HARMONIC FUNCTIONS

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Contents

1. Harmonic functions

A C^2 (twice continuously differentiable) real valued function u is said to be harmonic if it satisfies the Laplace equation $\triangle u = 0$ that is,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = 0 \tag{1.1}$$

Example 1.1. (Harmonic functions)
i.The function $u(x, y) = x^2 - y^2$ is harmonic.
ii. The function $u(x, y) = x^3$ is not harmonic.
iii. The function $u(x, y) = \sin x - \cos y$ is not harmonic.

A complex valued function is harmonic iff its real and imaginary parts are each harmonic. [1, Chapter 9, p.250]

1.1. **Use of Harmonic mappings.** If a function is harmonic (that is, it satisfies Laplace's equation $\nabla^2 f = 0$) over a plane domain (which is two-dimensional), and is transformed via a conformal map to another plane domain, the transformation is also harmonic. For this reason, any function which is defined by a potential can be transformed by a conformal map and still remain governed by a potential.
Examples in physics of equations defined by a potential include the electromagnetic field, the gravitational field, and, in fluid dynamics, potential flow, which is an approximation to fluid flow assuming constant density, zero viscosity, and irrotational flow. One example of a fluid dynamic application of a conformal map is the Joukowsky transform.

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1.2. **Harmonic functions and holomorphicity.** If D is a region in the plane, a real-valued function $u(x, y)$ having continuous second partial derivatives is said to be harmonic on D if it satisfies Laplaces equation on D.
There is an intimate relationship between harmonic functions and analytic functions.

If $f = u + iv$ is a holomorphic function then both u and v are harmonic. To see, apply

$$\frac{\partial}{\partial \overline{z}} f = 0 \tag{1.2}$$

Further, take partial derivative w.r.to z

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = 0 \tag{1.3}$$

Making use of the differential operators

$$\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{1.4}$$

$$\frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{1.5}$$

we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[u + iv] = 0 \tag{1.6}$$

Equating real and imaginary parts to zero, we conclude that u and v are harmonic.

Example 1.2. The function $f(z) = (z + z^2)^2$ is holomorphic and hence real and imaginary parts of f are each harmonic.

Exercise 1. Show that ‘real part of an analytic function is harmonic’.

1.3. **Harmonic Conjugate.** If U is a simply connected region (with no holes) and u is harmonic on U, then there is a holomorphic function F on U such that $\operatorname{Re} F = u$, that is, there exists a harmonic function v defined on U such that $F = u + iv$ is holomorphic on U.
v is called **harmonic conjugate** for u.

Example 1.3 (5). Consider the function $u(x, y) = x^2 - y^2 - x$ on the square $U = \{(x, y) : |x| < 1, |y| < 1\}$. Find harmonic conjugate for u.

[Hint: Make use of C-R equations.]
Ans. $v(x, y) = 2xy - y + C$, $F = z^2 - z + iC$.

Exercise 2. Find the harmonic conjugate for the function $u(x, y) = x^3 - 3xy^2$. Write down the corresponding analytic function G for which $\operatorname{Re} G = u$.

Exercise 3. Show that if u and v are harmonic conjugate to each other in some domain, they are constant.

1.4. **Potential function.** Let $F : U \to \mathbb{R}^2$ be the map (often called vector field) such that $\mathbf{F}(x, y) = (u(x, y), -v(x, y))$. $\mathbf{F}$ is called vector field associated with f. (See, for details, [4, Chapter 1, article 6]).

The potential function ϕ for vector field $\mathbf{F}$ is defined as

$$\frac{\partial \phi}{\partial x} = u \quad \texttt{and} \quad \frac{\partial \phi}{\partial y} = -v$$

Remark 1. ϕ is potential function of vector field $\mathbf{F} = u\hat{\imath} - v\hat{\jmath}$ if $\mathbf{F} = \nabla\phi = \dfrac{\partial\phi}{\partial x}\hat{\imath} + \dfrac{\partial\phi}{\partial y}\hat{\jmath}$.

Theorem 1.4. [4, p. 243] *Let g be primitive for an analytic function f on U, i.e. $g' = f$. Write g in terms of real and imaginary parts $g = \phi + i\psi$. Then ϕ is potential function for* $\mathbf{F}$.

