① 31.10.23

SEMINAR S-132

$$f(e_0 \circ e_0) = f(e_0) \Delta f(e_0) \Longleftrightarrow \Delta f(e_0)'^{\Delta}$$

$$\| \quad f(e_0)$$

$$f(e_0) \Delta f(e_0)'^{\Delta} = \left(f(e_0) \Delta f(e_0)\right) \Delta f(e_0)'^{\Delta}$$

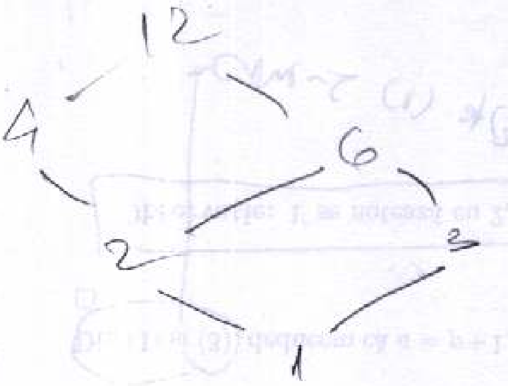
$$\| \quad e_\Delta \qquad\qquad \| \quad f(e_0) \Delta \left(f(e_0) \Delta f(e_0)'^{\Delta}\right)$$

$$\| \quad f(e_0) \Delta e_\Delta = f(e_0)$$

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Δ	α	β	γ
α	α	β	γ
β	β	γ	α
γ	γ	α	β

$$\varphi(1 \oplus 2) = \varphi(0) = \alpha = \beta \Delta \gamma = \varphi(1) \Delta \varphi(2)$$



Fie $f_k: \mathbb{Z} \to \mathbb{Z}$, $f_k(x) = kx$, ($k \in \mathbb{Z}$ fixat)

Fie $a, b \in \mathbb{Z}$

Atunci $f(a+b) = k(a+b) = \overset{ka+kb}{} = f(a)+f(b)$

Ca urmare, f_k e morfism de grupuri.

Fie $f \in End_{Grp}(\mathbb{Z})$.

Notăm $\alpha = f(1)$

Atunci: $f(2) = f(1+1) = f(1)+f(1) = 2\alpha$.

$f(3) = f(2+1) = f(2) + f(1) = 3\alpha$

inductiv (~~temat~~) $f(n) = n\alpha \quad \forall n \in \mathbb{N}$

Fie $n \in \mathbb{Z}_-$. Atunci

$$\boxed{0 = f(0) = f(n+(-n)) = f(n) + \underbrace{f(-n)}_{-\alpha n} = f(n) - \alpha n}$$

$\Downarrow$

$f(n) = \alpha \cdot n$

Concluzie: $\forall n \in \mathbb{Z} \quad f(n) = \alpha \cdot n$

③

Ca urmare,

$End_{Gp}(\mathbb{Z}) = \{f_\alpha : \alpha \in \mathbb{Z}\}$

unde, pt $\alpha \in \mathbb{Z}$ fixat,

$f_\alpha : \mathbb{Z} \to \mathbb{Z}$, $f_\alpha(n) = \alpha \cdot n$

Pe $M \overset{not}{=} End_{Gp}(\mathbb{Z})$ considerăm operația de compunere.

$(M, \circ)$ e monoid comutativ

Dem: Fie $f, g \in M$.

Fie $m, n \in \mathbb{Z}$

Atunci $(f \circ g)(m+n) = f(g(m+n)) \underset{g \in End(\mathbb{Z})}{=} f(g(m) + g(n)) \underset{f \in End(\mathbb{Z})}{=}$

$f(g(m)) + f(g(n)) = (f \circ g)(m) + (f \circ g)(n)$.

Deci $f \circ g \in End(\mathbb{Z})$.

Ca urmare, operația $\circ$ e corect definită pe M.

Fie $f, g, h \in M$. Fie $n \in \mathbb{Z}$. Atunci

$[(f \circ g) \circ h](n) = (f \circ g)(h(n)) = f(g(h(n)))$

$[f \circ (g \circ h)](n) = f((g \circ h)(n)) = f(g(h(n)))$ $\Big\} \Rightarrow$

$(f \circ g) \circ h = f \circ (g \circ h)$.

