STAT340 Lecture 07: Estimation part 2

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Introduction

In these notes, we will continue our discussion of the problem of estimation.

Learning objectives

After this lesson, you will be able to

- Explain the concept of confidence intervals
- Use Monte Carlo methods to construct a confidence interval for the parameter of a distribution.
- Explain the central limit theorem and use it to construct a confidence interval for the parameter of a distribution
- Explain the connection between confidence intervals and hypothesis testing

Recap: estimation so far

Last week, we began discussing the problem of estimation, where we are interested in determining the value of some quantity out there in the world (most commonly the parameter of a distribution). We focused in our previous lectures on point estimation, where we are concerned with simply giving a single number as our “best guess” of the true value of the parameter of interest. Still, we mentioned several times in lecture the fact that our “best guess”, i.e., our estimate, would itself be random because the data is random. The result of this randomness is that for any estimate that we might produce, we have level of “uncertainty” associated to that estimate.

Example: gone fishing I

Suppose we are working for the Department of Natural Resources (DNR), tasked with estimating how many fish live in Lake Mendota. Our colleagues in ecology go out in the field and come up with an estimate of 10,848 fish in Lake Mendota (fictional).

Presumably, the exact number of fish in Lake Mendota changes from day to day.

So perhaps we would not object much if someone came along and revised this number from, say, 10,848 to 10,306.

Perhaps we would not even object much if this estimate were revised by a thousand fish in one direction or the other.

Still, there are a range of values around 10,848 that seem like “reasonable” guesses, and there are other values that seem pretty clearly “unreasonable”.

For example, revising our estimate from approximately 10,000 fish down to 1,000 fish seems extreme.

Example: gone fishing II

What we would like, ideally, is to provide a range of values for our estimate that capture our level of uncertainty (or, conversely, our level of certainty, i.e., . . . confidence) about this number. Specifically, we would like to produce an interval that is somehow "usually" correct.

But what does it mean for an interval to be "usually right"?

Enter: Confidence intervals I

Suppose we observe a particular value for our statistic S. Remember, $S = S(X_1, X_2, \ldots, X_n)$ is a function of our data and estimates our parameter of interest, say, θ.

We'll assume further, for simplicity, that $\mathbb{E}S = \theta$ (our estimator is unbiased for θ).

Note that this isn't always the case, but further discussion of the case where S is biased will have to wait (and there's no need to make things more complicated than they already are!).

We would like to be able to give an interval of values around our observed value of S such that θ is (with some level of certainty, anyway), in that interval.

That is, roughly speaking, the definition of a confidence interval.

Enter: Confidence intervals II

First things first: $S = S(X_1, X_2, \ldots, X_n)$ is a random variable, thus it has a distribution.

- If we knew that distribution, we could compute $\mathbb{E}S = \theta$ exactly
- Of course, in practice, we don't know that distribution

But let's imagine for now that we knew the true distribution of S.

Example I

Let's say that S is the sample mean of $n = 25$ independent draws from Normal$(\mu = 1, \sigma = 1)$.

The true value of $\mathbb{E}S$ in this case is

$$\mathbb{E}S = \mathbb{E}\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}X_i = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu = 1,$$

Example II

Where we have used the facts that

1. S is the sample mean, i.e., $S = \sum_{i=1}^{n} X_i/n$
2. The expectation operator is linear: $\mathbb{E}aX = a\mathbb{E}X$ and expectation of a sum is the sum of expectations
3. $\mathbb{E}X_i = \mu$ (because $X_i \sim \text{Normal}(1,1)$ for all i)

In fact, in this case, we know the exact distribution of S.

- S is the mean of $n = 25$ independent Normal$(1,1)$ RVs
- $S \sim \text{Normal}(\mu = 1, \sigma = 1/\sqrt{n})$
- Plugging in $n = 25$, $S \sim \text{Normal}(\mu = 1, \sigma = 1/5)$

Example III

Let's use R to compute the exact quantiles of this distribution.

Reminder: We do that with the function `qnorm`. Analogously to `rnorm`, `pnorm` and `dnorm`, think "q for quantiles"

```
# The first argument to qnorm is a vector of the quantiles we want
# (given as probabilities). Asking for the 2.5% and 97.5% quantiles.
qnorm( c(0.025, 0.975), mean=1, sd=1/5)
## [1] 0.6080072 1.3919928
```

What this says is that if $Z \sim \text{Normal}(\mu = 1, \sigma = 1/5)$, then 2.5% of the time Z will be less than ≈ 0.608, and 97.5% of the time Z will be less than ≈ 1.392.

Putting those facts together, we conclude that

$$\Pr\left[0.608 \leq S \leq 1.392\right] = 0.95.$$

95% of the time, S is between 0.608 and 1.392.

Aside: wait, that wasn't a confidence interval!

No, this is not a confidence interval.

The definition of a $(1-\alpha)$-confidence interval for the parameter θ is

- a data-dependent interval $C(X_1, X_2, \ldots, X_n) = (L, U)$ (L, U for "lower" and "upper"; they are functions of the data!),
- with the property that when the data are drawn under the distribution with parameter θ (note: we write $\Pr[\ldots; \theta]$ to drive home the fact that this is the probability under the model where θ is the true parameter),

$$\Pr\left[\theta \in C(X_1, X_2, \ldots, X_n); \theta\right] = \Pr\left[L \leq \theta \leq U\right] = 1 - \alpha.$$

Do you see the issue? In our statements about S above, all the randomness was in S, and our interval $(0.608, 1.392)$ was not random at all. Here, on the other hand, all the randomness is in the interval C.

Capturing uncertainty in our estimate I

So let's return to our illustrative example above. Recall that

- S is the sample mean of $n = 25$ independent draws from Normal($\mu = 1, \sigma = 1$).
- $\mathbb{E}S = \mu = 1$.
- $S \sim$ Normal($\mu = 1, \sigma = 1/5$)

In our experiment above, we computed an exact interval, $C = (0.6080072, 1.3919928)$, such that with probability 0.95, $S \in C$.

