Representations of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$

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Introduction

In contrast to my talk, in which I discussed the representations of $\mathfrak{sl}_2(\mathbb{C})$, these notes will be focused on the $\mathfrak{sl}_n(\mathbb{C})$-case, the *special linear Lie algebra* over $n \times n$-matrices, with complex entries and trace zero. Even tough approaching this needs a few more abstract tools and methods of representation theory, than the 2×2-case, the concept is quite similar and should always be kept in mind. In the sense of a certain brevity there are a few inaccuracies, but these are usually marked as such. Especially the *universal enveloping algebra*, mentioned in section 3 is affected by this and I recommend using either [Ser92] or [Hal15] as references for filling those gaps.

If not explicitly defined otherwise, $\mathfrak{sl}_n$ will denote $\mathfrak{sl}_n(\mathbb{C})$ and all $\mathfrak{sl}_n$-modules are assumed to be finite dimensional.

Recall

Let V be a vector space, $X, Y \in \mathfrak{g}$

1. A representation of a Lie algebra is a linear map:

$$\pi : \mathfrak{g} \to \mathrm{End}(V) \quad \text{satisfying the relation} \quad \pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

2. The pair (V, π) is called a $\mathfrak{g}$-module.

By abuse of terminology one usually writes "V is a $\mathfrak{g}$-module" and $\pi(X) \cdot v = X \cdot v = Xv$ are used in an equivalent way. In some references the linear map, which effectively defines the representation is only implied or V itself is even referred to as the representation, but as it is possible to identify endomorphisms with matrices and they are not studied explicitly in many cases this small inaccuracy is not uncommon. Especially when studying well-understood Lie-algebras like $\mathfrak{sl}_2$ or $\mathfrak{sl}_n$ this does not pose a problem, since the representations are essentially identified by the eigenvalues or rather weights of H and $\mathfrak{h}$ respectively.

The canoncical basis for $\mathfrak{sl}_2(\mathbb{C})$ is given by:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

satisfying the bracket relations: $[X,Y] = H \qquad [H,X] = 2X \qquad [H,Y] = -2Y$.

As the given elements are, up to multiplication, the only three possible ways to define 2×2 matrices with trace zero, their choice arises naturally, but, as we will see, there is a much more methodical approach to defining the basis of $\mathfrak{sl}_n$ for any n. This will be the first step we take.

1 Constructing a basis for $\mathfrak{sl}_n$

In order to do this we shall not discuss all matrices with trace zero explicitly (as we did in the $\mathfrak{sl}_2$-case) but rather classify them with regards to the position of their entries and then analyse the three subalgebras of $\mathfrak{sl}_n$ generated by them:

$$\begin{aligned}
\mathfrak{h} &= \text{Lie algebra of diagonal matrices } H = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \text{ with } \Sigma\ \lambda_i = 0, \lambda_i \in \mathbb{C} \\
\mathfrak{x} &= \text{Lie algebra of superdiagonal matrices} \\
\mathfrak{y} &= \text{Lie algebra of infradiagonal matrices.}
\end{aligned}$$

$\mathfrak{sl}_n$ may then be decomposed into their direct sum[1]: $\mathfrak{sl}_n = \mathfrak{h} \oplus \mathfrak{x} \oplus \mathfrak{y}$.
Note that $\mathfrak{h}$ is the cartan subalgebra of $\mathfrak{sl}_n$ (hence abelian[2]), $\mathfrak{x}$ resp. $\mathfrak{y}$ are nilpotent and $\mathfrak{h} \oplus \mathfrak{x}$ is the canonical borel algebra[3].

1.1 Roots of $\mathfrak{sl}_n$

Definition 1.1
Let $\mathfrak{h}^*$ be the dual[4] of $\mathfrak{h}$, then elements $\chi \in \mathfrak{h}^*$ are of the form: $\chi = \sum_{i=1}^{n} u_i \lambda_i$, with $u_i \in \mathbb{C}$ and λ_i being the entries of some $h \in \mathfrak{h}$.