Deci, $\circ$ e asociativă

Fie $f \in \mathcal{U}$,

Atunci $f \circ 1_{\mathbb{Z}} = f$

$1_{\mathbb{Z}} \circ f = f$,

deci $1_{\mathbb{Z}}$ e element neutru pt $\circ$.

Ca urmare, $(\mathcal{U}, \circ)$ e monoid comutativ.

Considerăm $\Phi : \mathcal{U} \to \mathbb{Z}$,

$\Phi(f) = f(1)$.

Atunci:

Fie $f, g \in \mathcal{U}$. Obținem:

$$\left.\begin{array}{l} \Phi(f \circ g) = (f \circ g)(1) = f(g(1)) = f(g(1) \cdot 1) = g(1) \cdot f(1) \\ \Phi(f) \cdot \Phi(g) = f(1) \cdot g(1) \end{array}\right\} \Rightarrow$$

$\Rightarrow \Phi(f \circ g) = \Phi(f) \cdot \Phi(g)$. (1)

$\Phi(1_{\mathbb{Z}}) = 1_{\mathbb{Z}}(1) = 1$. (2)

(1), (2) $\Rightarrow$ Φ e morfism de monoizi. (10)

Considerăm $\Psi : \mathbb{Z} \to \mathcal{U}$,

$\Psi(k)(n) = k \cdot n$.

Fie $k, m, n \in \mathbb{Z}$. Atunci

$\psi(k)(m+n) = k\cdot(m+n) = km+kn =$

$= \psi(k)(m) + \psi(k)(n)$,

deci $\psi(k) \in \mathrm{End}_{Grp}(\mathbb{Z}) = \mathcal{M}$.

Ca urmare, ψ e corect definită.

Fie $k, l \in \mathbb{Z}$. Fie $n \in \mathbb{Z}$. Atunci:

$\psi(kl)(n) = (kl)\cdot n$

$(\psi(k)\circ\psi(l))(n) = \psi(k)(\psi(l)(n)) = \psi(k)(ln) = k(ln)$

$\xrightarrow[\text{de pe } \mathbb{Z}]{\text{asoc.}}$ $\psi(kl)(n) = (\psi(k)\circ\psi(l))(n)$.

Dar n a fost ales arbitrar, iar fct. au același domeniu și aceleași codomenii, deci obținem

$\psi(kl) = \psi(k)\circ\psi(l)$. (3)

În plus,

$\psi(1)(n) = 1\cdot n = n$, deci $\psi(1) = 1_{\mathbb{Z}}$ (4)

(3), (4) $\Rightarrow$ ψ e morfism de monoizi (11)

$\phi\circ\psi: \mathbb{Z}\to\mathbb{Z}$,

$\phi\circ\psi(k) = \phi(\psi(k)) = \phi($acea fct. $\mathbb{Z}\to\mathbb{Z}$ ce duce fiecare n în $kn) =$

$=($acea fct. ... ce duce n în $kn)(1) = k$.

Deci, $\phi\circ\psi = 1_{\mathbb{Z}}$ (5)

$\psi \circ \phi : \mathcal{M} \to \mathcal{M}$,

$(\psi \circ \phi)(f) = \psi(\phi(f)) = \psi(f(1))$.

Deci, pt $n \in \mathbb{Z}$,

$[(\psi \circ \phi)(f)](n) = [\psi(f(1))](n) =$

$= f(1) \cdot n \xlongequal{f \in End(\mathbb{Z})} f(n)$;

Ca urmare, $(\psi \circ \phi)(f) = f$.

Deci, $\psi \circ \phi = 1_{\mathcal{M}}$ (6)

(5), (6) $\Rightarrow \psi = \phi^{-1}$. (12).

Din (10), (11) și (12) obținem că ϕ e izomorfism de monoizi.

Ca urmare, $(\mathcal{M}, \circ) \simeq (\mathbb{Z}, \cdot)$.