Class 3/27

These notes contain all the technical details of the proofs for the statements that we considered on Friday.

1 Poisson kernel

Definition. The function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

for $0 \leq r < 1$ and $-\infty < \theta < \infty$ is called the Poisson kernel.

Lemma 1. *If $z = re^{i\theta}$ with $0 \leq r < 1$, then*

$$P_r(\theta) = \Re\left(\frac{1+z}{1-z}\right).$$

Proof. We have

$$\frac{1+z}{1-z} = (1+z)(1+z+z^2+...) = 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta},$$

and taking real parts gives

$$\Re\left(\frac{1+z}{1-z}\right) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta).$$

Use $\cos(n\theta) = 2^{-1}(e^{in\theta} + e^{-in\theta})$ to get the claimed identity. □

The real part can be evaluated explicitly to get the representation

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Proposition 1. *For $0 \leq r < 1$ and $\theta \in \mathbb{R}$ we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1,$$
$$P_r(\theta) > 0 \text{ for all } \theta,$$
$$P_r(-\theta) = P_r(\theta),$$
$$P_r(\theta) < P_r(\delta) \text{ if } 0 < \delta < |\theta| \leq \pi,$$

and for each $\delta > 0$

$$\lim_{r\to 1-} P_r(\theta) = 0$$

uniformly in θ for $\pi \geq |\theta| \geq \delta$.

Proof. If $r < 1$ is fixed, then the series defining the Poisson kernel converges uniformly in θ, hence the integration can be performed termwise, which gives the first claim. The second claim follows from

$$P_r(\theta) = \frac{1-r^2}{|1-re^{i\theta}|^2}$$

and $r < 1$, the third claim follows since $\cos\theta$ is even.

For the fourth claim we observe that

$$P_r'(\theta) = -\frac{(1-r^2)2r\sin\theta}{(1-2r\cos\theta+r^2)^2} < 0$$

for $\theta \in [\delta, \pi]$, and finally, the limit relation holds pointwise for $\theta \neq 0$, and by the previous statement, uniformly in the stated region. □

The Dirichlet problem has a solution for the unit disk.

Theorem 1. *Let D be the open unit disk. Assume that $f : \partial D \to \mathbb{R}$ is continuous. Then there exists a continuous function $u : \overline{D} \to \mathbb{R}$ such that $u = f$ on ∂D and u is harmonic in D. Moreover, u is unique and can be defined by*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})dt$$

for $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$.

Proof. Define u by the integral formula in the open unit disk and define $u = f$ on the unit circle. We need then to show that u is continuous on the closed disk and harmonic on D.

To show that u is harmonic on D, we note

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re\left[\frac{1+re^{i(\theta-t)}}{1-e^{i(\theta-t)}}\right] f(e^{it})dt \\ &= \Re\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+re^{i\theta}}{e^{it}-e^{i\theta}} f(e^{it})dt\right] \end{aligned}$$

Write the fraction as $(e^{it}+z)/(e^{it}-z)$ and note that

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} f(e^{it})dt$$

is analytic in the open unit disk and satisfies $\Re g = u$. Hence, u is harmonic in D.

The main difficulty lies in establishing continuity on the unit circle.

Lemma 2. *Let $\alpha \in [-\pi, \pi]$ and $\varepsilon > 0$. There exists $1 > \rho > 0$ and an arc A of the unit circle with center $e^{i\alpha}$ such that for $r > \rho$ and any $e^{i\theta} \in A$ the inequality*

$$|u(re^{i\theta}) - f(e^{i\alpha})| < \varepsilon$$

holds.

The lemma implies continuity of u on the boundary. □

Proof of the lemma. With a rotation we may assume that $\alpha = 0$. We note that

$$u(re^{i\theta}) - f(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)(f(e^{it}) - f(1))dt,$$

and we split the integration into two parts, over $|t| < \delta$ and $\delta \leq |t| \leq \pi$, where δ remains to be chosen. The first condition for δ is that

$$|f(e^{it}) - f(1)| < \varepsilon/3$$

for $|t| < \delta$. This is possible since f is continuous, and allows us to estimate the first integral as

$$< \varepsilon/3 \int_{|t|<\delta} < \varepsilon/3$$

by extending the integration over the whole range. (This estimate is independent of θ.)

For the second integral $f(e^{it})$ and $f(1)$ are far away from each other. Defining M to be the maximum of f on the unit circle, we use the trivial estimate

$$|f(e^{it}) - f(1)| \leq 2M.$$

We therefore need to use properties of P_r to make the second integral small when θ is close to zero.

Assume that $|\theta| \leq \delta/2$ and $|t| \geq \delta$. Then $|t-\theta| \geq \delta/2$. From the previous proposition we can find $\rho < 1$ so that

$$P_r(\eta) < \varepsilon/(3M)$$

for all $|\eta| \geq \delta/2$ and $\rho \leq r < 1$. Inserting this into the second integral with $\eta = t - \theta$ gives the claim.

To show uniqueness, note that for another solution v we have that $u-v = 0$ on the boundary of the unit disk, hence the maximum principle implies that $u - v$ is the zero function. $\square$

We note that since

$$P_r(\theta) = \Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right),$$

any function that is continuous on the closed unit disk and harmonic in the open unit disk may be written as the real part of

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it})dt.$$