Warning

What you are viewing is the **PDF version** of the presentation. Note that due to how animations were constructed, the content you see here might not be everything that you see in the presentation. Furthermore, slides might be confusing to look at in the PDF since elements that belong to the next slides can be present on a slide beforehand (again, this was necessary for the animations to work).

TL;DR: Look at the PowerPoint version instead of this PDF.

Warning

Install the font “**Calluna**” on your computer
in order to properly view some contents of this presentation.
This is *not* required to display any math content.

The repo alongside this presentation can be found here:
https://github.com/Splines/lean-continuous

Felix Lentze & Dominic Plein
July 16th, 2024

Continuous Functions

formalized in **Lean4**

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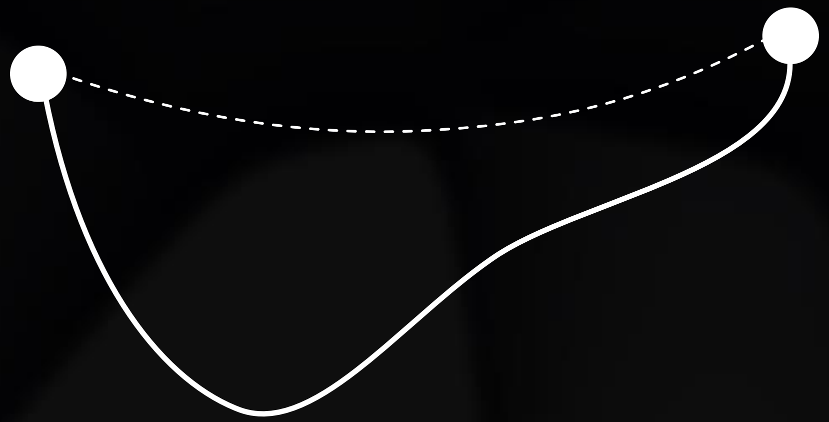

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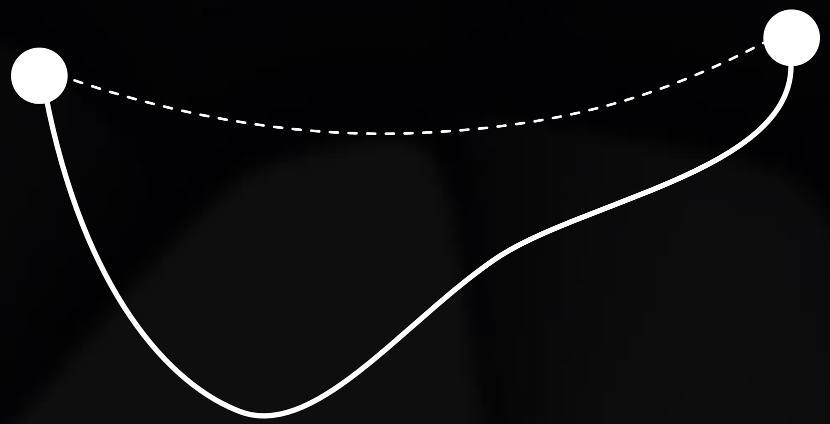

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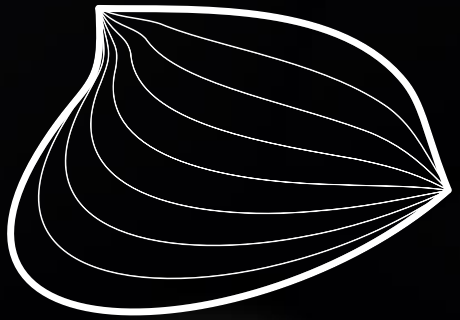
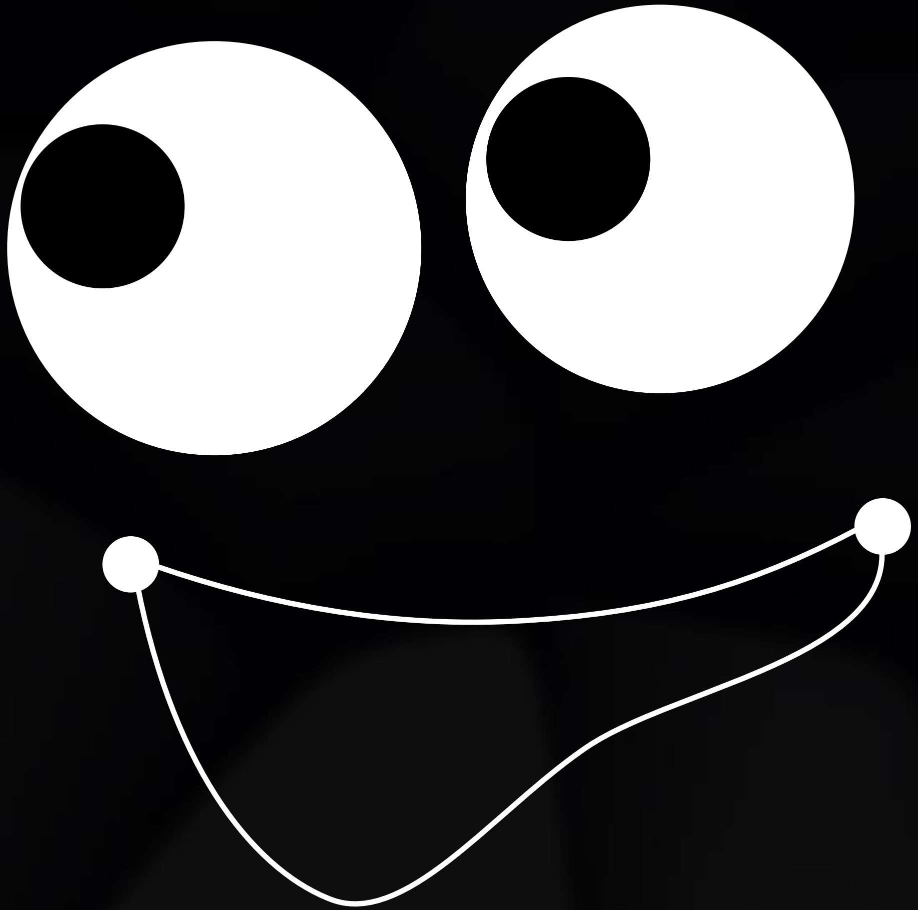
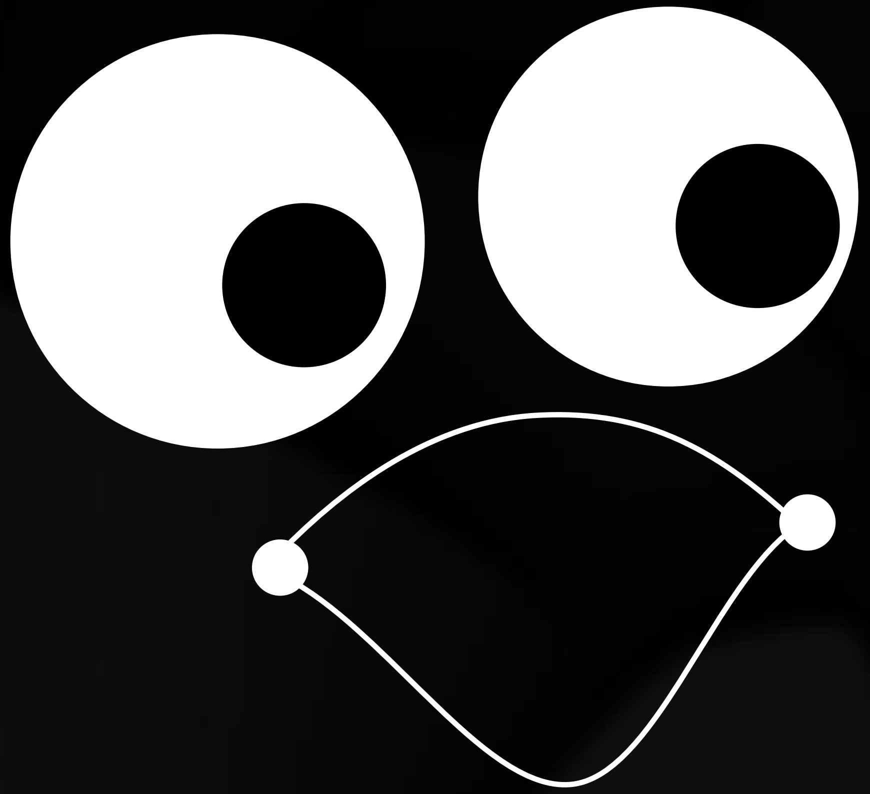
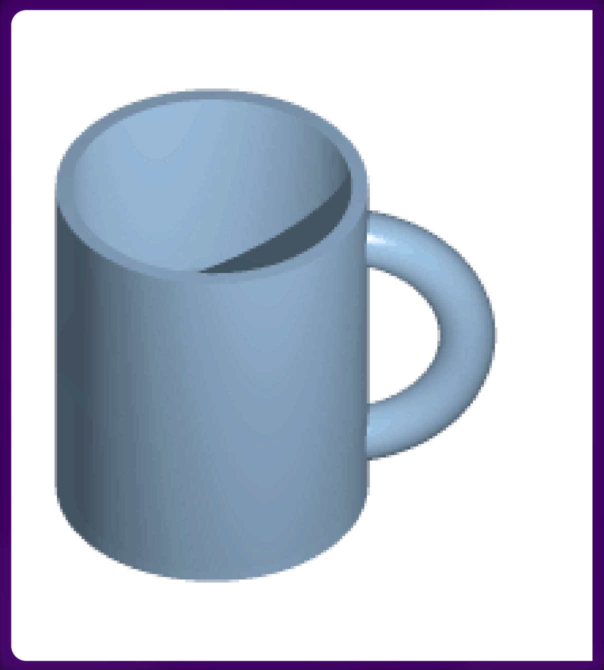
Homotopy of paths
in Topology









By Lucas Vieira - Own work, Public Domain. Wikipedia.

Outline

Continuous Functions

Examples

Constant function

$x \mapsto c$

Line

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Continuous functions

Definition. Continuous at a point.

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous at the point* $a \in D$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \quad \Big(|x - a| < \delta \ \Rightarrow \ |f(x) - f(a)| < \epsilon \Big)$$

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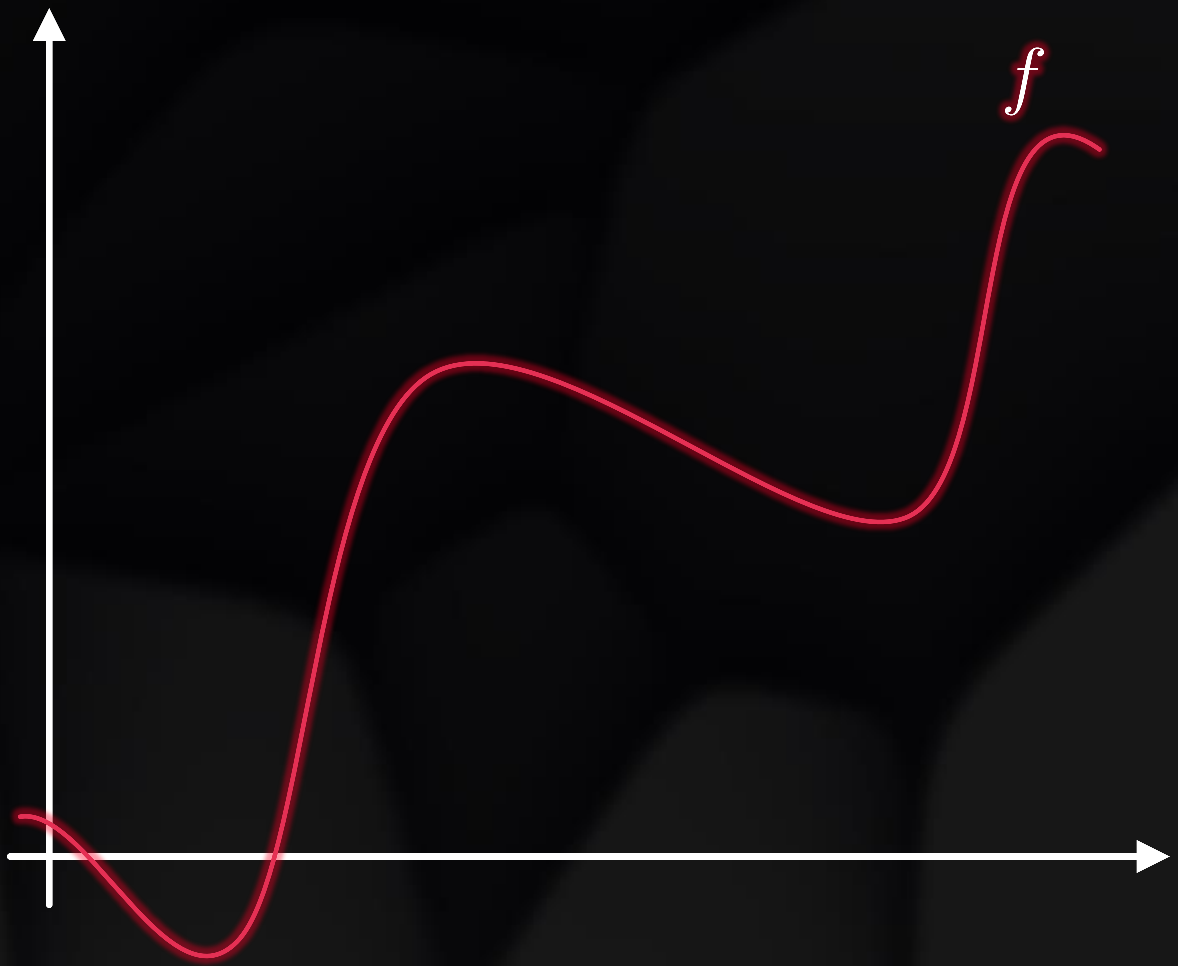

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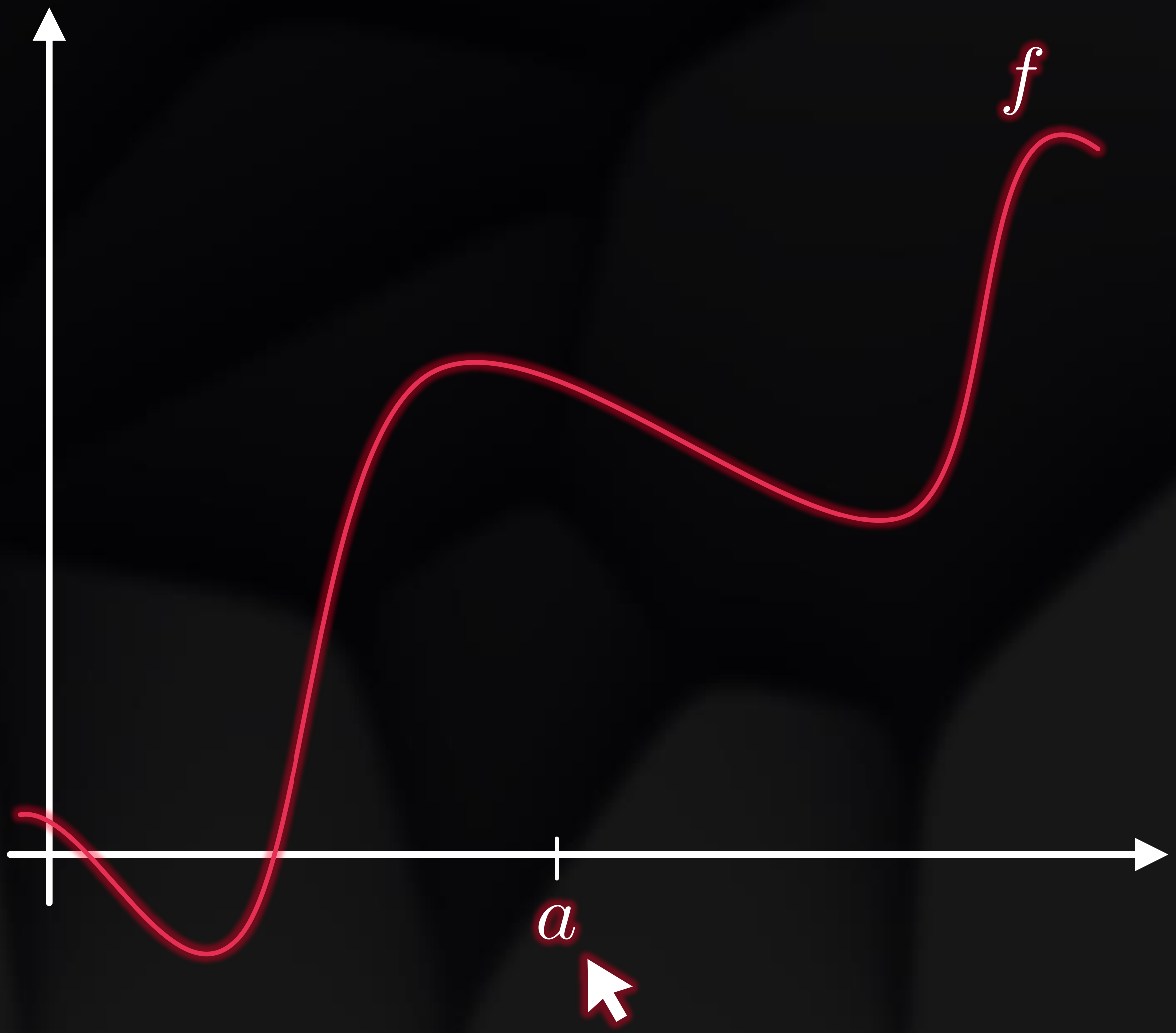