How does that help us to express (un)certainty?

Well, the standard deviation of S is $\sigma/\sqrt{n}$. If we increase n, the standard deviation of S decreases, and our resulting confidence interval gets narrower.

Capturing uncertainty in our estimate II

For example, if $n = 100$, then $S \sim \text{Normal}(\mu = 1, \sigma = 1/10)$, and our quantiles become

```
qnorm( c(0.025, 0.975), mean=1, sd=1/sqrt(100) )
## [1] 0.8040036 1.1959964
```

Compare that with our interval $(0.6080072, 1.3919928)$ when $n = 25$.

This is in keeping with the fact that as we collect more data (i.e., observe more samples), we get more confident about our predictions. Again, a narrower interval corresponds to a higher level of confidence.

Okay, but we don't know $\mathbb{E}S$ I

We have been assuming in our example above that we know the expectation of S, which is... well, never true in practice. Moreover, we know the full distribution of S even less often.

If we knew the true parameter (i.e., $\theta = \mathbb{E}S$), we could generate data from the "true" distribution like above.

In particular, we could see what data and/or what values of S are "typical" given that value of the parameter θ.

Solution: estimate our model (i.e., estimate the parameter θ, presumably using $\hat{\theta} = S(X_1, X_2, \ldots, X_n)$ as our estimate). Then generate data from our estimated model and see what values of our estimator are "typical", by computing our statistic on the generated data.

Said another way, we pretend to believe that our estimated model is the truth and generate data from it many times to get a sense of the "typical" behavior of S.

Okay, but we don't know $\mathbb{E}S$ II

In essence, we are doing the following:

- Pretend for now that our estimated parameter is the truth
- Generate data from our model with that parameter
- Compute the statistic S on that data.

This gives us multiple replicates of simulated data under the setting where our observed value of S is the truth.

In a sense, these samples produce values of our statistic that would be "reasonable" if the true parameter were equal to $s = S(x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n$ are our original observed data.

Example: Intervals for the Poisson rate parameter I

Suppose that we have data $X_1, X_2, \ldots, X_n$ drawn i.i.d.$\sim$from a Poisson with unknown rate parameter λ, and our goal is to estimate λ.

Since $\mathbb{E}X_i = \lambda$ for all $i = 1, 2, \ldots, n$, a reasonable choice of estimator is the sample mean $\bar{X} = n^{-1} \sum_i X_i$. Indeed, the law of large numbers guarantees that $\bar{X}$ is close to $\lambda = \mathbb{E}X_1$ with high probability.

Now, let's generate data from this model, this time imagining that we don't know the parameter λ.

Instead, we'll estimate λ from our data, and to get a "confidence interval" for λ, we will pretend that our estimate $\bar{X}$ is the truth and generate new copies of the data using $\lambda = \bar{X}$.

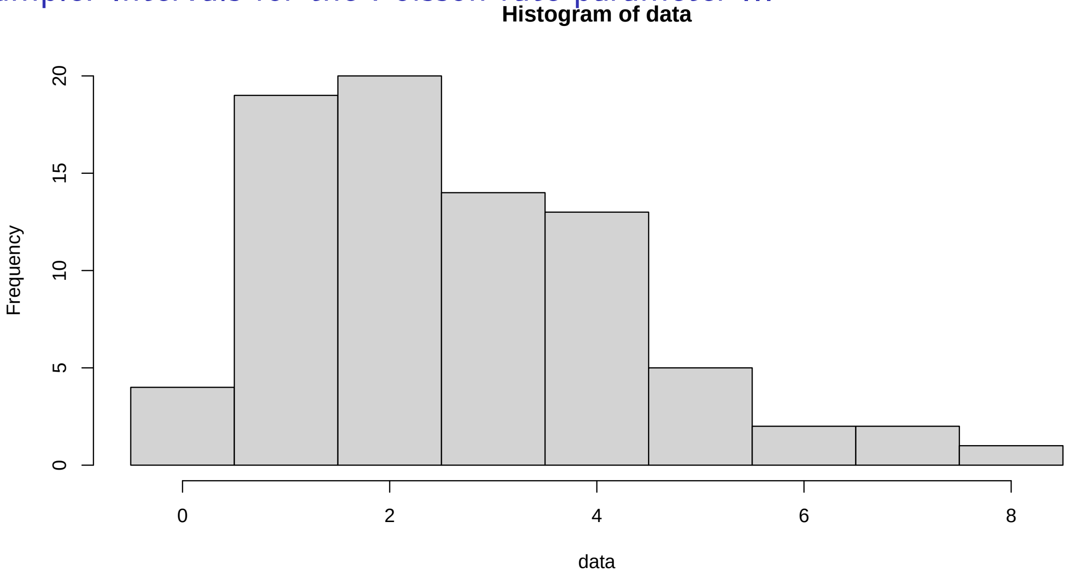
Example: Intervals for the Poisson rate parameter II

```r
n <- 80; # Sample size of our data
lambda_true <- 2.718; # True (but unknown!) Poisson parameter

data <- rpois(n=n, lambda=lambda_true);

hist(data, breaks=seq(-.5,max(data)+.5,1));
```

