Sum of Squares in Lean

Using Lean 3.0

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Theorem Definition

[1/2] Statement

The two statements are equivalent:

a) If a sum of squares is 0, then all the elements of that sum are 0.

b) -1 is not a sum of squares in $\mathbb{R}$

Notation: List = Set, but allows multiplicity

Note: We do not want to show that **a)** or **b)** is actually true!

[2/2] Statement

So now we would like to formalize the statements using mathematical notations

a) If a sum of squares is **0**, then all the elements of that sum are **0**.

$$\forall\ L : \text{List}\ \mathbb{R},\ x_i \in L : \quad \sum_{i \ldots n} x_i^2 = 0 \Rightarrow \quad x_i = 0\ \ \forall x_i \in L$$

b) **-1** is not a sum of squares in $\mathbb{R}$

$$\forall\ L : \text{List}\ \mathbb{R},\ x_i \in L : \quad \sum_{i \ldots n} x_i^2 \neq -1$$

And we can write those together in an equivalence relation **a)** $\leftrightarrow$ **b)** as follows :

$$\forall\ L : \text{List}\ \mathbb{R},\ x_i \in L : \quad \sum_{i \ldots n} x_i^2 = 0 \Rightarrow \quad x_i = 0\ \ \forall x_i \in L \leftrightarrow \sum_{i \ldots n} x_i^2 \neq -1$$

Propositional Logic - Example

[1/1] Propositional Logic - Example

Given the following statements

Q : The weather is nice
P : I'm at the zoo

We can express the relation between **Q** and **P** like this:

$Q \rightarrow P$ (If the weather is nice, I'm at the zoo)

The **contraposition** of this would be:

$Q \rightarrow P = \neg P \rightarrow \neg Q$ (If I'm not at the zoo, the weather is bad)

(And *not* $Q \rightarrow P = \neg Q \rightarrow \neg P$ (If the weather is bad, I'm not at the zoo))

Proof of the Sum of Squares Theorem

[1/4] Proof

We will start the proof of **a)** $\leftrightarrow$ **b)** by proving the '$\rightarrow$' - direction: **b)** $\rightarrow$ **a)**.
We do a proof by contraposition, meaning we show that $\neg$**a)** $\rightarrow$ $\neg$**b)** is true.

First, let's negate the two statements:

$$\text{a)}\quad \neg(\forall\ L : \text{List}\ \mathbb{R},\ x_i \in L :\ \textstyle\sum_{i\ldots n} x_i^2 = 0 \rightarrow \forall\ x_i \in L :\ x_i = 0$$

$$\exists\ L : \text{List}\ \mathbb{R},\ x_i \in L :\ \textstyle\sum_{i\ldots n} x_i^2 = 0 \rightarrow \exists\ x_i \in L :\ x_i \neq 0 \qquad = \neg\text{a)}$$

$$\text{b)}\quad \neg(\forall\ L : \text{List}\ \mathbb{R},\ x_i \in L :\ \textstyle\sum_{i\ldots n} x_i^2 \neq -1)$$

$$\exists\ L : \text{List}\ \mathbb{R},\ x_i \in L :\ \textstyle\sum_{i\ldots n} x_i^2 = -1 \qquad = \neg\text{b)}$$

[2/4] Proof

We show ¬a) → ¬b):

Assume ¬a).

Let **L** be a list over ℝ whose sum of squares is equal to **0** and without loss of generality, let $x_1 \neq 0$.

Then :

$$\sum_{i=1}^{n} x^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = 0$$

We can now divide all the terms by x_1^2 and continue with the equality, because division of 0 by any number is still equal to zero.

[3/4] Proof

So now we have:

$$\frac{\sum_{i=1}^{n} x^2}{x_1^2} = \frac{x_1^2}{x_1^2} + \frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2}$$
$$= 1 + \frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2} = 0$$

By adding (-1) to both sides, we get :

$$\frac{x_2^2}{x_1^2} + \cdots + \frac{x_n^2}{x_1^2} = -1$$

Which confirms our initial assumption **¬a)**

Thus we have proven **b)** → **a)**

[4/4] Proof

We will now proceed to show the '←' - direction : **a) → b) of a) ↔ b)**.

$$\exists\ L : \text{List}\ \mathbb{R}, x_i \in L :\ \sum_{i\dots n} x_i^2 = 0 \rightarrow\ \exists\ x_i \in L\ ,\ x_i \neq 0 \qquad = \neg a)$$

$$\exists\ L : \text{List}\ \mathbb{R}, x_i \in L :\ \sum_{i\dots n} x_i^2 = -1 \qquad = \neg b)$$

We will prove the contraposition **¬b) → ¬a)**.

Assuming **¬b)**, we want to show **¬a)**.

From our assumption of **¬b)**, let **L** be that list.

By appending **1** to **L**, the sum of squares of **{ 1 :: L }** must be now **0**.

Example in C

[1/1] Example in $\mathbb{C}$

The lean proof of the equivalence holds for any field with decidable equalities.
Since $\mathbb{C}$ is such a field, we can show that the equivalence still holds:

$$\neg a) \quad \leftrightarrow \quad \neg b)$$

$$\exists L : \text{List } \mathbb{R}, x_i \in L : \sum_{i \ldots n} x_i^2 = 0 \rightarrow \exists x_i \in L : x_i \neq 0 \leftrightarrow \exists L : \text{List } \mathbb{R}, x_i \in L : \sum_{i \ldots n} x_i^2 = -1$$

To show ¬b), we can choose $L := \{ i \},\{ -i \},\{ 1/i \}$, or $\{ 1/-i \}$, giving us a sum of squares = -1.
Since ¬a) must hold as well, we construct $L2 := \{ 1 :: L \}$ i.e. 1 appended to L.

Since $1 \in L2$ and the sum of squares of L2 is 0 by construction, we have found a list that satisfies ¬a) as well.

Therefore, a) ↔ b) remains (negatively) equivalent in $\mathbb{C}$

Ty for your time 🤡

Any Questions?

Resources for further reading:

https://leanprover-community.github.io/mathlib_docs/