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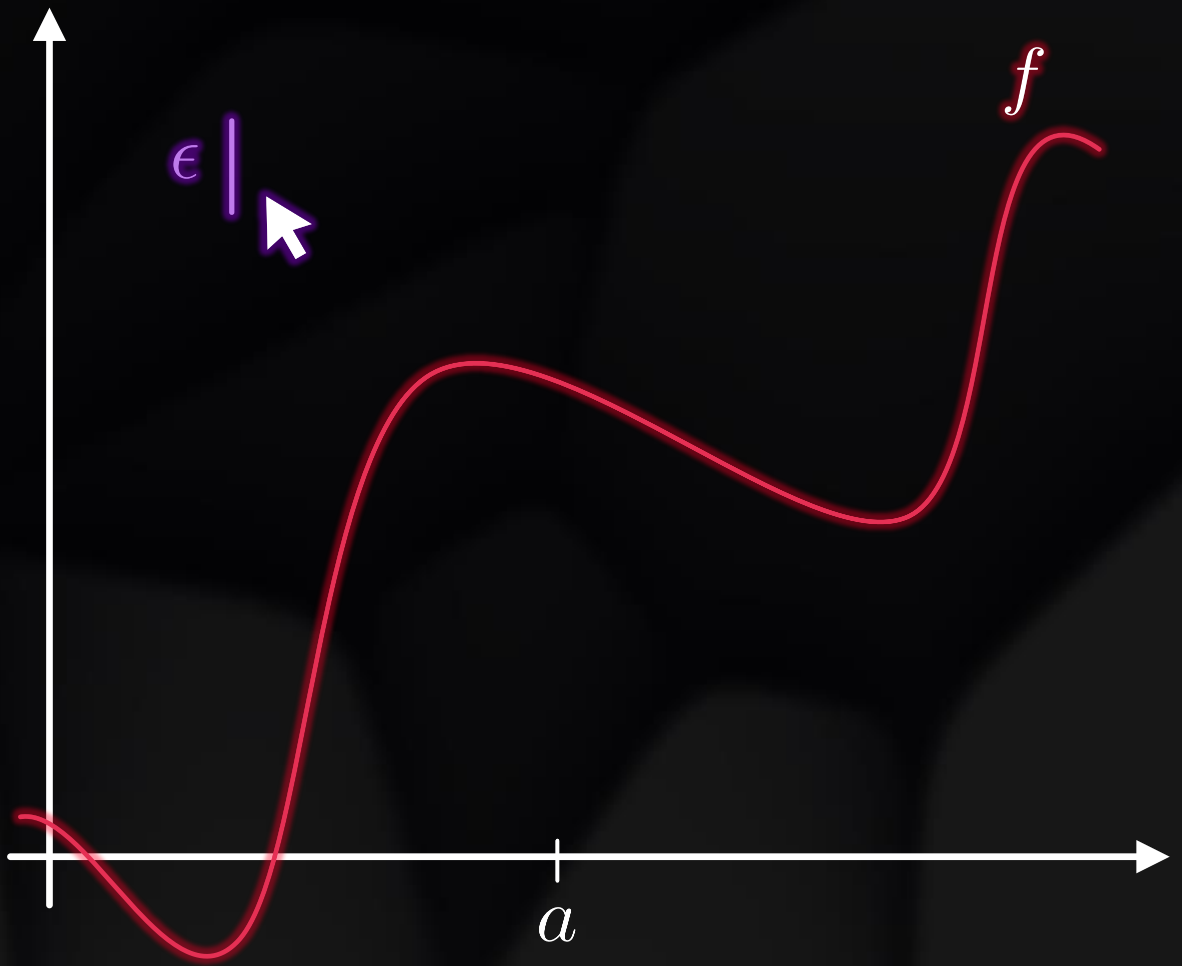

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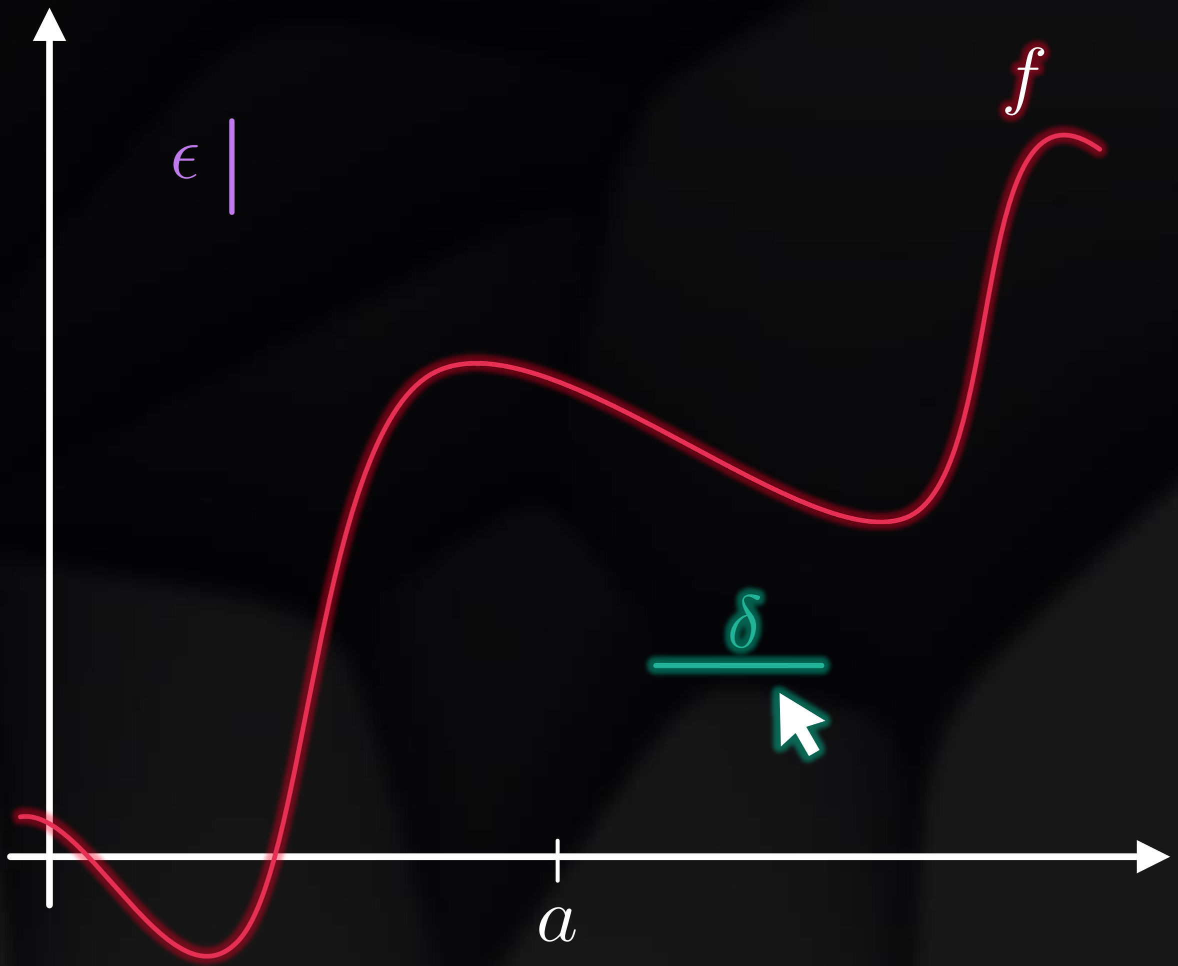

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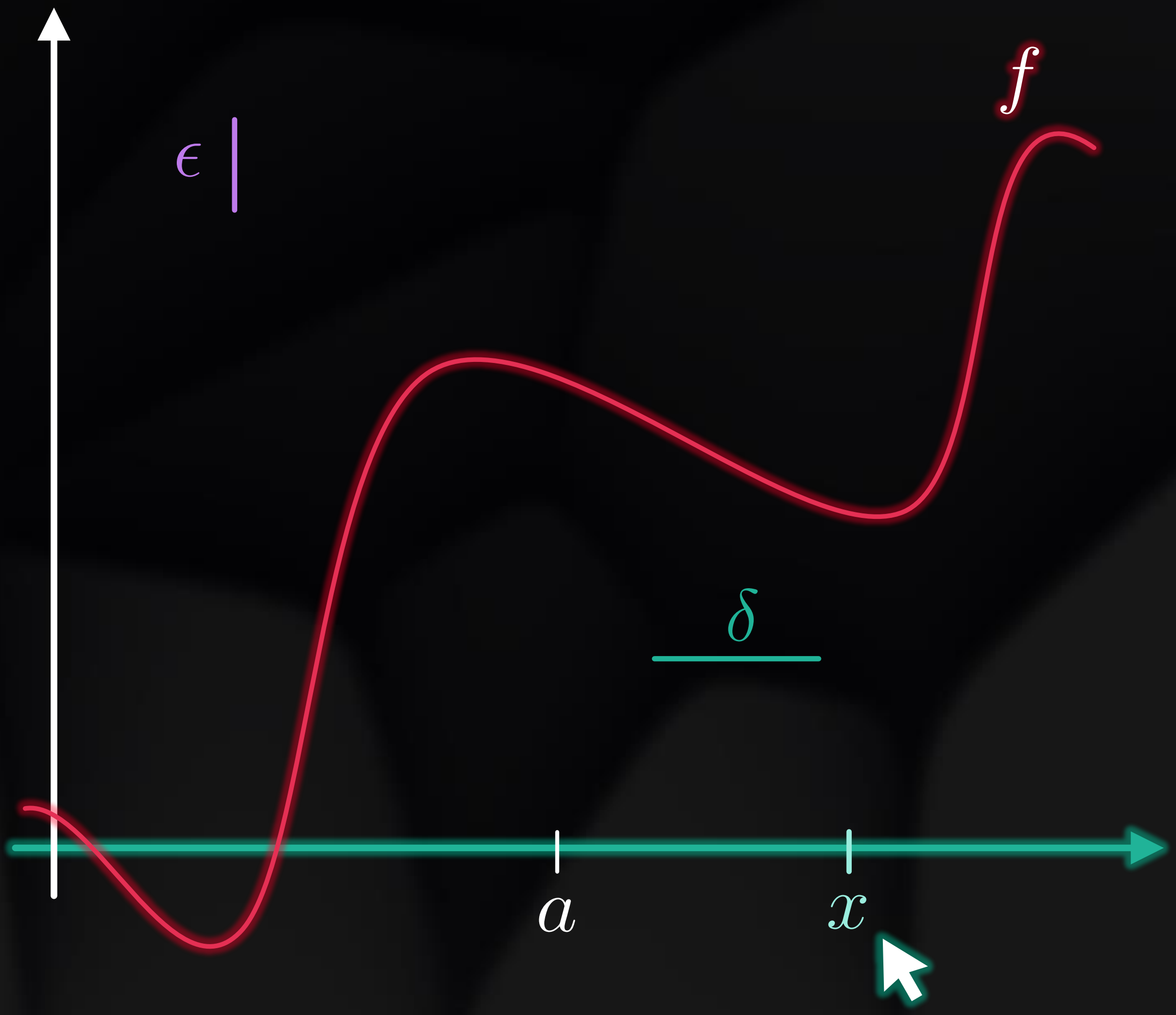

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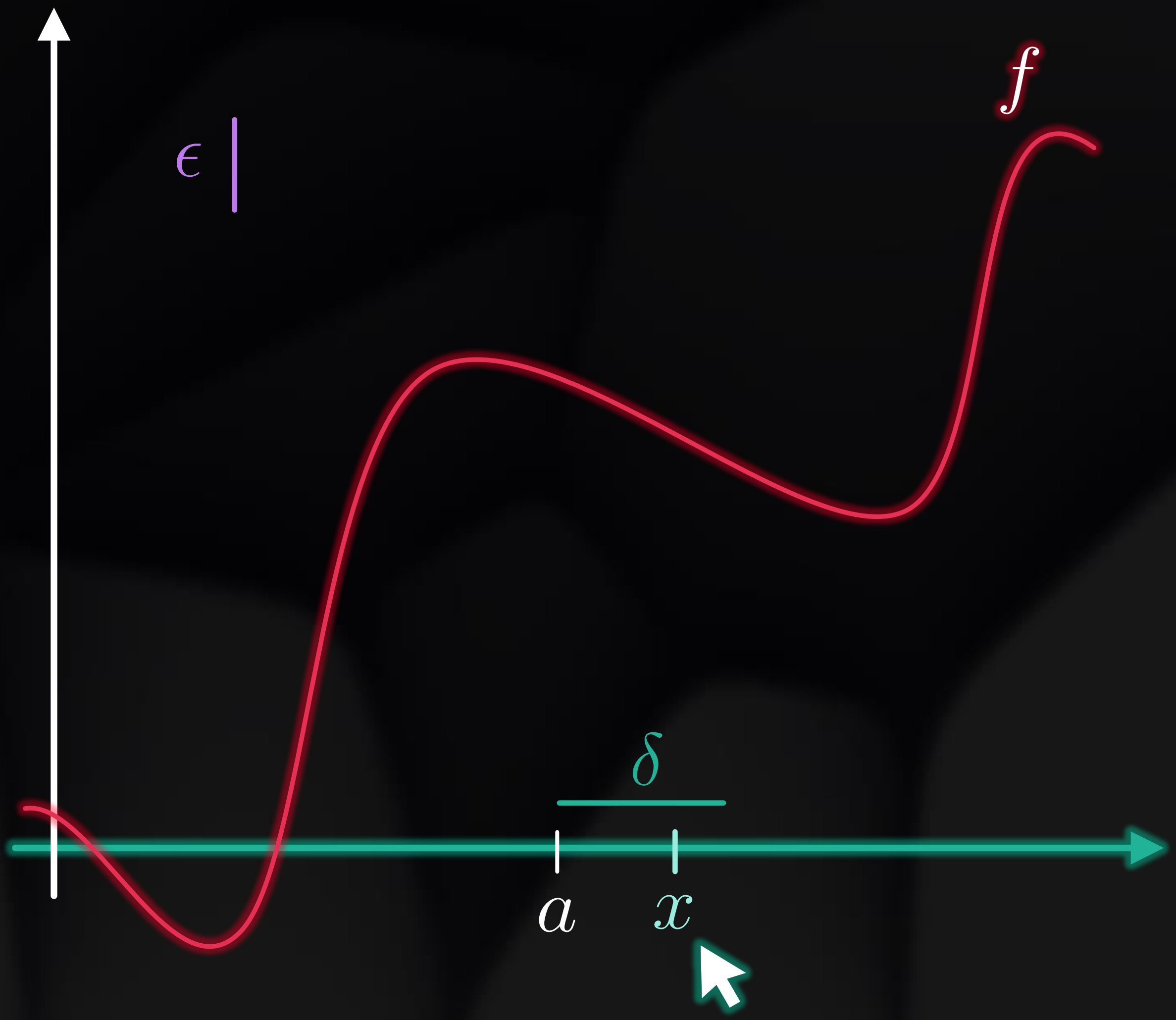

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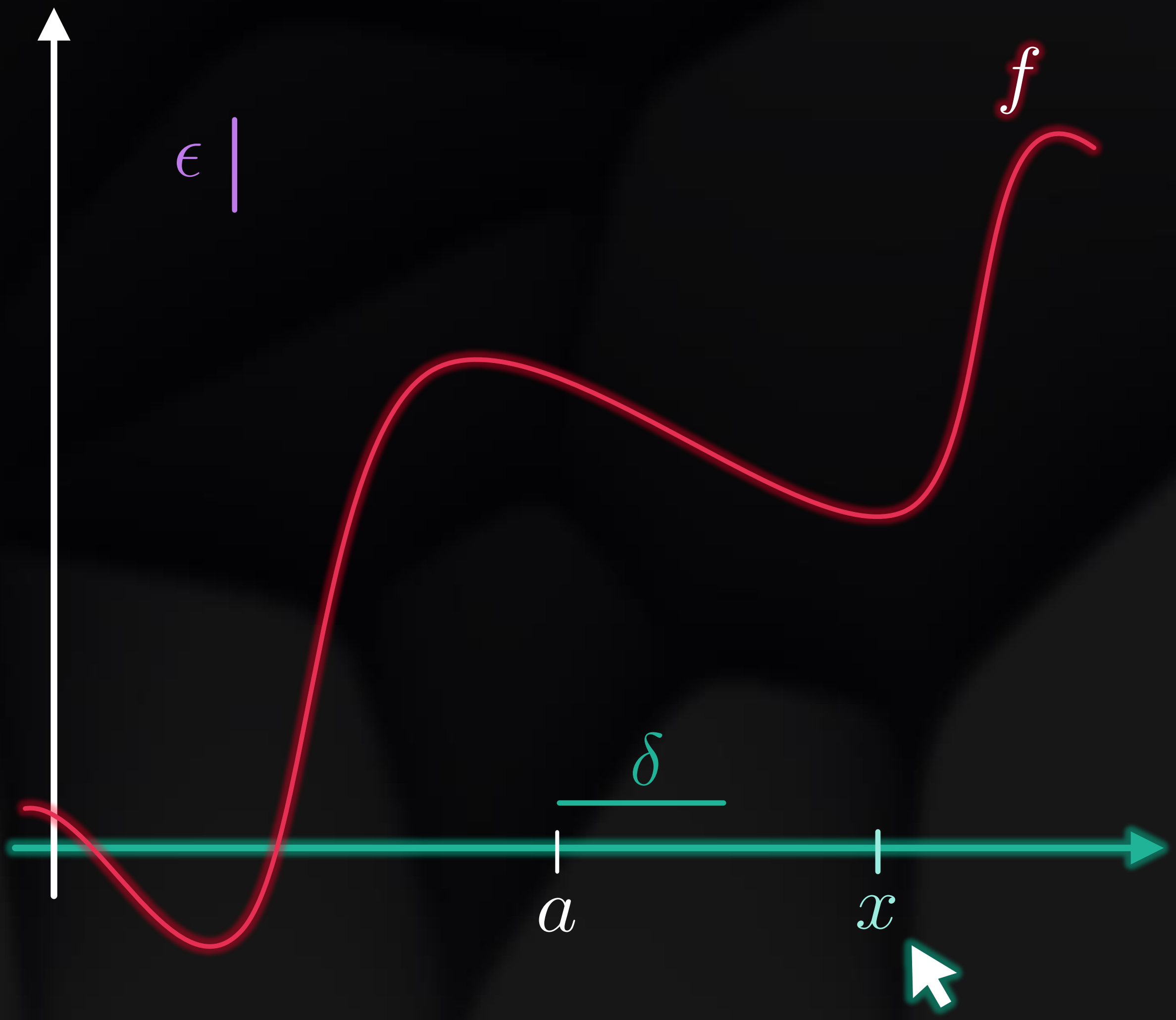