Proof: By definition $g' = u + iv$ is analytic. Primitive (antiderivative) of analytic function f is also analytic. Therefore making use of C-R equations for g and comparing the derivatives of g with f, we follow that $\dfrac{\partial\phi}{\partial x} = u$ and $\dfrac{\partial\phi}{\partial y} = -v$ as desired.

Remark 2. The problem of finding a primitive for an analytic function is equivalent to the problem of finding a potential function for the associated vector field.

2. Poisson Kernel

In potential theory, the Poisson kernel is an integral kernel, used for solving the two-dimensional Laplace equation, given Dirichlet boundary conditions on the unit disc. The kernel can be understood as the derivative of the Green's function for the Laplace equation. It is named for *Simon Poisson*.

Poisson kernels commonly find applications in control theory and two-dimensional problems in electrostatics. In practice, the definition of Poisson kernels are often extended to n-dimensional problems.

Our aim in this section is to solve the Dirichlet problem for a disk, that is, to construct a solution of Laplaces equation in the disk subject to prescribed boundary values. The basic tool is the Poisson integral formula, which may be regarded as an analog of the Cauchy integral formula for harmonic functions. We will begin by extending Cauchys theorem and the Cauchy integral formula to functions continuous on a disk and analytic on its interior [5, Section 4.7].

For $z \in D(0,1)$, define functions P_z and Q_z on the real line R by $P_z(t) = \dfrac{1-|z|^2}{|e^{it}-z|^2}$ and $Q_z(t) = \dfrac{e^{it}+z}{e^{it}-z}$; $P_z(t)$ is called the Poisson kernel and $Q_z(t)$ the Cauchy kernel. We have

$$\mathrm{Re}[Q_z(t)] = \mathrm{Re}\,\frac{(e^{it}+z)(e^{-it}-\overline{z})}{|e^{it}-z|^2} = \mathrm{Re}\,\frac{1-|z|^2+ze^{-it}-\overline{z}e^{it}}{|e^{it}-z|^2} = P_z(t).$$

Note also that if $z = re^{i\theta}$, then

$$P_z(t) = \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} = \frac{1-r^2}{|e^{i(t-\theta)}-r|^2} = P_r(t-\theta). \tag{2.1}$$

Since $|e^{i(t-\theta)} - r|^2 = 1 - 2rcos(t-\theta) + r^2$, we see that

$$P_r(t-\theta) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} = P_r(\theta-t). \tag{2.2}$$

Thus for $0 \le r < 1$, $P_r(x)$ is an even function of x. Note also that $P_r(x)$ is positive and decreasing on $[0,\pi]$.

In the complex plane, the Poisson kernel for the unit disc is given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta+r^2} = \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right), \quad 0 \le r < 1. \tag{2.3}$$

Since

$$\frac{1+z}{1-z} = (1+z)\sum_{n=0}^{\infty} z^n = 1 + 2\sum_{n=1}^{\infty} z^n$$

$$P_r(\theta) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

This can be thought of in two ways: either as a function of r and , or as a family of functions of indexed by r.

Theorem 2.1. *The Poisson Kernel* $P_z(t) = \dfrac{1-|z|^2}{|e^{it}-z|^2}$ *is harmonic function of* z*. Moreover* $P_z(t) \geq 0$ *and*

$$\frac{1}{2\pi i}\int_0^{2\pi} P_r(t-\theta)\,\mathrm{d}\theta = 1 \tag{2.4}$$

We revisit one of the applications of Cauchy integral formula, known as Poisson integral formula from the view point of harmonic functions.

Theorem 2.2 (Poisson integral formula [5, 7])**.** *Let* f *be a holomorphic function defined on unit disk* $\mathbb{D}$ *and continuous on* $\bar{\mathbb{D}}$*. Then we have*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t) f(e^{it})\,\mathrm{d}t, \quad 0 \leq r < 1 \tag{2.5}$$

Proof: For holomorhic function f defined on unit disk, the Cauchy integral formula gives:

$$f(0) = \frac{1}{2\pi i}\int_{|w|=\rho} \frac{f(w)}{w}\,\mathrm{d}w = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\theta})\,\mathrm{d}\theta \quad (0 < \rho < 1) \tag{2.6}$$