Example: Intervals for the Poisson rate parameter III



Example: Intervals for the Poisson rate parameter IV

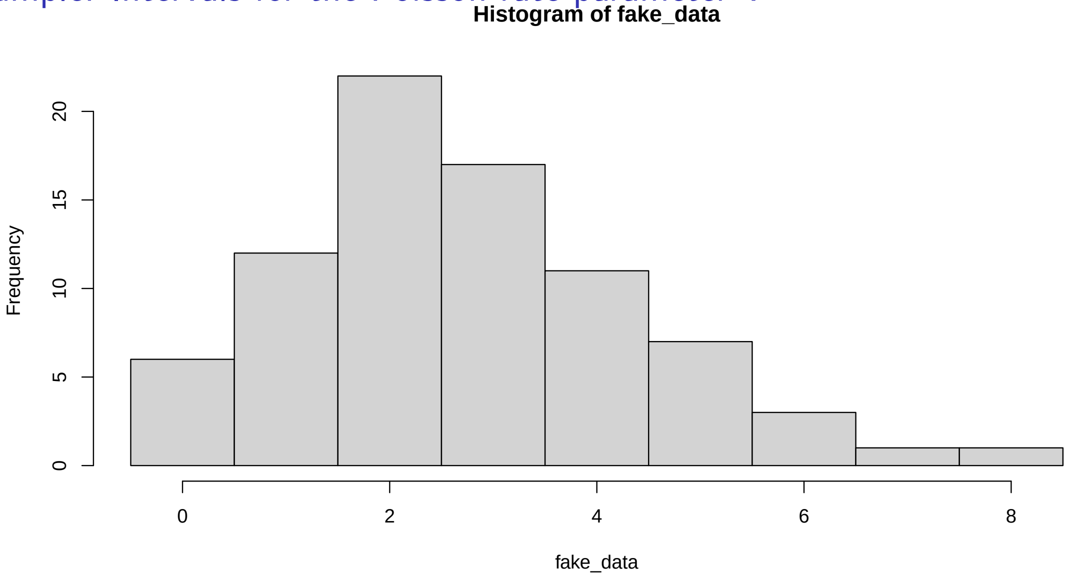
Now, let's get our estimate (just the sample mean!), pretend it's the true value of λ, and generate data.

```
# Reminder: we say lambdahat to remind ourselves this is
# an estimate of lambda.
lambda_hat <- mean(data); # Sample mean is our estimate of lambda.

# Generate data under the distribution where lambda_hat is the rate
# parameter. To reiterate, we're pretending that our estimate is the
# truth and seeing what our data would look like.
fake_data <- rpois(n=n, lambda=lambda_hat);

hist(fake_data, breaks=seq(-.5,max(fake_data)+.5,1))
```

Example: Intervals for the Poisson rate parameter V



Example: Intervals for the Poisson rate parameter VI

The math is more complicated to work out the exact sampling distribution of the sample mean under the Poisson. Instead we'll estimate using Monte Carlo.

Again, here's the recipe:

- Estimate our distribution's parameter from the data (we've done that with our sample mean `lambda_hat`)
- Generate data from the distribution with parameter equal to our estimate (we just did that above)
- Repeat the previous step many times, computing a new estimate on each new data sample. Each repetition gives us a replicate of our statistic, under the distribution where $\lambda = \text{lambda\_hat}\}$.
- We can use that to construct a confidence interval for λ, by looking at the quantiles of our sample.

Example: Intervals for the Poisson rate parameter VII

```r
# Reminder: sample size and true parameter
n<- 80; lambda_true <- 2.718; set.seed(1)
data <- rpois( n=n, lambda=lambda_true); # Generate data
lambda_hat <- mean(data); # Just estimating this again to remind you

Nrep <- 2000; # Number of repetitions (i.e., replicates)
replicates <- rep(NA,Nrep); # We will store replicates here.
for ( i in 1:Nrep) {
  fake_data <- rpois(n=n, lambda=lambda_hat);
  replicates[i] <- mean( fake_data );
}
```

Example: Intervals for the Poisson rate parameter VIII

```
# Now construct the confidence interval
CI <- quantile( replicates, probs=c(0.025, 0.975) );
cat(CI)
## 2.4625 3.175
```

On most simulations when we run the code above we should find that $\lambda = 2.718$ is inside of that interval.

Sometimes, though, due to randomness in our data, the interval will not contain the true value of λ.

Now, how often does that happen? Well, let's try creating lots of intervals like the one above and count how often our interval contains the true value of λ.

Implementation I

Run one instance of our experiment above, wherein we

1) Generate data from Pois(lambda_true)
2) Estimate lambdahat from that data.
3) Repeatedly generate data from Pois(lambdahat)
4) Use those repetitions to get an interval.

The inputs are:

- `lambdatrue` : the true lambda parameter for Poisson
- `n` : sample size for the data
- `Nrep` : repetitions to use when computing the interval

Implementation II

```
run_trial <- function(lambdatrue, n, Nrep) {
  data <- rpois(n=n, lambda=lambdatrue);
  lambdahat <- mean(data);
  replicates <- rep(NA,Nrep); # We will store replicates here.
  for ( i in 1:Nrep) {
    fake_data <- rpois(n=n, lambda=lambdahat);
    replicates[i] <- mean( fake_data );
  }

  # Now construct the confidence interval
  # the names=FALSE tells R to return the quantiles with no header
  CI <- quantile( replicates, probs=c(0.025, 0.975), names=FALSE );
  return( (CI[1] < lambdatrue) & (lambdatrue < CI[2]) );
}
```

Implementation III

Now, let's try repeating the experiment many times by calling `run_trial(lambdatrue=2.718, n=80, Nrep=1000 )` a bunch.

```
set.seed(1); Nexpt <- 500;
expt_results <- rep(NA,Nexpt); # Vector to store results
for (i in 1:Nexpt) {
  expt_results[i] <- run_trial(lambdatrue=2.718, n=80, Nrep=1000)
}

sum(expt_results)/length(expt_results)
## [1] 0.94
```

This is very close to .95. Recall we chose the quantiles (0.025, 0.975) to mimic a 95% confidence interval. Our experiment shows that the intervals that we constructed above based on simulations are (approximately) 95% confidence intervals!

Example: estimating the rate in the exponential distribution I

Suppose that we observe data $X_1, X_2, \ldots, X_n$ i.i.d. from an exponential distribution with rate parameter λ, so that $\mathbb{E}X_1 = 1/\lambda$, and our goal is to estimate λ.

The plug-in principal and/or the method of moments suggests that we use $1/\bar{X}$ as our estimate of λ.