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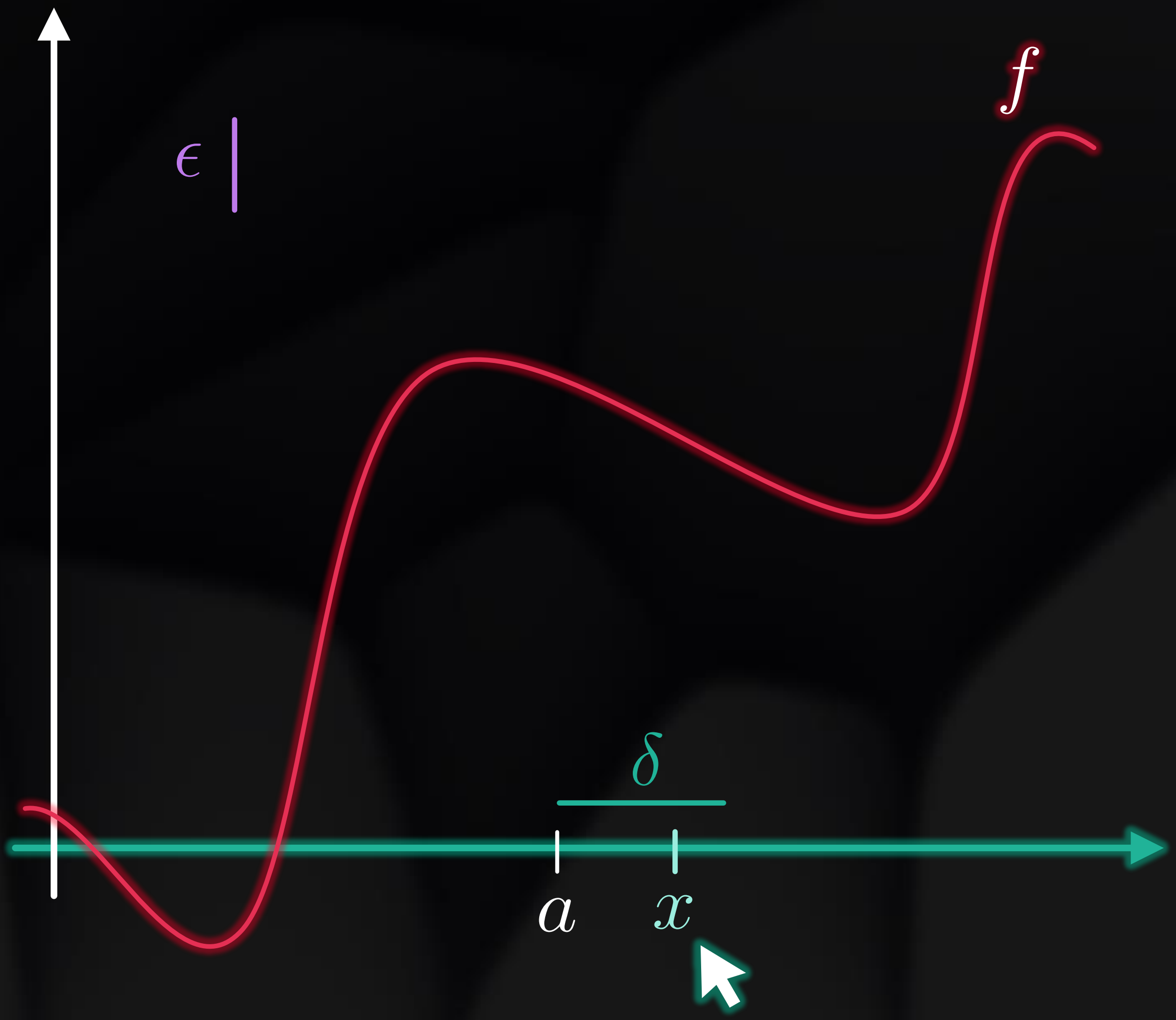

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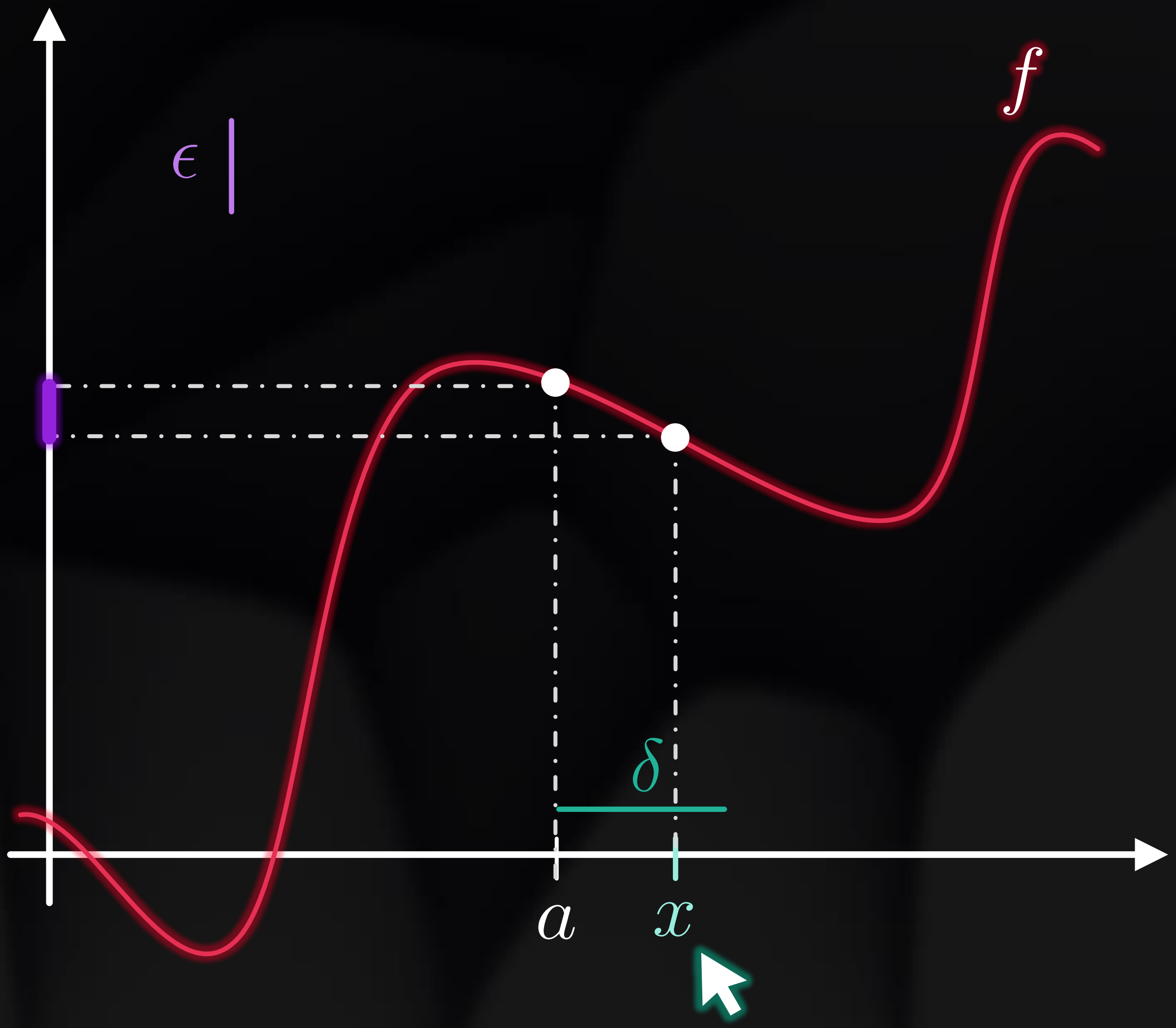

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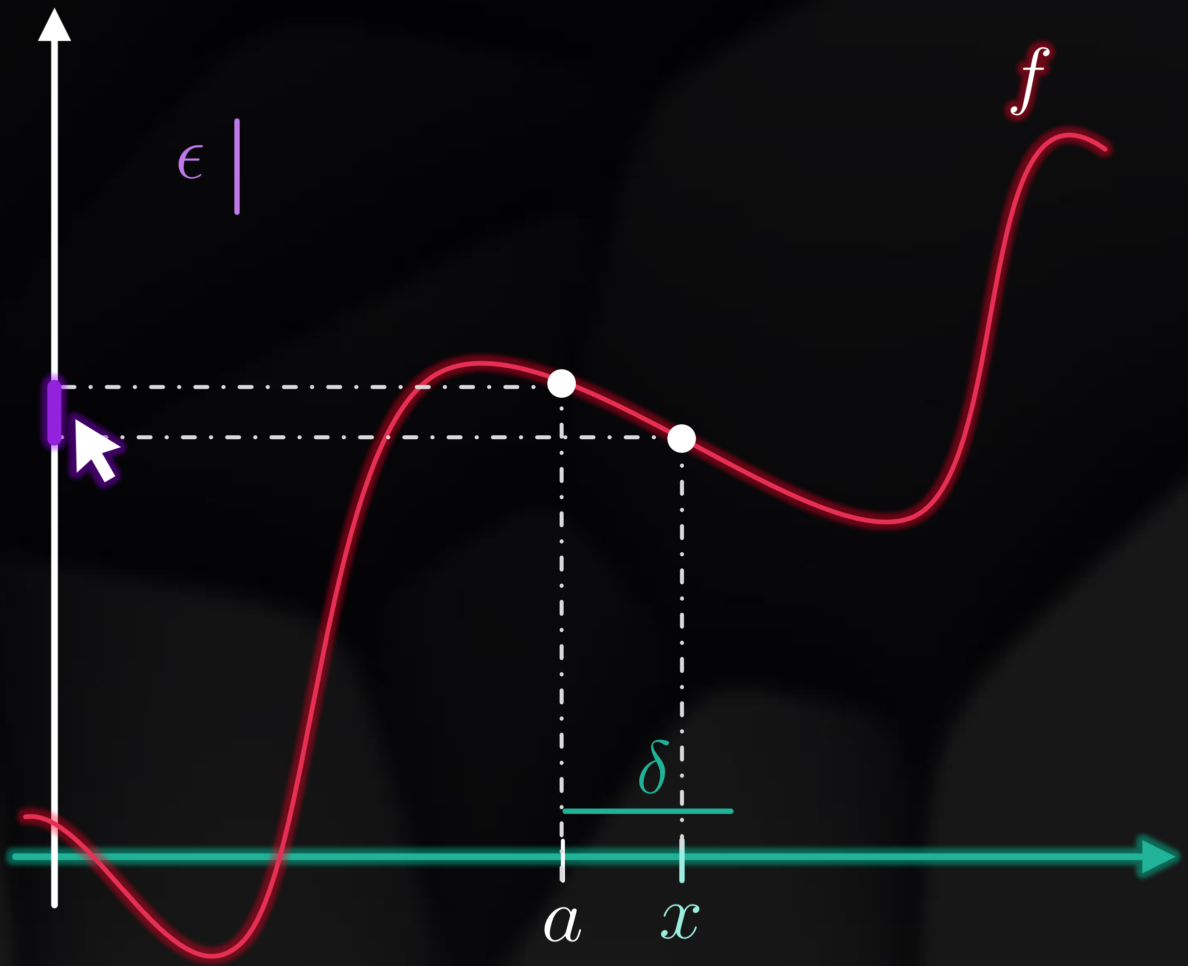

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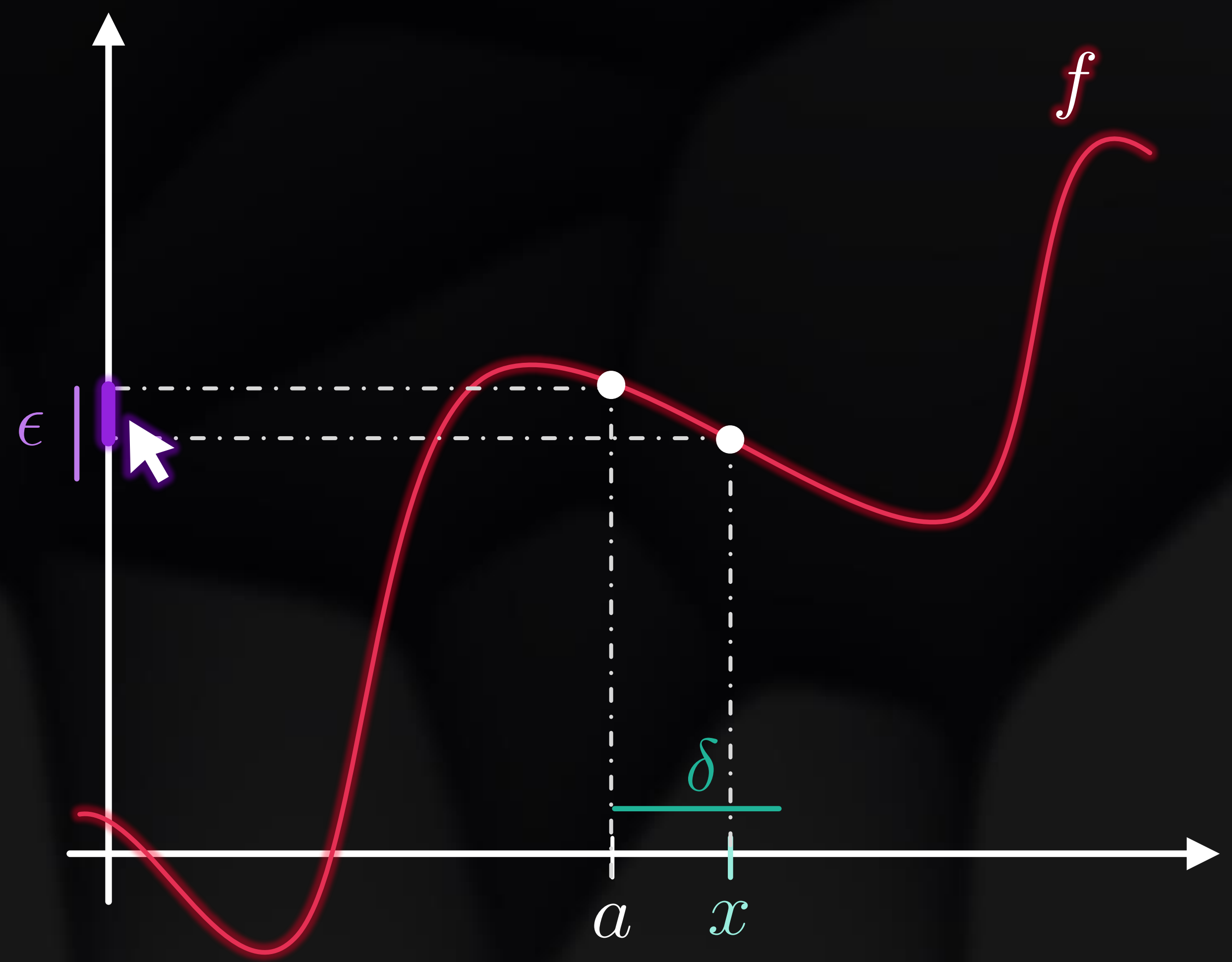

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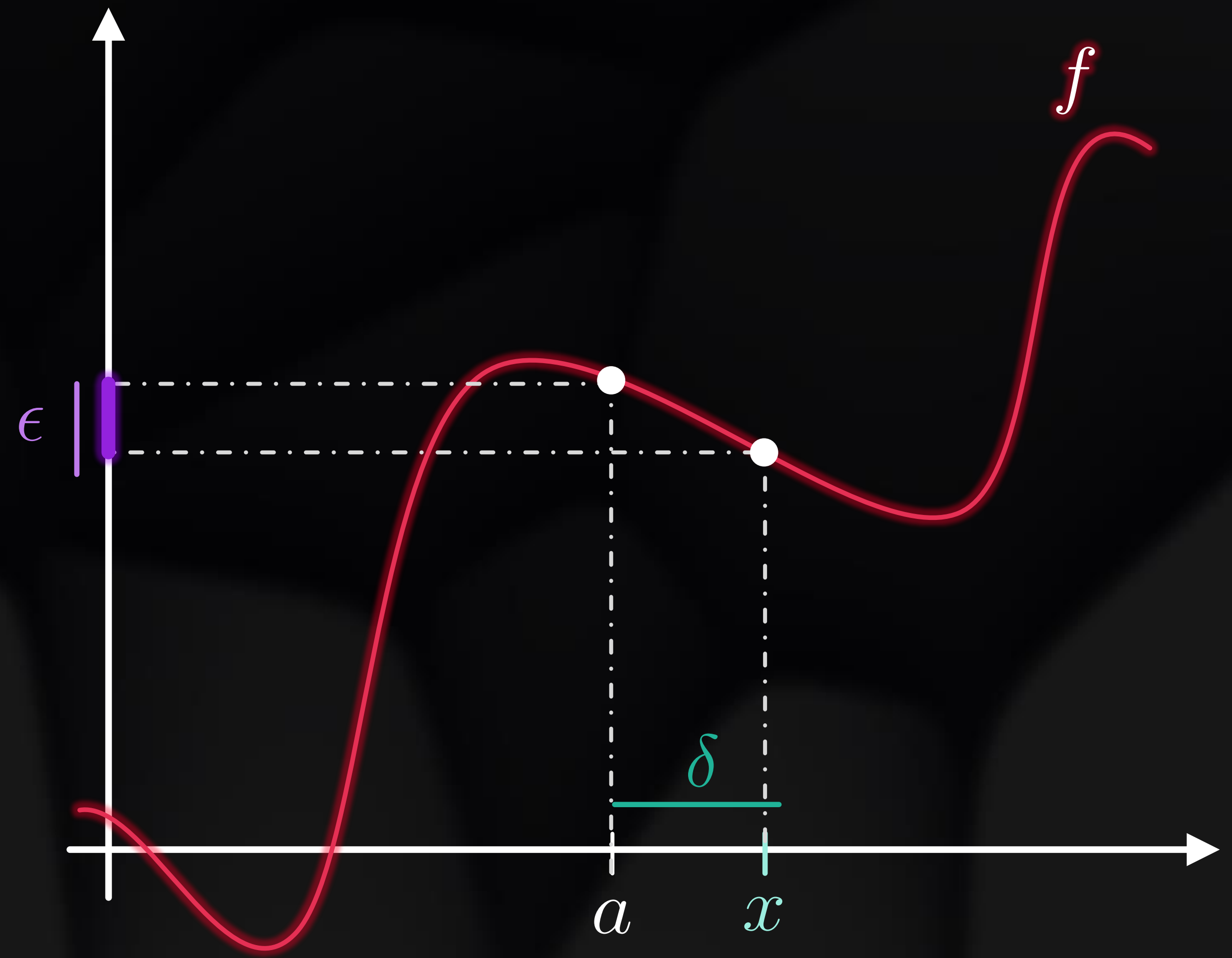

