Tema 6

1)

$$\begin{array}{ccc} B & \xrightarrow{S} & B' \\ \downarrow & & \downarrow \\ A_f & & A'_f \end{array}$$

$A_f = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$

$S = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

$A'_f = S^{-1} \cdot A_f \cdot S$

$S^{-1} = \frac{1}{-1} \cdot \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$

$A'_f = \begin{pmatrix} -4 & 3 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -10 \\ 19 & 16 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} =$

$= \begin{pmatrix} -24 & -31 \\ -10 & -7 \end{pmatrix}$

2) $A_f^n = \tilde{S} \cdot D^n \cdot \tilde{S}^{-1}$

$D^n = \begin{pmatrix} (-2)^n & 0 \\ 0 & 7^n \end{pmatrix}$ $\quad \tilde{S} = \begin{pmatrix} -4 & 1 \\ 5 & 1 \end{pmatrix}$

$\tilde{S}^{-1} = \frac{1}{-9} \begin{pmatrix} 1 & -1 \\ -5 & -4 \end{pmatrix}$

$A_f^n = \begin{pmatrix} -4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} (-2)^n & 0 \\ 0 & 7^n \end{pmatrix} \begin{pmatrix} -\frac{1}{9} & \frac{1}{9} \\ \frac{5}{9} & \frac{4}{9} \end{pmatrix} =$

$= \begin{pmatrix} -4 \cdot (-2)^n & 7^n \\ 5 \cdot (-2)^n & 7^n \end{pmatrix} \begin{pmatrix} -\frac{1}{9} & \frac{1}{9} \\ \frac{5}{9} & \frac{4}{9} \end{pmatrix} =$

$= \begin{pmatrix} \frac{4}{9}(-2)^n + \frac{5}{9} \cdot 7^n & -\frac{4}{9}(-2)^n + \frac{4}{9} 7^n \\ -\frac{5}{9}(-2)^n + \frac{5}{9} 7^n & \frac{5}{9}(-2)^n + \frac{4}{9} 7^n \end{pmatrix}$

3) $f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(x, y, z) = (x + 4y,\ 2y + 3z,\ y)$

dem că f e endomorfism

fie $v_1, v_2 \in \mathbb{R}^3$

$v_1 = (x_1, y_1, z_1)$ $\quad v_2 = (x_2, y_2, z_2)$

fie $\alpha, \beta \in \mathbb{R}$

$f(\alpha v_1 + \beta v_2) = f(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2,\ \alpha z_1 + \beta z_2) =$

$= (\alpha x_1 + \beta x_2 + 4\alpha y_1 + 4\beta y_2,\ 2\alpha y_1 + 2\beta y_2 + 3\alpha z_1 + 3\beta z_2,$
$\alpha y_1 + \beta y_2) =$

$= \alpha(x_1 + 4y_1,\ 2y_1 + 3z_1,\ y_1) + \beta(x_2 + 4y_2,\ 2y_2 + 3z_2,$
$y_2) =$

$= \alpha f(v_1) + \beta f(v_2) \Rightarrow f$ e endomorfsm

4) $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x,y,z) = (2x - y + 2z,\ -x + 2y - z,\ x + y + z)$

$\forall (x,y,z) \in \mathbb{R}^3$

a) $A_f = ?$ în rap. cu B_0

b) det val. pr. și vec. pr. coresp.

c) verif dacă f e diagonalizabilă

d) dacă da, $D = ?$, $B = ?$

a) $A_f = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

b) $P(\lambda) = 0$

$\det(A_f - \lambda I_3) = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (4 + \lambda^2 - 4\lambda)(1-\lambda) - \cancel{2} + 1 - (4 - 2\lambda) + \cancel{2} - \lambda - (1-\lambda) =$$

$$\begin{matrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \end{matrix}$$

$= \lambda^2 - 4\lambda + \cancel{4} - \lambda^3 + 4\lambda^2 - 4\lambda + \cancel{1} - \cancel{4} + 2\lambda - \cancel{\lambda} - \cancel{1} + \cancel{\lambda} =$

$= -\lambda^3 + 5\lambda^2 - 6\lambda = -\lambda(\lambda^2 - 5\lambda + 6) =$

$= -\lambda \cdot (\lambda - 2) \cdot (\lambda - 3)$

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 2 \quad \text{val. pr.} \\ \lambda_3 = 3 \end{cases}$$

$Spec(f) = \{0, 2, 3\}$

$m_a(\lambda_i) = 1 \quad i = \overline{1,3}$

$V_{\lambda_1} = \{ v \in \mathbb{R}^3 \mid A_f \cdot v = \lambda_1 \cdot v \}$

$(A_f - \lambda_1 I_3) \cdot v = O_{3,1}$ $\quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Delta_1 = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = -3 \neq 0 \Rightarrow rg\, A_f = 2$

y, z nec pr.

