Algorithms and Data Structures (II)

Gabriel Istrate

June 9, 2024

Minimum Spanning Tree

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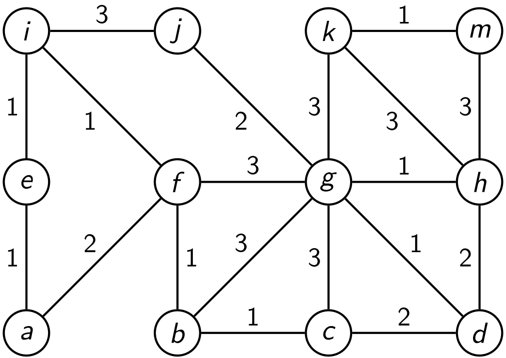
- T's total weight of the tree is minimal

$$w(T) = \sum_{(u,v)\in T} w(u, v)$$

  - a ***minimum-weight spanning tree***, or "minimum spanning tree"

Example



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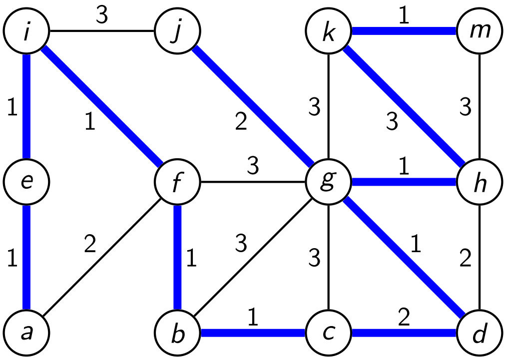

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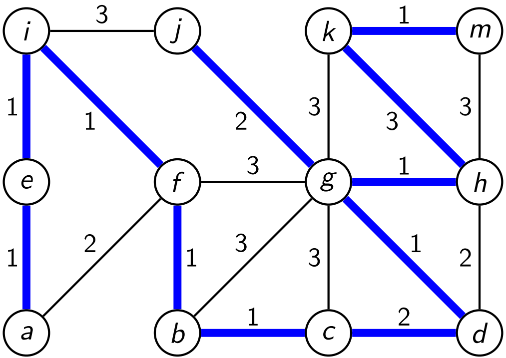

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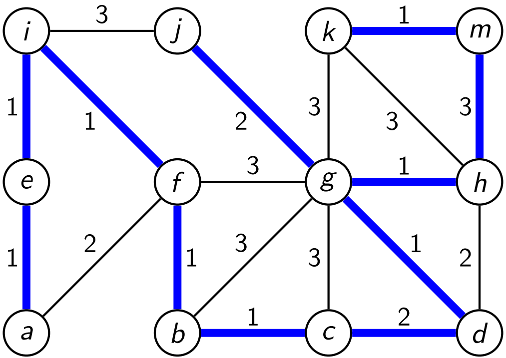

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- Does it work?

Refined Idea

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Generic-MST(G, w)
    A = ∅
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  - more or less the *definition* of a greedy algorithm

Preliminary Definitions

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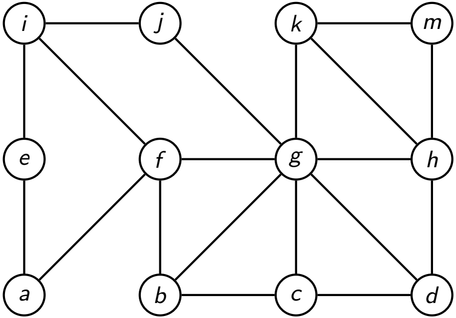

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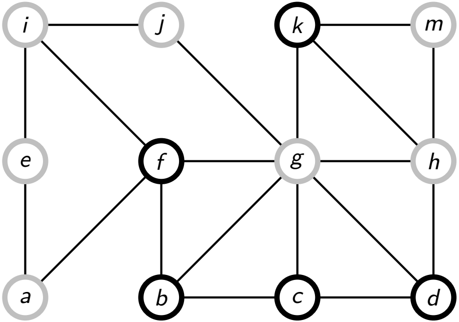

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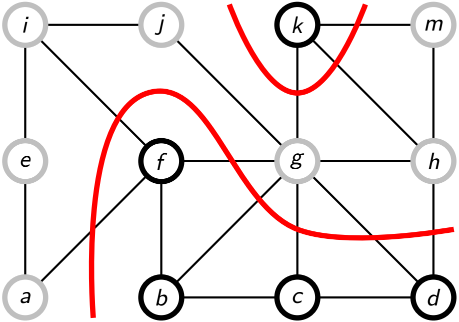

