CS 577

Assignment 9 – Network Flow

Fall 2022

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

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Network Flow

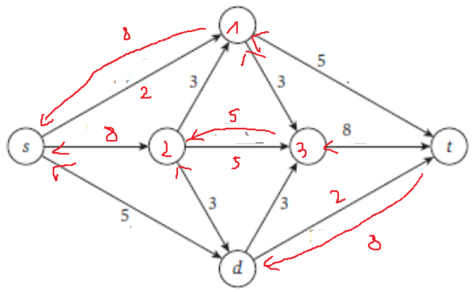
1. *Kleinberg, Jon. Algorithm Design (p. 415, q. 3a)* The figure below shows a flow network on which an $s - t$ flow has been computed. The capacity of each edge appears as a label next to the edge, and the flow is shown in boxes next to each edge. An edge with no box has no flow being sent down it.

   FlowGraph.png

   (a) What is the value of this flow?

   Solution: $5 + 8 + 5 = 18$

   (b) Please draw the **residual graph** associated with this flow.



Solution:

(c) Is this a maximum $s-t$ flow in this graph? If not, describe an augmenting path that would increase the total flow.

Solution: Solutions: s-1-3-2-d-t the max flow 21

2. *Kleinberg, Jon. Algorithm Design (p. 419, q. 10)* Suppose you are given a directed graph $G = (V, E)$. This graph has a positive integer capacity c_e on each edge, a source $s \in V$, a sink $t \in V$. You are also given a maximum $s - t$ flow through G: f. You know that this flow is *acyclic* (no cycles with positive flow all the way around the cycle), and every flow $f_e \in f$ has an integer value.

   Now suppose we pick an edge e^* and reduce its capacity by 1 unit. Show how to find a maximum flow in the resulting graph G^* in time $O(m + n)$, where $n = |V|$ and $m = |E|$.

Solution:

We assume flow f is integer value. We have 2 possibilities for edge e*(v,w)

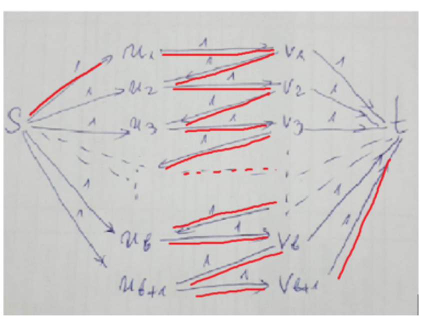
- f(e*) < ce* than lowering the capacity for 1 will not contributed to flow change
- f(e*) = ce* than we construct path w –t such that all the edges carry flow, and we reduce the flow by 1unit on each such edge. Also the same on the pat from v to s. There are not cycles so all the edges on the pat are reduced by 1 and the capacity is restored. New flow f'=f-1. We have to decide if current flow f' is maximum flow or can be increased. We attempt to find an augmented path from s to t in the residual graph G. If we do not find one, f' is maximum, else the flow is augmented to have value at least f. Since the current flow maximal cannot have a larger maximum flow value than the original one, this is a maximum flow.

3. *Kleinberg, Jon. Algorithm Design (p. 420, q. 11)* A friend of yours has written a very fast piece of code to calculate the maximum flow based on repeatedly finding augmenting paths. However, you realize that it's not always finding the maximum flow. Your friend never wrote the part of the algorithm that uses backward edges! So their program finds only augmenting paths that include all forward edges, and halts when no more such augmenting paths remain. (Note: We haven't specified *how* the algorithm selects forward-only augmenting paths.)

   When confronted, your friend claims that their algorithm may not produce the maximum flow every time, but it is guaranteed to produce flow which is within a factor of b of maximum. That is, there is some constant b such that no matter what input you come up with, their algorithm will produce flow at least $1/b$ times the maximum possible on that input.

   Is your friend right? Provide a proof supporting your choice.