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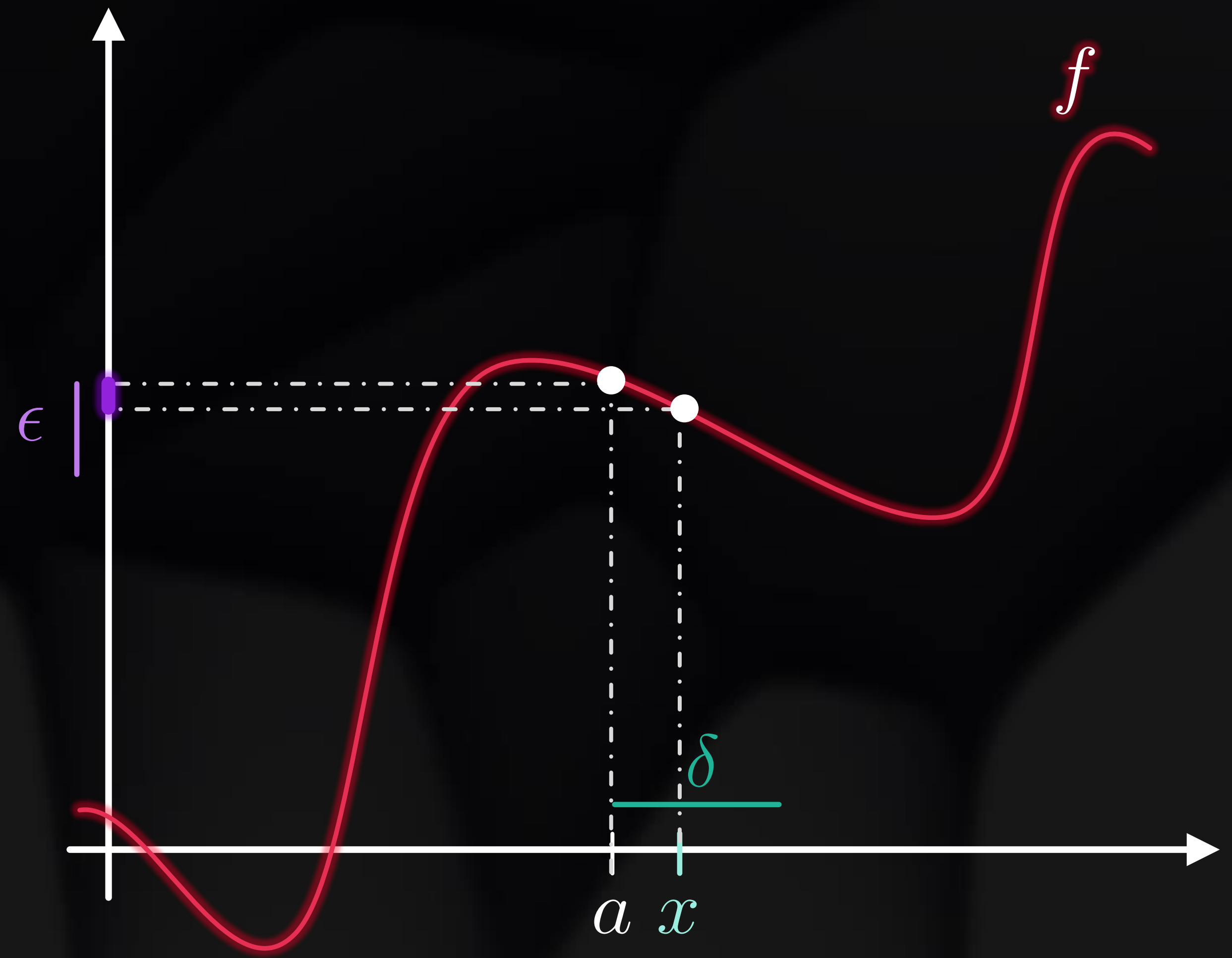

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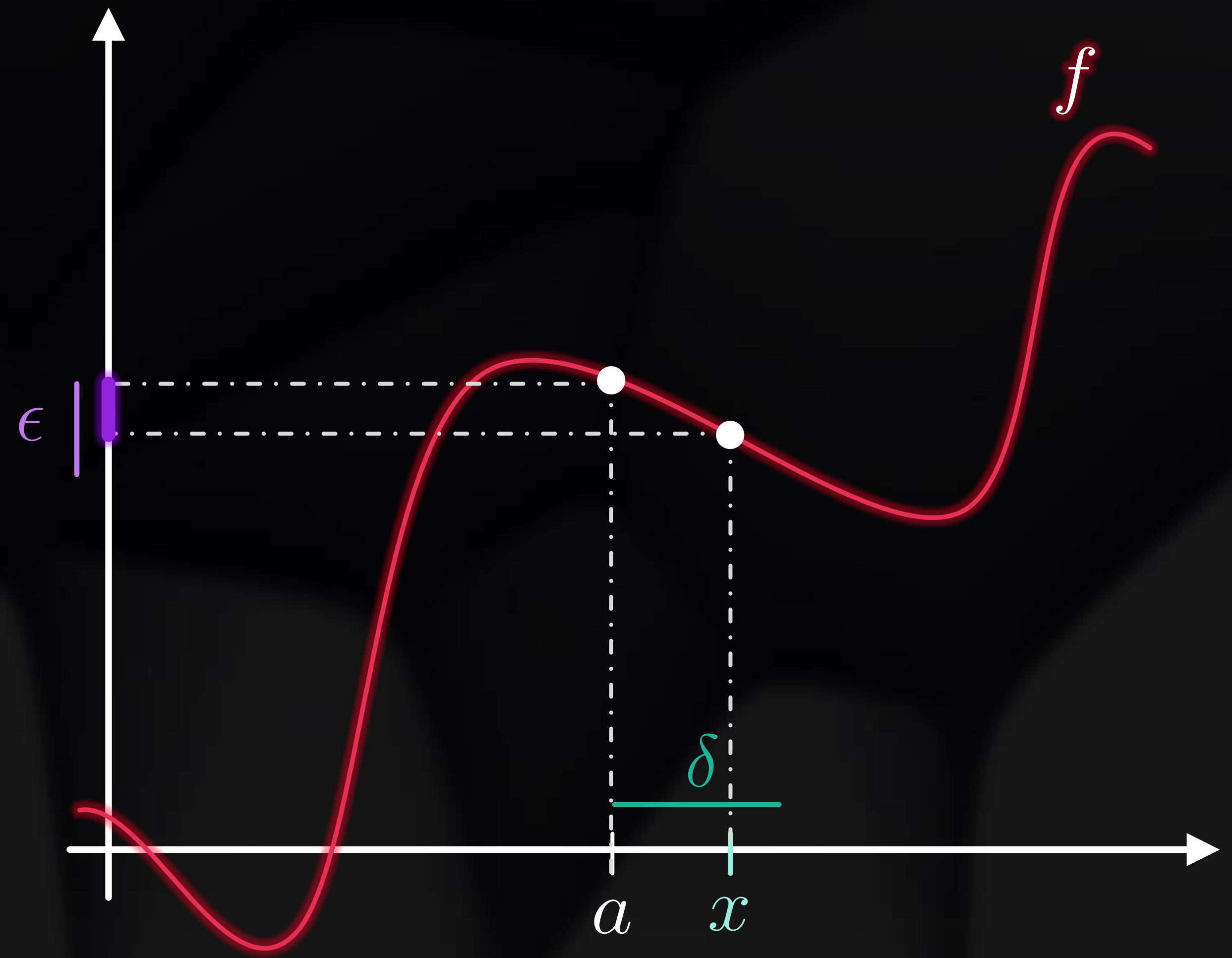

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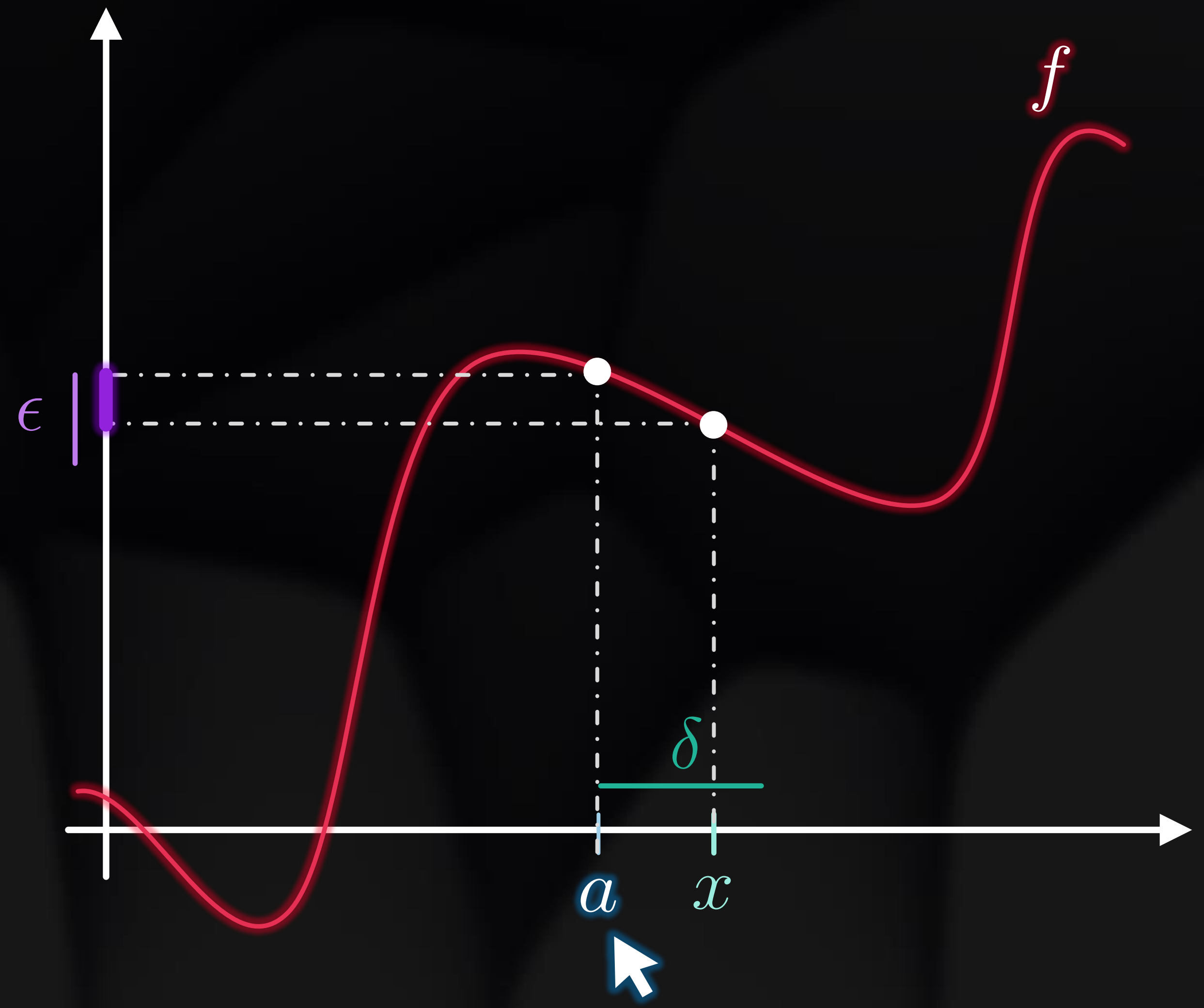

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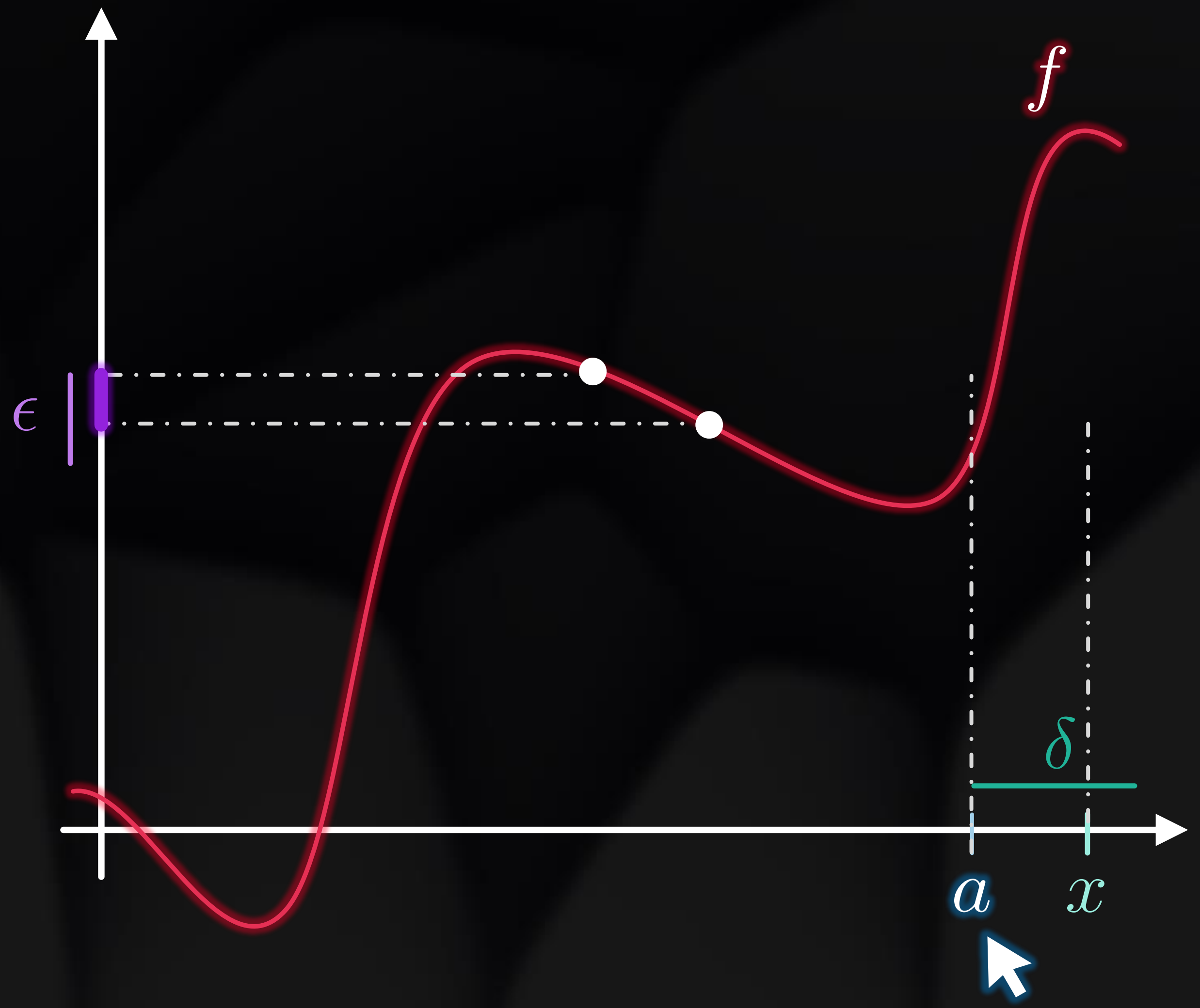

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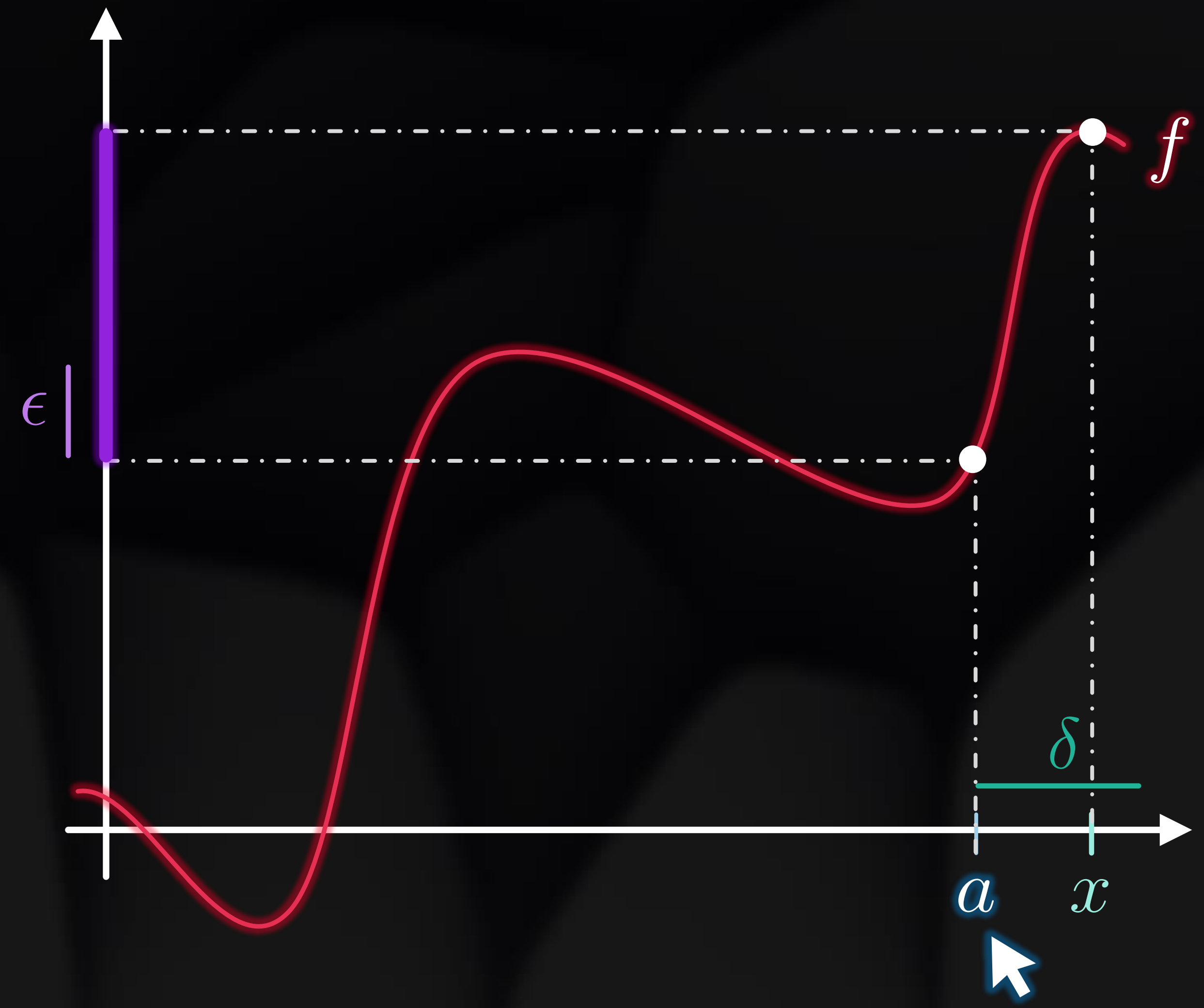