(i) A linear form $\alpha = \lambda_i - \lambda_j$ $(i < j)$ is called *root*

(ii) The set of *positive roots* is denoted $R_+ = \{\alpha \in \mathfrak{h}^* \mid \alpha = \lambda_i - \lambda_j\ ,\ (i < j)\}$

(iii) The set of *roots* $R = R_+ \cup (-R_+)$

(iv) Positive roots of the form: $\alpha_i = \lambda_i - \lambda_{i+1}$ are called *fundamental roots.*

Using the previous definitions we will now construct more explicit classes of matrices, which will then prove useful to find bases of the subalgebras of $\mathfrak{sl}_n$, making it possible to study them in a way, similar to the case of $\mathfrak{sl}_2$.

Definition 1.2
Let $\alpha = \lambda_i - \lambda_j \in R, (i \neq j)$ and $H_\alpha, X_\alpha \in \mathfrak{sl}_n$

$X_\alpha := X_{(i,j)} = 1$ and zero elsewhere

$H_\alpha :=\ H \in \mathfrak{h}$ with entries $H_{(i,i)} = 1, H_{(j,j)} = -1$ and zero elsewhere

Proposition 1.3.

(i) The X_α's make a basis of $\mathfrak{x}$ and the $X_{-\alpha}$'s make a basis of $\mathfrak{y}$

(ii) If $H \in \mathfrak{h}, \alpha \in R$ then $[H, X_\alpha] = \alpha(H) X_\alpha$

(iii) $[X_\alpha, X_{-\alpha}] = H_\alpha$

Proof.

(i) The way the X_α and $X_{-\alpha}$ are defined, the claim follows directly from the conditions $i < j$ and $i \neq j$.

(ii) Consider $(\lambda_1, \dots, \lambda_n)$, the entries on the diagonal of H and $\alpha = \lambda_i - \lambda_j$. Due to the definition of H and X_α we know: $H \cdot X_\alpha = \lambda_i \cdot X_\alpha$ and $X_\alpha \cdot H = \lambda_j \cdot X_\alpha \implies [H, X_\alpha] = (\lambda_i - \lambda_j) X_\alpha = \alpha(H) X_\alpha$

(iii) $X_\alpha \cdot X_{-\alpha}$ yields a matrix A with $(A_{i,i}) = 1$ and $X_{-\alpha} \cdot X_\alpha = A'$ with $A'_{(j,j)} = 1$. Then we get: $A - A' = H_\alpha$ □

Remark
The statement $\alpha(H_\alpha) = 2$ is always true.

Example 1.4
Returning to the the case of $\mathfrak{sl}(2; \mathbb{C})$ we only have one positive root: $\alpha = \lambda_1 - \lambda_2 = 2$, so H is unique and we can construct the canonical basis:

$$H_\alpha = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_\alpha = X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad X_{-\alpha} = Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[1] Using the concept of *rootspace decomposition* (talk 5) it's possible to construct the following decomposition "from scratch" so to say, but for my talk I will assume it as given
[2] Cf. [Ser87] Ch.3.5 Th.3
[3] The maximal solvable subalgebra
[4] The space of linear forms $\chi : \mathfrak{h} \to \mathbb{C}$

2 Weights and primitive elements

Similar to the way we studied $\mathfrak{sl}_2$ in my talk we will now be analysing the eigenvalues, weights and primitive elements of $\mathfrak{sl}_n$-modules, to gain insight of their structure.

Definition 2.1
Let V be a finite dimensional $\mathfrak{sl}_n$ module, $v \in V,\ \chi \in \mathfrak{h}^*,$ and $H \in \mathfrak{h}$

(i) For any χ we denote the corresponding space of simultaneous eigenvectors (i.e. $H \cdot v = \chi(H) \cdot v \ \ \forall H \in \mathfrak{h}$) as V_χ and call it *weight space*

(ii) The elements of the weightspace V_χ are said to have *weight* χ

(iii) Elements χ with non-empty weight-space V_χ are called *weights* of V

(iv) The dimension of V_χ is called *multiplicity of* χ

Proposition 2.2. *Let $\chi \in \mathfrak{h}^*,\ v \in V_\chi,\ \alpha \in R$.
If v has weight χ, then $X_\alpha v$ has weight $\chi + \alpha$*

Proof. This is a simple calculation:
$HX_\alpha v = [H, X_\alpha]v + X_\alpha H v = \alpha(H)X_\alpha v + \chi(H)X_\alpha v = (\alpha + \chi)(H)X_\alpha v \implies X_\alpha v \in V_{\chi+\alpha}$ □