Applying our simulation-based CI recipe, we should:

1. Estimate λ as $\hat{\lambda} = 1/\bar{X}$.
2. Repeatedly generate data as though $\hat{\lambda}$ were the true value of the parameter, and "re-estimate" λ on each such data set.
3. Use the histogram of the resulting "fake" estimates to estimate our interval.

Example: estimating the rate in the exponential distribution II

```r
true_rate <- 5; nsamp <- 25;NMC <- 2000;
obsd_data <- rexp(n=nsamp, rate=true_rate);
estd_rate <- 1/mean(obsd_data); #estimate the rate parameter

replicates <- rep(NA, NMC);   expt_results <- rep(NA, NMC);
for ( i in 1:NMC) {
  fake_data <- rexp(nsamp, rate=estd_rate);
  replicates[i] <- 1/mean(fake_data);
}
# Now construct the confidence interval
# the names=FALSE tells R to return the quantiles with no header
CI <- quantile( replicates, probs=c(0.025, 0.975), names=FALSE );
(CI[1] < true_rate) & (true_rate < CI[2]) ;
## [1] TRUE
```

A function to check ‘coverage rate’ I

A function to check ‘coverage rate’ II

```
run_exprate_expt <- function( nsamp, true_rate ) {
  obsd_data <- rexp(n=nsamp, rate=true_rate);
  estd_rate <- 1/mean(obsd_data);

  # Now generate "fake" data sets and estimate lambda on each.
  NMC <- 2000;
  replicates <- rep(NA, NMC);
  expt_results <- rep(NA, NMC);
  for ( i in 1:NMC) {
    fake_data <- rexp(nsamp, rate=estd_rate);
    replicates[i] <- 1/mean(fake_data);
  }
  CI <- quantile( replicates, probs=c(0.025, 0.975), names=FALSE );
  return( (CI[1] < true_rate) & (true_rate < CI[2]) );
}
```

A function to check ‘coverage rate’ III

Now let’s run the above experiment a few thousand times and see how often we “catch” the true value of lambda.

```
lambda_true <- 2;
nsamp <- 100;
M <- 2000;
reps <- rep(NA, M);
for (i in 1:M) {
  reps[i] <- run_exprate_expt( nsamp, lambda_true )
}

sum(reps)/M
## [1] 0.9445
```

A function to check ‘coverage rate’ IV

One important thing to bear in mind is that while the simulation-based recipe demonstrated above is a fairly general approach, it only produces an *approximately correct* confidence interval, which relies on how well we can estimate the model parameter to begin with.

Exercise:

Try playing around with `nsamp` and `lambda_true` in the code above. Consider what happens if `nsamp` is small (e.g., 20 or 25): $\bar{X}$ will not necessarily be close to $\mathbb{E}\bar{X} = 1/\lambda$, and thus our estimate of λ, $1/\bar{X}$ will likely be far from λ, and thus our "fake" data will not actually look very much like it should.

In another direction, try playing around with the parameter $\lambda > 0$. What happens if you make λ really big or really close to zero?

In your discussion section and homework, you'll get more practice with this framework.

The important takeaway is this: our goal in producing lots of replicates is to get a sense of what "reasonable" other values of our statistic might have been.

Confidence intervals another way: the CLT

Now, how do we relate the above framework to the version of confidence intervals that you might have learned in your intro courses?

The central limit theorem states that if $X_1, X_2, \ldots$ are iid random variables with shared mean $\mu = \mathbb{E} X_1$ and variance $\sigma^2 = \operatorname{Var} X_1$, then as $n \to \infty$, the standardized random variable

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\sigma^2/n}}$$

is well approximated by a standard normal.

Specifically, what we mean by "well approximated" is that, letting $Z \sim \text{Normal}(0,1)$, for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \Pr\left[\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\sigma^2/n}} \le t\right] = \Pr[Z \le t].$$

Aside

If you've seen *convergence in distribution* before, this should look familiar– this says that the sample mean (after centering and scaling) converges in distribution to a standard normal.

Said another way, the central limit theorem says that for large sample size, the distribution of the sample mean $\bar{X}$ is well approximated by a normal random variable with mean $\mu = \mathbb{E}X_1$ and variance $\sigma^2/n = n^{-1} \operatorname{Var} X_1$.

The CLT in action I

Let's quickly do an experiment that you've probably seen before: watching the CLT in action.

Again, the CLT says that if we take a sample of i.i.d. random variables, take their mean, center it about their expectation, and rescale them by their standard deviation, they should be *approximately* normal.

Let's have a look.

The CLT in action II

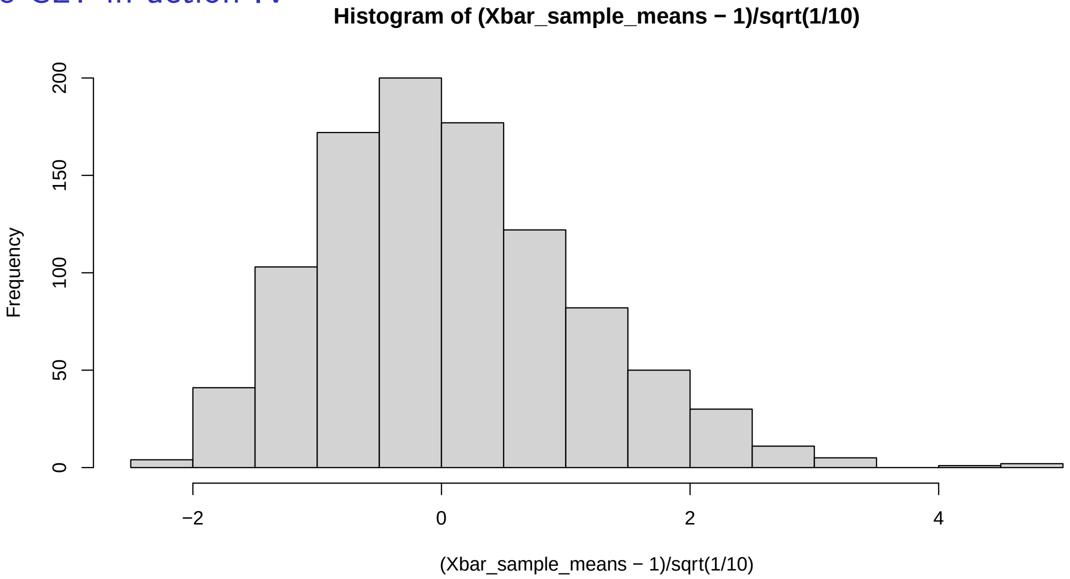
```r
NMC <- 1e3;
# Code to draw n iid RVs from exp(1) and compute their sample mean.
generate_sample_mean <- function( n ) {
  X <- rexp(n=n, rate=1);
  # X has mean 1, variance 1
  # See https://en.wikipedia.org/wiki/Exponential_distribution
  return( mean(X) )
}
```

