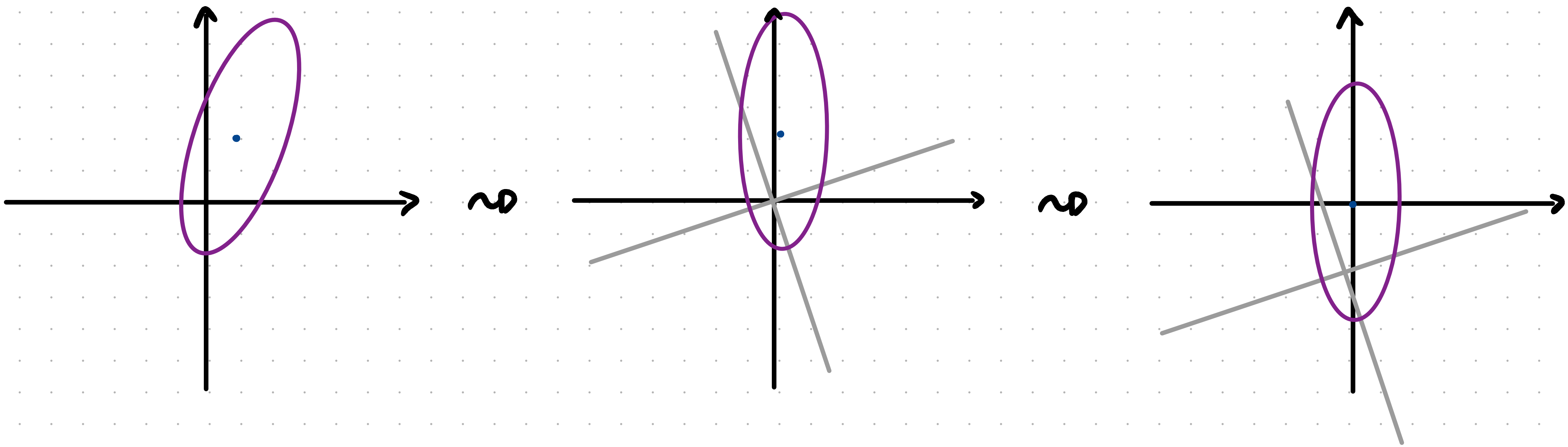
Principal Axis Transformation

Seminar on computer-assisted mathematics

Janina Planeta, Julia Renner



Principal axis theorem

The principal axis theorem states that the principal axes of quadrics are perpendicular and gives a procedure to find them.

$\leadsto$ **Algorithm**: transforming coordinates of a given equation
$\leadsto$ **Main tool:** diagonalization of symmetric matrices through the multiplication with orthogonal matrices

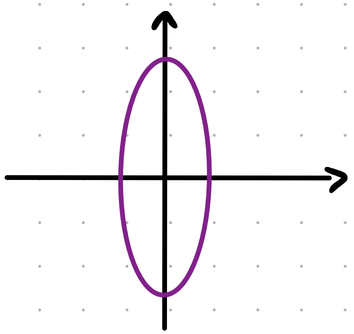
Visually: rotations, translations and reflections

Through this method we can determine the type of a given quadric and its geometrical characteristics, such as center, axes, vertex,...

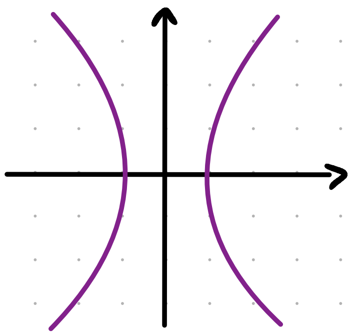
Reminder

Quadric

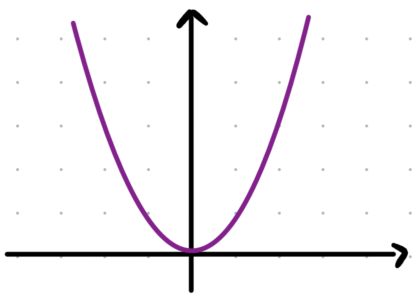
A quadric is a point set in $\mathbb{R}^n$ which satisfies $Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid q(x_1, \dots, x_n) = 0\}$ where $q(x_1, \dots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j + 2 \cdot \sum_{i=1}^{n} b_i x_i + c$.



$$Q_e = \left\{ x \in \mathbb{R}^2 \;\middle|\; \frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} = 1 \right\}$$