Proposition 2.3. *The module V is a direct sum of weightspaces V_χ:*

$$V = \bigoplus_{\chi \in \mathfrak{h}^*} V_\chi$$

Proof. The eigenvectors corresponding to distinct eigenvalues are linearly independent, hence the sum of all weightspaces is direct. We also know that the module V' generated by the sum is stable under $\mathfrak{sl}_n$, due to it being stable by the X_α's and $\mathfrak{h}$. This yields: $V' \subseteq V$. Assume now, that there exists a different, non-zero V'', such that $V' \oplus V'' = V$:
With $\mathfrak{h}$ being abelian[5] and $\mathbb{C}$ being algebraically closed we know that V'' contains an eigenvector $v \neq (0)$ of $\mathfrak{h}$, which by definition should be contained in some V_χ, contradicting the assumption: $V'' \cap V' = 0$. This implies $V' = V$ □

Definition 2.4
$e \in V \backslash \{0\}$ is called *primitive element* if and only if e is an eigenvector to $\mathfrak{h}$ and $X_\alpha e = 0\ \forall \alpha \in R_+$

Proposition 2.5. *Any non-zero $\mathfrak{sl}_n$-module contains a primitive element*

Proof. If $V \neq 0$ we know that the set of weights is non-empty and finite, hence there are elements χ which can not be "moved" to a space of higher weight. These elements are primitive elements (if they are non-zero)[6]. □

Observation
By applying X_α to elements of a certain weight we can "move" them to a space of higher (or lower) weight. This is the first instance, in which we can (albeit still rather heuristical) observe that if any module contains weights with respect to $\mathfrak{h}$ and is stable under the action of X_α it must contain elements of every possible weight. This is one of the goals of the following section.

[5]meaning the elements are simultaneous diagonalizable
[6]Alternatively this can be proved using Lie's theorem (Talk 1?)

3 Irreducible $\mathfrak{sl}_n$ modules

To gain better insight on how to distinctly classify representations of $\mathfrak{sl}_n$, we are going to study modules, that are generated by primitive elements. These are again very similar to the ones mentioned in my presentation, but as before, we need another, more general concept which will be introduced in the following. As mentioned in the introduction, this is an almost philosophical explanation of the idea, rather than a rigorous definition[7].

3.1 The universal enveloping algebra

Definition 3.1
A universal enveloping algebra $(U\mathfrak{g}, \pi)$ of $\mathfrak{g}$ is a pair of an associative algebra with unit, and a linear map π, satisfying the following properties:

1. $\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)\ \forall X, Y \in \mathfrak{g}$
2. $U\mathfrak{g}$ is π-invariant and especially generated by elements $\pi(X)$ $(X \in \mathfrak{g})$, in the sense that there is no Lie algebra, properly contained in $U\mathfrak{g}$ which also contains every $\pi(X)$.
3. For every other associative Lie algebra $\mathfrak{a}$ with unit and a linear map ρ, which satisfies the given "commutator condition", there exists a homomorphism $\phi : U\mathfrak{g} \to \mathfrak{a}$ such that $\phi(1) = 1$ and $\phi(\pi(X) = \rho(X)$.

Fact 3.2
The representations of $U\mathfrak{g}$ correspond to those of $\mathfrak{g}$.

Example 3.3
Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the basis as defined above. The universal algebra $U\mathfrak{g}$ is then given by the associative algebra with unit, generated by three elements x, y, h, satisfying *only* the relations:

$$\begin{aligned} hx - xy &= 2x \\ hy - yh &= -2y \\ xy - yx &= h \end{aligned}$$

and a linear map $\pi : \mathfrak{sl}_n \to U\mathfrak{g}$, such that $\pi(x) = X,\ \pi(y) = Y,\ \pi(H) = h$

3.2 Modules generated by primitive elements

Let V be an arbitrary $\mathfrak{sl}_n$-module, $e \in V_\chi$ a primitive element and $V_1 = (U\mathfrak{sl}_n) \cdot e$ the module generated by e.

Fact 3.4
The weights of V_1 are of the form $\chi - \sum_{i=1}^{n-1} m_i\alpha_i$ $(m_i \geq 0)$.

Proof. We will not prove this rigorously but it follows from the fact, that $U\mathfrak{sl}_n$ can be decomposed into the tensor product of the universal algebra of $\mathfrak{y}$ and the borel-algebra $\mathfrak{b}$, and that the universal algebra of $\mathfrak{y}$ is generated by *monomials*. Cf. [Ser92] Ch.7, Th. 3.1 □

Theorem 3.5.