The CLT in action III

```
# Generate a bunch of such sample means, each on n=10 data points
Xbar_sample_means <- rep( NA, NMC);
for( i in 1:NMC ) {
  Xbar_sample_means[i] <- generate_sample_mean(10);
}

# We center about the mean 1,
# and rescale by the standard deviation of the sample mean,
# sqrt( ( Var X_1 )/10 ) = sqrt(1/10)
hist( (Xbar_sample_means-1)/sqrt(1/10) )
```

The CLT in action IV

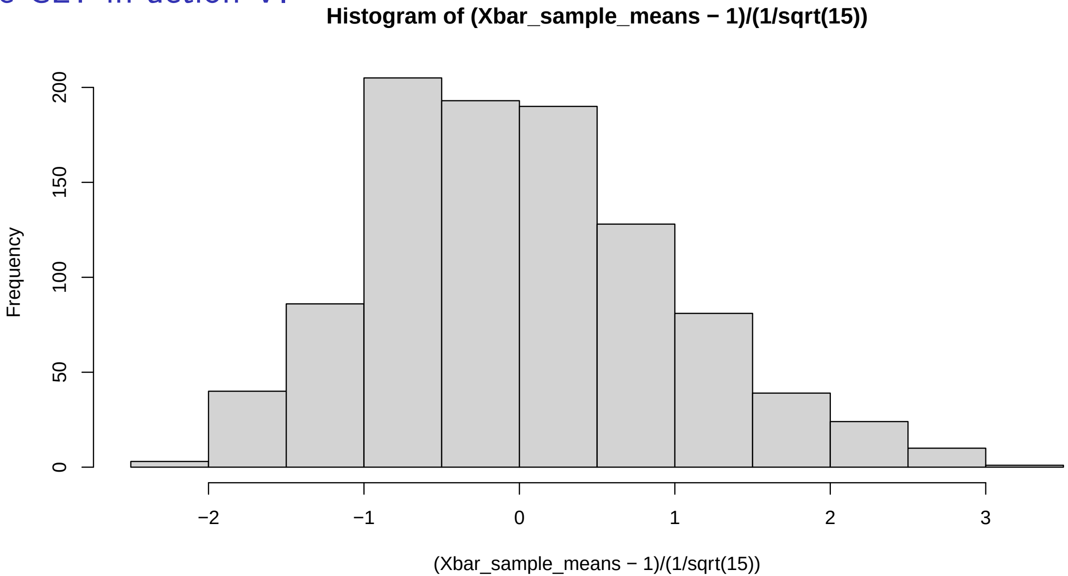


The CLT in action V

```
# Now let's try again, but with a larger sample size.
# Changing n=10 to n=15.

Xbar_sample_means <- rep( NA, NMC);
for( i in 1:NMC ) {
  Xbar_sample_means[i] <- generate_sample_mean(15);
}
hist( (Xbar_sample_means-1)/(1/sqrt(15)) )
```

The CLT in action VI

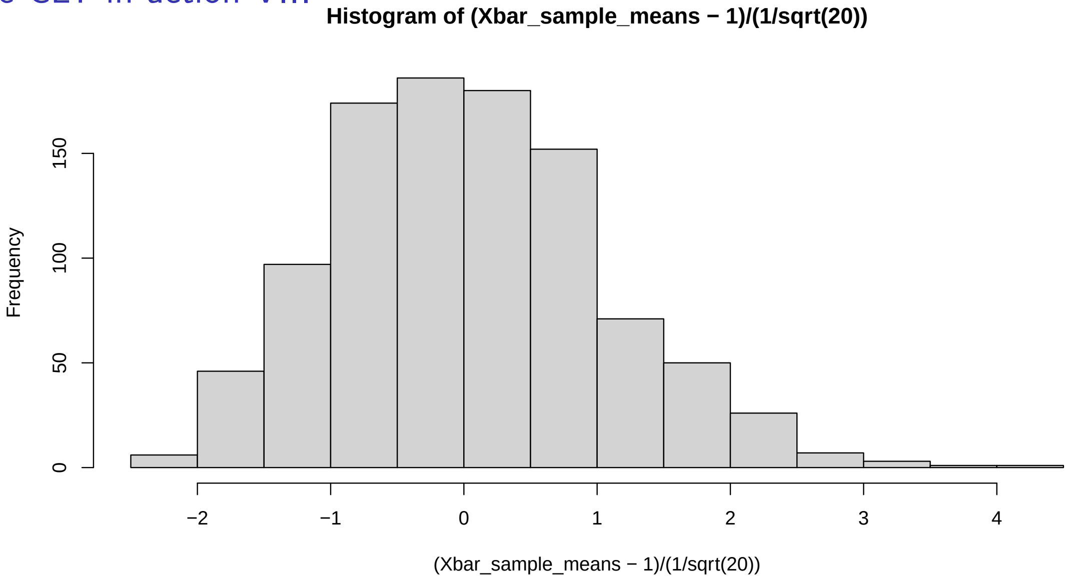


The CLT in action VII

Let's try one more, this time with $n = 20$ data points.

```
Xbar_sample_means <- rep( NA, NMC);
for( i in 1:NMC ) {
  Xbar_sample_means[i] <- generate_sample_mean(20);
}
hist( (Xbar_sample_means-1)/(1/sqrt(20)) )
```