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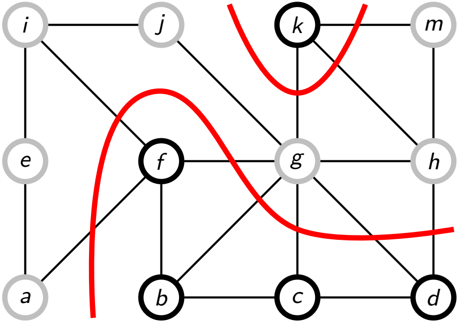

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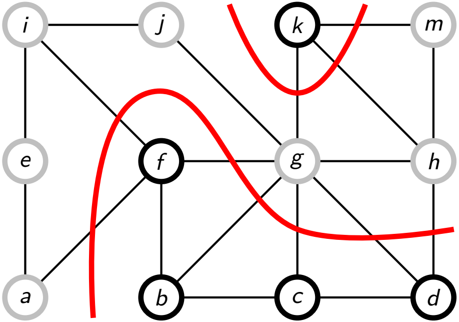

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- A cut $(S, V - S)$ *respects* a set of edges A if no edge in A crosses the cut

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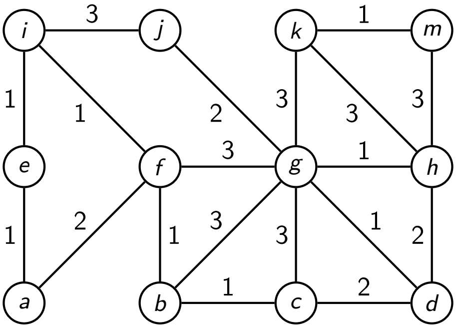

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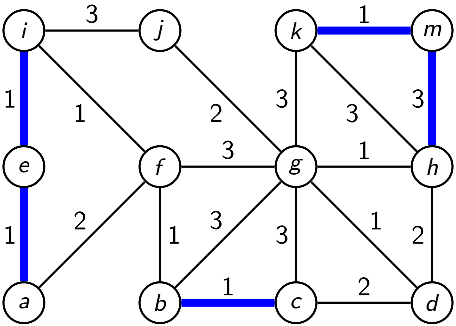

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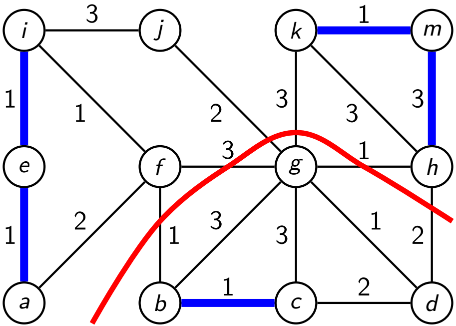

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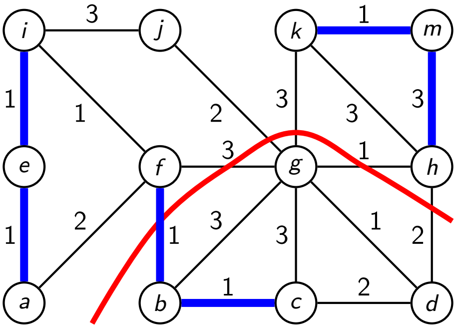

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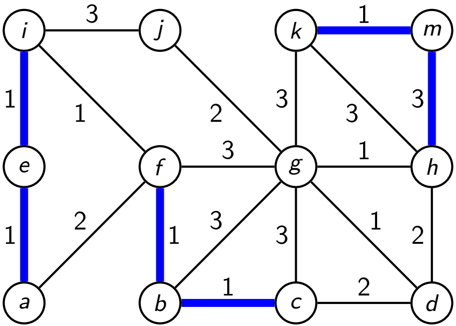

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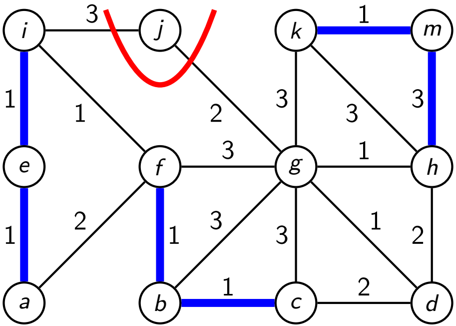

