THE ROW-ECHELON FORM AND ELEMENTARY MATRICES

Proseminar – Computer-Assisted Mathematics

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What we want to do:

- check if a given matrix A is in row-echelon form
- use Gaussian elimination to reduce a matrix A into row-echelon form
- output an elementary matrix E so that $A' = E * A$ with A' in row-echelon form

The concrete problem

We start with a matrix A and check if it is in row-echelon form.

If it is not, we use row operations to reduce A to row-echelon form.

We also need the exact operations to get the elemental matrix E.

$$\begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & \frac{5}{4} & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Check for Row-Echelon Form

We check for row-echelon form by seeing if the submatrix running from under a pivot to the next one and spanning to the last row is a zero matrix.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The mathematical solution

- We perform Gaussian elimination by using the following operations, which can also be represented by left-multiplication with the corresponding elementary matrix.
- First define a matrix $E^{(i,j)} \in \mathrm{Mat}(n \times n;\ \mathbb{Q})$, where $l, k \in \{1, \dots, n\}$:

  $$E^{(i,j)} := \left(e^{(i,j)}_{p,q}\right)_{\substack{1\le p\le n \\ 1\le q\le n}} \text{ where } e^{(i,j)}_{p,q} := \begin{cases} 1, \text{if } p = k \text{ and } q = l \\ \quad 0, \text{else} \end{cases}$$

- The First operation is **S**wapping two rows $R_i \leftrightarrow R_j$:

  $$S^{(i,j)} := 1_n - E^{(i,i)} - E^{(j,j)} + E^{(i,j)} + E^{(j,i)}$$

- The Second operation is **M**ultiplying a row by a nonzero number $\lambda R_i \to R_i$:

  $$M^{(i)}_{\lambda} := 1_n + (\lambda - 1)\ E^{(i,i)}$$

- The Third operation is **A**dding a multiple of one row to another row $\lambda R_i + R_j \to R_j$:

  $$A^{(i,j)}_{\lambda} := 1_n + \lambda E^{(i,j)}$$

- Let now $F := \begin{cases} S^{(i,j)} \\ M_\lambda^{(i)} \\ A_\lambda^{(i,j)} \end{cases}$
- With a equal to the total number of row operations and $F_b, b \epsilon \{1, \dots, a\}$ we get

$$A' = F_a * \left(F_{a-1} * (\dots * (F_1 * A) \dots)\right)$$

$$= (F_a * F_{a-1} * \cdots * F_1) * A = E * A$$

with $(F_a * F_{a-1} * \cdots * F_1) =: E$

- Now we have found a matrix E to satisfy $A' = E * A$ with A' in row-echelon form

How our program works