \in FL^{\Box}([\alpha]\phi)$,atunci $FL(\sigma) \in FL^{\Box}([\alpha]\phi) \cup FL(\phi)$.

Consecintele lemei 7.1.1.:

1. Daca $[\alpha]\psi \in FL(\phi)$, atunci $\psi \in FL(\phi)$.
2. Daca $[\rho?]\psi \in FL(\phi)$, atunci $\rho \in FL(\phi)$.

3. Daca $[\alpha \cup \beta]\psi \in FL(\phi)$, atunci $[\alpha]\psi \in FL(\phi) si [\beta]\psi \in FL(\phi)$.
4. Daca $[\alpha;\beta]\psi \in FL(\phi)$, atunci $[\alpha][\beta]\psi \in FL(\phi)$ si $[\beta]\psi \in FL(\phi)$.
5. Daca $[\alpha^*]\psi \in FL(\phi)$, atunci $[\alpha][\alpha^*]\psi \in FL(\phi)$.

Lema 7.1.2.:

1. $|FL(\phi)| \leqslant |\phi|, \forall\ \phi \in \Phi$
2. $|FL^{\Box}([\alpha]\phi)| \leqslant |\alpha|, \forall\ [\alpha]\phi \in \Phi$

Modele nonstandard

Un model nonstandard este o structura $\mathfrak{N} = (N, m_{\mathfrak{N}})$ care respecta toate proprietatile structurilor Kripke definite anterior, in afara de faptul ca $m_{\mathfrak{N}}(\alpha^*)$ nu este inchiderea reflexiv tranzitiva a lui $m_{\mathfrak{N}}(\alpha)$, ci doar o relatie binara reflexiva si tranzitiva ce satisface axiomele operatorului de iteratie.

O structura nonstandard Kripke este *standard* daca indeplineste urmatoarea conditie:

$$m_{\mathfrak{K}}(\alpha^*) \;=\; \bigcup_{n\geq 0} m_{\mathfrak{K}}(\alpha)^n$$

7.2. Filtrarea modelelor nonstandard

Fie ϕ o formula si $k = (K, m_{\mathfrak{K}})$ o structura Kripke. Definim $k/FL(\phi) = (K/FL(\phi), m_{\mathfrak{K}/FL(\phi)})$, numita **filtrarea lui k prin FL(ϕ)**:

Mai intai definim relatia binara $\equiv$ astfel:

$$u \equiv v \Longleftrightarrow \forall \psi \in FL(\phi), (u \in m_k(\psi) \Longleftrightarrow m_k(\psi))$$

Altfel spus, eliminam din K toate starile u si v care sunt echivalente. Definim:

$$\hat{u} = \{v | v \equiv u\}$$

$$K/FL(\phi) = \{\hat{u} | u \in K\}$$

$$m_{\mathfrak{K}/FL(p)} = \{\hat{u} | u \in m_{(K)}(p)\},\ p \in \Phi_0$$

$$m_{\mathfrak{K}/FL(a)} = \{(\hat{u}, \hat{(v)}) | (u, v) \in m_{(K)}(a)\},\ a \in \Pi_0$$

Teorema 7.2.1: (Teorema de filtrare a modeleor nonstandard)
Fie $\mathfrak{N}$ o structura nonstandard Kripke si u, v doua stari din $\mathfrak{N}$.

- *i)* $\forall\ \psi \in FL(\phi), u \in m_{\mathfrak{N}}(\psi)$ *daca si numai daca* $\hat{u} \in m_{\mathfrak{N}/FL(\phi)}(\psi)$
- *ii)* $\forall\ [\alpha]\phi \in FL(\phi)$
  - Daca $(u, v) \in m_{\mathfrak{N}}(\alpha)$, atunci $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha)$;
  - Daca $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha)$ si $u \in m_{\mathfrak{N}}([\alpha]\phi)$, atunci $v \in m_{\mathfrak{N}}(\phi)$.