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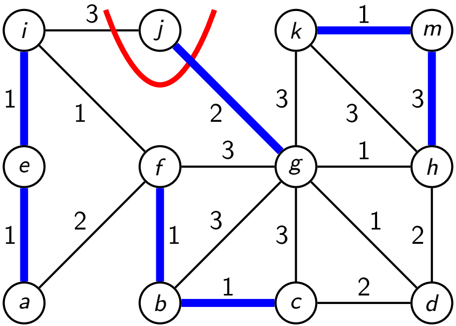

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Concrete Algorithms

- Kruskal's algorithm (1956)
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- Prim's algorithm (1957)
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  - incrementally builds a ***single tree*** A

Disjoint-Set Data Structure

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- *Union*(x, y) joins the sets containing x and y

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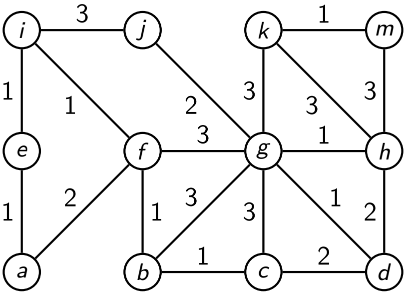

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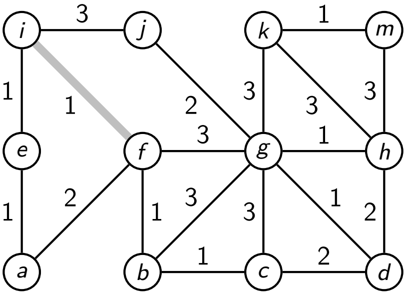

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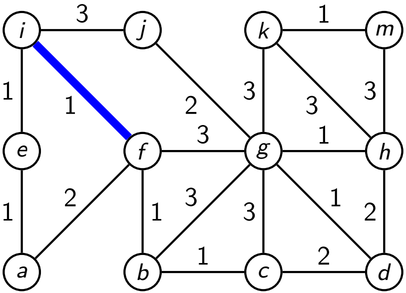

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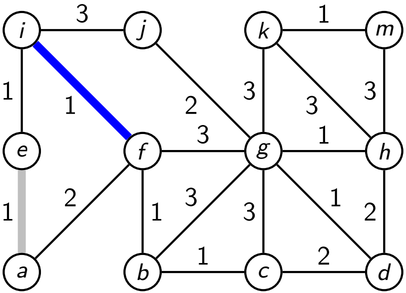

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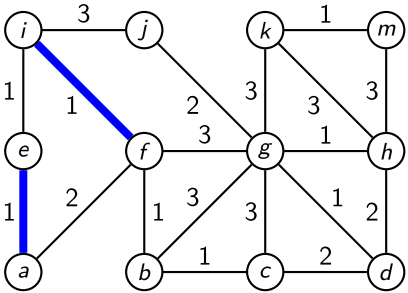

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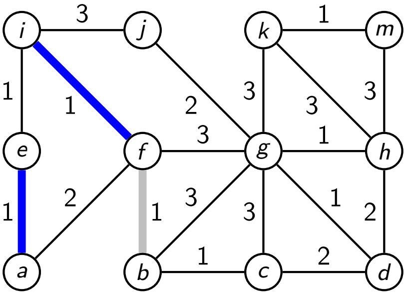

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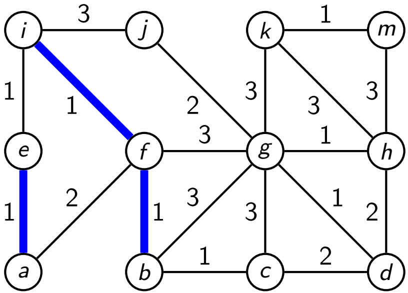

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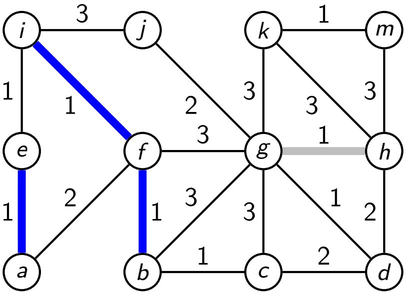