$x = \alpha$ nec sec.

$$\begin{cases} -y + 2z = -2\alpha \\ 2y - z = \alpha \\ x = \alpha \end{cases} \Rightarrow \begin{cases} -y + 2(2y - \alpha) = -2\alpha \\ z = 2y - \alpha \\ x = \alpha \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 3 \cdot y = 0 \Rightarrow y = 0 \\ z = -\alpha \\ x = \alpha \end{cases}$$

$V_{\lambda_1} = \{ \alpha \underbrace{(1, 0, -1)}_{v_1} \mid \alpha \in \mathbb{R} \} = \langle v_1 \rangle$

$V_{\lambda_2} = \{ v \in \mathbb{R}^3 \mid A_f \cdot v = \lambda_2 \cdot v \}$

$(A_f - \lambda_2 I_3) \cdot v = O_{3,1}$

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Delta_1 = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1 \neq 0 \Rightarrow rg\, A_f = 2$

y, z nec pr.

$x = \alpha$ nec sec.

$$\begin{cases} -y+2z=0 \\ -z=\alpha \\ x=\alpha \end{cases} \Rightarrow \begin{cases} x=\alpha \\ y=-2\alpha \\ z=-\alpha \end{cases}$$

$$V_{\lambda_2} = \{\alpha \underbrace{(1,-2,-1)}_{v_2} \mid \alpha \in \mathbb{R}\} = \langle v_2 \rangle$$

$$V_{\lambda_3} = \{ v \in \mathbb{R}^3 \mid A_f \cdot v = \lambda_3 \cdot v \}$$

$$(A_f - \lambda_3 I_3) \cdot v = O_{3,1}$$

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$O_p = \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = -1 \neq 0 \Rightarrow rg A_f = 2$$

y, z nec pr

$x = \alpha$ nec sec

$$\begin{cases} -y+2z=\alpha \\ -y-z=\alpha \\ \cancel{x}=\alpha \end{cases} \Rightarrow \begin{cases} x=\alpha \\ y=-\alpha \\ z=0 \end{cases}$$

$$V_{\lambda_3} = \{\alpha \underbrace{(1,-1,0)}_{v_3} \mid \alpha \in \mathbb{R}\} = \langle v_3 \rangle$$

c) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \Rightarrow A_f$ e diagonalizzabile

$\left(\sum_{i=1}^{3} m_a(\lambda_i) = 3 = \dim \mathbb{R}^3\right)$

$\left(m_a(\lambda_i) = m_g(\lambda_i) = 1 \quad i = \overline{1,3} \right)$

d) $$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$B_0 \xrightarrow{S} B = B_1 \cup B_2 \cup B_3 = \{v_1, v_2, v_3\}$$

$$\begin{matrix} B_0 & & B \\ \downarrow & & \downarrow \\ A_f & & D \end{matrix}$$

$D = S^{-1} A_f S$ $\qquad S = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \Rightarrow$

$\Rightarrow S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

$S^{-1} = \dfrac{1}{\det S} \cdot S^*$

$S^t = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ $\quad \cancel{\det}$

$S^* = \begin{pmatrix} +(-1) & -0 & +(-1) \\ -(-1) & +(-2) & -(+1) \\ +(-2) & -0 & +0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -2 & -1 \\ -2 & 0 & 0 \end{pmatrix}$

$\det S = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ -1 & -1 & 0 \end{vmatrix} = 0 + 0 + \cancel{1} - 2 - \cancel{1} - 0 =$
$\begin{matrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{matrix}$
$= -2$

$S^{-1} = -\dfrac{1}{2} \cdot S^* = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$

$D = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ -1 & -1 & 0 \end{pmatrix} =$

$= \begin{pmatrix} \frac{3}{2} & 0 & \frac{3}{2} \\ -\frac{3}{2} & 4 & -\frac{3}{2} \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ -1 & -1 & 0 \end{pmatrix} =$

$$= \begin{pmatrix} 0 & 0 & \frac{3}{2} \\ 0 & -8 & -\frac{11}{2} \\ 0 & 2 & 3 \end{pmatrix}$$