Dem:

Cele doua puncte ale teoremei de filtrare se demonstreaza prin inductie simultana.

Pentru i) exista 4 cazuri, depinzand de de forma lui ψ.

- Cazul I)

  Fie $p \in \Phi_0$,$p \in FL(\phi)$. Daca $u \in m_{\mathfrak{K}}(p)$. atunci din definitia $k/FL(\phi)$ avem $\hat{u} \in m_{k(\phi)}(p)$.
  Daca $\hat{u} \in m_{k(\phi)}(p)$, atunci $\exists\ u'$ a.i. $u'^{\equiv u}$ si $u'^{\in m_{\mathfrak{K}}(p)} \Rightarrow u \in m_{\mathfrak{K}}(p)$.

- Cazul II)

  Daca $\psi \to \phi \in FL(\phi)$ atunci din *Lema 7.1.1.* avem $\psi \in FL(\phi)$ si $\sigma \in FL(\phi)$. Din ipoteza de inductie, i) e adevarata pentru ϕ si σ, deci:
  $s \in m_{\mathfrak{K}}(\psi \to p) \iff (s \in m_{\mathfrak{K}}(\psi) \implies s \in m_{\mathfrak{K}}(p) \iff (\hat{s} \in m_{k(\phi)}(\phi) \implies \hat{s} \in m_{k(\phi)}(p)) \iff \hat{s} \in m_{k(\phi)}(\phi \to p)$.

- Cazul IV)

  Daca $[\alpha]\psi \in FL(\phi)$ atunci aplicam ipoteza de inductie pentru α si ψ. Din *Lema 7.1.2. (1)* avem $\psi \in FL(\phi)$. Din ipoteza de inductie si *Lema 7.1.2 (2)* avem:

  $$s \in m_{\mathfrak{K}}([\alpha]\phi) \implies \forall t,\ ((\hat{s},\hat{t}) \in m_{k/FL(\phi)}(\alpha) \implies t \in m_{\mathfrak{K}}(\psi)$$

  din a doua parte din ii);

  $$\forall t,\ ((\hat{s},\hat{t}) \in m_{k/FL(\phi)}(\alpha) \implies t \in m_{\mathfrak{N}}(\psi) \implies \forall\, t, ((s,t) \in m_{\mathfrak{K}}(\alpha) \implies t \in m_{\mathfrak{N}}(\psi) \implies s \in m_{\mathfrak{N}}([\alpha]\psi)\$$$

  din prima parte din ii).
  $s \in m_{\mathfrak{N}}([\alpha]\psi) \iff \forall\ t, ((\hat{s},\hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha) \implies t \in m_{\mathfrak{N}}(\psi)) \iff \forall\ t, ((\hat{s},\hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha) \implies \hat{t} \in m_{\mathfrak{N}/FL(\phi)}(\psi)) \longleftrightarrow \hat{s} \in m_{\mathfrak{N}/FL(\phi)}([\alpha]\psi)$.

Pentru *ii*) exista cinci cazuri, depinzand de forma lui α.

- Cazul I) Pentru $\alpha = a \in \Pi_0$:

  Prima parte a lui *ii*) reiese direct din definitie. Pentru a demonstra a doua parte, consideram $(\hat{s},\hat{t}) \in m_{\mathfrak{N}}/FL(a)$. Atunci din definitia lui $m_{\mathfrak{N}/FL(a)}$ stim ca $\exists\ s'^{,t'}$ cu $s'^{\equiv s}$ si $t'^{\equiv t}$ a.i. $(\hat{s},\hat{t}) \in m_{\mathfrak{N}}(a)$. Daca $s \in m_{\mathfrak{N}}([\alpha]\psi)$, atunci datorita faptului ca $s'^{\equiv s}$ si $[\alpha]\psi \in FL(\phi)$, avem $s'^{\in m_{\mathfrak{K}}([\alpha]\psi)}$ si $t'^{\in m_{\mathfrak{N}}([\alpha]\psi)}$. Dar cum $\psi \in FL(\phi)$ din *Lema 7.1.2. (1)* si $t'^{\equiv t}$, avem $t \in m_{m\mathfrak{N}}(\psi)$.

