① SEMINAR 11_132 12.11.23

$\hat{a} \in U(\mathbb{Z}_n) \iff \exists b \in \mathbb{Z} \quad \hat{a}\cdot\hat{b} = \hat{1} \iff$

$(\exists b \in \mathbb{Z} \quad n \mid ab-1) \iff (\exists b, s \in \mathbb{Z} \quad ab-1 = ns)$

$\iff (\exists b, s \in \mathbb{Z} \quad ab - ns = 1) \iff (a, n) = 1$

Deci:

$U(\mathbb{Z}_n) = \{\hat{a} \in \mathbb{Z}_n : (a, n) = 1\}$.

Deci, $U(\mathbb{Z}_{18}) = \{\hat{a} \in \mathbb{Z}_{18} : (a, 18) = 1\} =$

$= \{\hat{1}, \hat{5}, \hat{7}, \hat{11}, \hat{13}, \hat{17}\}$.

$\hat{a} \in \mathcal{D}(\mathbb{Z}_n) \iff (\exists b \in \mathbb{Z} \quad (\hat{b} \neq \hat{0} \wedge \hat{a}\hat{b} = \hat{0})) \iff$

$\iff (\exists b \in \mathbb{Z} \quad (n \nmid b \wedge n \mid ab)) \Rightarrow (n, a) \neq 1$

$\Leftarrow$: $b \overset{\text{not}}{=} \dfrac{n}{(a, n)}$

Deci,

$\mathcal{D}(\mathbb{Z}_n) = \{\hat{a} \in \mathbb{Z}_n : (a, n) \neq 1\}$.

d.ex, $\mathcal{D}(\mathbb{Z}_{18}) = \{\hat{0}, \hat{2}, \hat{3}, \hat{4}, \hat{6}, \hat{8}, \hat{9}, \hat{10}, \hat{12}, \hat{14}, \hat{15}, \hat{16}\}$

$\hat{a} \in N(\mathbb{Z}_n) \iff \exists k \in \mathbb{N}^* \quad \hat{a}^k = \hat{0} \iff$

$\exists k \in \mathbb{N}^* \quad n \mid a^k \quad (1)$

② Fie $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ descompunerea standard a lui n. Atunci

$(1) \iff \exists\, k \in \mathbb{N}^* \quad p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} \mid a^k \iff$

$p_1 p_2 \dots p_r \mid a.$

Ca urmare,

$$N(\mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}}) = \widehat{p_1 p_2 \dots p_r}\, \mathbb{Z}_{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}}.$$

d.ex, $N(\mathbb{Z}_{18}) = \hat{6} \cdot \mathbb{Z}_{18} = \{\hat{0}, \hat{6}, \widehat{12}\}.$

$N(\mathbb{Z}_{400}) = \widehat{2 \cdot 5}\, \mathbb{Z}_{400} = \widehat{10}\, \mathbb{Z}_{400}$

$N(\mathbb{Z}_{30}) = \widehat{2 \cdot 3 \cdot 5} \cdot \mathbb{Z}_{30} = \widehat{30}\, \mathbb{Z}_{30} = \{\hat{0}\}.$

deci, $\mathbb{Z}_{30}$ este inel redus.

$\boxed{\text{Curs}}$ $\boxed{\text{T}}$ $\mathbb{Z}_m \times \mathbb{Z}_n \underset{Rng}{\cong} \mathbb{Z}_{mn} \iff (m,n) = 1.$

$$(\bar{a}, \tilde{a}) \longrightarrow \hat{a}$$

Dacă $R_{1,2}$ sunt inele

• $Idemp(R_1 \times R_2) = Idemp(R_1) \times Idemp(R_2)$

dem: Fie $a \overset{not}{=} (a_1, a_2) \in R_1 \times R_2$. Atunci:

$a \in Idemp(R_1 \times R_2) \iff a^2 = a \iff$

$(a_1, a_2)^2 = (a_1, a_2) \iff (a_1^2, a_2^2) = (a_1, a_2) \iff$

$(a_1^2 = a_1 \wedge a_2^2 = a_2) \iff a \in Idemp(R_1) \times Idemp(R_2)$ □

Cu demonstrații similare se probează: ③

• $\boxed{N(R_1 \times R_2) = N(R_1) \times N(R_2)}$ TD

și

• Dacă $R_{1,2}$ sunt unitare,

$U(R_1 \times R_2) = U(R_1) \times U(R_2)$

$\boxed{PS_1}$ Cum arată $\mathcal{I}(R_1 \times R_2)$?

$\boxed{PS_2}$ E adevărat că $N\left(\prod_{\alpha \in A} R_\alpha\right) = \prod_{\alpha \in A} N(R_\alpha)$?

($A \neq \emptyset$; R_α inele comutative pt orice $\alpha \in A$)

Fie p prim și $\alpha \in \mathbb{N}^*$. Atunci

$\mathrm{Idemp}(\mathbb{Z}_{p^\alpha}) = \{\hat{0}, \hat{1}\}$

Dem: Fie $\hat{a} \in \mathbb{Z}_{p^\alpha}$.

$\hat{a} \in \mathrm{Idemp}(\mathbb{Z}_{p^\alpha}) \Leftrightarrow \hat{a}^2 = \hat{a} \Leftrightarrow \widehat{a^2} = \hat{a} \Leftrightarrow$

$p^\alpha \mid a^2 - a \Leftrightarrow p^\alpha \mid a(a-1)$ (1)

Dar p nu poate divide simultan a și $a-1$ (căci ar rezulta $p \mid a-(a-1) = 1$, $\not\#$), deci

(1) $\Leftrightarrow (p^\alpha \mid a \vee p^\alpha \mid a-1) \Leftrightarrow (\hat{a} = \hat{0} \vee \hat{a} = \hat{1}) \Leftrightarrow$

$\hat{a} \in \{\hat{0}, \hat{1}\}$ □

• Determinați $\mathrm{Idemp}(\mathbb{Z}_{360})$.

Sol: $\mathbb{Z}_{360} \xrightarrow{\sim} \mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5$

$\hat{a} \longmapsto (\bar{a}, \bar{\bar{a}}, \tilde{a})$

Elem. idempotente: 1)

$\hat{0}$	$(\bar{0}, \bar{\bar{0}}, \tilde{0})$
$\widehat{216}$	$(\bar{0}, \bar{\bar{0}}, \tilde{1})$
$\widehat{280}$	$(\bar{0}, \bar{\bar{1}}, \tilde{0})$
$\widehat{136}$	$(\bar{0}, \bar{\bar{1}}, \tilde{1})$
$\widehat{225}$	$(\bar{1}, \bar{\bar{0}}, \tilde{0})$
$\widehat{81}$	$(\bar{1}, \bar{\bar{0}}, \tilde{1})$
$\widehat{145}$	$(\bar{1}, \bar{\bar{1}}, \tilde{0})$
$\hat{1}$	$(\bar{1}, \bar{\bar{1}}, \tilde{1})$

1) Am aplicat:

- Idemp $(R_1 \times R_2 \times R_3) = $ Idemp $(R_1) \times$ Idemp $(R_2) \times$ Idemp (R_3)
- Idemp $(\mathbb{Z}_{p^\alpha}) = \{\hat{0}, \hat{1}\}$ pt orice $p \in \mathbb{N}$ prim $\forall$ orice $\alpha \in \mathbb{N}^*$.

Întrucât statutul de idempotent al unui element se conservă prin izomorfisme de inele, elementele idempotente din $\mathbb{Z}_{360}$ sunt cele de pe coloana din stânga a tabelului de mai sus.

[TD] Determinați Idemp $(\mathbb{Z}_{720})$