Group Theory

Involutions

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Chapter 01

Fundamentals

Definition

A **Group** is an ordered pair $(G, *)$ of a set G and a binary operator

$$* : \begin{cases} G \times G \to G \\ (a, b) \mapsto a * b \\ \end{cases}$$

that satisfies the group axioms:

- **Associativity**

  $$\forall a, b, c \in G : \quad (a * b) * c = a * (b * c)$$

- **Identity element**

  $\exists e \in G$ such that $\forall a \in G : \quad a * e = e * a = a$

- **Inverse element**

  $$\forall g \in G \; \exists a^{-1} \in G : \quad a * a^{-1} = a^{-1} * a = e$$

Definition

Given two groups, $(G, *)$ and $(H, \cdot)$, a **group homomorphism** from $(G, *)$ to $(H, \cdot)$ is a function

$$f:\ G \rightarrow H$$

such that $\forall u, v \in G$ it holds that

$$f(u * v) = f(u) \cdot f(v)$$

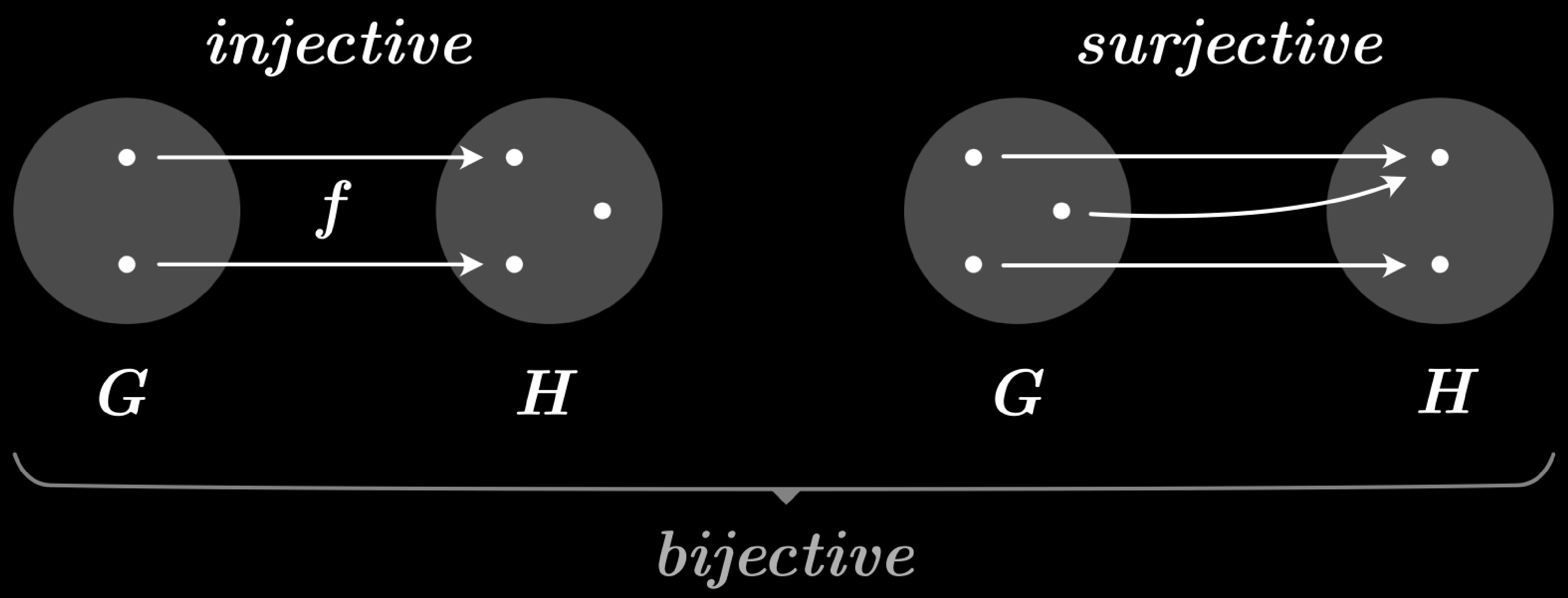
