Learning, Hypothesis Testing and Nash Equilibrium

Foster & Young (2003)

Mateusz Pyla

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Introduction

1. Context
2. Paper
   - Main Result
   - Helper results
   - Tools
   - Lemma
3. Getting further

Previously in Game Theory

- If players are rational, good predictors, and learn deterministically, there are many games for which neither beliefs nor actions converge to a Nash equilibrium.
- Trivial algorithm (search the space of mixed strategies) dynamically converging to NE
- For some games, using fictitious play leads to Nash equilibrium.
- Knowing the appropriate Bayesian NE prior makes all limit points of the posterior to be also Bayesian NE.
- Assigning positive probabilities in players' prior belief to all events with actual positive probability leads to guarantee of reaching ε-equilibrium with probability one.
- Predicting the past i.e. conditional regret minimization.

Main result

Theorem (Foster & Young, 1998-2003)

Let G be a finite, normal-form, n-person game and let $\varepsilon \geq 0$. If the players are almost rational, use sufficiently powerful hypothesis tests with comparable amount of data, and are flexible in their adoption of new hypothesis, then at least $1 - \varepsilon$ of the time:

I *Their repeated-game strategies are ε-close to subgame perfect equilibrium,*

II *All players are ε-good predictors.*

Implied the assumption of continuity of $A_i^{\sigma_i}$

Towards the proof

1. Powerful family of tests
2. Upper bounds on errors
3. Fairly good model
4. Guarantees for responses

Then prove convergence in probability and ε-closeness to beliefs.

Two points of anchor:

- Fixed-point theorem
- Hypothesis testing tools

Bounds

$$\alpha_{i,s_i} \leq k_i(\tau)e^{-r_i(\tau)s_i} \qquad \text{(Upper bound on type-I error)}$$

$$\beta i, s_i, \tau \leq k_i(\tau)e^{-r_i(\tau)s_i} \qquad \text{(Upper bound on type-II error)}$$

$$\alpha_{i,s} \leq \sum_{j=\lceil\kappa/2\rceil}^{\kappa} \binom{\kappa}{j} \alpha^j (1-\alpha)^{\kappa-j} \leq [...] \leq \alpha(4\alpha(1-\alpha))^{\lceil\kappa/2\rceil}$$

Diffusion: $\forall\tau > 0 \quad \alpha_{i,s_i,\tau} \leq (1+\frac{\tau}{\xi})^{s_i}\alpha_{i,s_i}$

$q(R) = \sum_{\bar{\omega}^{t-1}\in R} \prod_t q(\omega^t|\bar{\omega}^{t-1}) \leq \sum_{\bar{\omega}^{t-1}\in R} \prod_t p(\omega^t|\bar{\omega}^{t-1}) \cdot (1+\frac{\tau}{\xi}) = (1+\frac{\tau}{\xi})^{s_i} p(R)$

Why we needed that?

$$\beta_{i,s_i,\tau} \leq k_i(\tau)e^{-r_i(\tau)s_i/2}$$

There exists $c_i(\tau) \leq \tau$ such that

$$\alpha_{i,s_i,c_i(\tau)} \leq k_i(\tau)e^{-r_i(\tau)s_i/2}$$

Lemma

We call model vector $\overrightarrow{\phi}$ to be fairly good if it is good for all responsive players. We denote it to be good if it is good for all players, meaning $|\phi_i - P_i(A^{\overrightarrow{\sigma}}(\overrightarrow{\phi}))| \leq \tau$ holds for player i (otherwise it is bad for i). Finally, we call it bad if it is not fairly good.

Theorem

The model vector $\overrightarrow{\phi}^t$ is fairly good at least $1 - \varepsilon$ of the time.

Consider the following process to reach a great state from a bad state.

1. Player 1 alone is testing and rejects his hypothesis and accepts a model reasonably close to ϕ_i^* but still wrong.
2. Then one after another the other players test their hypothesis and adopt a model within γ of ϕ_j^*
3. Player 1 starts a new test and now adopts ϕ_1^*
4. Don't start a test for $(n+2)s_*$

Almost done

Theorem

We assume that for the conditions described in the main theorem, the participants use M memory hypotheses for which they employ suitable hypothesis tests with comparable amount of data.
Moreover we assume that the engaged players have σ_i-smoothed best response functions.
Let $\varepsilon > 0$. There exist functions $\sigma(\varepsilon)$, $\tau(\varepsilon,\sigma)$ and $s(\varepsilon,\sigma,\tau)$ bounding the corresponding parameters.
Then at least $1-\varepsilon$ of the time t:

- I $|a^t - A^{\vec{\sigma}}(P(\vec{a}^{\,t}))| \leq \varepsilon/2$ *the responses are close to being a fixed point*
- II $|U_i^t(a_i^t, P_i(\vec{a}^{\,t})) - max_{a_i'} U_i^t(a', P_i(\vec{a}^{\,t}))| \leq \varepsilon$ *for all players i* *the responses are ε-optimal*
- III $|\phi_i^t - P_i(A^{\sigma_i}(\vec{\phi}^{\,t}))| \leq \varepsilon$ *for all players i* *the models are within ε of being correct*

Due to the previous result, we have that $\vec{\phi}^{\,t}$ is fairly good at least $1-\varepsilon$ of the time. Thus $|\phi_i - P_i(A^{\vec{\sigma}}(\vec{\phi}))| \leq \tau$. Now use continuity!

Several extra ideas

- More practical guarantees
- Unscrupulous diner's dilemma (gambit)
- Better bounds

Room for discussion