- Cazul II) Pentru $\alpha = \rho?$:

  Din *Lema 7.1.2. (2)* avem $\rho \in FL(\phi)$. Prima parte a lui *ii*) deriva direct din *i*) . Pentru a doua parte avem:
  $(\hat{s}, \hat{s}) \in m_{\mathfrak{N}/FL(\phi)}(\rho)$ si $s \in m_{\mathfrak{N}}([\rho?]\psi) \Longrightarrow \hat{s} \in m_{\mathfrak{N}/FL(\phi)}(\rho)$ si $s \in m_{\mathfrak{N}}(\rho \to \psi) \Longrightarrow s \in m_{\mathfrak{N}}(\psi)$.

- Cazul III)
  Pentru $\alpha = \beta \cup \gamma$:

  Demonstam *a*):

  $(u, v) \in m_{\mathfrak{N}}(\beta \cup \gamma) \to (u, v) \in m_{\mathfrak{N}}(\beta)$ sau $(u, v) \in m_{\mathfrak{N}}(\gamma) \to (\hat{u}, \hat{v}) \in m_{\mathfrak{N}}(beta)$ sau $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}}(\gamma) \to (\hat{u}, \hat{v}) \in m_{\mathfrak{N}}(\beta \cup \gamma)$.

  Demonstram *b*):

  $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\beta \cup \gamma)$ si $u \in m_{\mathfrak{N}([\beta]\psi) \to (\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\beta)}$ sau $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\gamma)$ si $u \in m_{\mathfrak{N}([\beta]\psi) \to v \in (\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\psi)}$ din partea a doua din ii) aplicata pentru $[\beta]\psi$.

- Cazul IV) Pentru $\alpha = \beta; \gamma$:

  Demonstram mai intai prima parte:

  Din *7.1.2.(4)* avem $[\beta][\gamma]\psi \in FL(\phi)$ si $[\gamma]\psi \in FL(\phi)$;

  $(s, t) \in m_{\mathfrak{N}}(\beta; \gamma) \Longrightarrow \exists\, u,\ (s, u) \in m_{\mathfrak{N}}(\beta)$ si $(u, t) \in m_{\mathfrak{N}}(\gamma) \Longrightarrow \exists u,\ (\hat{s}, \hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\beta)$ si $(\hat{t}, \hat{u}) \in m_{\mathfrak{N}/FL(\phi)}(\gamma) \Longrightarrow (\hat{s}, \hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\beta; \gamma)$.
  Demonstram a doua parte. Avem:

  $(\hat{s}, \hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\beta; \gamma)$ si $s \in m_{(N)}([\beta; \gamma]\psi) \Longrightarrow \exists\ u,\ (\hat{s}, \hat{u}) \in m_{\mathfrak{N}/FL(\phi)}(\beta),\ (\hat{u}, \hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\gamma)$ si $s \in m_{(N)}([\beta][\gamma]\psi) \Longrightarrow \exists\ u,\ (\hat{u}, \hat{t}) \in m_{\mathfrak{N}/FL(\phi)}(\gamma)$ si $u \in m_{(N)}([\gamma]\psi) \Longrightarrow t \in m_{\mathfrak{N}}(\psi)$.

- Cazul V) Pentru $\alpha = \alpha^*$;

  Fie $\mathfrak{N}$ o structura nonstandard Kripke si $(u, v) \in m_{\mathfrak{N}}(\alpha)$. Vrem sa aratam ca $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha^*)$ (sau, echivalent, ca $u \in E$, unde $E = \{t \in \mathfrak{N} \mid (\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha^*)\}$).

