Given: • V is a real vector space with odd dimension

• A_1 and A_2 are commuting linear operators on V

Proof: Since V has an odd dimension, its characteristic polynom has a real root and with that a non-zero eigenvector $v_1 \in V$.

• Consider the eigenspace E_1 corresponding to the eigenvalue λ_1 of T_1 associated with v_1. Since T_1 is a linear operator, E_1 is a T_1-invariant subspace.

• Now, since T_1 and T_2 commute, T_2 maps the eigenspace E_1 into itself, i.e., $T_2(E_1) \subseteq E_1$.

If T_2 is the zero operator on E_1, then any vector in E_1 is a common eigenvector for T_1 and T_2. So, we have a common eigenvector and there is nothing left to prove

Otherwise, if T_2 is not the zero operator on E_1, it has an eigenvector $V_2 \in E_1$ associated with eigenvalue λ_2.

• Since $V_2 \in E_1$, it is also an eigenvector of T_1, associated with eigenvalue λ_1.

• Since V_2 is a non-zero vector, there exists a scalar k such that $v_1 + k \cdot v_2 \neq 0$ (assuming the field does not have characteristic 2)

• Consider $u = v_1 + k\, v_2$. We claim that u is a common eigenvector for T_1 and T_2.

• To prove this, we apply T_1 and T_2 on u:

$T_1(u) = T_1(v_1) + k \cdot T_1(v_2) = \lambda_1 \cdot v_1 + k \cdot (\lambda_1 \cdot v_2) = \lambda_1 \cdot (v_1 + k \cdot v_2) = \lambda_1 u$

$T_2(u) = T_2(v_1) + k \cdot T_2(v_2) = \lambda_2 \cdot v_1 + k \cdot (\lambda_2 \cdot v_2) = \lambda_2 \cdot (v_1 + k \cdot v_2) = \lambda_2 u$

Hence, u is an eigenvector for both T_1 and T_2, with eigenvalues λ_1 and λ_2 respectively.

- Therefore, u is a common eigenvector for T_1 and T_2

$\blacksquare$