That is the value of f at the centre of the circle $|w| = \rho$ is the mean value of f on $|w| = \rho$. Taking $\rho \to 1^-$, we get

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,\mathrm{d}t \tag{2.7}$$

If $z \neq 0$, by Cauchy-Goursat theorem

$$f(z) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w-z}\,\mathrm{d}w \tag{2.8}$$

and

$$0 = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)}{w-1/\bar{z}}\,\mathrm{d}w \tag{2.9}$$

since $\bar{z} \notin \mathbb{D}$. Subtracting (2.9) from (2.8), we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|w|=1}\left(\frac{1}{w-z} - \frac{1}{w-1/\bar{z}}\right) f(w)\,\mathrm{d}w \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{e^{it}-z} - \frac{1}{e^{it}-1/\bar{z}}\right) e^{it} f(e^{it})\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{it}}{e^{it}-z} + \frac{e^{it}\bar{z}}{1-\bar{z}e^{it}}\right) f(e^{it})\,\mathrm{d}t \qquad (2.10) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{it}}{e^{it}-z} + \frac{\bar{z}}{e^{-it}-\bar{z}}\right) f(e^{it})\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it})\,\mathrm{d}t \end{aligned}$$

Writing $z = re^{i\theta}$, using (2.1), we obtain the desired result (2.5).

This Poisson Integral Formula is obtained for the circular disc. Note that this Poisson Integral Formula for Harmonic functions can be obtained from the corresponding formula for the analytic functions by comparing the real parts. We now provide the series representation for the Poisson Integral Formula.

Corollary 2.3 (Fourier point of view). *Let f be a holomorphic function defined on unit disk $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$. Then we have*

$$f(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n \sin n\theta) \quad 0 \le r < 1 \qquad (2.11)$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi}\cos(nt)f(re^{it})\,\mathrm{d}t$, $b_n = \frac{1}{2\pi}\int_0^{2\pi}\sin(nt)f(re^{it})\,\mathrm{d}t$.

Proof: Using equation (2.3), we get

$$P_r(\theta - t) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta - t) \quad 0 \le r < 1. \qquad (2.12)$$

which further implies

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\left(1 + 2\sum_{n=1}^{\infty} r^n\cos n(\theta - t)\right) f(re^{it})\,\mathrm{d}t, \quad 0 \le r < 1 \qquad (2.13)$$

that is,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\,\mathrm{d}t + 2\sum_{n=1}^{\infty} r^n\left(\frac{1}{2\pi}\int_0^{2\pi}\cos n(\theta - t) f(re^{it})\,\mathrm{d}t\right) \qquad (2.14)$$

This can be written as

$$f(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta) \qquad (2.15)$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi}\cos(nt)f(re^{it})\,\mathrm{d}t$, $b_n = \frac{1}{2\pi}\int_0^{2\pi}\sin(nt)f(re^{it})\,\mathrm{d}t$.

Remark 3. Note that this representation is same as the Fourier representation.

Exercise 4. Prove the Poisson integral formula for arbitrary disks. Let $f \in A$ and $z = re^{i\theta}$ in a domain D_R that contains $|z| < R$: Then

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2\pi R \cos(\theta - \phi) + r^2} f(Re^{i\phi})\, \mathrm{d}\phi$$

Exercise 5. Deduce the Poisson integral when $f(z) = 1$.

If $\mathbb{D} = \{z : |z| < 1\}$ is the open unit disc in C, T is the boundary of the disc, and f a function on T that lies in L1(T), then the function u given by

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\, \mathrm{d}t, \quad 0 \leq r < 1$$

is harmonic in D and has a radial limit that agrees with f almost everywhere on the boundary T of the disc.

3. Schwarz integral formula

In complex analysis, a branch of mathematics, the Schwarz integral formula, named after Hermann Schwarz, allows one to recover a holomorphic function, up to an imaginary constant, from the boundary values of its real part.

Let us decompose f into real and imaginary parts as $f = u + iv$. Corollary of Poisson integral formula The formula follows from Poisson integral formula applied to u:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \operatorname{Re} \frac{e^{it} + z}{e^{it} - z}\, dt \text{ for } |z| < 1$$

Let $f = u + iv$ be a function which is holomorphic on the closed unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Then

$$f(z) = u(z) + iv(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{e^{it} + z}{e^{it} - z}\, dt + i\lambda \tag{3.1}$$

Setting $z = 0$, the value of the integral on R.H.S. is $u(0)$ and hence $\lambda = v(0)$.

