The Poisson kernel (P_r) and conjugate Poisson kernel (Q_r)

Recall: for $F(z) = \frac{1+z}{1-z}$ $(|z|<1)$ we have

$$P(z) = \Re F(z) = \frac{1-|z|^2}{|1-z|^2}$$

$$Q(z) = \Im F(z) = \frac{2\,\Im z}{|1-z|^2}.$$

Moreover, for $r \in [0,1)$ we define

$$P_r(t) := P(re^{2\pi i t}) = \frac{1-r^2}{1-2r\cos(2\pi t)+r^2}$$

$$Q_r(t) := Q(re^{2\pi i t}) = \frac{2r\sin(2\pi t)}{1-2r\cos(2\pi t)+r^2}.$$

We saw that $\{P_r;\, r\nearrow 1\}$ was an approximate identity. On the other hand, $\{Q_r;\, r\nearrow 1\}$ is not: use that

$$\int Q_r(t)\,dt = \frac{1}{2\pi}\log\left[1-2r\cos(2\pi t)+r^2\right].$$

Other differences between P_r and Q_r: for $t\in[0,1)$ we have

$$\lim_{r\nearrow 1} P_r(t) = \begin{cases} 0; & t\neq 0 \\ \infty; & t=0 \end{cases}$$

The Dirac „function" (measure). We have $C \subset \underset{\text{w.r.t. this measure}}{L^1}$ (and $\int f = f(0)$).

$$\lim_{r\nearrow 1} Q_r(t) = \begin{cases} \cot(\pi t); & t\neq 0 \\ 0; & t=0. \end{cases}$$

Non-integrable: even $\{\text{constants}\} \not\subset \underset{\text{w.r.t. this measure}}{L^1}$

Note: $(r,t) \longmapsto Q_r(t)$ does not extend to a cont. fct on $[0,1]\times[-1/2,1/2]$, because there is a problem at $t=0, r=1$:

$$\lim_{r\nearrow 1} Q_r(0) = 0, \quad \text{but} \quad \lim_{t\to 0}\cot(\pi t) \text{ does not exist.}$$

Still, we may find a way to define (calculate) boundary values of $v_r(t) = (f * Q_r)(t)$. We use the fact that Q_r is odd.

<u>Proposition</u> If $f:\mathbb{T}\to\mathbb{C}$ is differentiable then for $\forall t\in\mathbb{T}$ the limit

$$\lim_{\delta\downarrow 0}\int_{\delta\le|\tau|\le 1/2} \cot(\pi\tau)\, f(t-\tau)\,d\tau$$

exists.

Proof We have

$$\int_{\delta\le|\tau|\le 1/2} \cot(\pi\tau)\, f(t-\tau)\,d\tau = \int_{\delta}^{1/2} \cot(\pi\tau)\left[f(t-\tau)-f(t+\tau)\right]d\tau$$

Use: $\begin{cases} \bullet\ \int = \int_{\delta}^{1/2} + \underbrace{\int_{-1/2}^{-\delta}}_{\text{new variable}: -\tau} \\ \bullet\ \cot \text{ odd} \end{cases}$

$$= \int_{\delta}^{1/2} \underbrace{\tau\cot(\pi\tau)}_{\text{continuous}} \underbrace{\left[\frac{f(t-\tau)-f(t)}{\tau} - \frac{f(t+\tau)-f(t)}{\tau}\right]}_{=: g_t(\tau)} d\tau$$

continuous, since is differentiable

$$\longrightarrow \int_{0}^{1/2} \tau\cot(\pi\tau)\, g_t(\tau)\,d\tau \quad \text{as } \delta\downarrow 0. \qquad \square$$

This is called the principal value of the integral

$$\int_{\mathbb{T}} f(t-\tau)\cot(\pi\tau)\,d\tau$$

and denoted

$$\text{p.v.}\int_{\mathbb{T}} f(t-\tau)\cot(\pi\tau)\,d\tau.$$

Also, it is called the Hilbert transform of f and denoted $Hf(t)$.

<u>Proposition</u> If $f:\mathbb{T}\to\mathbb{C}$ is differentiable, then for $\forall t\in\mathbb{T}$

$$\lim_{r\nearrow 1}(f * Q_r)(t)$$

exists and equals $(Hf)(t)$.

