Temă 10. 4. 2024.

Tema 5

1) $f : \mathbb{R}^3 \to \mathbb{R}^3$ $f(x,y,z) = (x+y-2z,\ 2x+y+z,\ 3x+2y-z)$

dem $f$ e apl. lin.

fie $v_1 = (x_1, y_1, z_1),\ v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$

$$f(v_1+v_2) = f(x_1+x_2,\ y_1+y_2,\ z_1+z_2) =$$

$$= (x_1+x_2+y_1+y_2-2z_1-2z_2,\ 2x_1+2x_2+y_1+y_2+z_1+z_2,$$

$$3x_1+3x_2+2y_1+2y_2-z_1-z_2) =$$

$$= (x_1+y_1-2z_1,\ 2x_1+y_1+z_1,\ 3x_1+2y_1-z_1) +$$

$$+ (x_2+y_2-2z_2,\ 2x_2+y_2+z_2,\ 3x_2+2y_2-z_2) =$$

$$= f(v_1) + f(v_2)$$

fie $v = (x, y, z) \in \mathbb{R}^3$ fie $\alpha \in \mathbb{R}$

$f(\alpha v) = f(\alpha x, \alpha y, \alpha z) =$

$= \cancel{(\alpha x + \alpha y - \alpha 2 z, 2\alpha x}$

$= (\alpha x + \alpha y - 2\alpha z,\ 2\alpha x + \alpha y + \alpha z,\ 3\alpha x + 2\alpha y - \alpha z) =$

$= \alpha (x + y - 2z,\ 2x + y + z,\ 3x + 2y - z) =$

$= \alpha f(v)$

$\Rightarrow f$ e apl lin

2) $V_1 = \{ (x, y, 0) \mid x, y \in \mathbb{R} \}$

$V_2 = \{ (t, 0, t) \mid t \in \mathbb{R} \}$

dem că $V_1, V_2$ : sp vect

fie $\alpha, \beta \in \mathbb{R}$

fie $v_1, v_2 \in V_1$ $\quad v_1 = (x_1, y_1, 0)$

$v_2 = (x_2, y_2, 0)$

$\alpha v_1 + \beta v_2 = \alpha (x_1, y_1, 0) + \beta (x_2, y_2, 0) =$

$= (\underbrace{\alpha x_1 + \beta x_2}_{x}, \underbrace{\alpha y_1 + \alpha y_2}_{y}, 0) \in V_1$

~~$v_1 + v_2 = (x_1, y_1, 0) + (x_2, y_2, 0) =$~~

$\Rightarrow V_1$ e sp vect

fie $\alpha, \beta \in \mathbb{R}$

fie $v_1, v_2 \in V_2$ $\quad v_1 = (x_1, 0, x_1)$

$v_2 = (x_2, 0, x_2)$

$\alpha v_1 + \beta v_2 = \alpha (x_1, 0, x_1) + \beta (x_2, 0, x_2) =$

$= (\alpha x_1 + \beta x_2,\ 0,\ \alpha x_1 + \beta x_2) \in V_2$

$\Rightarrow V_2$ e sp vect

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17.4.2023.

Temă

(Tema 5) 3)a) $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; \begin{cases} a-b+c-d=0 \\ a+2b-c+3d=0 \end{cases} \right\}$

Dem că $V \subset M_2(\mathbb{R})$ ssp

fie $\alpha, \beta \in \mathbb{R}$

fie $v_1, v_2 \in V$ $\quad v_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$\alpha v_1 + \beta v_2 = \alpha \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \beta \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \overbrace{\alpha a_1 + \beta a_2}^{a} & \overbrace{\alpha b_1 + \beta b_2}^{b} \\ \underbrace{\alpha c_1 + \beta c_2}_{c} & \underbrace{\alpha d_1 + \beta d_2}_{d} \end{pmatrix}$$

$$a-b+c-d = \alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2 + \alpha c_1 + \beta c_2 -$$

$$- \alpha d_1 - \beta d_2 = \alpha(\underbrace{a_1 - b_1 + c_1 - d_1}_{0}) + \beta(\underbrace{a_2 - b_2 + c_2 - d_2}_{0}) = 0$$

3) b) $\begin{cases} a - b + c - d = 0 \\ a + 2b - c + 3d = 0 \end{cases}$

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$$\bar{A} = \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & -1 & 3 & 0 \end{array}\right) \longrightarrow \quad \begin{array}{l} rg\, A = 2 \\ rg\, \bar{A} = 2 \end{array}$$

$$\xrightarrow{L_2 \leftarrow L_2 - L_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 3 & -2 & 4 & 0 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 \cdot \frac{1}{3}}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} & 0 \end{array}\right)$$