The CLT in action VIII



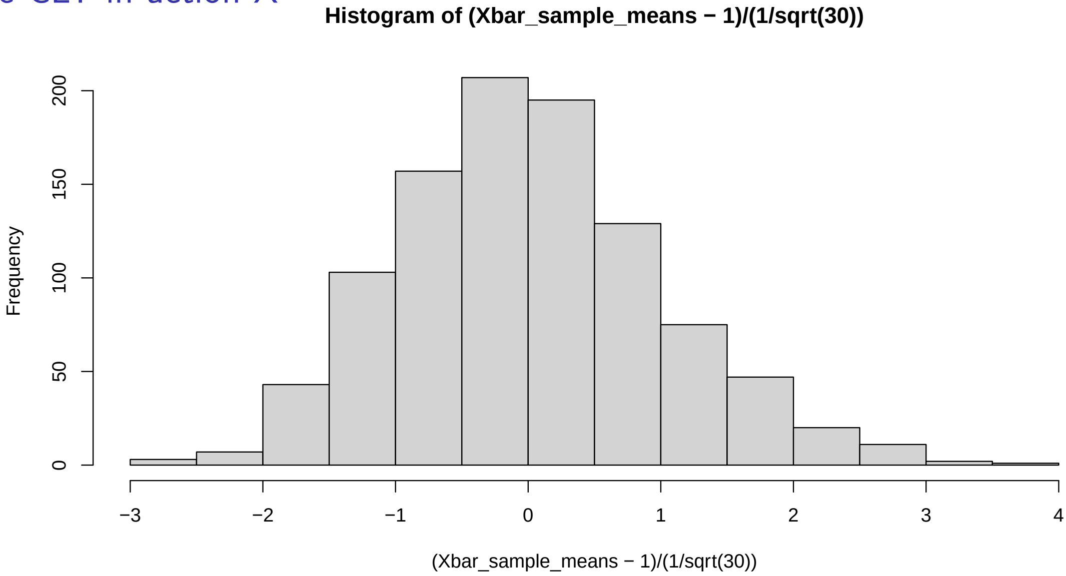
The CLT in action IX

Now let's crank it all the way up to $n = 30$.

```
Xbar_sample_means <- rep( NA, NMC);
for( i in 1:NMC ) {
  Xbar_sample_means[i] <- generate_sample_mean(30);
}

hist( (Xbar_sample_means-1)/(1/sqrt(30)) )
```

The CLT in action X



The CLT in action XI

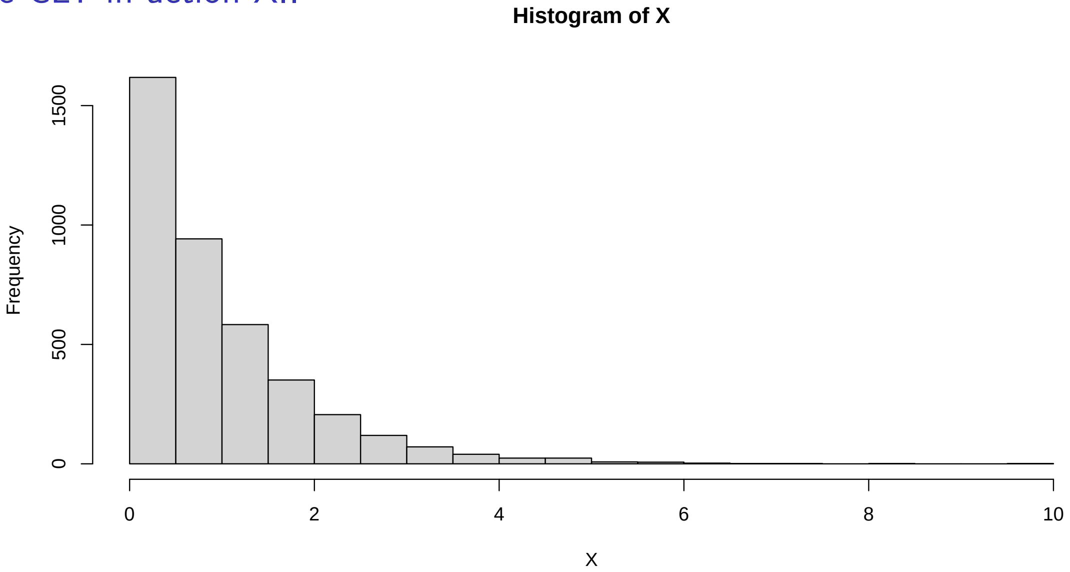
With just 20 to 30 samples, the CLT is really kicking in.

Just to reiterate, the underlying distribution we are sampling from here look nothing like a bell curve:

```
X <- rexp(n=4000, rate=1);
# X has mean 1, variance 1
# See https://en.wikipedia.org/wiki/Exponential_distribution

hist( X , breaks=30)
```

The CLT in action XII



The CLT in action XIII

Let's look at a different example where the variance is not 1 to see what that implies

