PROSEMINAR / SEMINAR ON COMPUTER-ASSISTED MATHEMATICS

HEIDELBERG UNIVERSITY, SUMMER SEMESTER 2024

Target audience: Bachelor students.
Instructors: Judith Ludwig, Florent Schaffhauser.
Language of instruction: English.
Schedule: Tuesday, 16:00-18:00.

OBJECTIVES

Through a learning-by-doing approach, the purpose of this proseminar / seminar is to give an introduction to computer-assisted mathematics. It will be split into the following two topics:

(1) Linear algebra (in *SageMath*).
(2) Formal proofs (in *Lean 4*).

In the first part, we will for instance write a short program that asks the user to input the matrix of a linear transformation T in the canonical basis of $\mathbb{R}^2$ or $\mathbb{R}^3$, then input a family of vectors, check that it is a basis and, if so, compute the matrix of T in that new basis.

In the second part, the goal is to learn what it means to formalise a proof in order to ask a computer to check it, or even assist us with it. An example of goal for a class project would be: to implement the basic properties of Gaussian integers in Lean.

STRUCTURE AND CONTENT

This proseminar / seminar is mathematically-oriented and intended for beginners: there are no pre-requisites beyond a first course in linear algebra (essentially the Gaussian elimination algorithm and its applications) and familiarity with basic arithmetic. No previous knowledge of computer algebra systems or proof assistants will be required.

The proseminar / seminar will be project-based and students will work in groups. Those already familiar with the material or computer software can act as tutors. We will devote approximately seven weeks to each of the main topics (first linear algebra, then proof assistants). Active participation in the group meetings is expected from all students looking to obtain credits for the seminar. The schedule for each part will be organised as follows:

- Weeks 1-2: Presentation of the material and choice of projects.
- Weeks 3-6: Group meetings to discuss the understanding of the mathematical content and the progress of computer implementation.
- Weeks 7-8: Project presentations.

Linear algebra. The goal is to understand how *SageMath* can assist us with the following topics:

(1) Computer algebra systems. Representations of vectors and matrices.
(2) Row operations. Gaussian elimination. Row-reduced echelon form of a matrix.
(3) Invertible matrices. Elementary matrices. Determinant.
(4) Linear independence. Bases for the kernel and the image of a linear transformation.
(5) Rank-nullity theorem and the row space of matrix. Basis for the row space.
(6) Base change. Coordinates of a vector, matrix of a linear transformation.
(7) Eigenvalues and the characteristic polynomial. Diagonalisation.
(8) The Gram-Schmidt process. Least-square approximation.

Optional additional topics include:

- Gaussian reduction of quadratic forms.
- The Smith normal form (over $\mathbb{Z}$ or $\mathbb{K}[x]$, for $\mathbb{K}$ a field).
- The Jordan normal form.

Formal proofs. The goal is to complete the following steps:

(1) Installation of a proof assistant. Familiarisation with the interface.
(2) Play the natural number game (Peano axioms and the induction principle).
(3) Understand tactics to prove algebraic identities.
(4) Understand logical reasoning and proof tactics in *Lean 4*.
(5) Formalize a piece of mathematics.

REFERENCES

The instructors will provide a guided introduction to the required notions as the seminar progresses. No experience in programming will be required.