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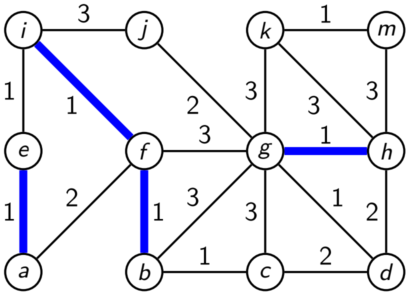

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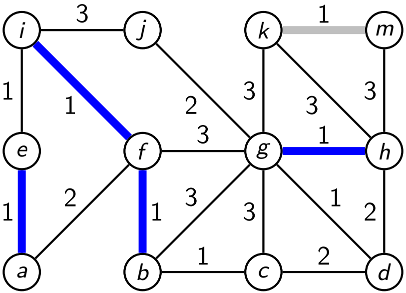

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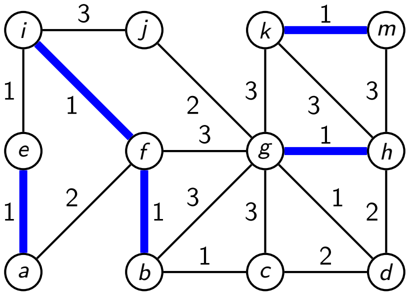

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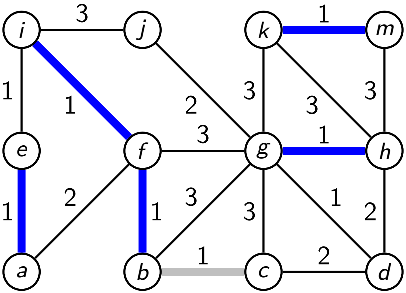

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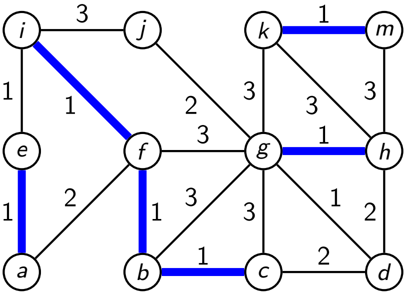

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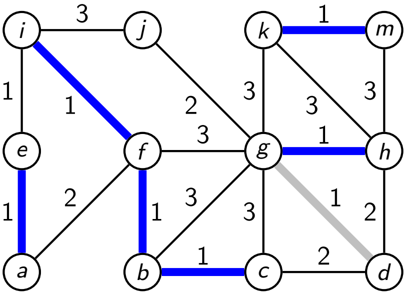

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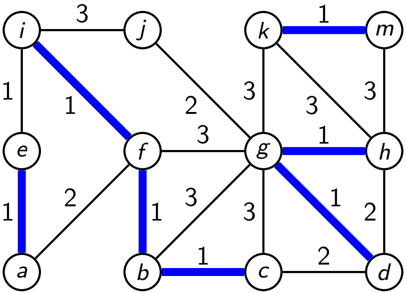
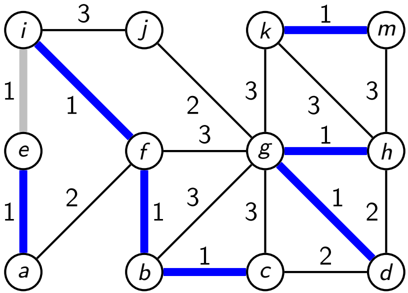


Kruskal's Algorithm

MST-Kruskal(G, w)

1. $A = \emptyset$
2. **for** each vertex $v \in V(G)$
3.   **Make-Set**(v) // *disjoint-set* data structure
4. sort E in non-decreasing order by weight w
5. **for** each edge $(u, v) \in E$, taken in non-decreasing order by w
6.   **if Find-Set**$(u) \neq$ **Find-Set**(v)
7. $A = A \cup \{(u, v)\}$
8.   **Union**(u, v)



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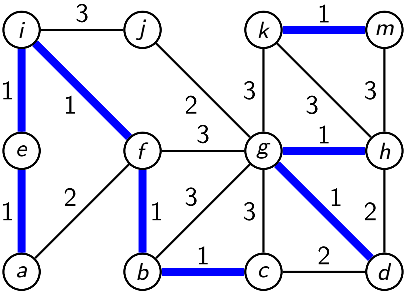

