STATISTICĂ

POPULAȚIE $\sim F_\theta$

STIU

VREAU SĂ AFLU

$\theta = (\theta_1, \theta_2, \dots \theta_k)$

ESANTION

$X_1, X_2, X_3 \dots X_n$ v.a. i.i.d. (VIRTUAL)

$x_1, x_2, x_3, \dots x_n$ (EFECTIV)

(observații)

$$f(x) = x^2$$
$$f'(5) = ?$$

$$\mathbf{I} \quad \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \dots$$

$$\lim_{x \to 7} \frac{f(x) - f(7)}{x - 7} = \dots$$

$\mathbf{II} \quad f'(x) = 2x$

$f'(5) = 2 \cdot 5 = 10$

$f'(7) = 2 \cdot 7 = 14$

$\hat{\theta}(X_1, X_2, X_3 \dots X_n) \rightarrow$ ESTIMATOR pt. $\hat{\theta}$

Construcția estimatorilor

$\rightarrow$ v.a.

$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \rightarrow$ media de selecție (empirică)

$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

$f(x) = \frac{x^2}{54\theta^3} \cdot e^{-\frac{x}{3\theta}}$, $x > 0$, $\underline{\theta > 0}$

Fie $X_1, X_2, \dots X_n$ un eșantion cu valorile de selecție $x_1, x_2 \dots x_n$

$\underline{\mathbf{I} \quad \text{Metoda momentelor}}$

$$E(X) = \overline{X}$$

media teoretică → media empirică

$$\Gamma(a) = \int_0^\infty x^{a-1} \cdot e^{-x} \, dx, \quad a > 0$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx = \int_0^\infty x \cdot \frac{x^2}{54\theta^3} \cdot e^{-\frac{x}{3\theta}} \, dx = \int_0^\infty \frac{x^3}{54\theta^3} \cdot e^{-\frac{x}{3\theta}} \, dx$$

s.v. $\frac{x}{3\theta} = t \Rightarrow x = 3\theta t \Rightarrow dx = 3\theta \, dt$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$$E(X) = \int_0^\infty \frac{(3\theta t)^3}{54\theta^3} \cdot e^{-t} \cdot 3\theta \, dt = \frac{1}{2} 3\theta \cdot \int_0^\infty t^3 \cdot e^{-t} \, dt = \frac{3}{2} \theta \cdot \Gamma(4) = \frac{3}{2} \theta \cdot 4! = 9\theta$$

$E(X) = \overline{X} \quad (\Leftrightarrow) \quad 9\theta = \overline{x} \Rightarrow \boxed{\hat{\theta} = \frac{\overline{X}}{9}} \rightarrow$ estimator

$\underline{\mathbf{II} \quad \text{Metoda verosimilității maxime}}$

Pas 1 I. de verosimilitate

$$L(\theta \mid x_1, x_2 \dots x_n) = \prod_{i=1}^{n} f(x_i) = \begin{cases} \prod_{i=1}^{n} \frac{x_i^2}{54\theta^3} \cdot e^{-\frac{x_i}{3\theta}}, & x_i > 0 \; \forall i \\ 0, & x_i = 0 \end{cases}$$

variabile / parametru

$$L(\theta) = \left(\frac{1}{54\theta^3}\right)^n \cdot \prod_{i=1}^{n} x_i^2 \cdot e^{-\frac{1}{3\theta} \cdot \sum_{i=1}^{n} x_i}$$

$$\ln L(\theta) = -n \ln 54\theta^3 + \ln\left(\prod_{i=1}^{n} x_i^2\right) - \frac{1}{3\theta} \underbrace{\left(\sum_{i=1}^{n} x_i\right)}_{n \cdot \overline{x}}$$

Pas 2 Logaritmarea

$$\ln L(\theta) = -n \ln 54 - 3n \ln \theta + \ln\left(\prod_{i=1}^{n} x_i^2\right) - \frac{1}{3\theta} \cdot n \cdot \overline{x}$$

Pas 3 Det. punctelor critice

$$\ln L'(\theta) = 0 \; (\Leftrightarrow) \; -3n \cdot \frac{1}{\theta} + \frac{n \overline{x}}{3} \cdot \frac{1}{\theta^2} = 0 \Big/ \cdot 3\theta^2 \; (\Leftrightarrow) \; -9n\theta + n\overline{x} = 0$$

$$(\Leftrightarrow) \boxed{\theta = \frac{\overline{x}}{9}} \Rightarrow \boxed{\hat{\theta} = \frac{\overline{X}}{9}} \rightarrow \text{estimatorul de verosimilitate maximă}$$

Pas 4 Verif. punct de maxim

$$\ln L''(\theta)\Big|_{\theta = \frac{\overline{x}}{9}} = 3n \cdot \frac{1}{\theta^2} + \frac{n\overline{x}}{3} \cdot (-2) \cdot \frac{1}{\theta^3}\Big|_{\theta = \frac{\overline{x}}{9}} = \frac{n}{\theta^2}\left(3 - \frac{2}{3} \cdot \overline{x} \cdot \frac{1}{\theta}\right)\Big|_{\theta = \frac{\overline{x}}{9}} = \underbrace{\frac{81n}{\overline{x}^2}}_{>0} \cdot \underbrace{\left(3 - \frac{2}{3} \cdot \overline{x} \cdot \frac{9}{\overline{x}}\right)}_{<0} < 0$$