  Cum E este o reuniune de clase de echivalenta definita prin asignarea unei valori de adevar elementelor din $FL(\phi)$, atunci exista ϕ_E care defineste E in N: $E = m_{\mathfrak{N}}(\phi_E)$. ϕ_E este o disjunctie de conjunctii de formule $\phi_{\hat{t}}$, cate una pentru fiecare clasa de echivalenta $\hat{t}$ din **E**. Astfel, $\phi_{\hat{t}}$ contine fie ρ, fie $\neg\rho$ $\forall \rho \in FL(\phi)$, depinzand de valoarea de adevar asignata lui $\hat{t}$.

Cum $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha^*)$, atunci $u \in E$ si E este inchisa sub $m_{\mathfrak{N}}(\alpha)$, in sensul in care $s \in E$ si $(s,t) \in m_{\mathfrak{N}}(\alpha) \Longrightarrow t \in E$.

Se poate observa ca daca $s \in E$ si $(s,t) \in m_{\mathfrak{N}}(\alpha)$,atunci $(\hat{s},) \in m_{(N)/FL(\phi)}(\alpha)$ prin ipoteza de inductie si $(\hat{u}, \hat{t}) \in m_{\mathfrak{N}}(\alpha)$ prin definitia lui E. Rezulta ca $(s,t) \in m_{\mathfrak{N}}(\alpha) \Longrightarrow t \in E$.

Aceste lucruri nu conduc direct la faptul ca $v \in E$, dat fiind faptul ca $m_{\mathfrak{N}}(\alpha^*)$ nu reprezinta in mod obligatoriu inchiderea reflexiv tranzitiva a lui $m_{\mathfrak{N}}(\alpha)$. Cu toate acestea, datorita faptului ca $E = m_{\mathfrak{N}}(\phi_E)$ este echivalent cu $\mathfrak{N} \vDash \phi_E \rightarrow [\alpha]\psi_E$. Aplicand regula LI, avem:

$$\mathfrak{N} \vDash \phi_E \rightarrow [\alpha^*]\psi_E$$

Cum LI este echivalenta cu inductia (demonstratia a fost facuta deductiv, asadar este valida si pentru structuri nonstandard). Avem acum ca $(u,v) \in m_{\mathfrak{N}}(\alpha*)$ (din ipoteza) si $u \in E$, asadar $v \in E$. Din definitia lui E reiese ca $(\hat{u}, \hat{v}) \in m_{\mathfrak{N}/FL(\phi)}(\alpha^*)$.

7.3. Teorema de completitudine

Lema 7.3.1 Fie Σ o multime de formule din LDP. Atunci:

1. Σ este consistenta daca si numai daca $Sigma \cup \{\psi\}$ este consistenta sau $Sigma \cup \{\neg\psi\}$ este consistenta.
2. Daca Σ este consistenta, atunci Σ este inclusa intr-o multime maximala consistenta.

   In plus, daca Σ *este* o multime maximala consistenta, avem:

3. Σ contine toate teoremele din LDP.
4. Daca $\phi \in \Sigma$ si $\phi \rightarrow \psi \in \Sigma$, atunci $\psi \in \Sigma$.
5. $\phi \vee \psi \in \Sigma$ daca si numai daca $\phi \in \Sigma$ sau $\psi \in \Sigma$.
6. $\phi \wedge \psi \in \Sigma$ daca si numai daca $\phi \in \Sigma$ si $\psi \in \Sigma$.
7. $\phi \in \Sigma$ daca si numai daca $\neg\phi \notin \Sigma$.
8. $\mathbf{0} \notin \Sigma$.

Lema 7.3.2. Fie Σ si Γ doua multimi maximale consistente de formule si fie α un program. Atunci:

1. a) Daca $\phi \in \Gamma$, atunci $< \alpha > \phi \in \Gamma$, $\forall\phi$.
2. b) Daca $[\alpha]\phi \in \Sigma$, atunci $\phi \in \Gamma$, $\forall\phi$.