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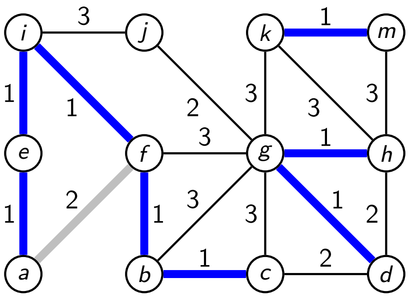

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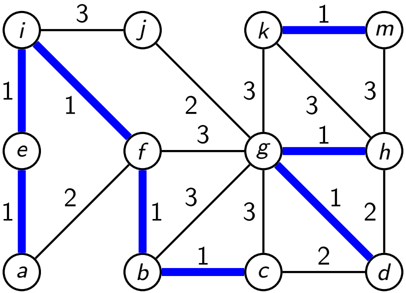

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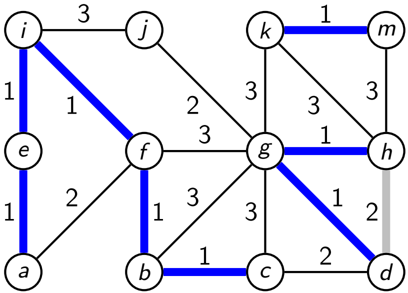

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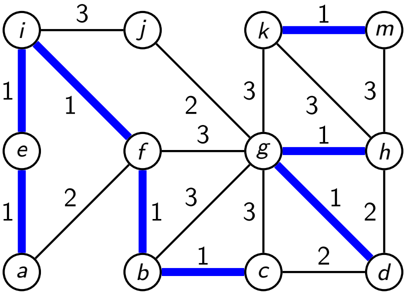

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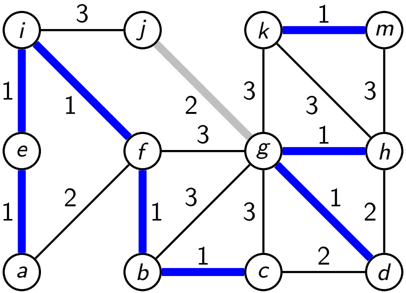

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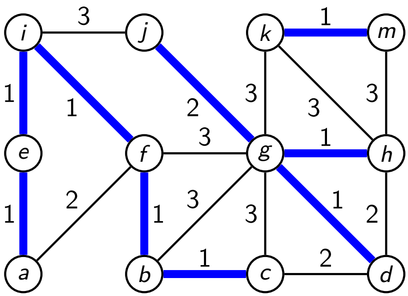

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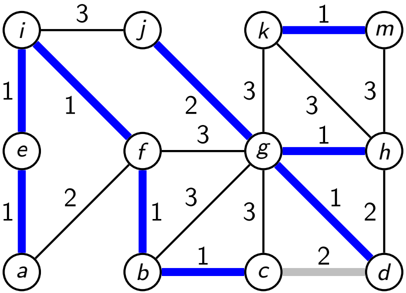

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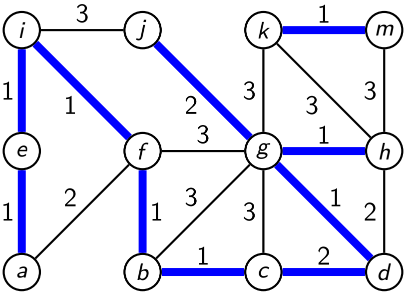

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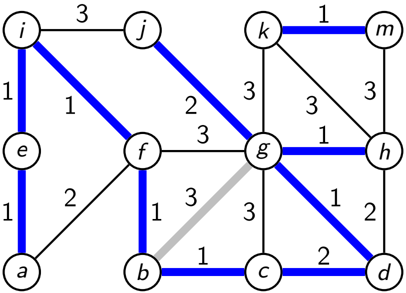

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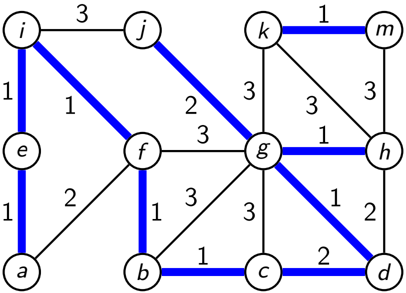

