The Fundamental Theorem of Algebra via Linear Algebra

Tilman Jacobs, Sandro Reinhard

Seminar on Computer Assisted Mathemathics

July 2023

Theorem (Fundamental Theorem of Algebra)

Any nonconstant polynomial with coefficients in $\mathbb{C}$ *has a complex root.*

Usual Proof via Complex Analysis.

Apply Liouville's theorem to show that the reciprocal of a polynomial without roots is bounded and therefore already constant. $\square$

However, since the FTA is a statement mostly associated with Linear Algebra we are interested in a proof via LA...

Theorem (Restatement of FTA)

For each $n \geq 1$, every $n \times n$ square matrix over $\mathbb{C}$ has an eigenvector in $\mathbb{C}^n$. Equivalently, for each $n \geq 1$, every linear operator on an n-dimensional complex vector space V has an eigenvector in V.

```
variable {m : ℕ} [Fintype (Fin m)] [Field ℂ]

def IsEigenvector (A : Matrix (Fin m) (Fin m) ℂ) (v : Fin
    m → ℂ) := (v ≠ 0) ∧ (∃ μ : ℂ, (mulVec A v) = μ · v)

theorem exists_eigenvector (A : Matrix (Fin m) (Fin m) ℂ)
    : (m ≥ 1) → (∃ v : Fin m → ℂ, IsEigenvector A v) :=
by sorry
```

Lemma

Fix an integer $n \geq 1$ and a field $\mathbb{F}$. Suppose that, for very $\mathbb{F}$-vector space V whose dimension is not divisible by m, every linear operator on V has an eigenvector in V. Then for every $\mathbb{F}$-vector space V whose dimension is not divisible by m, each pair of commuting linear operators has a common eigenvector in V.

```
universe u v w

variable {ℂ : Type v} {V : Type w} [Field ℂ] [AddCommGroup V] [Module ℂ V]

lemma comm_lin_opHasEigenvector [FiniteDimensional ℂ V] [Nontrivial V] (f :
    End ℂ V) (g : End ℂ V) (h : End ℂ V) : (m ≥ 1) ∧ ¬(m ∣ (finrank ℂ  V))∧ (∃ v
    : V , f.HasEigenvector μ v)→ (∃ v : V , g.HasEigenvector μ v ∧
    h.HasEigenvector ν v)  := sorry
```

Corollary

For every real vector space V whose dimension is odd each pair of of commuting linear operators on V has a common eigenvector in V.

```
theorem exists_eigenvector_odd (A : Matrix (Fin m) (Fin m) ℂ): (Odd(card)) →
    (∃ v :  Fin m → ℂ, IsEigenvector  A v) := by sorry
```

Main idea

Strong Induction on the highest power of 2 dividing dimension n of V

$$n = 2^k n' : k \geq 0, n' \text{odd}$$