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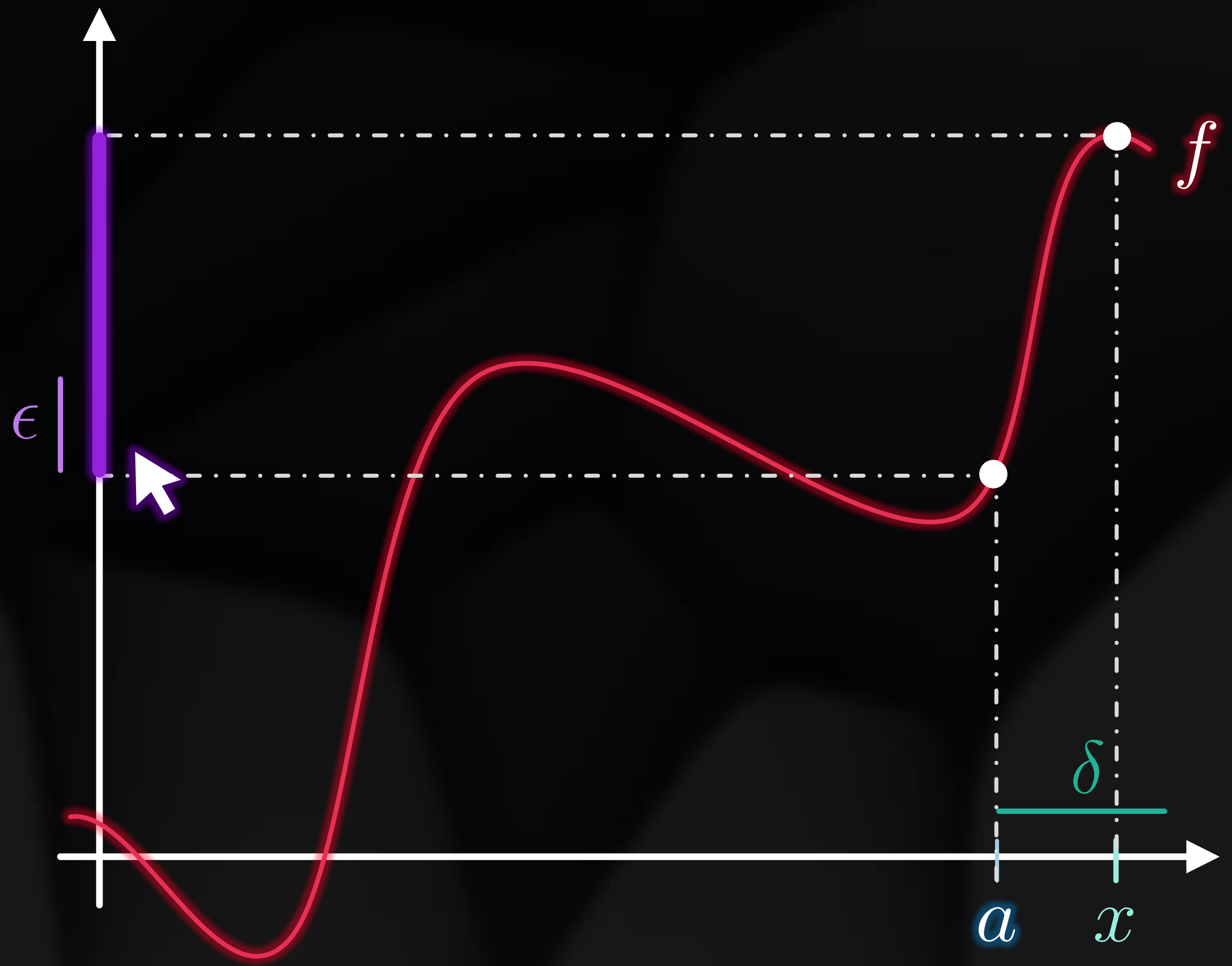

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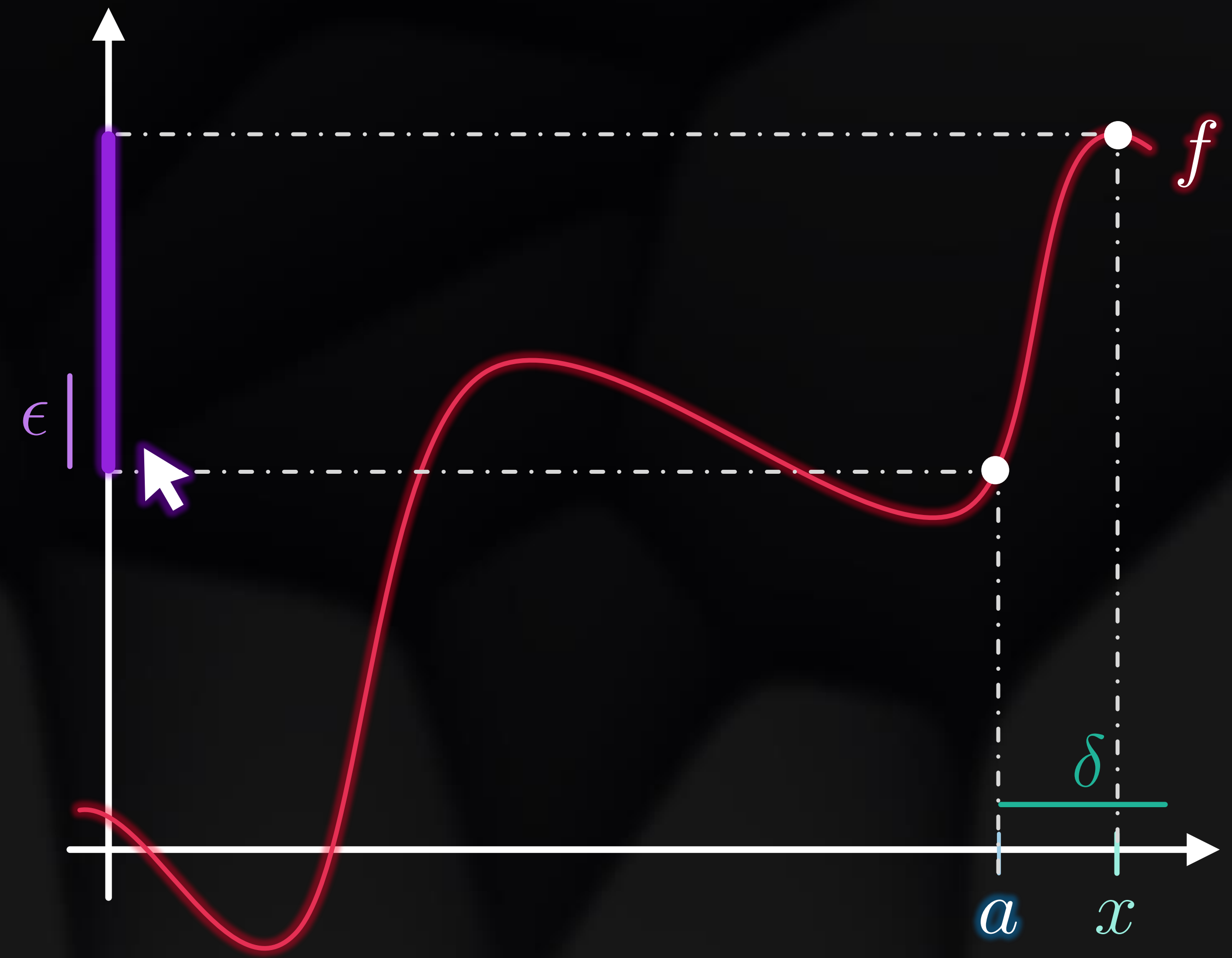

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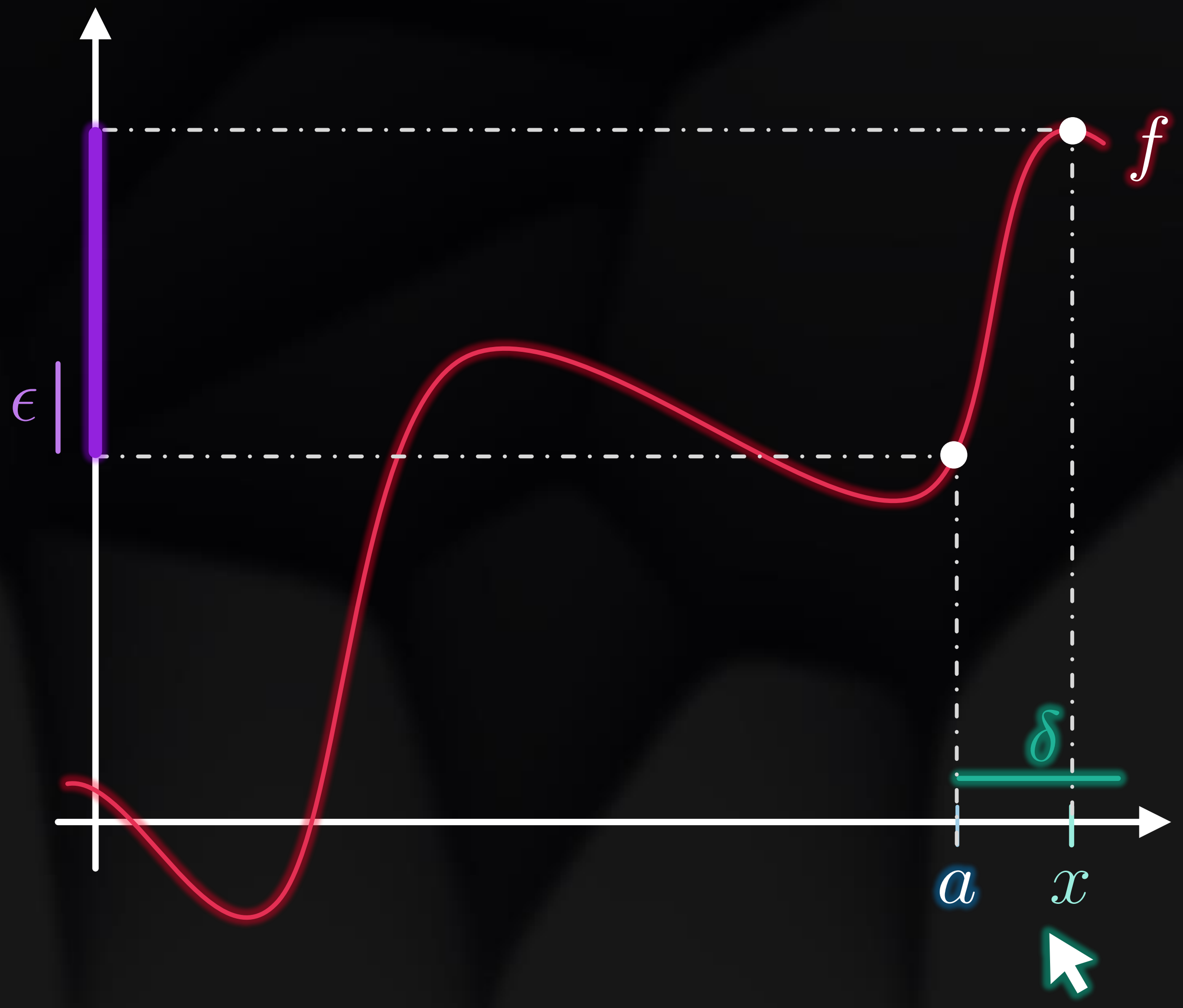

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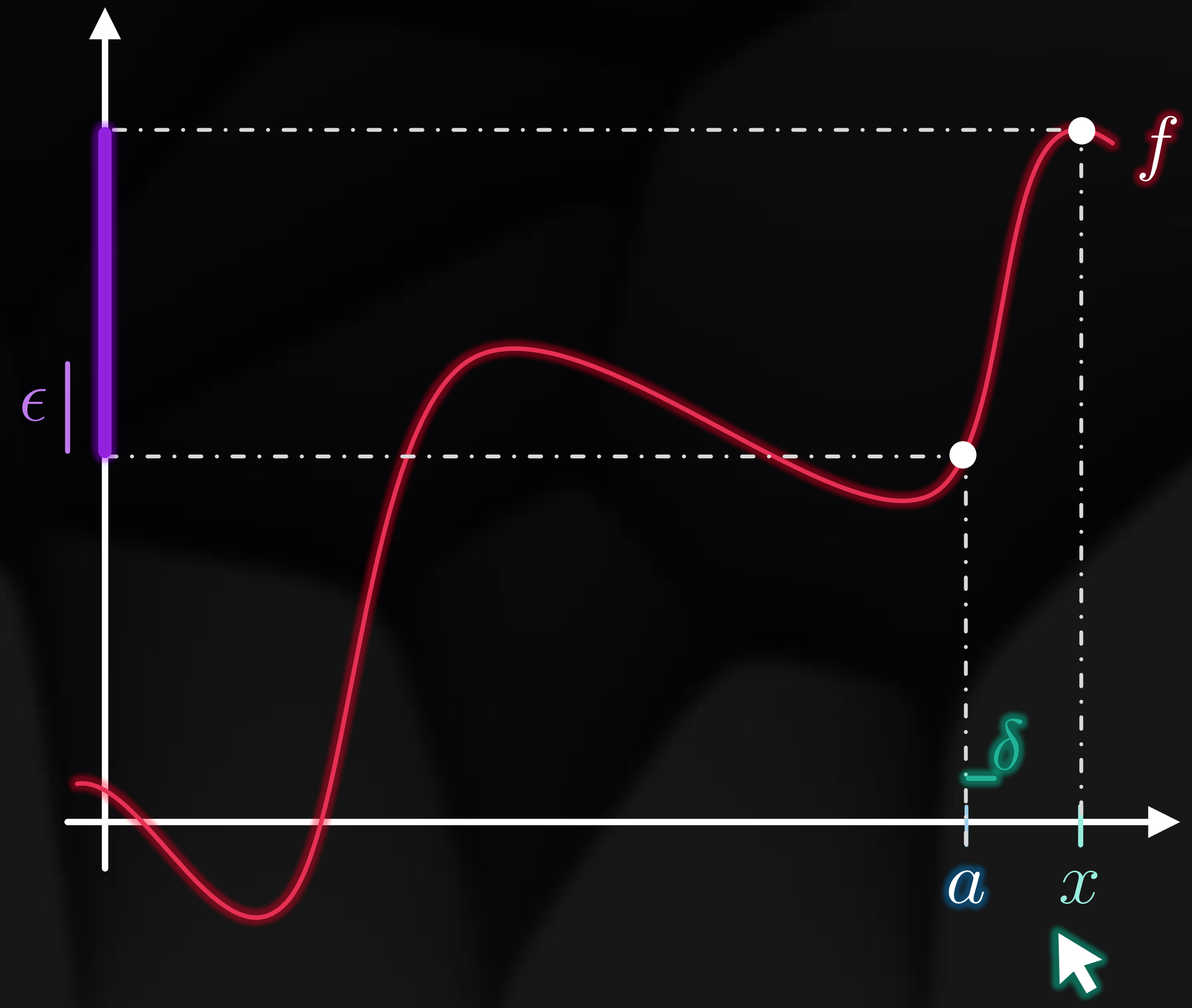

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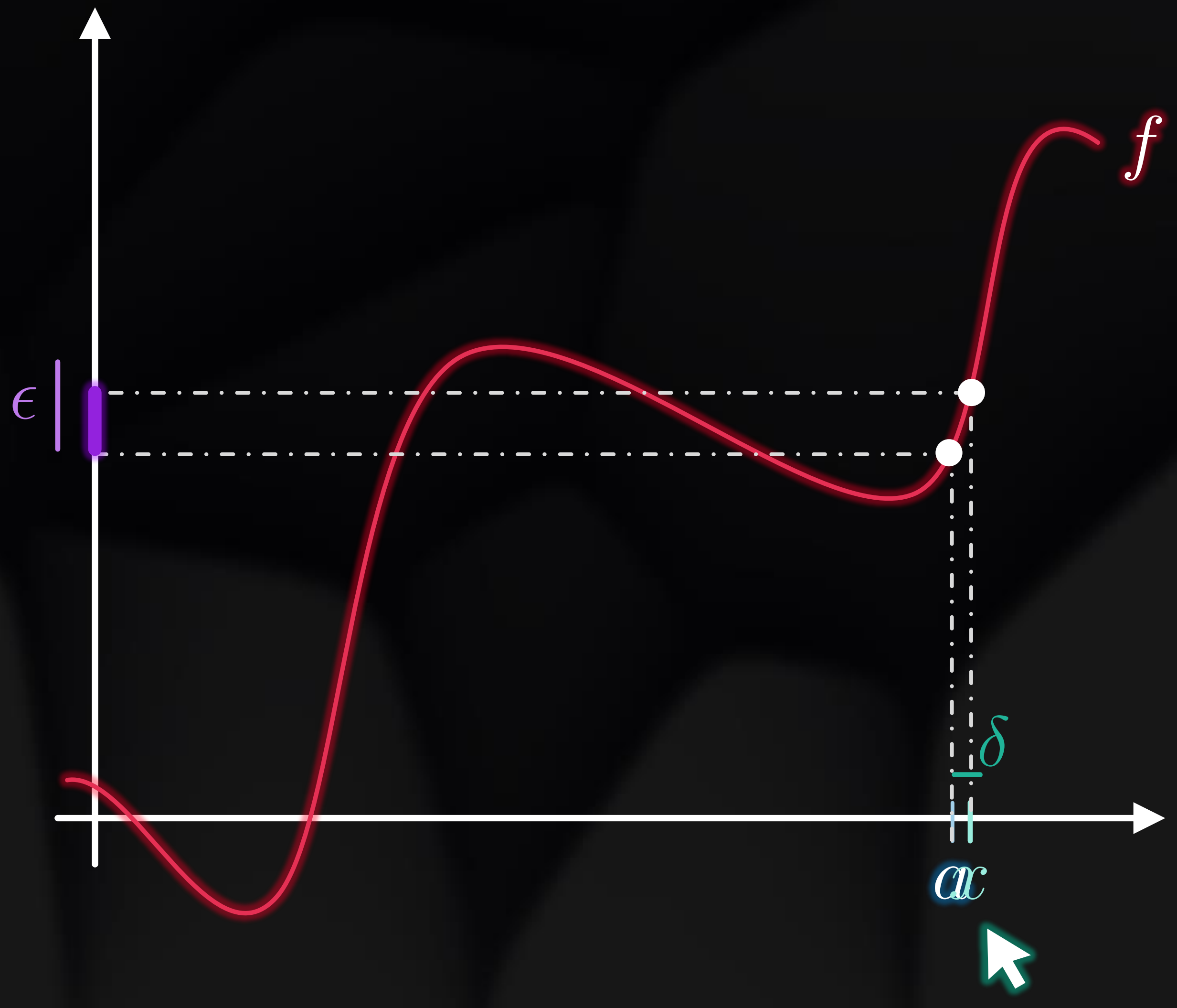

