Classification of Nilpotent Orbits

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1 Introduction

The goal of this exposition is to prove that the number of nilpotent orbits in a lie algebra $\mathfrak{g}$ is finite. We begin by showing there is a one to one correspondence between certain conjugacy classes of the adjoint action and nonzero nilpotent orbits. We further show that there is also a one to one correspondence between the aforementioned conjugacy classes and certain semisimple orbits, these semisimple orbits will ultimately be shown to be finite via their corresponding weighted Dynkin diagrams.

From now on, we let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ it's Cartan subalgebra.

Definition 1.1. *A* ***standard triple*** *in* $\mathfrak{g}$ *is a three dimensional subalgebra isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*.*

Remark 1.2. *Any root vector* X_α *for a given root* α *yields a standard triple,denoted* $\{H_\alpha, Y_\alpha, X_\alpha\}$ *such that*

- $[H_\alpha, Y_\alpha] = -2Y_\alpha$
- $[H_\alpha, X_\alpha] = 2X_\alpha$
- $[X_\alpha, Y_\alpha] = H_\alpha$

Definition 1.3. *We denote by* $\mathcal{A}_{triple}$ *the set of* G_{ad} *conugacy classes of standard triples in* $\mathfrak{g}$*.*
Define : $\mathcal{A}_{triple} \longrightarrow \{$*non zero nilpotent orbits in* $\mathfrak{g}$ $\}$ *via* $\Omega(\{H, X, Y\}) = \mathcal{O}_X$

2 Bijectivity of Ω

2.1 Surjectivity

Theorem 2.1. *(Jacobson-Morozov) if* $\mathfrak{g}$ *is complex semisimple,* X *a nonzero nilpotent element then there exists a standard triple* $\{H, X, Y\}$ *such that* X *corresponds to* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$*. We call* X *the nilpositive element.*

Proof. By induction on the dimension of $\mathfrak{g}$. If $dim\mathfrak{g} = 3$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and the result holds true. Assume now that the dimension is greater than three and that X does not lie in a proper subalgebra of $\mathfrak{g}$.
denoting by κ the Killing form of $\mathfrak{g}$, we recall that $\kappa(X, \mathfrak{g}^X) = 0$. This implies that $X \in (\mathfrak{g}^X)^\perp = [\mathfrak{g}, X]$.
Choose an $H' \in \mathfrak{g}$ such that $[H', X] = 2X$, this H' can be assumed to be semisimple, since any nilpotent element acting on $\mathbb{C}X$ has eigenvalues 0 and the Jordan decomposition of $H' = H'_s + H'_n$ implies that $[H'_S, X] = 2X]$ and $[H'_n, X] = 0$.
Picking $H = H'_s$, we claim that $H \in [\mathfrak{g}, X]$

Proof. of claim: assume $H \notin [\mathfrak{g}, X]$ i.e $\kappa(H, \mathfrak{g}^X) \neq 0$. We consider the eigenspace decomposition of $\mathfrak{g}^X$, this exists due to the facts that ad_H leaves $\mathfrak{g}^X$ invariant and acts semisimply, denote this decomposition as

$$\mathfrak{g}^X = \mathfrak{g}_0^X \oplus \bigoplus \mathfrak{g}_{\tau_i}^X \tag{1}$$

notice that $g_0^X = \{Z \in \mathfrak{g}^X | [H, Z] = 0\}$, allowing us to rewrite the decomposition

$$\mathfrak{g}^X = (\mathfrak{g}^X)^H \oplus \bigoplus \mathfrak{g}_{\tau_i}^X. \tag{2}$$

We can find some $Z \in (\mathfrak{g}^X)^H$ such that $\kappa(H, Z) \neq 0$, this follows from the fact that the Killing form is nondegenarte and that for any $Z_i \in \mathfrak{g}_{\tau_i}$

$$0 = \kappa(H, [H, Z_i]) = \kappa(H, \tau_i Z_i) = \tau_i \kappa(H, Z_i). \tag{3}$$