Proof For any $\delta \in (0, 1/2)$ we get

used that Q_r was odd

$$v_r(t) = (f * Q_r)(t) =$$

$$= \int_{|\tau|<\delta} \tau Q_r(\tau) \cdot \frac{f(t-\tau) - f(t)}{\tau} \, d\tau + \int_{|\tau| \geq \delta} [Q_r(\tau) - \cot(\pi\tau)] f(t-\tau) \, d\tau$$

$$+ \int_{|\tau| \geq \delta} \cot(\pi\tau) f(t-\tau) \, d\tau$$

$$= \underline{I} + \underline{II} + \underline{III}.$$

We proved in the last proposition that $\underline{III} \longrightarrow (Hf)(t)$ as $\delta \to 0$.

As for I, we claim

$$\sup_{\substack{r \in [0,1) \\ \tau \in [-1/2, 1/2]}} |\tau Q_r(\tau)| < \infty. \qquad (1)$$

Indeed,

$x = 2\pi\tau$

$$\tau Q_r(\tau) = \frac{\tau \cdot 2r \sin(2\pi\tau)}{1 - 2r\cos(2\pi\tau) + r^2} = \frac{1}{\pi} \cdot \frac{x \, r \sin x}{1 - 2r\cos x + r^2}.$$

If $\underline{r \sim 1}$ (e.g., $1/2 \leq r \leq 1$), then $x \, r \sin x \sim r \sin x \cdot r \sin x$,

thus $$\tau Q_r(\tau) \sim \frac{(r \sin x)^2}{1 - 2r\cos x + r^2} \sim \frac{|z - \bar{z}|^2}{|1 - z|^2} = \frac{|w - \bar{w}|^2}{|w|^2}$$

$w := 1 - z$

$z = re^{ix}$

$$= \left| \frac{w}{|w|} - \frac{\bar{w}}{|w|} \right|^2.$$

Hence for all $z = re^{2\pi i \tau} \neq 1$ and $|\tau| \lesssim 1$ we have

$$|\tau Q_r(\tau)| \lesssim 1.$$

If $r \sim 0$ (e.g., $0 \leq r \leq 1/2$), then $1 - 2r\cos(2\pi\tau) + r^2 \geq (1-r)^2 \sim 1$.

Hence $|\tau Q_r(\tau)| \lesssim 1$ again. This proves (1).

Now,

$$|I| \lesssim \int_{|\tau|<\delta} \underbrace{|\tau Q_r(\tau)|}_{\substack{\lesssim 1, \\ \text{by (1)}}} \cdot \underbrace{\left| \frac{f(t-\tau) - f(t)}{\tau} \right|}_{\substack{\text{continuous} \\ \text{in } \tau, \text{ since} \\ f \text{ differentiable}}} \, d\tau$$

$\Rightarrow$ bounded

$$\lesssim \delta \longrightarrow 0, \qquad \text{as } \delta \to 0.$$

Finally, cf. $\underline{\mathrm{II}}$ we calculate

$$Q_r(\tau) - \cot(\pi\tau) = \frac{2r\sin(2\pi\tau)}{1-2r\cos(2\pi\tau)+r^2} - \frac{\sin(2\pi\tau)}{1-\cos(2\pi\tau)}$$

$$\underbrace{x = 2\pi\tau}$$

$$= \sin x \cdot \frac{2r(1-\cos x) - (1-2r\cos x + r^2)}{1-2r\cos x + r^2}$$

$$= \sin x \cdot \frac{2r - 1 - r^2}{1-2r\cos x + r^2} = -\sin x \cdot \frac{(1-r)^2}{1-2r\cos x + r^2}$$

$$= -\sin x \cdot \frac{1-r}{1+r} \cdot \underbrace{P_r(\tau)}_{\text{Poisson kernel}}. \tag{2}$$

Hence $$|\underline{\mathrm{II}}| \leq \int_{\delta \leq |\tau| \leq \frac{1}{2}} |Q_r(\tau) - \cot(\pi\tau)| \max_{\mathbb{T}} |f| \, d\tau$$

$$\overset{(2)}{\lesssim} (1-r)\, \|f\|_\infty \underbrace{\int_{\mathbb{T}} P_r(\tau)\, d\tau}_{=1 \text{ (proved before)}}$$

$$\lesssim 1 - r.$$

Thus by choosing $\delta := 1-r$ we get

$$\left| v_r(t) - \underline{\mathrm{III}} \right| \lesssim 1-r \longrightarrow 0 \quad \text{as } r \nearrow 1.$$

Since we know that $\underline{\mathrm{III}} \longrightarrow (Hf)(t)$ as $r \nearrow 1$ (Prop. above), this concludes the proof. □