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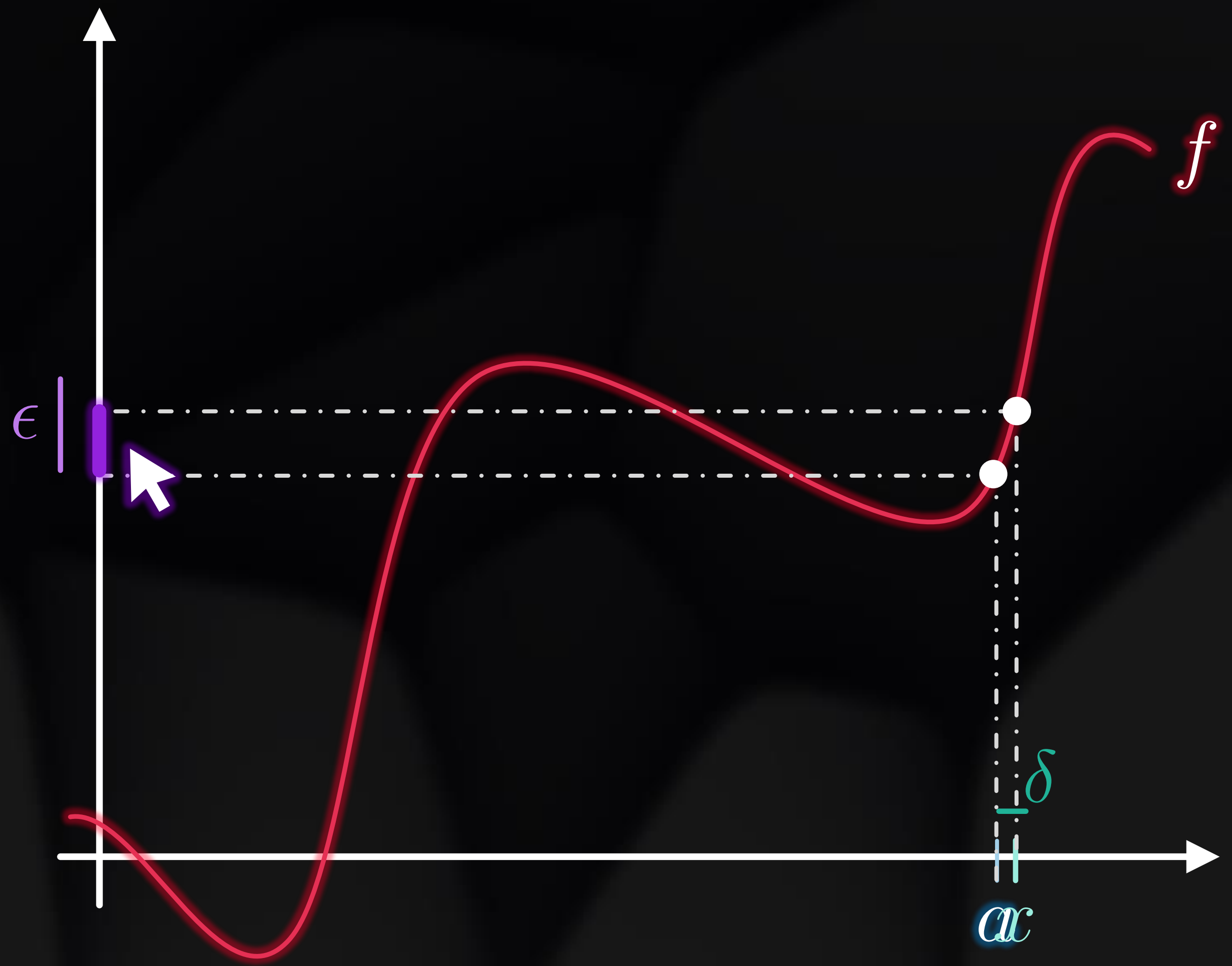

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Lean4: Programming language and Theorem Prover

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/-- Definition of a continuous function
`f : D → ℝ` at a point `a ∈ D`. -/
def IsContinuousAt
    (D : Set ℝ) (f : D → ℝ) (a : D) : Prop :=
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def IsContinuous
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Algebraic Properties

Left- and right-continuity

Outline

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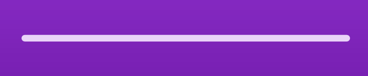
Outline

1. Continuous Functions
2. Examples
3. Algebraic Properties
4. Left- and right-continuity

Examples



Constant function

$x \mapsto c$

Line

$x \mapsto mx + y_0$

Parabola

$x \mapsto x^2$

Hyperbola

$x \mapsto {}^{1}\!/_{x}$

$c, m, y_0 \in \mathbb{R}$

Examples

Constant function

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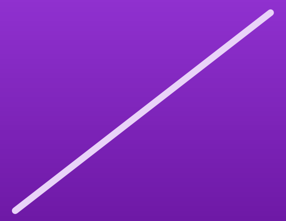

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Examples

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) := c$, where $c \in \mathbb{R}$. That is, f is a constant function. Then f is continuous at every point $a \in \mathbb{R}$.

Constant function

$x \mapsto c$

Examples

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \left(|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon \right)$$

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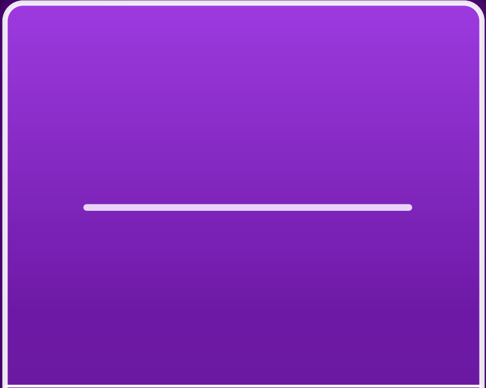

Constant function

$x \mapsto c$

Let $a \in \mathbb{R}$.

Examples

$$\boxed{\forall \epsilon > 0} \quad \exists \delta > 0 \quad \forall x \in D : \left(|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon \right)$$

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Let $a \in \mathbb{R}$. Let $\varepsilon > 0$.

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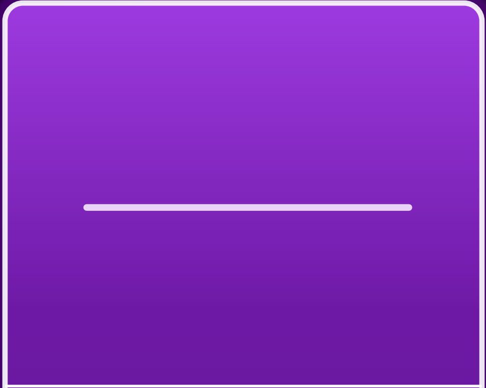

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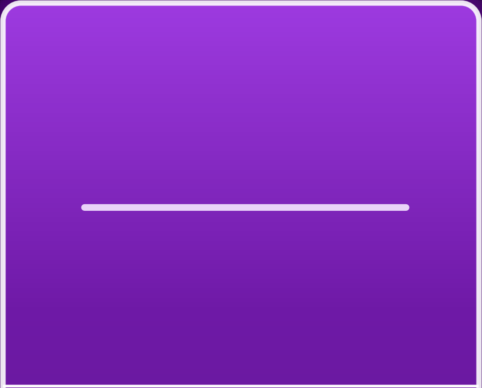

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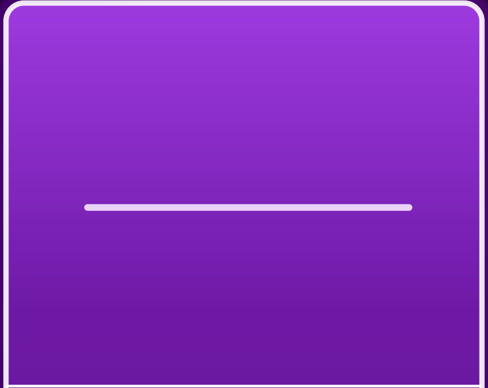
Then $|f(x) - f(a)|$



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Let $a \in \mathbb{R}$. Let $\varepsilon > 0$.
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Then $|f(x) - f(a)| = |c - c|$

Constant function
$x \mapsto c$

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Examples

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) := c$, where $c \in \mathbb{R}$. That is, f is a constant function. Then f is continuous at every point $a \in \mathbb{R}$.

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Constant function
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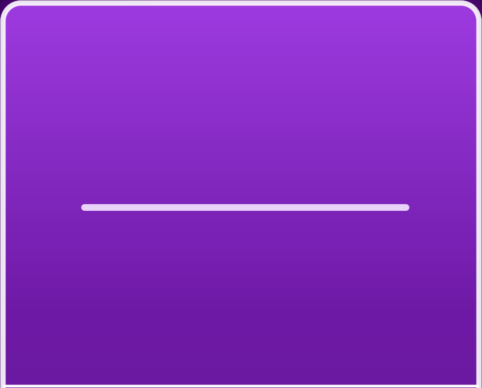

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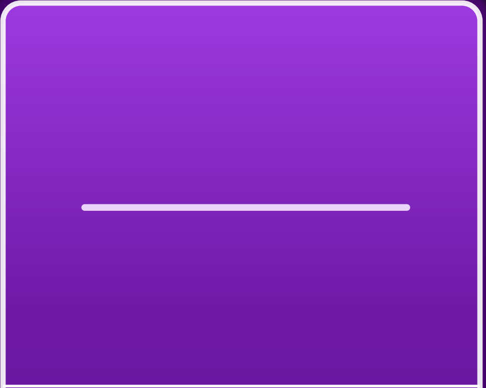

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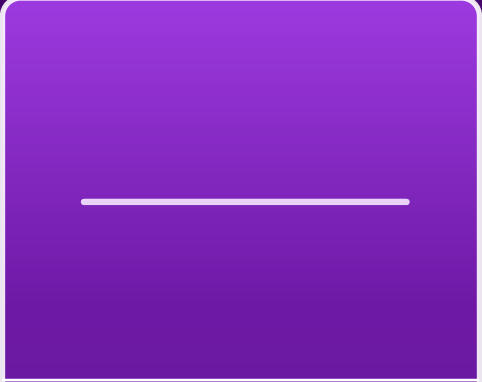

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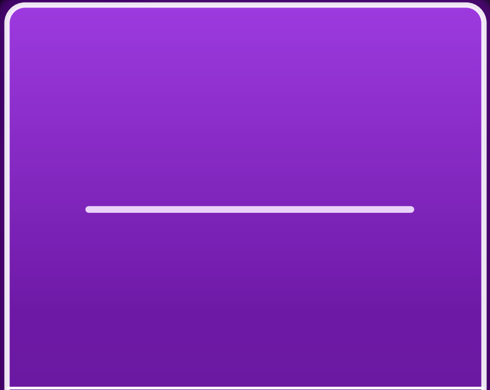
Therefore, f is continuous at a.



Constant function
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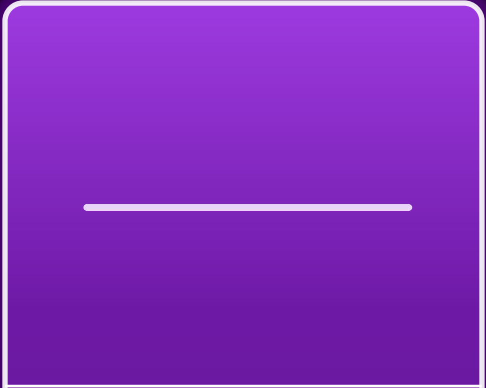

Constant function
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theorem constant_function_is_continuous_at_a_point
    (D : Set ℝ) (c : ℝ) (a : D)
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```
⊢ IsContinuousAt D (fun x ↦ c) a
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  simp only [sub_self, abs_zero]
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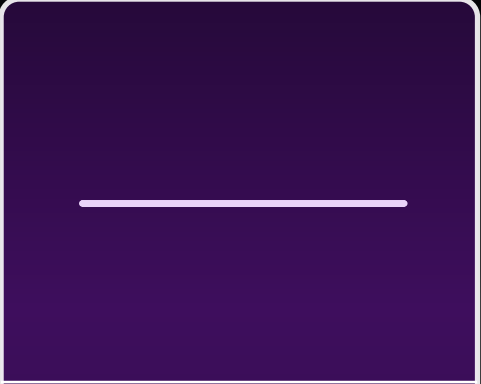
Examples



Constant function

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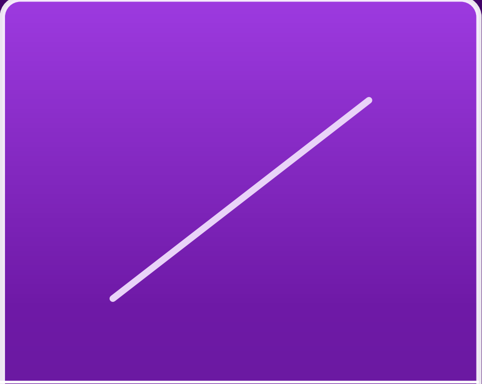
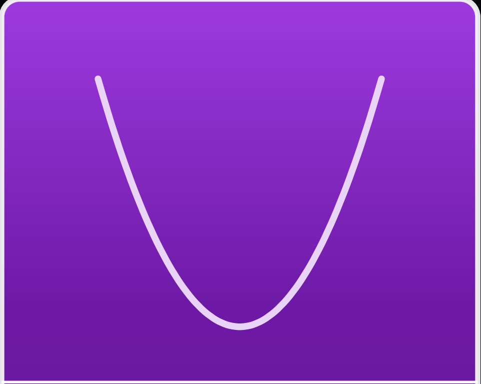


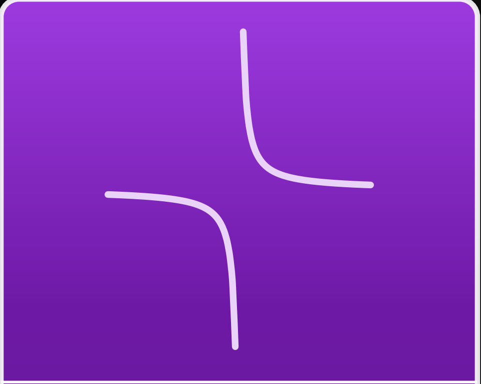
Line

$x \mapsto mx + y_0$



Parabola

$x \mapsto x^2$



Hyperbola

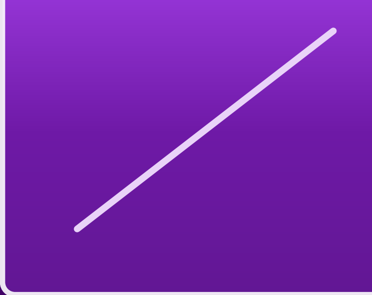
$x \mapsto {}^1\!/_x$

$c, m, y_0 \in \mathbb{R}$

Examples

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) := m \cdot x + y_0$, where $m, y_0 \in \mathbb{R}$. Then f is continuous at every point $a \in \mathbb{R}$.



Line

$x \mapsto mx + y_0$

```lean
theorem lines_are_continuous_at_a_point
   (D : Set ℝ) (m y₀ : ℝ) (a : D)
    : IsContinuousAt D (fun x ↦ m*x+y₀) a := by
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Examples

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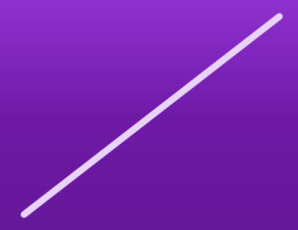

Line

$$x \mapsto mx + y_0$$

Let $a \in \mathbb{R}$.

```
theorem lines_are_continuous_at_a_point
   (D : Set ℝ) (m y₀ : ℝ) (a : D)
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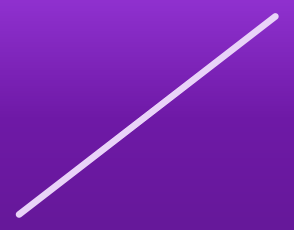

Line

$x \mapsto mx + y_0$

Let $a \in \mathbb{R}$.

Two cases:

$m = 0 \qquad m \neq 0$

```
theorem lines_are_continuous_at_a_point
    (D : Set ℝ) (m y₀ : ℝ) (a : D)
    : IsContinuousAt D (fun x ↦ m*x+y₀) a := by

  by_cases m_cases : m = 0

  -- m = 0
  · sorry

  -- m ≠ 0
  . sorry
```

Examples $x \mapsto mx + y_0$

Let $a \in \mathbb{R}$.

Two cases:

$m = 0$ $\quad$ $m \neq 0$

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$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D: \quad \Big(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon\Big)$$

Examples $\boxed{x \mapsto mx + y_0}$

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∿ Examples $\boxed{x \mapsto mx + y_0}$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D :$$
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```

Case $m = 0$.

⊢ IsContinuousAt D (fun x ↦ m * x + y₀) a

∿ Examples $\boxed{x \mapsto mx + y_0}$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D: \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)$$

Let $a \in \mathbb{R}$.

Case $m = 0$.

```
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    (D : Set ℝ) (m y₀ : ℝ) (a : D)
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  -- m = 0
  · subst m
```

⊢ IsContinuousAt D (fun x ↦ 0 * x + y₀) a

∿ Examples $\boxed{x \mapsto mx + y_0}$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D: \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)$$

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    simp only [zero_mul, zero_add]
```

```
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∿ Examples $\boxed{x \mapsto mx + y_0}$

Let $a \in \mathbb{R}$.

Case $m = 0$.

Then $f(x) = y_0$
is a constant function.

We have shown that
constant functions are continuous.

```
theorem lines_are_continuous_at_a_point
    (D : Set ℝ) (m y₀ : ℝ) (a : D)
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  -- m = 0
  · subst m
    simp only [zero_mul, zero_add]
    exact constant_function_is_continuous_at_a_point D y₀ a
```

⊢ Done with this case.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D :$$
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  · |
```

⊢ IsContinuousAt D (fun x ↦ m * x + y₀) a

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)$$

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  · intro ε h_εbigger0
    let δ := ε / |m|
    use δ
```

```
⊢ δ > 0 ∧ ∀ x : D, |x - a| < δ → |m * x + y₀ - (m * a + y₀)| < ε
```

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    have h_δbigger0 : δ > 0 := by positivity
    simp only [h_δbigger0, true_and]
```

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$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D :$

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    (D : Set ℝ) (m y₀ : ℝ) (a : D)
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Examples $x \mapsto mx + y_0$

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    simp
```

```
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$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D :$
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Examples $x \mapsto mx + y_0$

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    simp only [h_δbigger0, true_and]
    intro x h_xδ_criterion
    simp
```

```
⊢ |m * x - m * a| < ε
```

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D:$$
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∿ Examples $\boxed{x \mapsto mx + y_0}$

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```

```
⊢ |m*x-m*a| < ε
  |m*x-m*a| = …
```

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Examples $x \mapsto mx + y_0$

Let $a \in \mathbb{R}$.

Case $m \neq 0$.