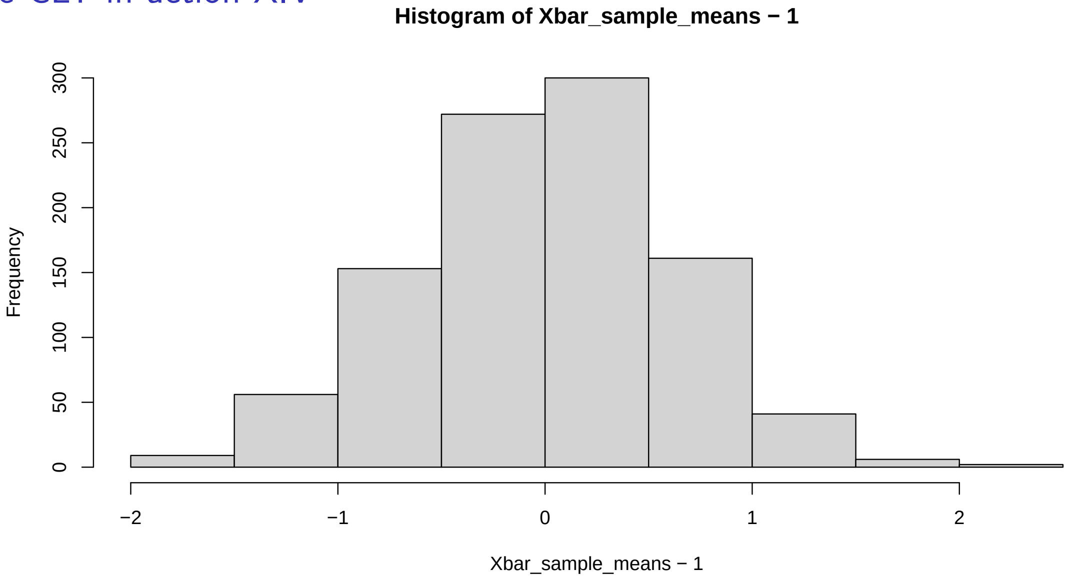
```
NMC <- 1e3;

generate_sample_mean <- function( n ) {
  X <- rnorm(n=10, mean=1, sd=2);
  return( mean(X) )
}

Xbar_sample_means <- rep( NA, NMC);
for( i in 1:NMC ) {
  Xbar_sample_means[i] <- generate_sample_mean(n);
}

hist( Xbar_sample_means-1 );
```

The CLT in action XIV



Exercise:

Repeat the above experiment with a different data distribution and see how large n has to be for the CLT to look like a reasonably good distribution. If you play around enough with different distributions and different choices of parameter values, you should be able to find two different data distributions that have decidedly different behavior in terms of when the centered, rescaled sample mean starts to “look” Gaussian.

Using the CLT to build a CI I

So the CLT says that the (recentered, rescaled) sample mean "looks like" a standard normal once n is large enough.

Well, looking like the standard normal is good– we know how to compute quantiles for the standard normal!

And that means we can compute (approximate) quantiles for the (recentered, rescaled) sample mean,

$$\frac{\bar{X} - \mu}{\sqrt{\sigma^2 / n}},$$

Using the CLT to build a CI II

For example, if $Z \sim \text{Normal}(0, 1)$, then

$$0.95 = \Pr[-1.96 \leq Z \leq 1.96] \approx \Pr\left[-1.96 \leq \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \leq 1.96\right].$$

Rearranging terms inside that second probability,

$$0.95 \approx \Pr\left[\bar{X} - 1.96\sqrt{\sigma^2/n} \leq \mu \leq \bar{X} + 1.96\sqrt{\sigma^2/n}\right]$$

so $\bar{X} \pm 1.96\sqrt{\sigma^2/n}$ would be an (approximate) 95% CI for μ if only we knew σ^2.

That's a pesky issue that we will ignore here, but you'll get to play around with estimating σ^2 in your homework (see below for an example of the case where we estimate the variance).

This confidence interval is only approximate, because it relies on the CLT, in much the same way as our simulation-based procedure above was only approximate.

Example: confidence interval for the mean using CLT I

Let's consider the problem of estimating the mean of a geometric random variable. For a geometric random variable with parameter p, the mean is $(1-p)/p$ and variance is $(1-p)/p^2$ (see `?rgeom`).

Suppose that we observe data $X_1, X_2, \ldots, X_n$ drawn i.i.d. according to a geometric distribution with parameter p. To obtain a confidence interval for the mean, we begin by noting that the expectation of a geometric random variable

$$\mathbb{E}\bar{X} = \frac{1-p}{p},$$

and the variance is $(1-p)/(np^2)$. Bear in mind that in R, the geometric RV counts the number of flips *before* the first heads, so it can take the value 0.

Example: confidence interval for the mean using CLT II

The central limit theorem says that the quantity

$$\frac{\bar{X} - (1-p)/p}{\sqrt{(1-p)/n}/p}$$

should be (approximately) normal with mean zero and variance one.

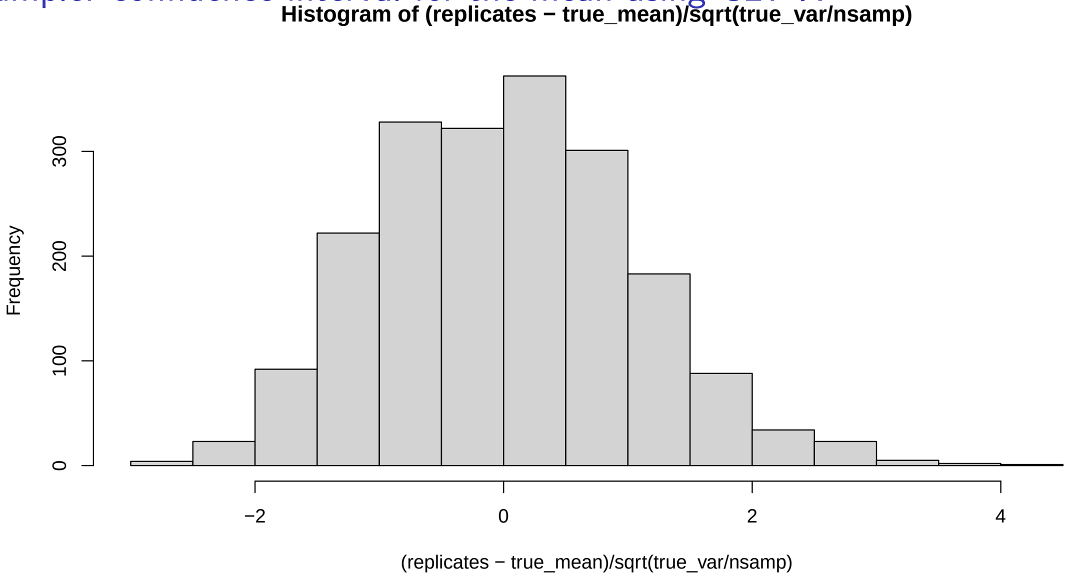
Let's first check that this is the case. Let $p = 0.33$.

```
ptrue <- 0.33;
nsamp <- 30;
M <- 2000;
replicates <- rep(NA, M);
for( i in 1:M) {
  data <- rgeom(n=nsamp, prob=ptrue);
  replicates[i] <- mean( data ); # = Xbar.
}
```