The Dirichlet problem on arbitrary domains

We saw that the harmonic extension of $f \in C(\mathbb{T})$ to D is given by

$$u(re^{2\pi i t}) = \int_0^1 P(re^{2\pi i t} e^{-2\pi i \tau}) f(e^{2\pi i \tau})\, d\tau,$$

that is,

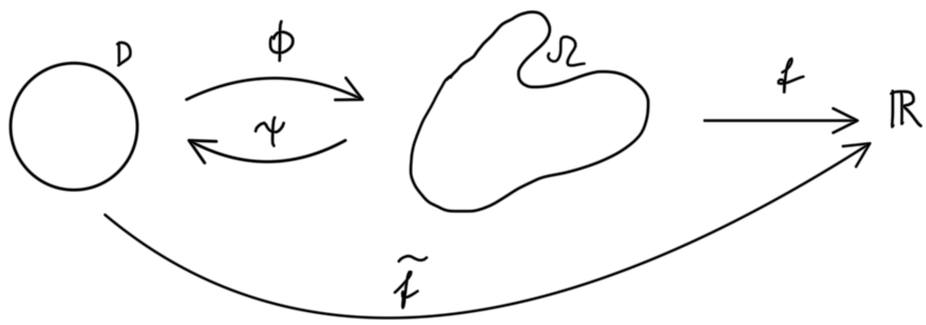
$$u(z) = \frac{1}{2\pi} \int_{\mathbb{T}} P(zw^{-1}) f(w)\, dw. \tag{3}$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with smooth boundary. Suppose we are given a real function $f \in C(\partial\Omega)$. We want to extend f to $u : \overline{\Omega} \longrightarrow \mathbb{R}$ in such a way that u is harmonic in Ω. Based on our experience with the case $\Omega = D$, we expect (and prove) a formula of the type (3), that is,

$$u(z) = \int_{\partial\Omega} P_{\Omega}(z, w) f(w)\, dw = \int_{\partial\Omega} P_{\Omega}(z, \cdot) f(\cdot)\, ds \qquad (4)$$

with P_{Ω} to be determined.

By the Riemann Mapping Theorem, there exists a biholomorphic isomorphism $\phi : D \longrightarrow \Omega$. Owing to our assumptions on Ω, we may extend it to $\phi : \overline{D} \longrightarrow \overline{\Omega}$ in a way such that $\phi : \partial D \longrightarrow \partial\Omega$ diffeomorphically. Write $\psi := \phi^{-1}$. So we have



and $\tilde{f} := f \circ \phi : \partial D \to \mathbb{R}$. We get a parametrization of $\partial\Omega$:

$$\partial\Omega = \{ \phi(e^{2\pi i \tau})\, ;\ \tau \in [0,1] \}.$$

Hence we may rewrite (4) as

$$u(z) = \int_0^1 \underline{P_{\Omega}(z, \phi(e^{2\pi i \tau}))}\, f(\phi(e^{2\pi i \tau})) \cdot \underline{|\phi'(e^{2\pi i \tau})|} \cdot 2\pi\, d\tau. \qquad (5)$$

We extend $\tilde{f}$ to $\tilde{u} : \overline{D} \longrightarrow \mathbb{R}$ by

$$\tilde{u}(\underbrace{re^{2\pi i t}}_{\tilde{z} \in \mathbb{T}}) = \int_0^1 P(re^{2\pi i t} \cdot e^{-2\pi i \tau})\, \tilde{f}(e^{2\pi i \tau})\, d\tau.$$

So for $u := \tilde{u} \circ \psi$, i.e., $\tilde{u} = u \circ \phi$ we get

$$u(\phi(\tilde{z})) = \int_0^1 P\left(\tilde{z} \cdot \overline{\phi^{-1}(\phi(e^{2\pi i \tau}))}\right) f(\phi(e^{2\pi i\tau}))\, d\tau,$$

so

$$\underset{\underset{\Omega}{\cap}}{u(z)} = \int_0^1 \underline{P(\phi^{-1}(z)\overline{\phi^{-1}(\phi(e^{2\pi i \tau}))})}\, f(\phi(e^{2\pi i\tau}))\, d\tau. \qquad (6)$$

A comparison between (5) and (6) gives, for $z \in \Omega$ and $w \in \partial\Omega$,

$$P_\Omega(z,w) \cdot |\phi'(\phi^{-1}(w))| \cdot 2\pi = P(\phi^{-1}(z) \cdot \overline{\phi^{-1}(w)}),$$

thus

$$\begin{aligned} P_\Omega(z,w) &= \frac{1}{2\pi} P(\psi(z)\overline{\psi(w)}) \cdot |\psi'(w)| \\ &= \frac{|\psi'(w)|}{2\pi} \cdot \frac{1 - |\psi(z)\overline{\psi(w)}|^2}{|1-\psi(z)\overline{\psi(w)}|^2} \\ &= \frac{|\psi'(w)|}{2\pi} \cdot \frac{1-|\psi(z)|^2}{|\psi(w)-\psi(z)|^2}. \end{aligned}$$