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⊢ |m*x-m*a| < ε
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    calc |m * x - m * a|

      _ = |m * (x - a)|
        := by ring_nf
```

$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D:$

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Examples $x \mapsto mx + y_0$

Let $a \in \mathbb{R}$.

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Since $\epsilon > 0$ and $|m| > 0$, we have $\delta > 0$.

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$\vdash |m * x - m * a| < \varepsilon$

$|m * x - m * a| = |m * (x - a)|$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D : \left(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \right)$$

Examples $\boxed{x \mapsto mx + y_0}$

Let $a \in \mathbb{R}$.

Case $m \neq 0$.

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```

⊢ | m * x - m * a | < ε
… = | m * (x - a) |

$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D :$

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∿ Examples $\boxed{x \mapsto mx + y_0}$

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  · …
    calc |m * x - m * a|

      _ = |m * (x - a)|
        := by ring_nf

      _ = |m| * |x - a|
        := abs_mul m (x - a)
```

⊢ | m * x - m * a | < ε

… = | m * (x − a) | = | m | * | x − a |

Examples $x \mapsto mx + y_0$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D:$$
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```
⊢ |m*x-m*a| < ε
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Let $a \in \mathbb{R}$.

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Since $\epsilon > 0$ and $|m| > 0$,
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⊢ |m*x-m*a|<ε
… = |m|*|x-a|

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        _ = |m| * |x - a|
          := abs_mul m (x - a)

        _ < |m| * δ
          := (mul_lt_mul_left (by positivity)).mpr
              h_xδ_criterion
```

∿ Examples $\boxed{x \mapsto mx + y_0}$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D:$$
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Let $x \in \mathbb{R}$ and $\boxed{|x - a| < \delta.}$

```
theorem mul_lt_mul_left
(_ : 0 < a) : a * b < a * c ↔ b < c
```

```
⊢ |m * x - m * a| < ε
… = |m| * |x - a|
```

```
theorem lines_are_continuous_at_a_point
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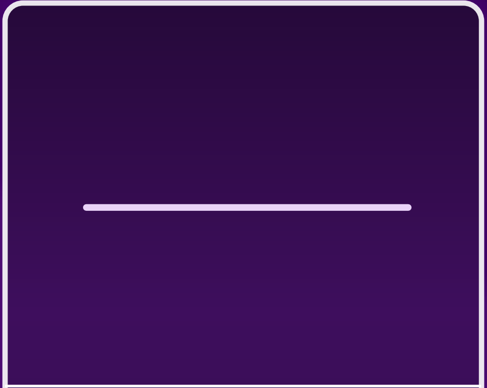
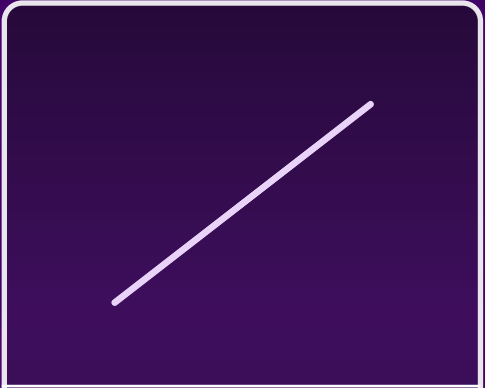
Examples



Constant function

$x \mapsto c$



Line

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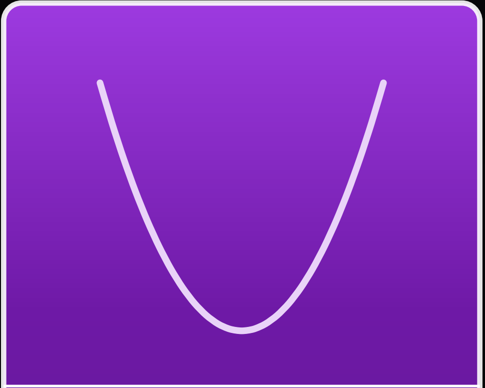
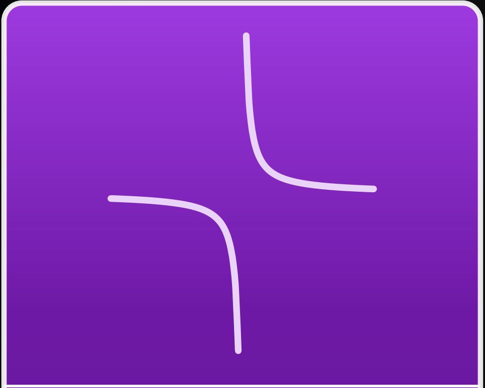


Parabola

$x \mapsto x^2$



Hyperbola

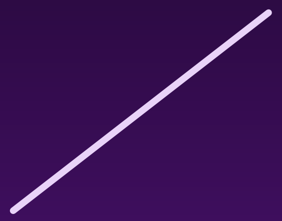
$x \mapsto {}^{1}\!/_{x}$

$c, m, y_0 \in \mathbb{R}$

Examples

See the proofs for these functions on GitHub

Constant function
$x \mapsto c$



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Outline

Continuous Functions → Examples → Algebraic Properties → Left- and right-continuity

Outline

- Continuous Functions
- Examples
- Algebraic Properties
- Left- and right-continuity

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Continuous Functions → Examples → Algebraic Properties → Left- and right-continuity

Properties

Theorem

If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f + g$ is continuous on D.

Sum

If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.

Product

Properties

If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.

Assume f and g are continuous on D. Let $a \in D$ and $\epsilon > 0$.

Properties

If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.

Assume f and g are continuous on D. Let $a \in D$ and $\epsilon > 0$.

By the continuity of f and g, we have:

"magical estimates"

- There $\exists \delta_1 > 0$ such that $\forall x \in D$, if $|x-a| < \delta_1$, then $|f(x) - f(a)| < \dfrac{\epsilon}{2|g(a)|+1}$.
- There $\exists \delta_2 > 0$ such that $\forall x \in D$, if $|x-a| < \delta_2$, then $|g(x) - g(a)| < \dfrac{\epsilon}{2(\epsilon + |f(a)|)}$.

Choose $\delta := \min(\delta_1, \delta_2)$. Then $\delta > 0$ since both $\delta_1 > 0$ and $\delta_2 > 0$.

Let $x \in D$. If $|x-a| < \delta$, we have:

$$|f(x) - f(a)| < \frac{\epsilon}{2|g(a)|+1}$$

$$|g(x) - g(a)| < \frac{\epsilon}{2(\epsilon + |f(a)|)}$$

Properties

If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.

$$
\begin{aligned}
|(f \cdot g)(x) - (f \cdot g)(a)| &= |f(x) \cdot g(x) - f(a) \cdot g(a)| \\
&= |f(x) \cdot g(x) \underline{- f(x) \cdot g(a) + f(x) \cdot g(a)} - f(a) \cdot g(a)| \\
&= \big|f(x) \cdot \big(g(x) - g(a)\big) + \big(f(x) - f(a)\big) \cdot g(a)\big| \\
&\leq \big|f(x) \cdot (g(x) - g(a))\big| + \big|(f(x) - f(a)) \cdot g(a)\big| \\
&= |f(x)| \cdot |g(x) - g(a)| + |f(x) - f(a)| \cdot |g(a)| \\
&\leq |f(x)| \cdot \frac{\epsilon}{2(\epsilon + |f(a)|)} + \frac{\epsilon}{2|g(a)| + 1} \cdot |g(a)|
\end{aligned}
$$

$$
\begin{aligned}
|f(x) - f(a)| &< \frac{\epsilon}{2|g(a)| + 1} \\
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&= \big|f(x)\cdot\big(g(x)-g(a)\big) + \big(f(x)-f(a)\big)\cdot g(a)\big| \\
&\le \big|f(x)\cdot(g(x)-g(a))\big| + \big|(f(x)-f(a))\cdot g(a)\big| \\
&= |f(x)|\cdot|g(x)-g(a)| + |f(x)-f(a)|\cdot|g(a)| \\
&\le |f(x)|\cdot\frac{\epsilon}{2(\epsilon+|f(a)|)} + \frac{\epsilon}{2|g(a)|+1}\cdot|g(a)| \\
&< \frac{\epsilon}{2}\cdot\frac{\epsilon+|f(a)|}{\epsilon+|f(a)|} + \frac{\epsilon}{2}\cdot\frac{|g(a)|}{|g(a)|+1/2} \\
&\le \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon
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$$

$$|f(x)-f(a)| < \frac{\epsilon}{2|g(a)|+1} \le \epsilon \quad \Rightarrow \quad |f(x)| < \epsilon + |f(a)|$$

$$\boxed{\begin{aligned} |f(x)-f(a)| &< \frac{\epsilon}{2|g(a)|+1} \\ |g(x)-g(a)| &< \frac{\epsilon}{2(\epsilon+|f(a)|)} \end{aligned}}$$

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If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.

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Hence, $f \cdot g$ is continuous at a. Since a was arbitrary, $f \cdot g$ is continuous on D.

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If f and g are continuous functions on a set $D \subseteq \mathbb{R}$, then $f \cdot g$ is continuous on D.

Outline

Continuous Functions

Examples

Algebraic Properties

Left- and right-continuity

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- Continuous Functions
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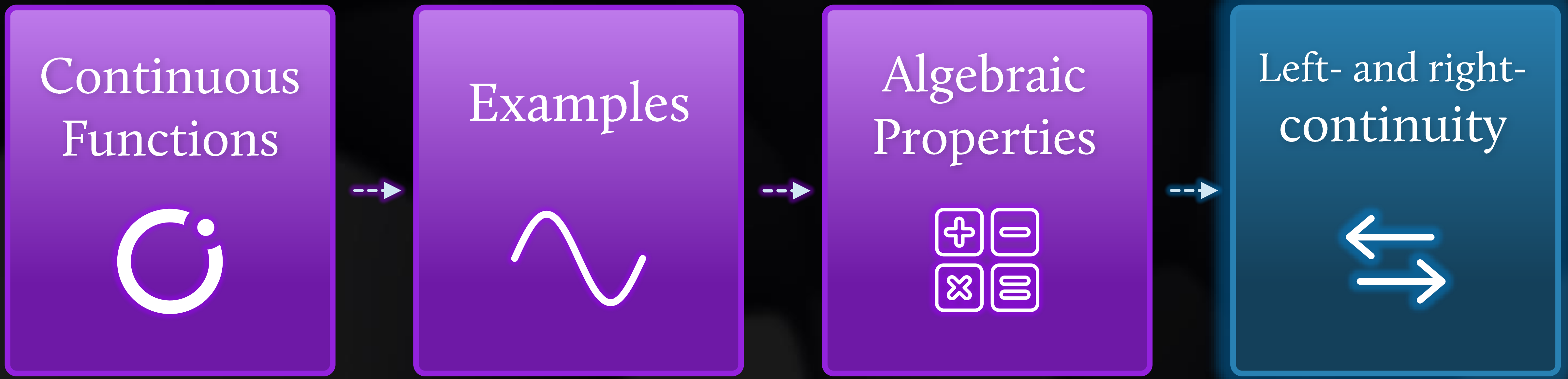
Continuous Functions
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Left- and right-continuity

A function $f : D \to \mathbb{R}$ is *left-continuous at a point* $a \in D$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D, \overset{x > a}{\boxed{x < a}}: \left(|x - a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \epsilon \right)$$

If f is left-continuous at every point in D, then we say that f is *left-continuous on D*.

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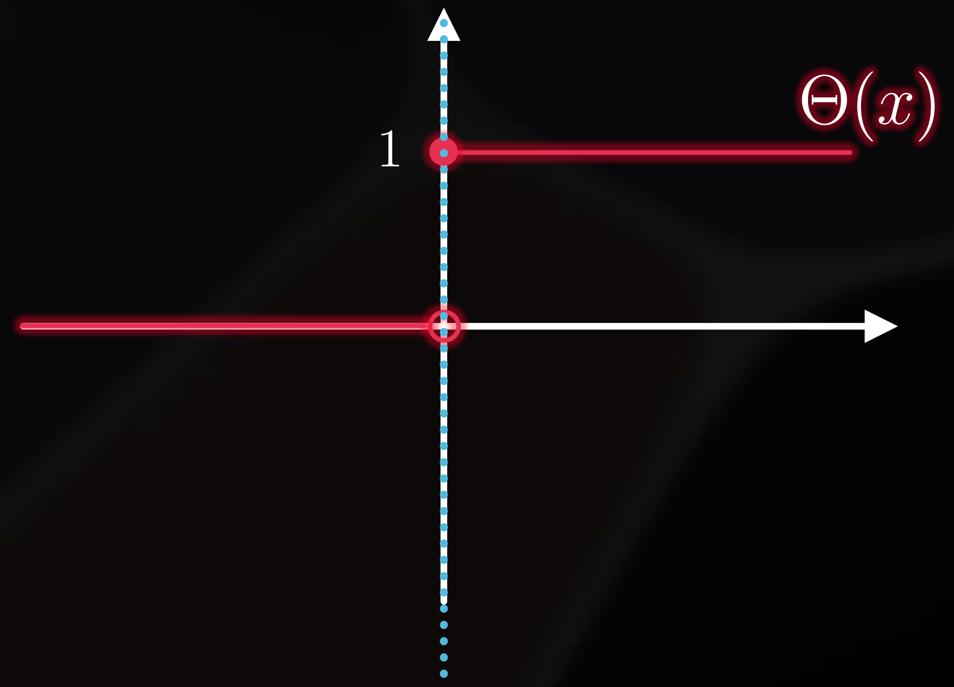
The Heaviside function $\Theta : \mathbb{R} \to \{0, 1\}$ is defined as

$$\Theta(x) := \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

```
noncomputable def Heaviside (x : ℝ) : ℝ :=
    if x < 0 then 0 else 1
```



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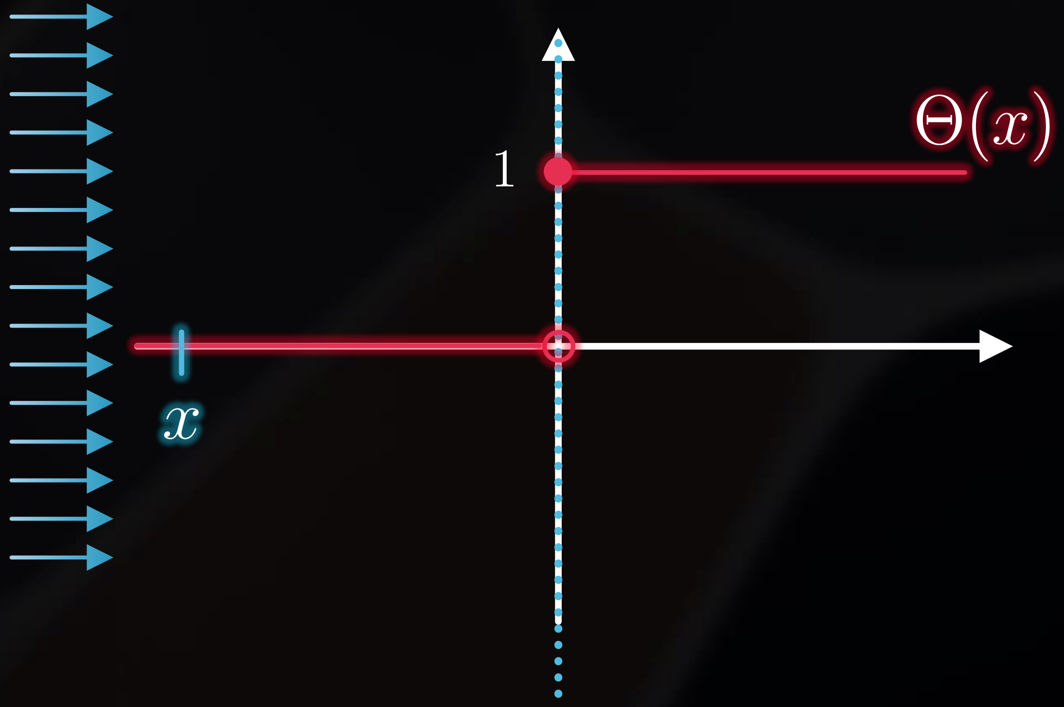

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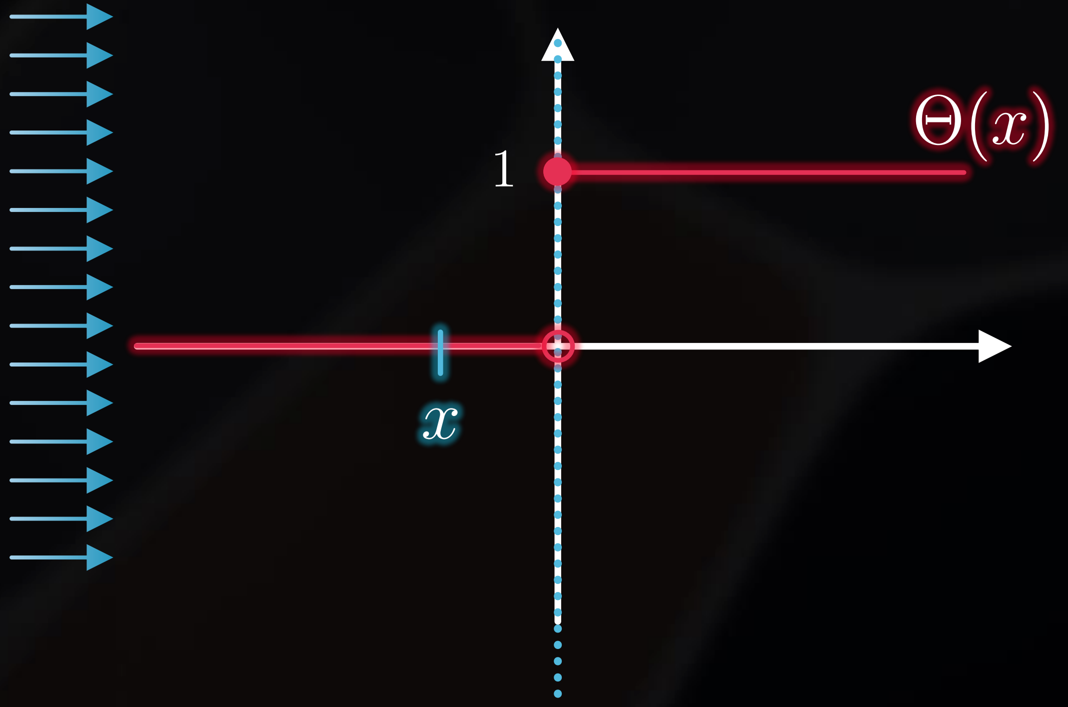