(i) Any primitive element $v \in V_1$ of weight χ is a multiple of e

(ii) V_1 is irreducible

Proof.

(i) Follows from the proof by construction of 3.4.

(ii) Suppose: $V_1 = V' \oplus V''$, and $v = v' + v''$.
Consider the weightspace $(V_1)_\chi = V'_\chi \oplus V''_\chi$ which implies that v' and v'' are both of weight χ. As we know from (i) they must be multiples of v, hence, one must be zero (we will choose $v'' = 0$). We then get: $v' = v \implies V' = V_1 \implies V'' = 0$

□

Theorem 3.6. *Let V, V', V'' be irreducible $\mathfrak{sl}_n$-modules*

(i) V contains a unique primitive element (up to multiplication by elements of $\mathbb{C}$). The weight of this element is specified to be the highest weight of V.

(ii) If V', V'' have the same highest weight, they are isomorphic.

[7] Cf. [Hal15] Sect. 9.3

Proof.

(i) V contains at least one primitive element (2.5). Let v, v' be primitive elements with weight χ and χ' respectively.
From 3.5 we have:

$$\chi - \chi' = \sum_{i=1}^{n-1} m_i \alpha_i \ , \tag{1}$$

$$\chi' - \chi = \sum_{i=1}^{n-1} m_i' \alpha_i \qquad (m_i, m_i' \geq 0 \forall i) \tag{2}$$

which implies $m_i = m_i' = 0 \implies \chi = \chi'$. The scalar multiplicity follows directly from 3.5.(ii).

(ii) Let $v' \in V'$ and $v'' \in V''$ be the respective primitive elements, each of weight χ. Consider $V' \oplus V''$ and the corresponding primitive element $v = (v', v'')$, which is also of weight χ. The $\mathfrak{sl}_n$-submodule W of $V' \times V''$ generated by v is irreducible (3.5) and the projection map $\pi_i : W \to V_i$ is non-zero. According to Schur's Lemma such an homomorphism between irreducible Lie algebra modules is either an isomorphism or zero, hence V', V'' are both isomorphic to W and therefore $V' \cong V''$.

□

4 Classification of Irreducible Modules

To classify all irreducible $\mathfrak{sl}_n$ modules the only thing left to discuss after stating Theorem 3.6 is to find a way of determining the highest weight of an arbitrary, irreducible $\mathfrak{sl}_n$-module.
Let $\chi \in \mathfrak{h}$, then $\chi(\lambda_1, \ldots, \lambda_n) = u_1\lambda_1 + \ldots + u_n\lambda_n$.

Theorem 4.1. *An irreducible $\mathfrak{sl}_n-$module with highest weight χ exists if and only if the difference of coefficients u_i and u_j is a positive integer for all $i < j$*

To prove this theorem there is a bit of groundwork to do first:

Proof of necessity

Let V be an irreducible $\mathfrak{sl}$-module with primitive element e of weight χ. We know, that there is an H_α such that $u_i - u_j = \chi(H_\alpha)$ for the positive root $\alpha = \lambda_i - \lambda_j \in R_+$. Using this it suffices to prove that $\chi(H_\alpha)$ is an integer under the given conditions. First off we will prove, that since V is an $\mathfrak{sl}_n$ module the same (or at least similar) formulas will hold, as we did when studying $\mathfrak{sl}_2$:

Lemma 4.2. *Let V be an irreducible $\mathfrak{sl}_n$-module, $e_0 \in V_\chi$ a primitive element and $e_m^\alpha = (\frac{1}{m!})X_{-\alpha}^m \cdot e_0$ then:*

(i) $H \cdot e_m^\alpha = (\chi - m\alpha)(H)e_m^\alpha$

(ii) $X_{-\alpha} \cdot e_m^\alpha = (m+1)e_{m+1}^\alpha$

(iii) $X_\alpha \cdot e_m^\alpha = (\chi(H_\alpha) - m + 1)e_{m-1}^\alpha$

Proof.