~~x, y nec pp~~

a, b nec pp

c, d nec sec   not $c = \alpha$, $d = \beta$

$$b = \frac{2}{3}\alpha - \frac{4}{3}\beta$$

$$a = \frac{2}{3}\alpha - \frac{4}{3}\beta - \alpha + \beta = \frac{-1}{3}\alpha + \frac{-1}{3}\beta =$$

$$= -\frac{1}{3}(\alpha + \beta)$$

$$S = \left\{ \left( \frac{-\alpha-\beta}{3}, \frac{2\alpha - 4\beta}{3}, \alpha, \beta \right) \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}$$

$a+2b-c+3d = \alpha a_1+\beta a_2+2\alpha b_1+2\beta b_2-\alpha c_1-\alpha c_2+$
$+3\alpha d_1+3\beta d_2 = \alpha(\underbrace{a_1+2b_1-c_1+3d_1}_{0})+\beta(\underbrace{a_2+2b_2-c_2+3d_2}_{0})$
$=0$

$\Rightarrow \alpha v_1+\beta v_2 \in V \Rightarrow V$ e sp vect

$V=\{(-\frac{1}{3}\alpha-\frac{1}{3}\beta, \frac{2}{3}\alpha-\frac{4}{3}\beta, \alpha, \beta) \mid \alpha,\beta\in\mathbb{R}\}=$
$=\{(-\frac{1}{3}\alpha, \frac{2}{3}\alpha, \alpha, 0)+(-\frac{1}{3}\beta, -\frac{4}{3}\beta, 0, \beta) \mid \alpha,\beta\in\mathbb{R}\}=$
$=\{\alpha\underbrace{(-\frac{1}{3}, \frac{2}{3}, 1, 0)}_{v_1}+\beta\underbrace{(-\frac{1}{3}, -\frac{4}{3}, 0, 1)}_{v_2} \mid \alpha,\beta\in\mathbb{R}\}$

$\Rightarrow B=\{v_1, v_2\}$ bază $\Rightarrow \dim_{\mathbb{R}^2} V=2$

c) completați baza B până la o bază în sp vect ambiental $M_2(\mathbb{R})$

$B_0=\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\}$ baza can. a lui $M_2(\mathbb{R})$

$\Rightarrow M_2(\mathbb{R}) \equiv \mathbb{R}^4$
$e_1, e_2, e_3, e_4$
$=(1,0,0,0)$

$B=\{v_1, v_2, x_1, x_2\}$ baza de completat

$v_1=(-\frac{1}{3}, \frac{2}{3}, 1, 0)$ $v_2=(-\frac{1}{3}, -\frac{4}{3}, 0, 1)$

$\det B = \begin{vmatrix} -\frac{1}{3} & -\frac{1}{3} & a_1 & a_2 \\ \frac{2}{3} & -\frac{4}{3} & b_1 & b_2 \\ 1 & 0 & c_1 & c_2 \\ 0 & 1 & d_1 & d_2 \end{vmatrix}$
$v_1 \quad v_2 \quad e_3 \quad e_4$

aleg să completez cu $e_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ și $e_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$\begin{vmatrix} -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -\frac{4}{3} & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{4}{3} \end{vmatrix} =$

$= \frac{4}{9}+\frac{2}{9}=\frac{6}{9}=\frac{2}{3}\neq 0 \Rightarrow B=\{\begin{pmatrix}-\frac{1}{3}&\frac{2}{3}\\1&0\end{pmatrix}, \begin{pmatrix}-\frac{1}{3}&-\frac{4}{3}\\0&1\end{pmatrix},$
$\begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\}$

Temă 4) $V = \{ P(x) \in \mathbb{R}_2[X] \mid a_0 + a_1 + a_2 = 0 \}$

$P(x) = a_0 + a_1 x + a_2 x^2$

a) dat. că $V \subset \mathbb{R}_2[X]$ sap vect

fie $\varphi, \gamma \in \mathbb{R}$

fie $P_1(x), P_2(x) \in V$

$P_1(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$

$P_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2$

$\varphi P_1(x) + \gamma P_2(x) = \varphi\alpha_0 + \varphi\alpha_1 x + \varphi\alpha_2 x^2 +$

$+ \gamma\beta_0 + \gamma\beta_1 x + \gamma\beta_2 x^2 =$

$= \varphi\alpha_0 + \gamma\beta_0 + (\varphi\alpha_1 + \gamma\beta_1)x + (\varphi\alpha_2 + \gamma\beta_2)x^2 \in$

$\in V \Leftrightarrow (\varphi\alpha_0 + \gamma\beta_0) + (\varphi\alpha_1 + \gamma\beta_1) + (\varphi\alpha_2 + \gamma\beta_2) =$

$= \varphi\alpha_0 + \varphi\alpha_1 + \varphi\alpha_2 + \gamma\beta_0 + \gamma\beta_1 + \gamma\beta_2 =$