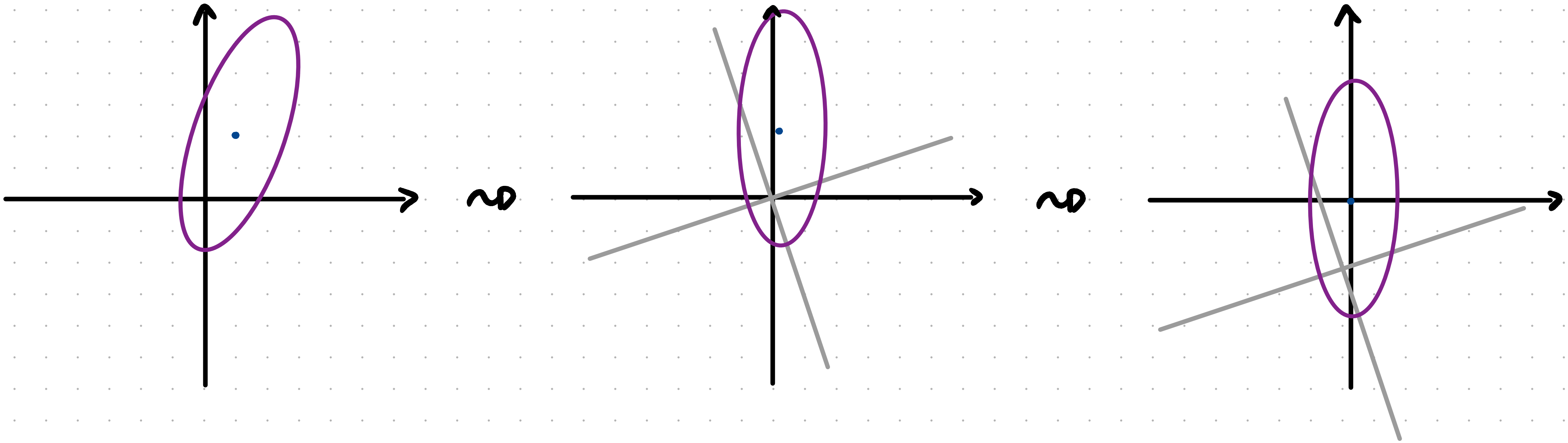
$$Q_h = \left\{ x \in \mathbb{R}^2 \;\middle|\; \frac{x_1^2}{\alpha^2} - \frac{x_2^2}{\beta^2} = 1 \right\}$$



$$Q_p = \left\{ x \in \mathbb{R}^2 \;\middle|\; \frac{x_1^2}{\alpha^2} - 2x_2 = 0 \right\}$$

Method

For this presentation we will only look at the two-dimensional case.
Visually, we will first rotate and then move the coordinate system in which our object is located.



We get an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ $(a, b, c, d, e, f \in \mathbb{R})$ and we want to bring it in the form $Q = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{\alpha^2} \pm \frac{y^2}{\beta^2} = 1\}$ or $Q = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{\alpha^2} - 2y = 0\}$.

Let $v := \begin{pmatrix} x \\ y \end{pmatrix}$. We write the equation in matrix form, that is

$$v^T \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} v + (d \quad e) v + f = 0$$

Step 1: Determine the matrix from the equation in matrix form

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

Step 2: Determine the eigenvalues λ_1 and λ_2

Step 3: Find the eigenvectors and normalize them

When we have found the eigenvectors v_1 and v_2, we also know the principal axes. In the two-dimensional case these are given by $\mathbb{R}v_1$ and $\mathbb{R}v_2$.

Step 4: Create a new matrix S whose columns correspond to the eigenvectors

A is a symmetric matrix in $\mathbb{R}$, which implies that it is diagonalizable.

$\Rightarrow A = SDS^T$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and S is orthonal.

Step 5: Getting rid of mixed terms by substitution (rotation)

We substitute $A = SDS^T$ and add $SS^T (= \mathbb{1}$, because S is orthogonal) in $v^T A v + (d\ e) v + f = 0$.

$$\Rightarrow v^T S D S^T v + (d\ e) v + f = 0$$

Write v in dependence of S and a new vector $w := \begin{pmatrix} s \\ t \end{pmatrix}$: $v = Sw$

$$\Rightarrow w^T \underbrace{S^T S}_{=\mathbb{1}} D \underbrace{S^T S}_{=\mathbb{1}} w + (d\ e) S \underbrace{S^T S}_{=\mathbb{1}} w + f = 0$$

$$\Rightarrow w^T D w + (d\ e) S w + f = 0$$

The new equation is now $d_1 s^2 + d_2 t^2 + g \cdot s + h \cdot t + f = 0$, g,h are new constants resulting from multiplication by S.

Visually, we rotate the coordinate system so that the symmetry axes of the object are parallel to the new coordinate axes afterwards. The normalized eigenvectors are the basis vectors of the new coordinate system.

If either λ_1 or λ_2 is zero, i.e. the object is a parabola, we do the same thing as in the following with only one variable.

Step 6: Getting rid of the non-square terms (shift)

The end goal is to get an equation of the form $\frac{(s-\xi)^2}{\alpha^2} \pm \frac{(t-\eta)^2}{\beta^2} = 1.$

Whereas (ξ, η) is the center of our object in the original coordinate system and α and β are the half-axes.

$$\lambda_1 \cdot s^2 + \lambda_2 \cdot t^2 + g \cdot s + h \cdot t + f = 0$$

$$\Rightarrow \lambda_1 \cdot \left(s^2 + \frac{g}{\lambda_1} \cdot s + \left(\frac{g}{2\lambda_1}\right)^2 - \left(\frac{g}{2\lambda_1}\right)^2\right) + \lambda_2 \cdot \left(t^2 + \frac{h}{\lambda_2} t + \left(\frac{h}{2\lambda_2}\right)^2 - \left(\frac{h}{2\lambda_2}\right)^2\right) = -f$$

$$\Rightarrow \lambda_1 \cdot \left(s + \frac{g}{\lambda_1}\right)^2 + \lambda_2 \cdot \left(t + \frac{h}{2\lambda_2}\right)^2 = -f + \frac{g^2}{4\lambda_1} + \frac{h^2}{4\lambda_2}$$

$$\Rightarrow \frac{(s-\xi)^2}{\alpha^2} \pm \frac{(t-\eta)^2}{\beta^2} = 1$$

Visually, we shift the coordinate system so that the center of the object is in zero.

Applications

The principal axis transformation is used in physics (for example to describe the kinematics of rigid bodies) and computer science (for example in pattern recognition).

References

https://de.wikipedia.org/wiki/Hauptachsentransformation%20
https://de.wikipedia.org/wiki/Quadrik%20
https://www.mathebibel.de/hauptachsentransformation