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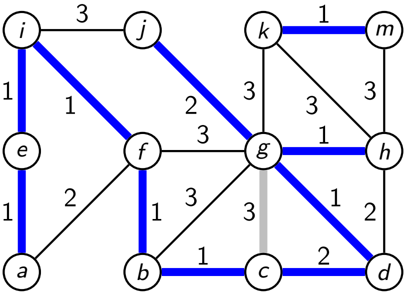

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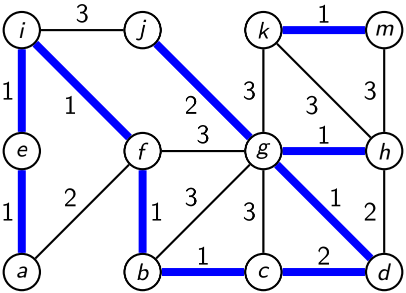

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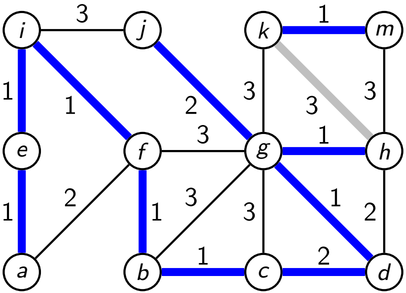

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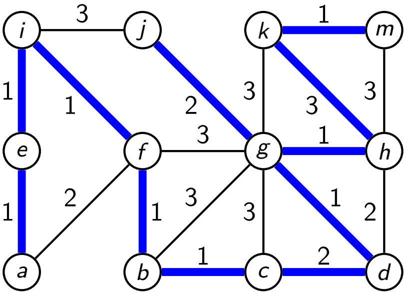

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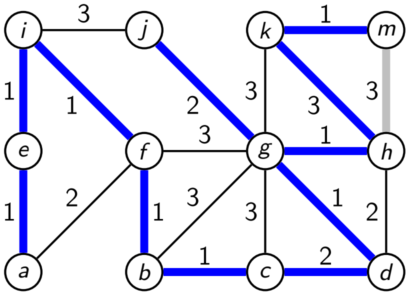

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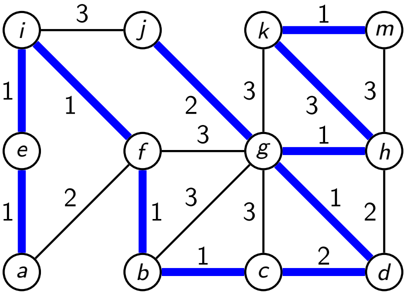

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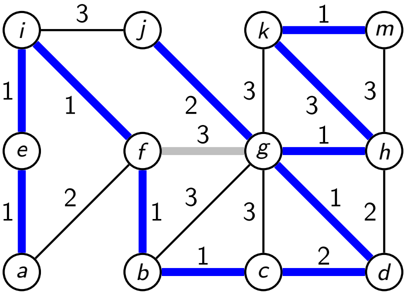

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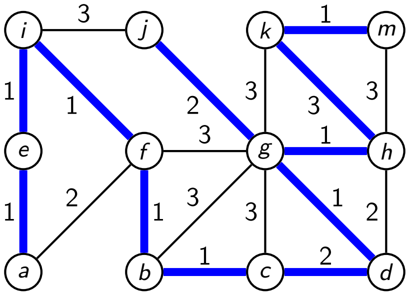

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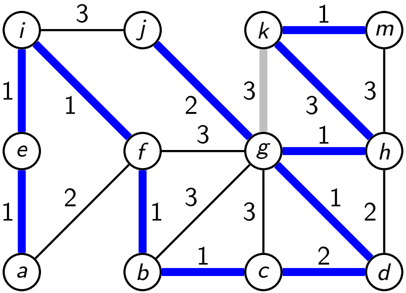

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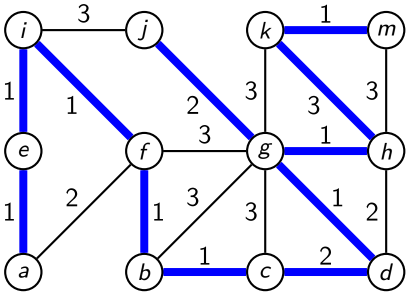