$= \varphi(\alpha_0 + \alpha_1 + \alpha_2) + \gamma(\beta_0 + \beta_1 + \beta_2) =$

$= \varphi \cdot 0 + \gamma \cdot 0 = 0$

$\Rightarrow \varphi P_1(x) + \gamma P_2(x) \in V \Rightarrow V$ e sap vect

b) det o bază; $\dim_{\mathbb{R}} V = ?$

$a_0 + a_1 + a_2 = 0$

not $a_1 = \alpha$, $a_2 = \beta$

$$\begin{cases} a_0 = -\alpha - \beta \\ a_1 = \alpha \\ a_2 = \beta \end{cases}$$

$S = \{(-\alpha-\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\} =$

$= \{(-\alpha, \alpha, 0) + (-\beta, 0, \beta) \mid \alpha, \beta \in \mathbb{R}\} =$

$= \{\alpha \underbrace{(-1, 1, 0)}_{v_1} + \beta \underbrace{(-1, 0, 1)}_{v_2} \mid \alpha, \beta \in \mathbb{R}\}$

$B = \{v_1, v_2\}$ bază $\Rightarrow \dim_{\mathbb{R}} V = 2$

c) completați baza $B$ până la o bază în spațiul ambient $\mathbb{R}_2[X]$

$B_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$ baza can a lui $\mathbb{R}^2[X]$

$B = \{v_1, v_2, e\}$ baza de completat

$v_1 = (-1, 1, 0)$

$v_2 = (-1, 0, 1)$ alg $e = 001$

$e \in B_0$

$$\det B = \begin{vmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow B = \{(-1,1,0), (-1,0,1), (0,0,1)\}$

5) fie $V_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

$V_2 = \{(u, 0, v) \mid u, v \in \mathbb{R}\}$

a) dem. că $V_2 \subset \mathbb{R}^3$ și $\dim_{\mathbb{R}} V_2 = ?$

b) dem. că $V_1 + V_2 = \mathbb{R}^3$

c) e valabil și pt $V_1 \oplus V_2 = \mathbb{R}^3$ ?

a) fie $\alpha, \beta \in \mathbb{R}$
fie $v_1 = (a_1, 0, b_1),\ v_2 = (a_2, 0, b_2) \in V_2$

$\alpha v_1 + \beta v_2 = (\alpha a_1, 0, \alpha b_1) + (\beta a_2, 0, \beta b_2) =$

$= (\underbrace{\alpha a_1 + \beta a_2}_{\in \mathbb{R}}, 0, \underbrace{\alpha b_1 + \beta b_2}_{\in \mathbb{R}})$ $\Rightarrow \alpha v_1 + \beta v_2 \in V_2$

$\alpha, \beta \in \mathbb{R}$
$a_1, a_2, b_1, b_2 \in \mathbb{R}$

$\Rightarrow V_2 \subset \mathbb{R}^3$ spvect

$V_2 = \{ u(1,0,0) + v(0,0,1) \mid u, v \in \mathbb{R} \} \Rightarrow$

$\Rightarrow B = \{ (1,0,0), (0,0,1) \} \Rightarrow \dim_{\mathbb{R}} V_2 = 2$

~~b) $\forall z \in \mathbb{R}^3 \ \exists! \ z_1 \in V_1, z_2 \in V_2$ a.i.~~

~~$z = z_1 + z_2$~~ ~~$z = (a, b, c)$~~

~~$z_1 = (x, y, 0)$~~

~~$z_2 = (u, 0, v)$~~

~~$\begin{cases} a = x + u \\ b = y \\ c = v \end{cases} \Rightarrow z = (x+u, y, v) \in \mathbb{R}^3$~~

b) fie $f \in \mathbb{R}^3$, $f = (a, b, c)$
fie $f_1 = (x, y, 0) \in V_1$, $f_2 = (u, 0, v) \in V_2$

$f = f_1 + f_2 \Leftrightarrow \begin{cases} a = x + u \\ y = b \\ v = c \end{cases}$

$\Rightarrow f \in \{ (x+u, y, v) \mid x, u, y, v \in \mathbb{R} \} \Rightarrow$

$\forall \in \mathbb{R}^3 \Rightarrow V_1 + V_2 = \mathbb{R}^3$

c) $V_1 \oplus V_2 = \mathbb{R}^3 \Leftrightarrow V_1 + V_2 = \mathbb{R}^3 \wedge V_1 \cap V_2 = \{0_{\mathbb{R}^3}\}$