(i) This formula basically states the fact, that $e_m^\alpha \in V_{\chi - m\alpha}$, which directly follows from the way $X_{-\alpha}$[8] acts on elements in V_χ

(ii) $X_{-\alpha}e_m^\alpha = X_{-\alpha} \cdot \frac{1}{m!}X_{-\alpha}^m \cdot e_0 = (m+1)\frac{1}{(m+1)!}X_{-\alpha}^{m+1}e_0 = (m+1)e_{m+1}^\alpha$

(iii) This is proved via induction on m:
For m=0 the formula holds[9].

$$\begin{aligned} m \cdot X_\alpha e_m^\alpha &= X_\alpha X_{-\alpha} e_{m-1}^\alpha \\ &= [X, Y]e_{m-1}^\alpha + X_{-\alpha}X_\alpha e_{m-1}^\alpha \\ &= (\chi(H_\alpha) - (m-1)\alpha(H_\alpha))e_{m-1}^\alpha + (m-1)(\chi(H_\alpha) - m + 2)e_{m-1}^\alpha \\ &= m(\chi(H_\alpha) - m + 1)e_{m-1}^\alpha. \end{aligned}$$

The last step uses the earlier remark: "$\alpha(H_\alpha) = 2$ is always true".

□

[8] Taking up the role Y had in the $\mathfrak{sl}_2$-case
[9] by convention $e_{-1} = 0$

Observation
As any module V is assumed to be finite dimensional, the number of possible weights is finite as well, hence there must be an integer m, such that $e^{\alpha}_{m+1} = 0$.
If we combine this observation with formula (iii) we get:

$$X_\alpha e^{\alpha}_{m+1} = 0 = (\chi(H_\alpha - m)e^{\alpha}_{m} \implies \chi(H_\alpha) = m.$$

Proof of sufficiency

Now we need to prove, that there is a $\mathfrak{sl}_n$-module of highest weight χ, under the assumption that the pairwise difference of all coefficients of χ is an positive integer.
We rewrite the definition of χ by introducing linear forms $\pi_1, \dots, \pi_{n-1}$ with $\pi_i = \sum_{k=1}^{i} \lambda_k$ and integers $m_1, \dots, m_{n-1}$:

$$\chi = \sum_{i=1}^{n-1} m_i \pi_i$$

Proposition 4.3. *Let χ, χ' be the highest weights of modules V and V', then:*

(i) $\chi + \chi'$ is the highest weight of an irreducible module $W \subseteq V \otimes V'$.

(ii) The set of highest weights is closed under addition

Proof. Let v, v' be the primitive elements of V, V' and corresponding weight χ, χ'.

(i) If v and v' are the primitive elements of V and V', then $v \otimes v'$ is a primitive element of $V \otimes V'$ of weight $\chi + \chi'$. Due to (Th.3.5) the submodule generated by $v \otimes v'$ is an irreducible $\mathfrak{sl}$-module with highest weight $\chi + \chi'$

(ii) this follows from (i)

□

We can apply this last proposition to the π_i's we defined earlier and prove that they are the highest weights of their respective submodules.

Proposition 4.4. *Let V be $\mathbb{C}^n$, and consider it a $\mathfrak{sl}_n$-module, with V_i being the i-th exterior Power of V $(1 \leq i \leq n-1)$. Then V_i is an irreducible $\mathfrak{sl}_n$-module of highest weight π_i*

Proof. The canonical basis of V is given by $e_1, \dots, e_n$ and we define $v_i = e_1 \wedge \dots \wedge e_i$, which is a primitive element of V_i, with weight π_i. For proving the irreducibility of V_i we apply monomials of the $X_{-\alpha}$'s to v_i, allowing us to obtain any term of the form $e_{m_1} \wedge \dots \wedge e_{m_i}$ which also concludes the proof of Theorem 4.1 □

I finished my presentation by proving, that the adjoint representation of $\mathfrak{sl}_2$ with dimension $2+1$ is isomorphic to the 2-nd symmetric power of the underlying vector space $\mathbb{C}^2$, and remarking, that this holds for all $m+1$ dimensional representations being isomorphic to the m-th symmetric power of $\mathbb{C}^2$. But from this last chapter we can draw an even stronger conclusion: The classes of irreducible $\mathfrak{sl}_n$-modules are in one-to-one correspondence with systems of $(n-1)$ integers: $(m_1, \dots, m_{n-1}$. In the case of $n = 2$ we only had one integer m, which made it possible to uniquely identify each module just by identifying the eigenvector with the highest corresponding eigenvalue.

References

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