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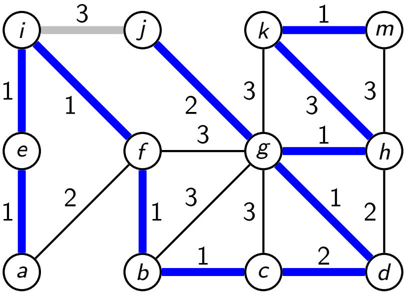

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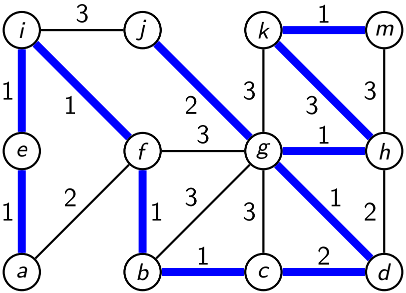

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- $O(|E|)$ times **Union**

Prim's Algorithm

```
MST-Prim(G, w, r)
for each vertex u ∈ V(G)
    key[u] = ∞
    π(u) = NIL
key[r] = 0
Q = V(G)
while Q ≠ ∅
    u = Extract-Min(Q) // min by key[u]
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
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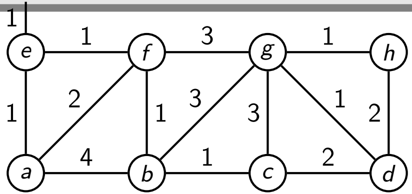

$$Q = \{(a, 0, \cdot), (b, \infty, \cdot), (c, \infty, \cdot), (d, \infty, \cdot), (e, \infty, \cdot), (f, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

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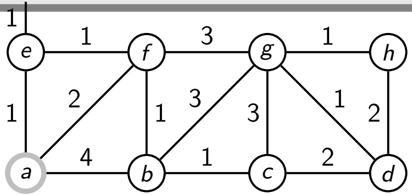

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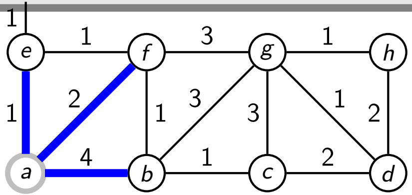

$Q = \{(e, 1, a), (f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

```
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for each vertex u ∈ V(G)
    key[u] = ∞
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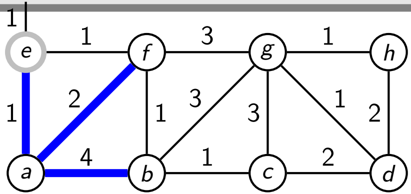
key[r] = 0
Q = V(G)
while Q ≠ ∅
    u = Extract-Min(Q) // min by key[u]
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
            key[v] = w(u, v)
```



$$Q = \{(f, 2, a), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

MST-Prim(G, w, r)

```
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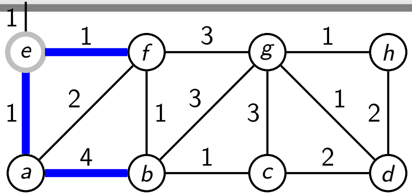
```



$$Q = \{(f, 1, e), (b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
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        if v ∈ Q ∧ w(u, v) < key[v]
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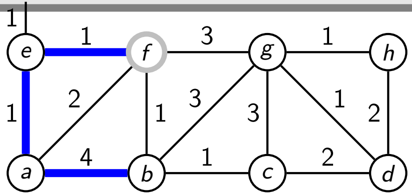
```



$$Q = \{(b, 4, a)(c, \infty, \cdot), (d, \infty, \cdot), (g, \infty, \cdot), (h, \infty, \cdot)\}$$

```
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        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
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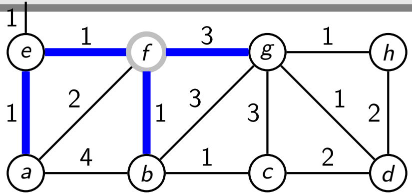
```



$$Q = \{(b, 1, f), (g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
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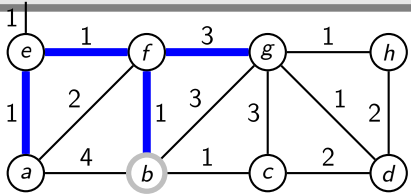
```



$$Q = \{(g, 3, f), (c, \infty, \cdot), (d, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

MST-Prim(G, w, r)