Solution:



We will prove that statement is false with following counter example. For b integer and b>1 we construct a graph as shown, with all edges having capacity 1.
Assume that the first augmenting path is s→u1→v1→u2→v2→... vb→ub+1 → vb+1 →t.
Then since all the backward edges are deleted from the residual graph according to the super-fast algorithm, the residual flow will be 1. But there is a flow of value b + 1 by using the horizontal edges (u→ v1).

4. *Kleinberg, Jon. Algorithm Design (p. 418, q. 8)* Consider this problem faced by a hospital that is trying to evaluate whether its blood supply is sufficient:

   In a (simplified) model, the patients each have blood of one of four types: A, B, AB, or O. Blood type A has the A antigen, type B has the B antigen, AB has both, and O has neither. Patients with blood type A can receive either A or O blood. Likewise patients with type B can receive either B or O type blood. Patients with type O can only receive type O blood, and patients with type AB can receive any of the four types.

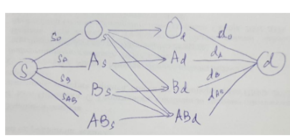
   (a) Let integers s_O, s_A, s_B, s_{AB} denote the hospital's blood supply on hand, and let integers d_A, d_B, d_O, d_{AB} denote their projected demand for the coming week. Give a polynomial time algorithm to evaluate whether the blood supply is enough to cover the projected need.

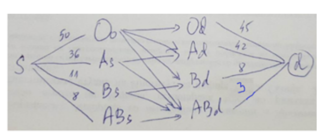
```
Input:   supply, s0, sA, sB, SAB      demand d0. dA, dB, dAB
Opuput:  T true, F  false
if  s0<d0                              return F
if  s0+sA <dA+d0                       return F
if  s0+sB <dB+d0                       return F
if  s0+sA +sB<dA+dB+d0                 return F
if  s0+sA+sB+sAB< dA+dB+sAB+d0  return F
return T
```

Solution:

   (b) Network flow is one of the most powerful and versatile tools in the algorithms toolbox, but it can be difficult to explain to people who don't know algorithms. Consider the following instance. Show that the supply is **insufficient** in this case, and provide an explanation for this fact that would be understandable to a non-computer scientist. (For example: to a hospital administrator.) Your explanation should not involve the words *flow*, *cut*, or *graph*.

blood type	supply	demand
O	50	45
A	36	42
B	11	8
AB	8	3





We have to find the maximal flow that will satisfied demand and saturated all demand edges.
Explanation for medical personal:
In presented case all demand is 90 units of blood of all the types. Also we have to satisfy the demand for particular blood types. In the case people with blood group O can receive only type O blood. Demand can be fulfilled with 42 units of type 0 having 50 units supply. Rest of 5 units supply can be used for people with blood type A. Together with supply of 36 units of this blood type (both 41) is not enough to cover demand of 42 units.

Solution:

5. Implement the Ford-Fulkerson method for finding maximum flow in graphs with only integer edge capacities, in either C, C++, C#, Java, or Python. Be efficient and implement it in $O(mF)$ time, where m is the number of edges in the graph and F is the value of the maximum flow in the graph. We suggest using BFS or DFS to find augmenting paths. (You may be able to do better than this.)

   The input will start with a positive integer, giving the number of instances that follow. For each instance, there will be two positive integers, indicating the number of nodes $n = |V|$ in the graph and the number of edges $|E|$ in the graph. Following this, there will be $|E|$ additional lines describing the edges. Each edge line consists of a number indicating the source node, a number indicating the destination node, and a capacity. The nodes are not listed separately, but are numbered $\{1 \dots n\}$.

   Your program should compute the maximum flow value from node 1 to node n in each given graph.

   A sample input is the following:

```
2
3 2
2 3 4
1 2 5
6 9
1 2 9
1 3 4
2 4 1
2 5 6
3 4 4
3 5 5
4 6 8
5 6 5
5 6 3
```

   The sample input has two instances. For each instance, your program should output the maximum flow on a separate line. Each output line should be terminated by a newline. The correct output for the sample input would be:

```
4
11
```