Dem:

1. a) $\rightarrow$ b)
   $[\alpha]\phi \in \Sigma \Longrightarrow < \alpha > \neg\phi \notin \Sigma \Longrightarrow \neg\phi \notin \Gamma \Longrightarrow \phi \in \Gamma$
2. b) $\rightarrow$ a)

   $\phi \in \Gamma \Longrightarrow \neg\phi \notin \Gamma \Longrightarrow \alpha]\neg\phi \notin \Gamma \Longrightarrow < \alpha > \phi \in \Sigma$

Construim o structura nonstandard Kripke $\mathfrak{N} = (N, m_{\mathfrak{N}})$. Elementele lui N vor fi multimi maximale consistente si le vom nume *stari*.

Obs: Daca $s \in N$, atunci $\phi \in s$.

Fie $\mathfrak{N} = (N, m_{\mathfrak{N}})$ definit prin:

1. $N =$ s — s multime maximala consistenta;
2. $m_{\mathfrak{N}}(\psi) = \{s|\psi \in s\}$;
3. $m_{\mathfrak{N}}(\alpha) = \{(s,t) \mid \forall\psi,\ daca\ \psi \in t,\ atunci\ \ < \alpha > \psi \in s\} = \{(s,t) \mid \forall\psi, daca < \alpha > \psi \in s,\ atunci\ \psi \in t\}$.

Obs: Definitiile date pentru $m_{\mathfrak{N}}(\psi)$ si $m_{\mathfrak{N}}(\psi)$ sunt valide pentru toate formulele si programele, nu doar pentru cele atomice.

Lema 7.3.3.

- $m_{\mathfrak{N}}(\psi \rightarrow \phi = (N - m_{\mathfrak{N}}(\psi)) \cup m_{\mathfrak{N}}(\phi)$
- $m_{\mathfrak{N}}(0) = \varnothing$
- $m_{\mathfrak{N}}([\alpha]\psi) = N - m_{\mathfrak{N}}(\alpha) \circ (N - m_{\mathfrak{N}}(\psi))$

Lema 7.3.4.

- $m_{\mathfrak{N}}(\alpha \cup \beta) = m_{\mathfrak{N}}(\alpha) \cup m_{\mathfrak{N}}(\beta)$
- $m_{\mathfrak{N}}(\alpha; \beta) = m_{\mathfrak{N}}(\alpha) \circ m_{\mathfrak{N}}(\beta)$
- $m_{\mathfrak{N}}(\phi?) = \{(s,s) \mid s \in m_{\mathfrak{N}}(\phi)\}$
- $m_{\mathfrak{N}}(\alpha^-) = m_{\mathfrak{N}}(\alpha)^-$

Dem: ii)

" $\supseteq$ "

$(u,v) \in m_{\mathfrak{N}}(\alpha) \circ m_{\mathfrak{N}}(\beta) \Longleftrightarrow \exists\ w\ a.i.\ (w,u) \in m_{\mathfrak{N}}(\alpha)\ si\ (w,v) \in m_{\mathfrak{N}}(\beta) \Longleftrightarrow \exists\ w\ a.i.\ \forall\psi \in v,\ < \beta > \psi \in w\ si\ \forall\psi \in w,\ < \alpha > \psi \in u \Longrightarrow \forall\psi \in v,\ < \alpha >< \beta > \in u \Longleftrightarrow \forall\psi \in v,\ < \alpha; \beta > \psi \in u \Longleftrightarrow (u,v) \in m_{\mathfrak{N}}(\alpha; \beta)$.

" $\subseteq$ "

Fie $(u,v) \in m_{\mathfrak{N}}(\alpha; \beta)$. Presupunem ca multimea $\{\phi|[\alpha]\phi \in u\} \cup \{< \beta > \psi|\psi \in v\}$. Fie:

1. $\{\phi_1, ..., \phi_k\} \subseteq \{\phi | [\alpha]\phi \in u\}$ si
2. $\{< \beta > \psi_1, ..., < \beta > \psi_l\} \subseteq \{< \beta > \psi | \psi \in v\}$

doua multimi finite arbitrare a.i. $\psi = \psi_1 \wedge \psi_2 \wedge ... \wedge \psi_k$ si $\phi = \phi_1 \wedge \phi_2 \wedge ... \wedge \phi_l$.

