Daniel Geisler, Daniel Fadin 26.07.2023

Lemma 3. *Fix an integer $m > 1$ and a field F. Suppose that, for every F-vector space V whose dimension is not divisible by m, every linear operator on V has an eigenvector in V. Then for every F-vector space V whose dimension is not divisible by m, each pair of commuting linear operators on V has a common eigenvector in V.*

Suppose that: Let V be a vector space over F

$m > 1$: $m \nmid \dim V$

Every linear operator on V has an eigenvector in V

Show that: If two linear operators commute

$\Downarrow$

they have a common eigenvector in V

Corollary 4. *For every real vector space V whose dimension is odd, each pair of commuting linear operators on V has a common eigenvector in V.*

Given : Let V be a vector space over $\mathbb{R}$

Dimension is odd $\Leftrightarrow$ $2 \nmid \dim V$

Show that: If two linear operators commute

$\Downarrow$

they have a common eigenvector in V

$\Rightarrow$ This is what we want to prove

Proof: Since V has an odd dimension, its characteristic polynom has a real root and with that a non-zero eigenvector $v_1 \in V$.

- Consider the eigenspace E_1 corresponding to the eigenvalue λ_1 of T_1 associated with v_1. Since T_1 is a linear operator, E_1 is a T_1-invariant subspace.
- Now, since T_1 and T_2 commute, T_2 maps the eigenspace E_1 into itself, i.e., $T_2(E_1) \subseteq E_1$.

If T_2 is the zero operator on E_1, then any vector in E_1 is a common eigenvector for T_1 and T_2. So, we have a common eigenvector and there is nothing left to prove.

Otherwise, if T_2 is not the zero operator on E_1, it has an eigenvector $v_2 \in E_1$ associated with eigenvalue λ_2.

- Since $v_2 \in E_1$, it is also an eigenvector of T_1, associated with eigenvalue λ_1.
- Since v_2 is a non-zero vector, there exists a scalar k such that $v_1 + k \cdot v_2 \neq 0$ (assuming the field does not have characteristic 2)
- Consider $u = v_1 + k\, v_2$. We claim that u is a common eigenvector for T_1 and T_2.
- To prove this, we apply T_1 and T_2 on u:

$$T_1(u) = T_1(v_1) + k \cdot T_1(v_2) = \lambda_1 \cdot v_1 + k \cdot (\lambda_1 \cdot v_2) = \lambda_1 \cdot (v_1 + k \cdot v_2) = \lambda_1 u$$

$$T_2(u) = T_2(v_1) + k \cdot T_2(v_2) = \lambda_2 \cdot v_1 + k \cdot (\lambda_2 \cdot v_2) = \lambda_2 \cdot (v_1 + k \cdot v_2) = \lambda_2 u$$

Hence, u is an eigenvector for both T_1 and T_2, with eigenvalues λ_1 and λ_2 respectively.

- Therefore, u is a common eigenvector for T_1 and T_2

$\blacksquare$

Source : https://kconrad.math.uconn.edu/blurbs/fundthmalg/fundthmalglinear.pdf