So we have

$$\boxed{\begin{aligned} u(z) &= \frac{1}{2\pi}\int_{\partial\Omega} P_\Omega(\psi(z)\overline{\psi(w)})\, |\psi'(w)|\, f(w)\, dw \\ &= \frac{1-|\psi(z)|^2}{2\pi}\int_{\partial\Omega} \frac{|\psi'(w)|}{|\psi(w)-\psi(z)|^2} f(w)\, dw. \end{aligned}}$$

Since $\tilde{u}$ is harmonic, ψ holomorphic and $u = \tilde{u} \circ \psi$, u is harmonic.

<u>Example</u>: Dirichlet problem on arbitrary balls.

If $\Omega = B(0,R)$ then $\phi(\tilde{z}) = R\tilde{z}$, so $\psi(z) = \frac{z}{R}$. Then

$$P_{B(0,R)}(z,w) = \frac{1/R}{2\pi} \cdot \frac{1-|z/R|^2}{|(w-z)/R|^2} = \frac{1}{2\pi R} \cdot \frac{R^2-|z|^2}{|w-z|^2},$$

thus

$$u(re^{2\pi i t}) = \int_{S(0,R)} \frac{1}{2\pi R} \cdot \frac{R^2 - r^2}{|w-z|^2} f(w)\, dw \qquad w = Re^{2\pi i \tau}$$

$$= \boxed{\int_0^1 \frac{R^2 - r^2}{R^2 - 2Rr\cos(2\pi(t-\tau)) + r^2} \cdot f(Re^{2\pi i \tau})\, d\tau}.$$

Clearly, by translation you may extend this to all balls:

$$u(a + re^{2\pi i t}) = \int_0^1 \frac{R^2 - r^2}{R^2 - 2Rr\cos(2\pi(t-\tau)) + r^2} \cdot f(a + Re^{2\pi i \tau})\, d\tau.$$

Example: Dirichlet problem in the upper half-plane:

ψ :

$$\psi(z) = \frac{az+b}{cz+d} \qquad \begin{aligned} 0 &\longmapsto -1 \\ 1 &\longmapsto -i \\ \infty &\longmapsto 1 \end{aligned} \qquad \psi(z) = \frac{z-i}{z+i} = 1 - \frac{2i}{z+i}$$

$$P_\Omega(z,w) = \frac{|\psi'(w)|}{2\pi} \cdot \frac{1 - |\psi(z)|^2}{|\psi(w) - \psi(z)|^2}$$

$$= \frac{1}{\pi |w+i|^2} \cdot \frac{1 - \left|\frac{z-i}{z+i}\right|^2}{\left|\frac{w-i}{w+i} - \frac{z-i}{z+i}\right|^2} \qquad \operatorname{Im} z > 0$$

$w \in \mathbb{R}$

$$= \frac{1}{\pi} \cdot \frac{|z+i|^2 - |z-i|^2}{|(w-i)(z+i) - (w+i)(z-i)|^2}$$

$$= \frac{1}{\pi} \cdot \frac{4 \operatorname{Im} z}{|2(w-z)i|^2}.$$

Thus,

$$u(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|w-z|^2} f(w)\, dw,$$

or

$$u(x,y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t)\, dt.$$

This is the Poisson integral for the upper half-plane. Thus

$$u(x,y) = (P_y * f)(x), \quad \text{where} \quad P_y(x) = \frac{1}{\pi} \cdot \frac{y}{x^2+y^2}.$$

We have

$$P_y(x) = \frac{1}{\pi} \operatorname{Re} \frac{i\bar{z}}{z\bar{z}} = \frac{1}{\pi} \operatorname{Re} \frac{i}{z}$$

$$Q_y(x) = \frac{1}{\pi} \operatorname{Im} \frac{i}{z} = \frac{1}{\pi} \operatorname{Re} \frac{1}{z}.$$

Hence

$$\left.\begin{array}{l} (x,y) \longmapsto P_y(x) \\ (x,y) \longmapsto Q_y(y) \end{array}\right\} \text{ are both harmonic.}$$

We have $Q_y(x) = \frac{1}{\pi} \cdot \frac{x}{x^2+y^2} \longrightarrow \frac{1}{\pi x}$ as $y \searrow 0$.