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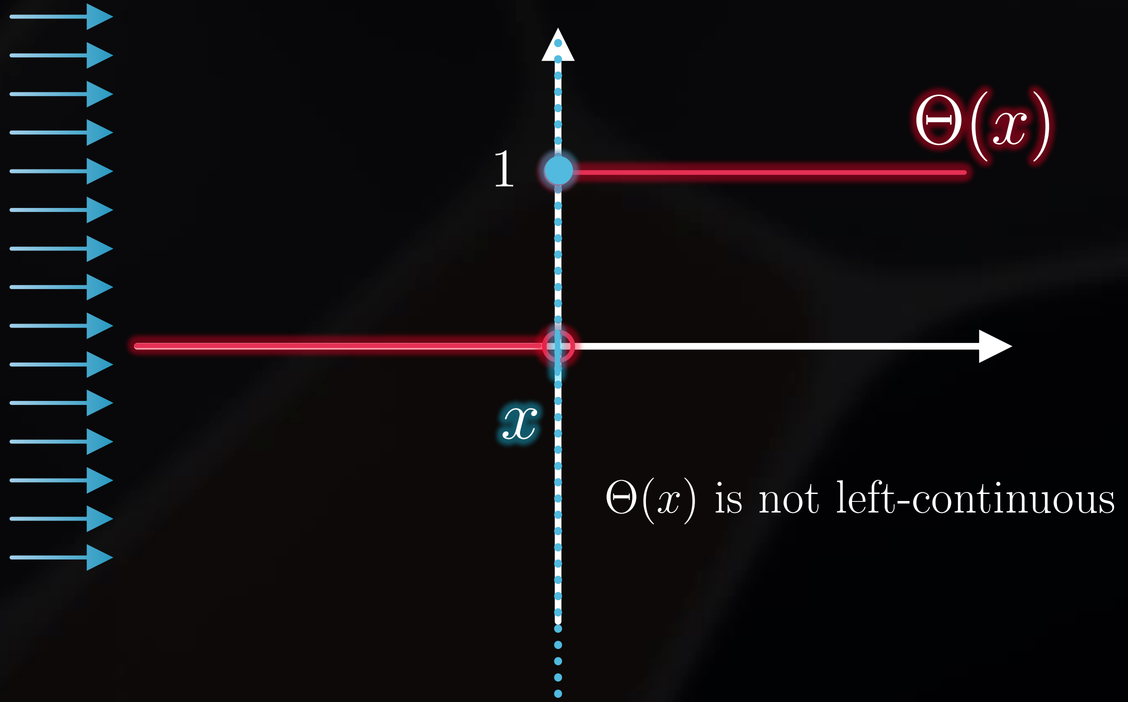

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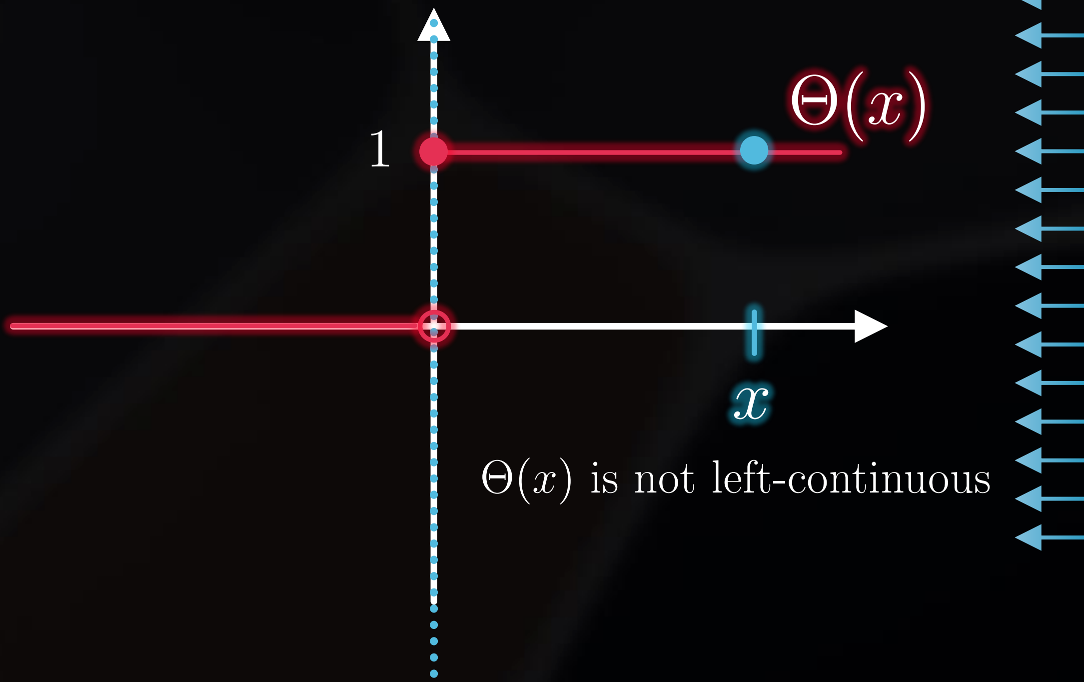

$\Theta(x)$ is not left-continuous

continuous at $x = 0$?

⇆ Left- and right-continuity

A function $f : D \to \mathbb{R}$
is *left-continuous at a point* $a \in D$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D, \overset{x > a}{\boxed{x < a}} : \left(|x - a| < \delta \; \Rightarrow \; |f(x) - f(a)| < \epsilon \right)$$

If f is left-continuous at every point in D, then we say that f is *left-continuous on* D.

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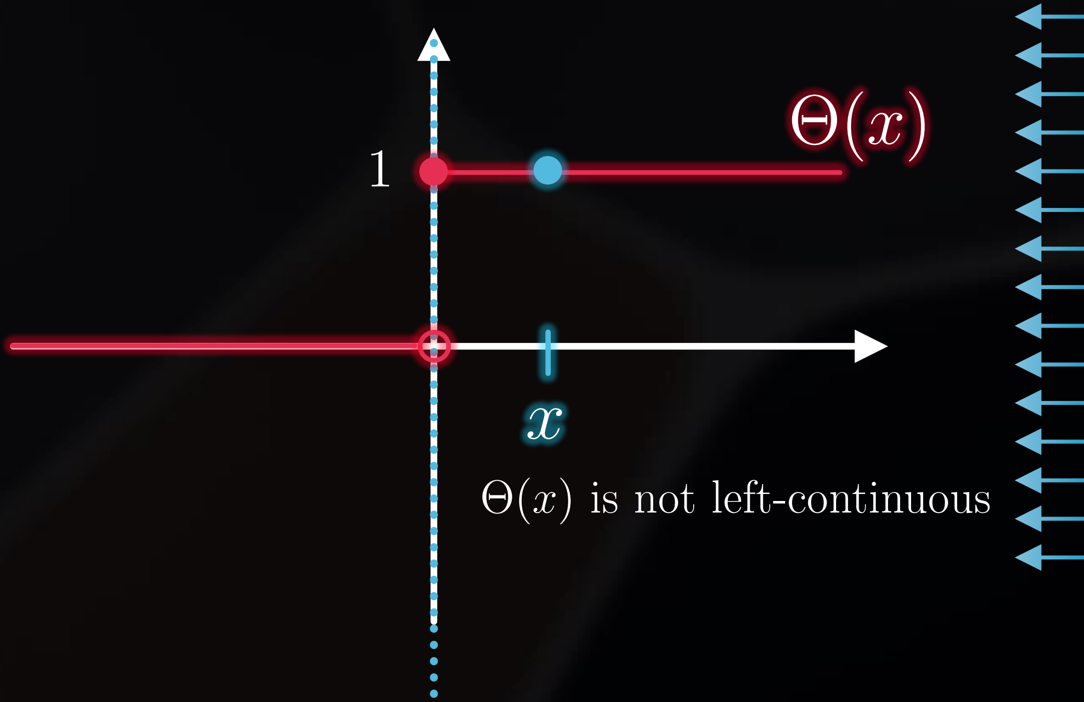

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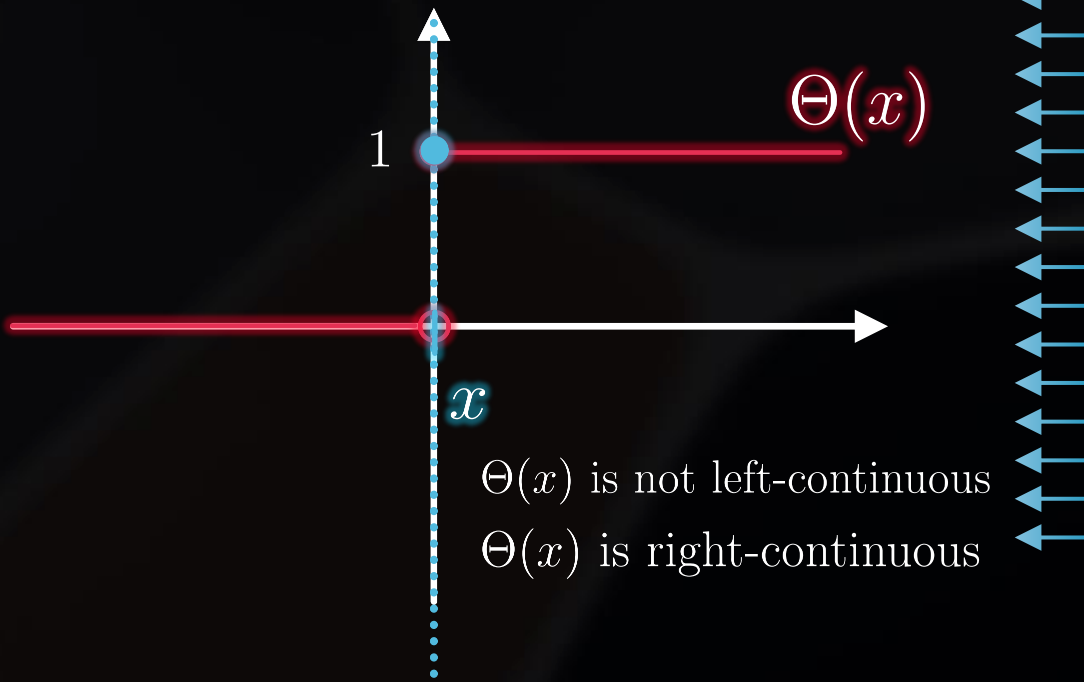

continuous at $x = 0$?

⇆ Left- and right-continuity

A function $f : D \to \mathbb{R}$
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$x > a$

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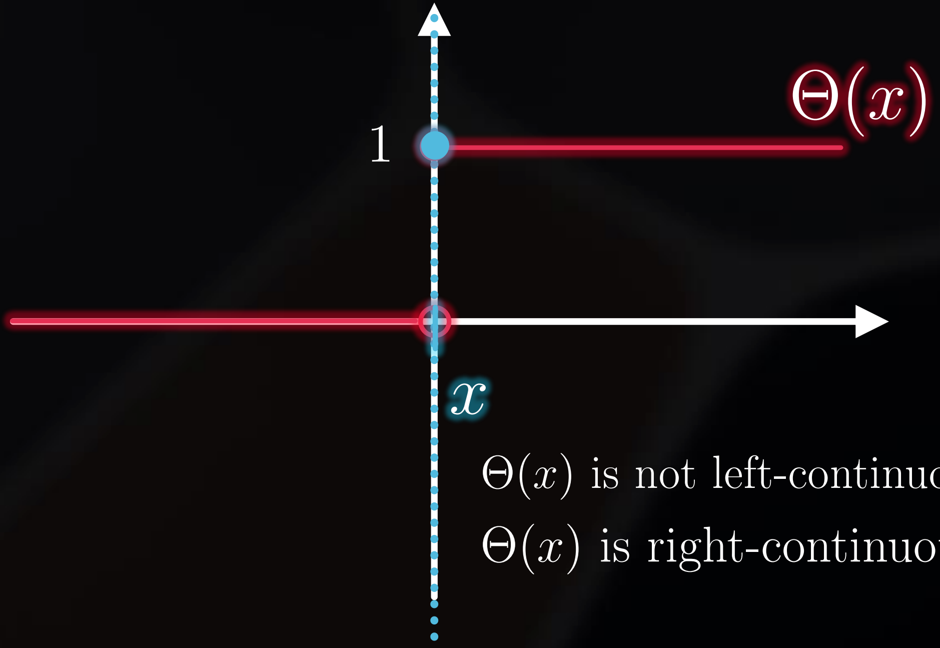

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$\Theta(x)$ is right-continuous

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⇆ Left- and right-continuity

Theorem

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at a point $a \in D$ if and only if it is both left-continuous and right-continuous at a.

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Let $a \in D$.

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constructor

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constructor

  -- Left side implies right side
  · intro h_continuous
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    -- Left-continuity
    · intro ε h_εbigger0
      obtain ⟨δ, h_δbigger0, h_δ⟩
          := h_continuous ε (by linarith)
      use δ
      use h_δbigger0
      intro x _h_x_smaller_a h_x_δ_criterion
      exact h_δ x h_x_δ_criterion
```

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    -- Right-continuity
    · …
```

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  · sorry
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$$\exists \delta_1 > 0 \quad \forall x \in D, x < a :$$
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$\leftrightarrows$ Left- and right-continuity

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    use min δ₁ δ₂
    use lt_min hδ₁ hδ₂
```

⇆ Left- and right-continuity

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Let $x \in D$ and $|x - a| < \delta$.

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    intro x h_x_δ_criterion
```

⇆ Left- and right-continuity

Let $a \in D$.

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$(|x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \epsilon)$
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Choose $\delta := \min(\delta_1, \delta_2)$.
Let $x \in D$ and $|x-a| < \delta$.

Due to our choice of δ, we also have $|x-a| < \delta_1$ and $|x-a| < \delta_2$.

- If $x < a$, then $|f(x) - f(a)| < \epsilon$ by left-continuity.

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    use min δ₁ δ₂
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    intro x h_x_δ_criterion

    by_cases h_a_value : x < a
    -- x < a (use left-continuity)
    · apply hδ₁_prop x h_a_value
      apply lt_of_lt_of_le h_x_δ_criterion
      apply min_le_left
```

Left- and right-continuity

Let $a \in D$.

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$$(|x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \epsilon)$$
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Left- and right-continuity

Let $a \in D$.

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    obtain ⟨δ₁, hδ₁, hδ₁_prop⟩
      := left_continuous ε (by linarith)
    obtain ⟨δ₂, hδ₂, hδ₂_prop⟩
      := right_continuous (ε) (by linarith)
    use min δ₁ δ₂
    use lt_min hδ₁ hδ₂

    intro x h_x_δ_criterion

    by_cases h_a_value : x < a
    -- x < a (use left-continuity)
    · apply hδ₁_prop x h_a_value
      apply lt_of_lt_of_le h_x_δ_criterion
      apply min_le_left

    -- x ≥ a
    · push_neg at h_a_value
      by_cases h_a_value' : x = a
      -- x = a
      · rewrite [h_a_value']
        simp [abs_zero, h_εbigger0]
```

⇆ Left- and right-continuity

Let $a \in D$.

$(\Leftarrow)$ Assume f is both left- and right-continuous at a.
Let $\epsilon > 0$.

$(|x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \epsilon)$
$(|x-a| < \delta_2 \Rightarrow |f(x) - f(a)| < \epsilon)$

Choose $\delta := \min(\delta_1, \delta_2)$.
Let $x \in D$ and $|x-a| < \delta$.

Due to our choice of δ, we also have $|x-a| < \delta_1$ and $|x-a| < \delta_2$.

- If $x < a$, then $|f(x) - f(a)| < \epsilon$ by left-continuity.
- If $x = a$, then $|f(x) - f(a)| = 0 < \epsilon$.
- If $x > a$, then $|f(x) - f(a)| < \epsilon$ by right-continuity.

```
constructor
  -- Left side implies right side
  · …

  -- Right side implies left side
  · …

    by_cases h_a_value : x < a
    -- x < a (use left-continuity)
    · apply hδ₁_prop x h_a_value
      apply lt_of_lt_of_le h_x_δ_criterion
      apply min_le_left

    -- x ≥ a
    · push_neg at h_a_value
      by_cases h_a_value' : x = a
      -- x = a
      · rewrite [h_a_value']
        simp [abs_zero, h_εbigger0]
```

Left- and right-continuity

Let $a \in D$.

$(\Leftarrow)$ Assume f is both left- and right-continuous at a.
Let $\epsilon > 0$.

$(|x - a| < \delta_1 \Rightarrow |f(x) - f(a)| < \epsilon)$

$(|x - a| < \delta_2 \Rightarrow |f(x) - f(a)| < \epsilon)$

Choose $\delta := \min(\delta_1, \delta_2)$.
Let $x \in D$ and $|x - a| < \delta$.

Due to our choice of δ, we also have $|x - a| < \delta_1$ and $|x - a| < \delta_2$.

- If $x < a$, then $|f(x) - f(a)| < \epsilon$ by left-continuity.
- If $x = a$, then $|f(x) - f(a)| = 0 < \epsilon$.
- If $x > a$, then $|f(x) - f(a)| < \epsilon$ by right-continuity.

```
constructor
  -- Left side implies right side
  · …

  -- Right side implies left side
  · …

    by_cases h_a_value : x < a
    -- x < a (use left-continuity)
    · apply hδ₁_prop x h_a_value
      apply lt_of_lt_of_le h_x_δ_criterion
      apply min_le_left

    -- x ≥ a
    · push_neg at h_a_value
      by_cases h_a_value' : x = a
      -- x = a
      · rewrite [h_a_value']
        simp [abs_zero, h_εbigger0]

      -- x > a (use right-continuity)
      · have h_x_bigger_a : x > a := by
          push_neg at h_a_value'
          exact lt_of_le_of_ne h_a_value
            (id (Ne.symm h_a_value'))

        apply hδ₂_prop x h_x_bigger_a
        apply lt_of_lt_of_le h_x_δ_criterion
        apply min_le_right
```

Left- and right-continuity

Theorem

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at a point $a \in D$ if and only if it is both left-continuous and right-continuous at a.

```
theorem LeftRightContinuousIffIsContinuous
    (D : Set ℝ) (f: D → ℝ) (a : D)
    : (IsContinuousAt D f a)
      ↔ (IsLeftContinuousAt   D f a
        ∧ IsRightContinuousAt D f a)
```

Left- and right-continuity

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⇆ Left- and right-continuity

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```

Outline

- Continuous Functions
- Examples
- Algebraic Properties
- Left- and right-continuity

Outline

Continuous Functions

Examples

Algebraic Properties

Left- and right-continuity

Comments

LaTeX document with "proof by hand" first

Formalization in Lean4 revealed mistakes in "proof by hand"

- Forgot $\forall\, \epsilon > 0$
- Inequality $<$ vs. $\leq$
- Silly mistakes $2x + 1 \neq 2 \cdot (x+1)$

Lean4: nice VSCode integration

Interactive goal solving.
Parser could be more stable locally-wise.

Lots of tactics to ease formalization

Some favorites include:
`simp`, `rewrite`, `use`, `linarith`, `ring_nf`, `field_simp`, `by_cases`, `splif_ifs`

Artificial intelligence

moogle.ai,
GitHub Copilot (GPT)
(also direct VSCode integration)

Thanks for listening!

Lean4: Programming language
and Theorem Prover

Thanks for listening!

Lean4: Programming language and Theorem Prover

You can become the next contributor!