

Lemma 3. Fix an integer $m > 1$ and a field F . Suppose that, for every F -vector space V whose dimension is not divisible by m , every linear operator on V has an eigenvector in V . Then for every F -vector space V whose dimension is not divisible by m , each pair of commuting linear operators on V has a common eigenvector in V .

Suppose that:

- Let V be a vector space over F
- $m > 1 \quad : m \nmid \dim V$
- Every linear operator on V has an eigenvector in V

Show that:

- If two linear operators commute
↓
they have a common eigenvector in V

Corollary 4. For every real vector space V whose dimension is odd, each pair of commuting linear operators on V has a common eigenvector in V .

Given :

- Let V be a vector space over \mathbb{R}
- Dimension is odd $\Leftrightarrow 2 \nmid \dim V$

Show that:

- If two linear operators commute
↓
they have a common eigenvector in V

⇒ This is what we want to prove

Proof: Since V has an odd dimension, its characteristic polynomial has a real root and with that a non-zero eigenvector $v_1 \in V$.

- Consider the eigenspace E_1 corresponding to the eigenvalue λ_1 of T_1 associated with v_1 . Since T_1 is a linear operator, E_1 is a T_1 -invariant subspace.
- Now, since T_1 and T_2 commute, T_2 maps the eigenspace E_1 into itself, i.e., $T_2(E_1) \subseteq E_1$.

If T_2 is the zero operator on E_1 , then any vector in E_1 is a common eigenvector for T_1 and T_2 . So, we have a common eigenvector and there is nothing left to prove. Otherwise, if T_2 is not the zero operator on E_1 , it has an eigenvector $v_2 \in E_1$ associated with eigenvalue λ_2 .

Since $v_2 \in E_1$, it is also an eigenvector of T_1 , associated with eigenvalue λ_1 .

- Since v_2 is a non-zero vector, there exists a scalar k such that $v_1 + k \cdot v_2 \neq 0$ (assuming the field does not have characteristic 2)
- Consider $u = v_1 + k \cdot v_2$. We claim that u is a common eigenvector for T_1 and T_2 .
- To prove this, we apply T_1 and T_2 on u :

$$T_1(u) = T_1(v_1) + k \cdot T_1(v_2) = \lambda_1 \cdot v_1 + k \cdot (\lambda_1 \cdot v_2) = \lambda_1 \cdot (v_1 + k \cdot v_2) = \lambda_1 u$$

$$T_2(u) = T_2(v_1) + k \cdot T_2(v_2) = \lambda_2 \cdot v_1 + k \cdot (\lambda_2 \cdot v_2) = \lambda_2 \cdot (v_1 + k \cdot v_2) = \lambda_2 u$$

Hence, u is an eigenvector for both T_1 and T_2 , with eigenvalues λ_1 and λ_2 respectively.

- Therefore, u is a common eigenvector for T_1 and T_2