As before (the case of unit disc), we may show that $Q_y *$ gives rise to a principal-value convolution operator:

<u>Theorem</u> If $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$, then the limit below exists:

$$(Hf)(x) := \lim_{y \searrow 0} (Q_y * f)(x) = \lim_{\delta \searrow 0} \frac{1}{\pi} \int_{\{|t|>\delta\}} \frac{f(x-t)}{t}\, dt$$

$$= \frac{1}{\pi}\, \text{p.v.} \int_{\mathbb{R}} \frac{f(x-t)}{t}\, dt = \frac{1}{\pi}\, \text{p.v.} \left(f * \frac{1}{t}\right)(x).$$

<u>Proof</u> We have

$$\int_{\{|t|>\delta\}} \frac{f(x-t)}{t}\, dt = \int_{\{\delta<|t|<1\}} + \int_{\{|t|\geq 1\}}$$

$|\cdot| \leq |f(\cdot - t)| \in L^1$

continuous on $[-1,1]$, since f is diffb.

$$= \int_{\{\delta<|t|<1\}} \frac{f(x-t)-f(x)}{t}\, dt + \int_{\{|t|>1\}} \frac{f(x-t)}{t}\, dt$$

$$\longrightarrow \int_{[-1,1]} \frac{f(x-t)-f(x)}{t}\, dt + \int_{\{|t|>1\}} \frac{f(x-t)}{t}\, dt \quad \text{as } \delta \searrow 0.$$

So $\boxed{\text{the limit}}$ p.v. $\left(f * \frac{1}{t}\right)$ exists.

We have to show that it equals $\lim_{y \downarrow 0} (Q_y * f)(x)$. We have

$$(Q_y * f)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{t}{t^2+y^2} f(x-t)\, dt$$

contributes nothing, as $t \mapsto \frac{t}{t^2+y^2}$ is odd

$$= \underbrace{\frac{1}{\pi} \int_{\{|t|<1\}} \frac{t^2}{t^2+y^2} \cdot \Big[\underbrace{\frac{f(x-t) - f(x)}{t}}_{= g_x(t)} \Big]\, dt}_{\mathrm{I}} + \underbrace{\frac{1}{\pi} \int_{\{|t| \geq 1\}} \frac{t}{t^2+y^2} f(x-t)\, dt}_{\mathrm{II}}$$

We want to show that

$$\mathrm{I} + \mathrm{II} \xrightarrow[y \to 0]{} \frac{1}{\pi} \int_{[-1,1]} \frac{f(x-t) - f(x)}{t}\, dt + \frac{1}{\pi} \int_{\{|t|>1\}} \frac{f(x-t)}{t}\, dt.$$

Indeed, cf. I we have

$$\left| \int_{[-1,1]} \frac{t^2}{t^2+y^2} g_x(t)\, dt - \int_{[-1,1]} g_x(t)\, dt \right|$$

$u = t/y$
$t = yu$

$$= \int_{[-1,1]} \left| \frac{t^2}{t^2+y^2} - 1 \right| \cdot |g_x(t)|\, dt \leq \|g_x\|_{L^\infty(-1,1)} \int_{-1}^{1} \frac{y^2}{t^2+y^2}\, dt$$

$$\leq \|g_x\| \cdot y \cdot \underbrace{\int_{\{|u|<1/y\}} \frac{du}{u^2+1}}_{= 2\operatorname{arctg}\frac{1}{y} < \pi}$$

$$\lesssim y,$$

hence $\longrightarrow 0$ as $y \to 0$.

Cf. II, on the other hand,

$$\left| \int_{\{|t| \geq 1\}} \frac{t}{t^2+y^2} f(x-t)\, dt - \int_{\{|t|>1\}} \frac{f(x-t)}{t}\, dt \right| \leq \int_{\{|t| \geq 1\}} \left| \frac{t}{t^2+y^2} - \frac{1}{t} \right| \cdot |f(x-t)|\, dt$$

$$= \int_{\{|t| \geq 1\}} \frac{y^2}{y^2+t^2} \left| \frac{f(x-t)}{t} \right| dt \leq \frac{y^2}{y^2+1} \underbrace{\int_{\{|t|\geq 1\}} |f(x-t)|\, dt}_{\leq \|f\|_{L^1(\mathbb{R})} < \infty}$$

$\frac{y^2}{y^2+1} \to 0$ as $y \to 0$

$\square$