It is evident that given such a $Z \in (\mathfrak{g}^X)^H$, the semisimple part Z_s of Z is not trivial, since otherwise Z is nilpotent and therefore $\kappa(H, Z) = 0$.
By [1], we know that $\mathfrak{g}^{Z_s}$ is reductive and therefore $[\mathfrak{g}^{Z_s}, \mathfrak{g}^{Z_s}]$ is a semisimple proper subalgebra of $\mathfrak{g}$.
This is ultimately a contradiction to X not being in any proper subalgebra of $\mathfrak{g}$, observe that both $H, X \in \mathfrak{g}^{Z_s}$, i.e $[H, X] = 2X \in [\mathfrak{g}^{Z_s}, \mathfrak{g}^{Z_s}]$.
In conclusion, $H \in [\mathfrak{g}, X]$, which proves our claim. $\square$

Pick some $Y \in \mathfrak{g}$ such that $H = [X, Y]$ and consider the eigenspace decomposition of $\mathfrak{g} = \bigoplus \mathfrak{g}_{\lambda_i}$. This allows us to write $Y = \bigoplus Y_i$ for $Y_i \in \mathfrak{g}_{\lambda_i}$.
Keeping in mind that $H \in \mathfrak{g}_0$, and by the theory of $\mathfrak{sl}_2(\mathbb{C})$

$$H = [X, Y] = [H, Y_{-2}] \implies [H, Y] = -2Y. \tag{4}$$

Therefore $\{H, X, Y\}$ is a standard triple and the map Ω is surjective. $\square$

2.2 Injectivity

Remark 2.2. • *The restriction of κ on $\mathfrak{g}^H$ is non degenerate*

- $\mathfrak{g}^X$ *admits a decomposition of the form* $\bigoplus_{i\geq 0} \mathfrak{g}_i^X$

Lemma 2.3. *Any two standard triples H, X, Y and H, X, Y' with identical H, X have identical Y.*

Proof. $Y - Y'$ is evidently in the -2-eigenspace of the ad_H, however it also in $\mathfrak{g}^X$, since $[X, Y - Y'] = [X, Y] - [X, Y'] = H - H = 0$. By the remark above we know that $\mathfrak{g}^X \cap \mathfrak{g}_{-2} = \{0\}$. This completes the proof □

The following results and definitions are important for the injectivity of Ω, however are above the scope of this presentation and deal with the theory of Lie groups and their geometric aspects, a detailed account of the theory can be found in [2]

Lemma 2.4. *let X be a nonzero nilpotent element of a semisimple $\mathfrak{g}$, defining $\mathfrak{u}^X = \mathfrak{g}^X \cap [\mathfrak{g}, X]$, it is an ad_H-invariant nilpotent idea lof $\mathfrak{g}^X$ and $\mathfrak{u}^X = \bigoplus_{i>0} \mathfrak{g}_i^X$.*

Lemma 2.5. *Denoting with U^X the lie algebra of u^X, there is a diffeomorphism between $\mathfrak{u}^X$ and U^X and $H + \mathfrak{u}^X = U^X H$*

Theorem 2.6. *(Kostant) For any two standard triples $\{H', X, Y'\}$ and $\{H, X, Y\}$ with the same nilpositive element X there exists an $x \in U^X$ such that $xH = H', xX = X, xY = Y'$, i.e Ω is injective.*

3 The Distinguished Semisimple Orbits

Definition 3.1. *Define the map $\mathcal{Y} : \mathcal{A}_{triple} \longrightarrow \{$**semisimple orbits**$\}$ via $\mathcal{Y}(\{H, X, Y\}) = \mathcal{O}_H$*

Definition 3.2. *We denote by $S_{dist} = Image(\mathcal{Y})$ the **distinguished semisimple orbits***

Theorem 3.3. *The map $\mathcal{Y}$ is a one to one correspondence between $\mathcal{A}_{triple}$ and S_{dist}*

3.1 $\mathcal{Y}$ is injective

Theorem 3.4. *(Mal'cev) Any two standard triples with the same neutral element H are conjugate by an element of $(G_{ad}^H)^\circ$*

Proof. Denote the two standard triples with $\{H, X, Y\}$ and $\{H, X', Y'\}$ and write $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$.