```
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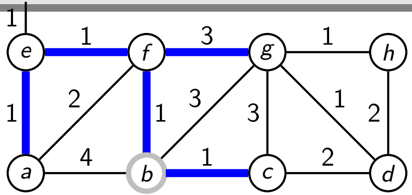
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
            key[v] = w(u, v)
```



$Q = \{(c, 1, b), (g, 3, f), (d, \infty, \cdot), (h, \infty, \cdot)\}$

Prim's Algorithm

```
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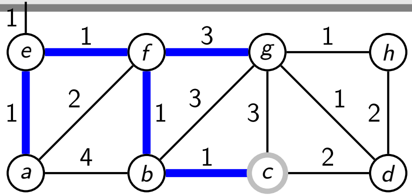

$$Q = \{(g, 3, f), (d, \infty, \cdot), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
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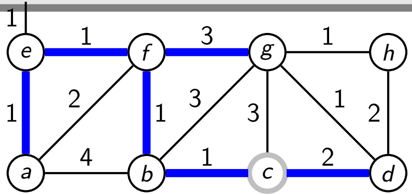
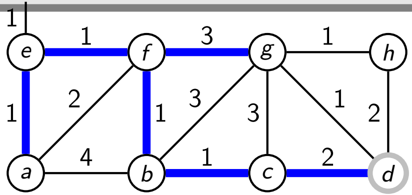
Q = V(G)
while Q ≠ ∅
    u = Extract-Min(Q) // min by key[u]
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
            key[v] = w(u, v)
```



$$Q = \{(d, 2, c), (g, 3, f), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
MST-Prim(G, w, r)
for each vertex u ∈ V(G)
    key[u] = ∞
    π(u) = NIL
key[r] = 0
Q = V(G)
while Q ≠ ∅
    u = Extract-Min(Q) // min by key[u]
    for each v ∈ Adj[u]
        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
            key[v] = w(u, v)
```



$$Q = \{(g, 3, f), (h, \infty, \cdot)\}$$

Prim's Algorithm

```
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    u = Extract-Min(Q) // min by key[u]
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        if v ∈ Q ∧ w(u, v) < key[v]
            π(v) = u
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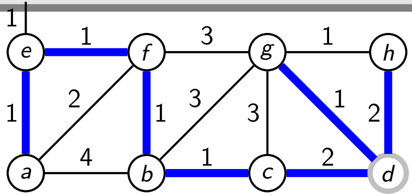
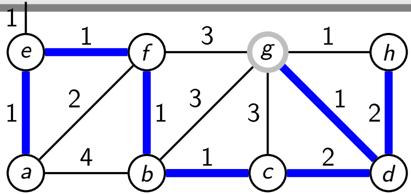


$$Q = \{(g, 1, d), (h, 2, d)\}$$

MST-Prim(G, w, r)

for each vertex $u \in V(G)$
    $key[u] = \infty$
    $\pi(u) = \text{NIL}$
$key[r] = 0$
$Q = V(G)$
while $Q \neq \emptyset$
    $u =$ **Extract-Min**(Q) // min by $key[u]$
    for each $v \in Adj[u]$
        if $v \in Q \wedge w(u, v) < key[v]$
            $\pi(v) = u$
            $key[v] = w(u, v)$



$Q = \{(h, 2, d)\}$

MST-Prim(G, w, r)

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            π(v) = u
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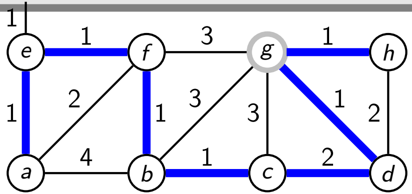

$Q = \{(h, 1, g)\}$

Prim's Algorithm

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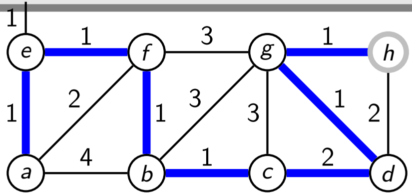

$Q = \{\}$

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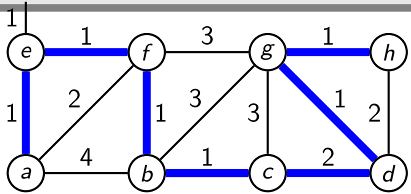

$Q = \{\}$