That is,

$$f(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{w + z}{w - z} \operatorname{Re}(f(w))\, \frac{dw}{w} + i\operatorname{Im}(f(0)) \tag{3.2}$$

for all $|z| < 1$.

Equating the imaginary part on both sides of the last integral (3.1), we get

$$v(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{2\operatorname{Im}(zw)}{|w - z|^2} u(w)\, \frac{dw}{w} + v(0) \tag{3.3}$$

Let $f = u + iv$ be a function that is holomorphic on the closed upper half-plane $\{z \in C : Im(z) \geq 0\}$ such that, for some $\alpha > 0, |z\alpha f(z)|$ is bounded on the closed upper half-plane. Then

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(w, 0)}{w - z}\, dw = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{Re(f)(w + 0i)}{w - z}\, dw$$

for all $Im(z) > 0$.

Note that, as compared to the version on the unit disc, this formula does not have an arbitrary constant added to the integral; this is because the additional decay condition makes the conditions for this formula more stringent.

4. Estimation of Function Bounds

Theorem 4.1 (Harnack's inequality for unit disk [7]). *Let $u = u(z)$ be harmonic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$. If $u(e^{i\theta}) \geq 0$ for all θ, then for $z = re^{i\theta} \in \mathbb{D}$, we have*

$$u(0)\frac{1-r}{1+r} \leq u(e^{i\theta}) \leq u(0)\frac{1+r}{1-r} \quad (r < 1) \tag{4.1}$$

Proof: The proof depends on the estimate

$$\frac{1-r}{1+r} \leq \frac{1-r^2}{1-2r\cos\theta+r^2} \leq \frac{1+r}{1-r} \quad (r < 1) \tag{4.2}$$

since $1-2r\cos\theta+r^2 \leq 1+2r+r^2 = (1+r)^2 > 0$ and $1-2r\cos\theta+r^2 \geq 1-2r+r^2 = (1-r)^2 > 0$. Multiplying both sides by $\frac{1}{2\pi}u(e^{i\theta})$ and integrating we obtain the desired result.

5. Dirichlet boundary condition

In mathematics, the Dirichlet (or first-type) boundary condition is a type of boundary condition, named after Peter Gustav Lejeune Dirichlet (18051859).[1] When imposed on an ordinary or a partial differential equation, it specifies the values that a solution needs to take on along the boundary of the domain.

Let D be a domain and let σ be a given function continuous on the boundary of D. The problem of finding a function φ harmonic on the interior of D and which agrees with σ on the boundary of D is called the Dirichlet problem.

The question of finding solutions to such equations is known as the Dirichlet problem. In engineering applications, a Dirichlet boundary condition may also be referred to as a fixed boundary condition.

Examples: For an ordinary differential equation, for instance:

$$y'' + y = 0$$

the Dirichlet boundary conditions on the interval $[a,\, b]$ take the form:

$$y(a) = \alpha \text{ and } y(b) = \beta$$

where α and β are given numbers.

PDE For a partial differential equation, for instance:

$$\nabla^2 y + y = 0$$

where ∇^2 denotes the Laplacian, the Dirichlet boundary conditions on a domain $\Omega \subset \mathbb{R}^n$ take the form: $y(x) = f(x) \quad \forall x \in \partial\Omega$ where f is a known function defined on the boundary $\partial\Omega$.

Engineering applications For example, the following would be considered Dirichlet boundary conditions:

In mechanical engineering (beam theory), where one end of a beam is held at a fixed position in space. In thermodynamics, where a surface is held at a fixed temperature. In electrostatics, where a node of a circuit is held at a fixed voltage. In fluid dynamics, the no-slip condition for viscous fluids states that at a solid boundary, the fluid will have zero velocity relative to the boundary.

Other boundary conditions Many other boundary conditions are possible, including the Cauchy boundary condition and the mixed boundary condition. The latter is a combination of the Dirichlet and Neumann conditions.

Exercise 6. List some physical problems that lead to the Laplace equation.

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