# THE ROW-ECHELON FORM AND ELEMENTARY MATRICES

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#### What we want to do:

- $\blacksquare$  check if a given matrix A is in row-echelon form
- lacktriangle use Gaussian elimination to reduce a matrix A into row-echelon form
- output an elementary matrix E so that A' = E \* A with A'in row-echelon form

### The concrete problem

We start with a matrix A and check if it is in row-echelon form.

If it is not, we use row operations to reduce *A* to row-echelon form.

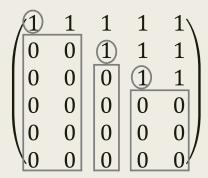
We also need the exact operations to get the elemental matrix *E*.

$$\begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{5}{4} & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

#### Check for Row-Echelon Form

We check for row-echelon form by seeing if the submatrix running from under a pivot to the next one and spanning to the last row is a zero matrix.



#### The mathematical solution

- We perform Gaussian elimination by using the following operations, which can also be represented by left-multiplication with the corresponding elementary matrix.
- First define a matrix  $E^{(i,j)} \in Mat(n \times n; \mathbb{Q})$ , where  $l, k \in \{1, ..., n\}$ :

$$E^{(i,j)} \coloneqq \left(e_{p,q}^{(i,j)}\right)_{\substack{1 \le p \le n \\ 1 \le q \le n}} \text{ where } e_{p,q}^{(i,j)} \coloneqq \begin{cases} 1, \text{ if } p = k \text{ and } q = l \\ 0, \text{ else} \end{cases}$$

■ The First operation is Swapping two rows  $R_i \leftrightarrow R_i$ :

$$S^{(i,j)} := 1_n - E^{(i,i)} - E^{(j,j)} + E^{(i,j)} + E^{(j,i)}$$

■ The Second operation is Multiplying a row by a nonzero number  $\lambda R_i \rightarrow R_i$ :

$$M_{\lambda}^{(i)} \coloneqq 1_n + (\lambda - 1) E^{(i,i)}$$

■ The Third operation is Adding a multiple of one row to another row  $\lambda R_i + R_j \rightarrow R_j$ :

$$A_{\lambda}^{(i,j)} \coloneqq 1_n + \lambda E^{(i,j)}$$

■ Let now 
$$F \coloneqq \begin{cases} S^{(i,j)} \\ M_{\lambda}^{(i)} \\ A_{\lambda}^{(i,j)} \end{cases}$$

■ With a equal to the total number of row operations and  $F_b$ ,  $b \in \{1, ..., a\}$  we get

$$A' = F_a * (F_{a-1} * (... * (F_1 * A) ...))$$
  
=  $(F_a * F_{a-1} * \cdots * F_1) * A = E * A$   
with  $(F_a * F_{a-1} * \cdots * F_1) =: E$ 

Now we have found a matrix E to satisfy A' = E \* A with A' in row-echelon form

## How our program works

