Let $m=2$, V be a vector space over $\mathbb{R}$ such that,

$m \nmid \dim V$ and all linear operators on V have an eigenvector in V

Let $A_1: V \xrightarrow{lin} V$, $A_2: V \xrightarrow{lin} V$

Now show:

$$A_1 \cdot A_2 = A_2 \cdot A_1 \quad \Rightarrow \exists \lambda_1, \lambda_2 \in \mathbb{R}, \exists v \in V: A_1 v = \lambda_1 v \wedge A_2 v = \lambda_2 v$$

Proof by induction over $\dim(V) = 2k+1 \quad k \in \mathbb{N}_0$:

<u>induction start:</u>

for $k=0 \qquad \dim V = 1 \quad \forall v \in V$

$$\ker(A_1 - \lambda_1) = \operatorname{span}(v) = \ker(A_2 - \lambda_2)$$

So they share a common eigenvector v, the whole dimension.

<u>induction assumption:</u>

Assume now that $\dim(V) > 1$ with $2 \nmid \dim V$ and all dimensions less than $\dim(V)$ are not divisible by 2. The above expression applies.

The assumption tells us now, if even A_1 and A_2 are only linear operators on V with $\dim V$ not divisible by 2, that A_1 has an eigenvalue in $\mathbb{R}$ with λ_1.

Let $V_1 = \operatorname{im}(A_1 - \lambda_1 I_{\dim V}) \neq \{0\}$ and $V_2 = \ker(A_1 - \lambda_1 I_{\dim V})$

$\forall v_1 \in V_1: A_1(v_1) \in V_1$ because it's not in V_2, v_1 is not an eigenvector

$\forall v_2 \in V_2: A_1(v_2) = \lambda_1 v_2 \in V_2$ because v_2 is an eigenvector.

So it's A_1-stable and we know that any lin. operator has at least an eigenvector so $\dim V_2 \geq 1$. V_1 and V_2 are A_2-stable because

A_1 and A_2 commute. Let $u = A_1(v_1) - \lambda_1 v_1 \in V_1$. Then $A_2(u) \in U$

$$A_2(u) = A_2(A_1(v_1) - \lambda_1 v_1) \overset{\text{lin}}{\underset{\text{op.}}{=}} A_1(A_2(v_1)) - \lambda_1(A_2(v_1)) = (A_1 - \lambda_1 I)(A_2(v_1)) \in U$$

Let $w = A_1(v_2) - \lambda_1 v_2 \in V_2$. Then $A_2(w) \in V_2$

$$A_2(w) = A_2(A_1(v_2) - \lambda_1 v_2) = A_2(0) = 0 \in V_2$$

We know that $\dim V_1 + \dim V_2 = \dim V$ because $V_1 \cap V_2 = \{0\}$

Because $\dim V$ is odd $\dim V_1$ or $\dim V_2$ is odd and then A_1, A_2 have a common eigenvector in V_1 or V_2 by induction.

The other case would be that V_1 or V_2 is empty and the other subspace is V, in that case $V_1 = \{0\}$ because the eigenvector should'nt be trivial. So $V_2 = V$ and $\forall v \in V$ is v an eigenvector for A_1 and one of them is eigenvector for A_2 since V has odd dimension