Further remarks

From this property, we can also deduce that

- $f(e_G) = e_H$
- $f(u^{-1}) = f(u)^{-1}$

Definition

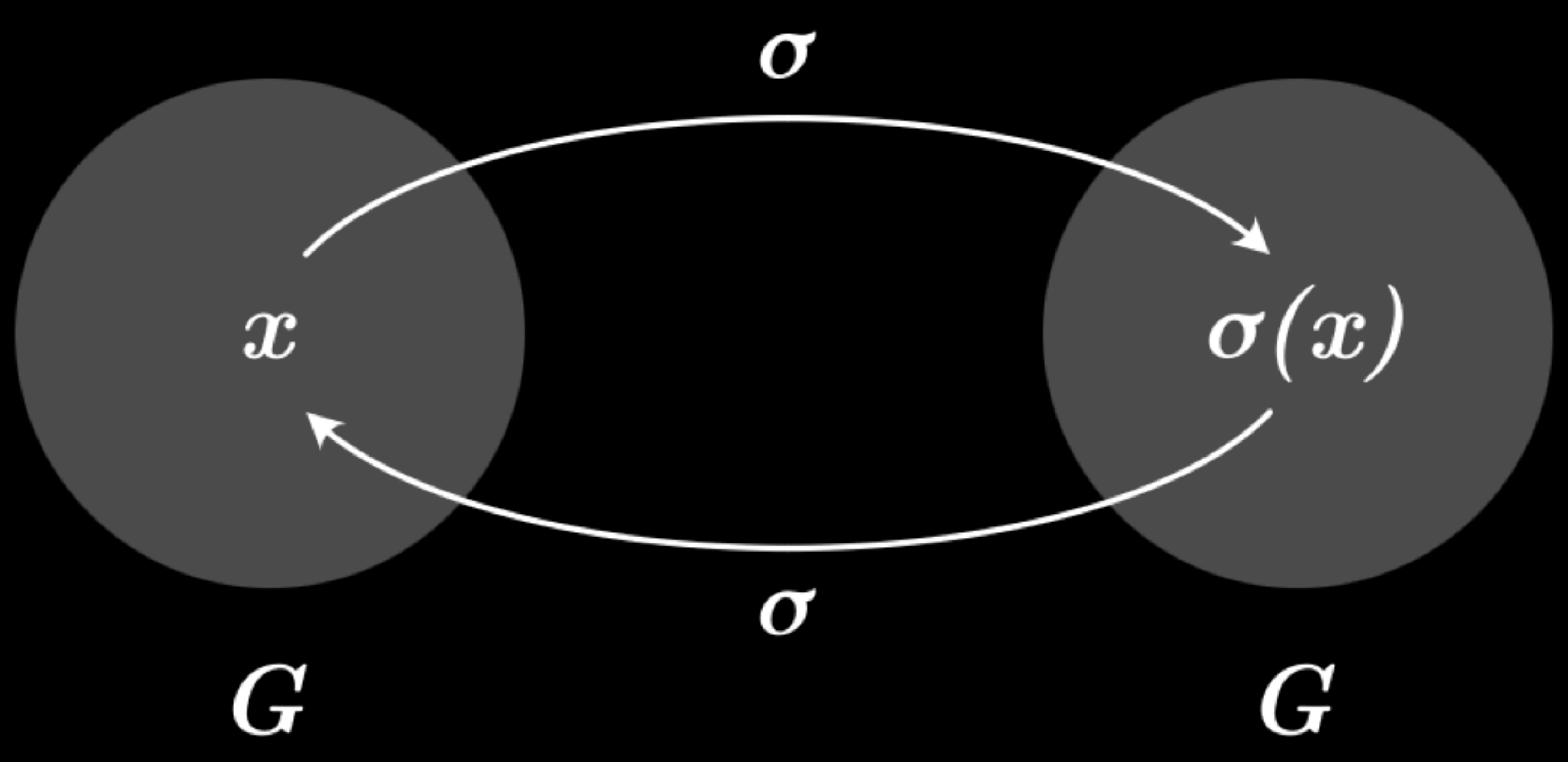
An **automorphism** is a bijective homomorphism of an object into itself.



Definiton

Given a group $(G, *)$, a group automorphism σ is an **involution**, if

$$\sigma(\sigma(x)) = x \qquad \forall x \in G$$



Definiton

An involution σ on a group $(G, *)$ has **<u>no non-trivial-fixpoints</u>** if the identity element $e \in G$ is the only fixpoint of σ:

$$\forall g \in G : \quad (\ \sigma(g) = g \Rightarrow g = e\ )$$

We call $e \in G$ a trivial fixpoint of σ.

Lemma

Every group $(G, *)$ has a trivial involution, namely the identity id.

proof:

Let $(G, *)$ be an arbitrary Group. For every $x \in G$:

$$x = \mathrm{id}(x) = \mathrm{id}(\mathrm{id}(x))$$

$\Rightarrow$ id is an involution.