Example: confidence interval for the mean using CLT III

```
true_mean <- (1-ptrue)/ptrue;
true_var <- (1-ptrue)/ptrue^2;
hist( (replicates-true_mean)/sqrt(true_var/nsamp) );
```

Example: confidence interval for the mean using CLT IV



Example: confidence interval for the mean using CLT V

The approximation isn't perfect– the fact that geometric RVs are non-negative means that the sampling distribution of the sample mean is skewed to the right. Further, the geometric distribution is discrete, which complicates things further. Still, let's press on and see if this is a good enough approximation.

Note: you may want to look at what happens if you decrease the number of samples (e.g., down to 25) or increase them (e.g., up to 35 or 40).

Now, let's use this fact to construct a CLT-based confidence interval for the mean $(1-p)/p$ under the geometric distribution.

Example With Data I

Suppose that we observe the following data:

```
data <- c( 2, 0, 1, 1, 0, 3, 2, 2, 0, 0, 1, 6, 0, 1, 0, 0, 0, 11, 0,
           1, 0, 1, 0, 2, 0, 2, 1, 2, 1, 0 );
```

Our sample mean is

```
xbar <- mean(data);
xbar
## [1] 1.333333
```

Example With Data II

Now, to construct our confidence interval for the mean $(1-p)/p$, we need to have an estimate for the variance. Remember,

$$\frac{\bar{X} - (1-p)/p}{\sqrt{(1-p)/n}/p}$$

will be approximately normal.

One option for estimating the variance would be to estimate p and then plug that into the expression for the variance. **Exercise:** try that!

Here, though, let's estimate the variance the easy way– computing the sample variance of the data.

```
varhat <- var( data );
varhat
## [1] 4.988506
```

Example With Data III

Plugging this into the CLT, a 0.95 confidence interval is

```
n <- length( data )
c( xbar - 1.96*sqrt( varhat/n ), xbar + 1.96*sqrt( varhat/n ))
## [1] 0.5340869 2.1325797
```

The true parameter was $p = 0.33$, so the true mean is

```
truep <- 0.33;
truemean <- (1-truep)/truep
truemean
## [1] 2.030303
```

Our CI caught the mean! Great!

Exercise:

Repeat our experiment from earlier to estimate how often the above CLT-based confidence interval contains the true mean. You'll want to repeatedly

1. Generate $n = 30$ random geometric RVs with success probability $p = 0.33$.
2. Compute the sample mean and sample variance.
3. Use these to build the confidence interval
4. Check whether or not the CI contains the true mean $(1 - p)/p = 2.03030$.

Repeat the above a couple of thousand times and check how often the CI contains the true mean. It *should* be about 0.95, though of course it won't be exact, both because of randomness and because the CLT is only an approximation.

Which confidence interval?

Now you know two completely different ways to construct confidence intervals! Unfortunately, which one is the “right one” is a tricky question.

Sometimes it’s a question of which one is easier (i.e., requires less computer time and/or thinking time).

Sometimes there are obvious reasons why one or the other will be more accurate than the other (e.g., because you know that the CLT is a good approximation even for small n for the data distribution).

Duality of testing and confidence intervals I

One important observation that we should make is that there is a connection between confidence intervals and the problem of testing.

Let's recall that the idea behind a confidence interval is that it is a random interval, and a confidence interval with confidence level $1 - \alpha$ should contain our parameter of interest with probability $1 - \alpha$.

Let's call our random interval $C = (L, U)$, bearing in mind that C is a random interval, which depends on our observed data. In particular, letting θ denote the true value of our parameter of interest, our confidence interval C obeys the property that

$$\Pr[\theta \in C] = \Pr[L \leq \theta \leq U] = 1 - \alpha.$$

Duality of testing and confidence intervals II

Suppose that we wish to test the hypothesis

$$H_0 : \theta = \theta_0,$$

where θ_0 is some particular value. For example, perhaps the parameter θ describes the size of the difference between two groups, and θ_0 corresponds to that difference being zero.

Consider the following test of H_0: given our data, compute the confidence interval C. If $\theta_0 \in C$, accept H_0. If $\theta_0 \notin C$, reject H_0.

It turns out that if C is a $100(1 - \alpha)$ confidence interval, then this test has level α. Let's see why this is.

Duality of testing and confidence intervals III

By the definition of our test,

$$\Pr[\text{ reject };H_0] = \Pr[\theta_0 \notin C; H_0] = 1 - \Pr[\theta_0 \in C; H_0].$$

Now, because C is a $100(1-\alpha)$-level confidence interval, $\Pr[\theta_0 \in C; H_0] = 1 - \alpha$. Therefore,

$$\Pr[\text{ reject };H_0] = \alpha.$$

That is, our test has level α.

It turns out that this argument runs the other way, too: given any test with level α, we can construct a $100(1-\alpha)$ confidence interval.

Duality of testing and confidence intervals IV

Challenge: sketch how we might do this. Imagine that we have a procedure such that for any value of θ_0, we can conduct a level-α test of the hypothesis

$$H_0 : \theta = \theta_0.$$

Explain how to turn this into a $100(1 - \alpha)\%$ confidence interval for θ.

Review

- Uncertainty in point estimates
- Probability intervals for estimator values
- MC Confidence interval for parameter value
- The Central Limit Theorem
- Parametric Confidence intervals (CLT)
- Comparison of MC vs CLT estimation assumptions
- Duality of testing & estimation