Din *Lema 7.3.1.* si din faptul ca $(u, v) \in m_{\mathfrak{N}}(\alpha; \beta)$ avem ca $< \alpha; \beta > \psi \in u$.
Cum $[\alpha]\phi \leftrightarrow [\alpha]\phi_1 \wedge ...[\alpha]\phi_k$ este o teorema a LDP si cum $[\alpha]\phi_1 \wedge ...[\alpha]\phi_k \in u$, avem ca $[\alpha]\phi \in u$. Stim ca $< \alpha(\phi \wedge < \beta > \psi) \in u >$ asadar, daca aplicam *GEN*, $\phi \wedge < \beta > \psi$ este consitenta.
Dar:

$$\vdash \phi \wedge < \beta > \psi \to \psi_1 \wedge \psi_2 \wedge ... \wedge \psi_k \phi_1 \wedge \phi_2 \wedge ... \wedge \phi_l$$

,asadar partea din dreapta a implicatei este consistenta. Cum aceasta este o conjunctie a unei submultimi finite arbitrare a $\{\phi | [\alpha]\phi \in u\} \cup \{< \beta > \psi | \psi \in v\}$, care este consistent, iar astfel se extinde la o mutlime maximala consistenta w.

Din definitia $m_{\mathfrak{N}}(\alpha)$ si a $m_{\mathfrak{N}}(\beta)$ avem $(u, v) \in m_{\mathfrak{N}}(\alpha)$ si $(w, v) \in m_{\mathfrak{N}}(\beta)$, deci $(u, v) \in m_{\mathfrak{N}}(\alpha) \circ m_{\mathfrak{N}}(\beta)$.

Teorema 7.3.5. $\mathfrak{N}$ este o structura nonstandard Kripke.
Dem: Lemele 7.3.3. si 7.3.4. arata ca operatori $\to$, 0, [ ], ;, $\cup$, $^-$ si ? se comporta in $\mathfrak{N}$ la fel ca in structurile standard. Trebuie sa mai demonstram urmatoarele doua proprietati:

1. $[\alpha^*]\psi \leftrightarrow \psi \wedge [\alpha; \alpha^*]\psi$
2. $[\alpha^*]\psi \leftrightarrow \psi \wedge [\alpha^*](\psi \to \alpha]\psi)$

Cum cele doua sunt proprietati ale LDP, Lema 7.3.1. ne spune ca trebuie sa fie adevarate in orice multime maximala consistenta. Astfel (N) satisface conditiile structurilor nonstandard Kripke:

1. $m_{\mathfrak{N}}([\alpha^*]\psi) = m_{\mathfrak{N}}(\psi \wedge [\alpha; \alpha^*]\psi)$
2. $m_{\mathfrak{N}}([\alpha^*]\psi) = m_{\mathfrak{N}}(\psi \wedge [\alpha^*](\psi \to [\alpha]\psi))$

Teorema 7.3.5. Daca $\vDash \psi$, atunci $\vdash \psi$.

Dem: Vrem sa aratam ca daca ψ este consistent, atunci el este satisfacut intr-o structura standard Kripke.

Daca ψ este consistenta, atunci din *Lema 7.3.1.* reiese ca este continut intr-o multime maximala consistenta u, care este o *stare* a structurii nonstandard Kripke $\mathfrak{N}$ construite anterior. Din *Lema filtrarii* avem ca ψ este satisfacut in starea $\hat{u}$ in structura Kripke $\mathfrak{N}/FL(\psi)$, care este, prin defitie, o structura standard Kripke.

8. Bibliografie

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