$\square$

Example

Real negation

$$-:\begin{cases}\mathbb{R}\to\mathbb{R},\\ x\mapsto -x\end{cases}$$

is an involution on $(\mathbb{R},+)$.

proof:

For $x, y \in \mathbb{R}$:

$$-(x+y) = -x + (-y)$$

and

$$x = -(-x) = -(-(x))$$

... $\Rightarrow$ real negation is an involution on $(\mathbb{R},+)$.

$\square$

Chapter 02

Main theorem

Theorem

Let $(G, *)$ be a finite group. If an involution with no non-trivial fixpoints on $(G, *)$ exists, then $*$ is commutative.

Thus making $(G, *)$ an abelian group.

proof: Later ...

Lemma

Let $(G, *)$ be a finite group and σ be an involution on G. If σ has no non-trivial fixpoints, then:

$$\forall g \in G \; \exists x \in G : \quad g = x^{-1} * \sigma(x)$$

proof:

In essence, we want to show surjectivity of a function:

$$x \mapsto x^{-1} * \sigma(x)$$

Because G is finite, we can conclude surjectivity by injectivity.

So, lets prove injectivity...

Suppose $x, y \in G$ with $x^{-1} * \sigma(x) = y^{-1} * \sigma(y)$.

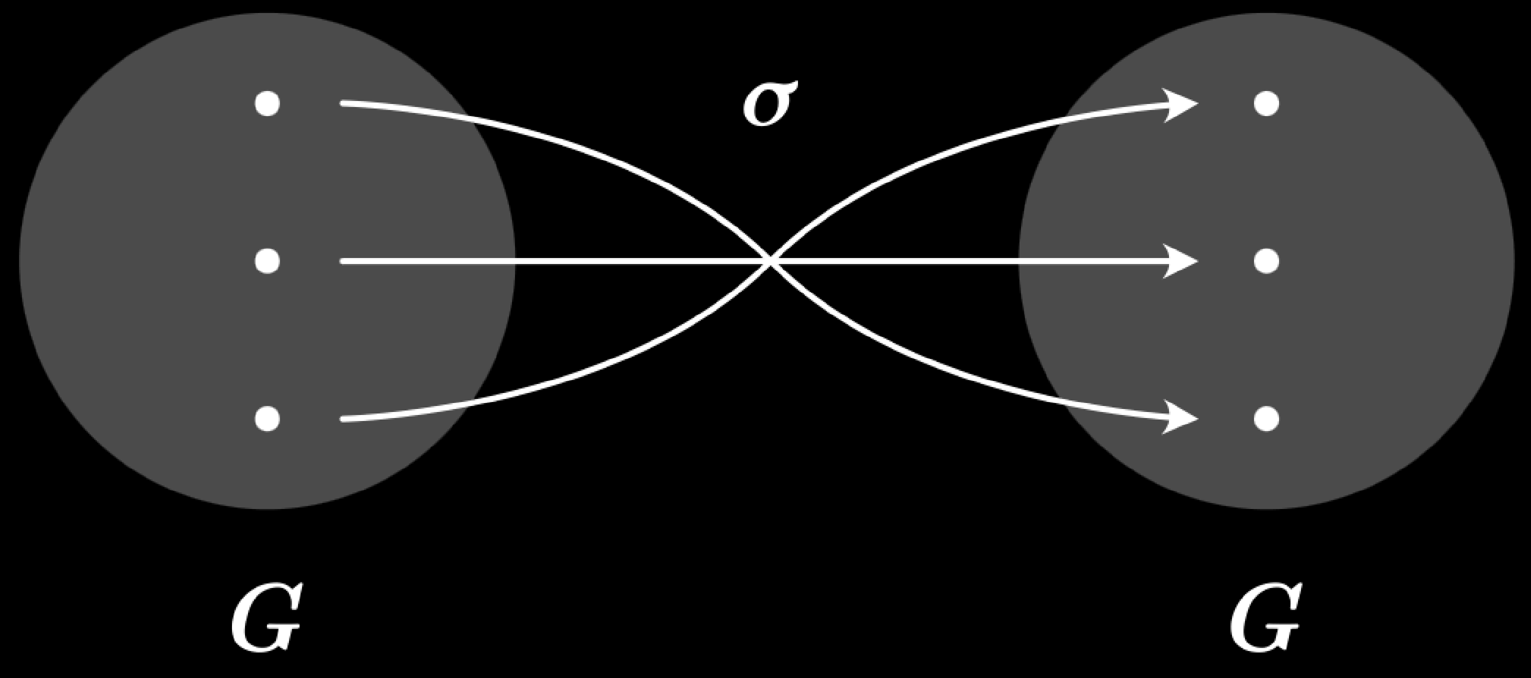
$$\begin{aligned} x &= \sigma(\sigma(x)) \\ &= \sigma(x * x^{-1} * \sigma(x)) \\ &= \sigma(x * y^{-1} * \sigma(y)) \\ &= \sigma(x) * \sigma(y^{-1}) * \sigma(\sigma(y)) \\ &= \sigma(x) * \sigma(y^{-1}) * y \end{aligned}$$