$V_1 \cap V_2 = \{\alpha(1,0,0) \mid \alpha \in \mathbb{R}\} \neq \{(0,0,0)\}$

$\Rightarrow V_1 \oplus V_2 \neq \mathbb{R}^3$

6) fie $V_1 = \{A \in \mathcal{M}_n(\mathbb{R}) \mid \operatorname{Tr} A = 0\}$

$V_2 = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = \lambda I_n, \lambda \in \mathbb{R}\}$

a) dem că $V_1, V_2$ sp. vect

b) dem că $V_1 \oplus V_2 = \mathcal{M}_n(\mathbb{R})$

c) verif th. dim în acest caz

~~a) fie $\alpha, \beta \in \mathbb{R}$~~

~~fie $A_1 \in V_1$, $A_2 \in V_2$~~

$$\cancel{\alpha A_1 + \beta A_2 = \begin{pmatrix} \alpha a_{11} + \beta\lambda & & \alpha \text{. elem lui } A \\ & \alpha a_{22} + \beta\lambda & \\ \alpha \text{. elem. lui } A & \ddots & \alpha a_{nn} + \beta\lambda \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}) \Rightarrow V}$$

a) fie $\alpha, \beta \in \mathbb{R}$ fie $A, B \in V_1$

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{nn} \end{pmatrix}$$

$a_{11} + \dots + a_{nn} = 0 \qquad b_{11} + \dots + b_{nn} = 0$

$$\alpha A + \beta B = \begin{pmatrix} \alpha a_{11} + \beta b_{11} & & \\ & \ddots & \\ & & \alpha a_{nn} + \beta b_{nn} \end{pmatrix}$$

$$\left.\begin{array}{l}\sum_{i=1}^{n} a_{ii} = 0 \\ \sum_{i=1}^{n} b_{ii} = 0\end{array}\right| \Rightarrow \sum_{i=1}^{n} a_{ii} + b_{ii} = 0 \Rightarrow$$

$\Rightarrow \alpha A + \beta B \in V_1 \Rightarrow V_1 \subset M_n(\mathbb{R})$ sp vect

fie $\alpha, \beta \in \mathbb{R}$ fie $A, B \in V_2$

$A = \lambda I_n$ , $B = \varphi I_n$

$$\alpha A + \beta B = \begin{pmatrix} \lambda+\varphi & & 0 \\ & \ddots & \\ 0 & & \lambda+\varphi \end{pmatrix} = (\lambda+\varphi) I_n \in V_2$$

$\Rightarrow V_2 \subset M_n(\mathbb{R})$ sp vect

b) $V_1 \oplus V_2 = M_n(\mathbb{R}) \Leftrightarrow V_1 + V_2 = M_n(\mathbb{R}) \wedge V_1 \cap V_2 = \{O_n\}$

fie $X \in M_n(\mathbb{R})$

fie $A \in V_1$, $B \in V_2$

~~$X = A + B \Leftrightarrow \sum_{i=1}^{n} x_{ii} = 0 + n\lambda$~~

fie $X = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & \ddots \\ & & & & x_n \end{pmatrix}$

$X = A + B \Leftrightarrow \sum_{i=1}^{n} x_i = 0 + \lambda \cdot n \Leftrightarrow$

$\Leftrightarrow \lambda = \dfrac{\sum_{i=1}^{n} x_i}{n}$

$\Rightarrow$ orice matrice de ord $n$ se poate scrie ca sumă dintre o matrice cu $Tr = 0$ și o matrice scalar astfel: se calculează media aritmetică a elem. de pe diag principală a matricii; aceasta este scalarul.

~~prin scăderea~~ scalarului din ~~matricea originală~~

prin scăderea matricei scalar din matricea originală obținem o matrice cu $Tr = 0$

$$\Rightarrow V_1 + V_2 = \mathcal{M}_n(\mathbb{R}) \quad \circledast$$

$$V_1 \cap V_2 = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \;\middle|\; \sum_{i=1}^{n} a_{ii} = 0 \;\land\right.$$

$$\left. \land\; a_{ii} = \lambda \right\} \Rightarrow V_1 \cap V_2 = \{ O_n \} \quad \circledast \qquad \Rightarrow V_1 \oplus V_2 = \mathcal{M}_n(\mathbb{R})$$

c) T dimensiunii Grassmann:

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

$$V_1 \cap V_2 = \{ O_n \} \Rightarrow \dim(V_1 \cap V_2) = 0$$

~~dim $V_2$~~ = dimensiunea e dată de nr. elementelor fără restricții

$$V_2 = \{ \lambda I_n \mid \lambda \in \mathbb{R} \} \Rightarrow \dim(V_2) = 1$$

$$V_1 = \{ A \mid Tr A = 0 \} \Rightarrow \dim(V_1) = n^2 - 1$$

$$V_1 + V_2 = \mathcal{M}_n(\mathbb{R}) \Rightarrow \dim(V_1 + V_2) = n^2$$

$$n^2 = n^2 - 1 + 1 - 0 \Rightarrow \text{Adevărat}$$

teorema se verifică