$$\begin{aligned} \Rightarrow x * y^{-1} &= \sigma(x) * \sigma(y^{-1}) \\ &= \sigma(x * y^{-1}) \end{aligned}$$

We have no non-trivial fixpoints, so $x * y^{-1}$ has to be the trivial fixpoint:

$$\begin{aligned} \Rightarrow x * y^{-1} &= e \\ \Leftrightarrow \quad x &= y \end{aligned}$$

$\Rightarrow x \mapsto x^{-1} * \sigma(x)$ is injective.



Since G is finite, we can conclude that $x \mapsto x^{-1} * \sigma(x)$ is also surjective on G.

$$\Rightarrow \forall g \in G \; \exists x \in G : \quad g = x^{-1} * \sigma(x)$$

$\square$

Lemma

Let $(G, *)$ be a finite group and σ be an involution on G. If σ has no non-trivial fixpoints, then:

$$\forall g \in G : \quad \sigma(g) = g^{-1}$$

proof:

In the previous Lemma, we showed that

$$\forall g \in G \; \exists x \in G : \quad g = x^{-1} * \sigma(x)$$

We can expand on that result:

$$\begin{aligned} \Rightarrow \sigma(g) &= \sigma(x^{-1} * \sigma(x)) \\ &= \sigma(x^{-1}) * \sigma(\sigma(x)) \\ &= \sigma(x^{-1}) * x \\ &= (\sigma(x))^{-1} * x \\ &= (x^{-1} * \sigma(x))^{-1} \\ &= g^{-1} \end{aligned}$$

$\square$

Theorem

Let $(G, *)$ be a finite group. If an involution with no non-trivial fixpoints on $(G, *)$ exists, then $*$ is commutative.

Thus making $(G, *)$ an abelian group.

proof:

Let $a, b \in G$.

$$\begin{aligned} a * b &= (a^{-1})^{-1} * (b^{-1})^{-1} \\ &= (b^{-1} * a^{-1})^{-1} \\ &= \sigma(b^{-1} * a^{-1}) \\ &= \sigma(b^{-1}) * \sigma(a^{-1}) \\ &= b * a \end{aligned}$$

$\Rightarrow$ $*$ is commutative, $(G, *)$ is abelian.

$\square$

Chapter 03

Counter example

Definiton

A **free group** $(Fs, *)$ over a given set S consists of all words that can be build by elements of S or their inverse.

Elements of S are called **generators**. Two constructed words are considered different unless their equality follows from the group axioms.

Universal property

Given any function f from S to a group $(G, *)$, there exists a unique homomorphism

$$\phi : \quad F_S \mapsto G$$

$$\begin{tikzcd}
 & F_S \arrow[dr, dashed, "\Phi"] & \\
S \arrow[ur] \arrow[rr, "f"'] & & G
\end{tikzcd}$$

Counter-Example

We will look at a free group $(F_2, *)$ on two generators $\{a, b\}$:

$$e, \quad a * b, \quad a^{-1} * b * b, \quad a^{-1} * b * b * a * a * b^{-1}a, \quad \dots$$

We can define an automorphism s that swaps the generators over a free group $(F_2, *)$.

$$s(x) := \begin{cases} a, & \text{if } x = b \\ b, & \text{if } x = a \\ s(u) * s(v), & \text{for } x = u * v, \quad u, v \in F_2 \end{cases}$$

Examples

$s(abba) = baab$ ***Note:*** *abba is short for* $a * b * b * a$.

By it's construction, s is a homorphism. It is therefore not difficult to show that s is indeed an involution on $(F_2, *)$.

s has no non trivial-fixed points

Let's assume that x is a fixed point of s:

$$s(x) = x$$

x has to look the same, with a and b swapped. This can only be the case if x is the empty word e, thus beeing a trivial fixed point.

$(F_2, *)$ is not abelian

By the very construction of any free group (on more than one generator), elements $a * b$ and $b * a$ are considered different, because there's no imposed property proving their equality.