Define $\rho = \{Z \in \mathfrak{g}_2 | \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0\}$ By the following computation it suffices to show that $(G_{ad}^H)^\circ$ acts transitively on ρ:

$$\begin{aligned} Z \in \rho &\iff \mathfrak{g}^Z \cap \mathfrak{g}_{-2} = 0 \iff x(\mathfrak{g}^Z \cap \mathfrak{g}_{-2}) \iff \mathfrak{g}^{xZ} \cap x\mathfrak{g}_{-2} = 0 \\ &\iff \mathfrak{g}^{xZ} \cap \mathfrak{g}_{-2} = 0 \iff xZ \in \rho \end{aligned}$$

claim 1: ρ is path connected and open in $\mathfrak{g}_2$

Proof. of claim 1: Define $T : \mathfrak{g}_2 \longrightarrow Hom(\mathfrak{g}_{-2}, \mathfrak{g}_0)$ via $T(Z) = ad_Z$.
We notice that $Image(T(z)) = [Z, \mathfrak{g}_{-2}]$ and $Kernel(T(z)) = \mathfrak{g}_{-2} \cap \mathfrak{g}^Z$, thus

$$\begin{aligned} Z \in \rho &\iff \mathfrak{g}_{-2} \cap \mathfrak{g}^Z = 0 \iff Kernel(T(z)) = 0 \\ &\iff dim(Image(T(z)) = dim(\mathfrak{g}_{-2}) \end{aligned}$$

In conclusion, $T(Z)$ must have full rank and is thus Zariski open in $\mathfrak{g}_{-2}$.
It is path connected due to the fact that $\mathbb{C} \setminus \{a_1, ..., a_n\}$ is path connected and there are only finitely many λ such that $\lambda A + (1 - \lambda)B \notin \rho$. $\square$

claim 2: Each $(G_a d^H)^\circ$ orbit in ρ is open and closed.

Proof. of claim 2: if $Z \subset \mathfrak{g}_{-2}$, then $[\mathfrak{g}^Z, Z] = 0$, thus

$$0 = \kappa([\mathfrak{g}^Z \cap \mathfrak{g}_0, Z, \mathfrak{g}_{-2}] = \kappa(\mathfrak{g}^Z \cap \mathfrak{g}_0, [Z, \mathfrak{g}_{-2}].$$

Meaning $[Z, \mathfrak{g}_{-2}] \in (\mathfrak{g}^Z \cap \mathfrak{g}_0)^\perp \cap \mathfrak{g}_0$, allowing us to compute that

$$dim((G_a d^H)^\circ Z) = dim(\mathfrak{g}_0 - dim(\mathfrak{g}_0) \cap \mathfrak{g}^Z) \geq dim(\mathfrak{g}_0) - dim(\mathfrak{g}_0) + dim(\mathfrak{g}_2) = dim(\mathfrak{g}_2)$$

where the last inequality follows from the dimension formula for vectorspaces and the computation above.
In conclusion, every orbit in ρ has full dimension, i.e it is open. Since the complement of an orbit is the union of orbits, they are also closed. $\square$

ρ is therefore a single orbit since it is path connected and the proof is complete. $\square$

4 Weighted Dynkin Diagrams

Fix a positive root system Φ^+ and a base of simple roots Δ

Definition 4.1. *The **Borel subalgebra** containing* $\mathfrak{h}$ *is* $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. *Denote by* $\overline{\mathfrak{b}}$ *the analog of the Borel subalgebra but going over the negative roots.*

Definition 4.2. *We call a* $Z \in \mathfrak{h}$ *Δ-**dominant** if* $\alpha(Z)$ *is real and nonnegative for all* $\alpha \in \Delta$.

By the nondegeneracy of κ, we can identify any $H \in \mathfrak{h}$ with an $H^* \in \mathfrak{h}^*$. Coupled with the fact that Δ is a basis of $\mathfrak{h}^*$, the values of $\alpha(H)$ for $\alpha \in \Delta$ completely determine H.
Since $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, H is both Δ-dominant and integral, therefore $\alpha(H) \in \mathbb{N}$ for all $\alpha \in \Delta$.
The fact that $Y \in \mathfrak{g}_{-2}$ implies that $[X_\alpha, Y] \in \overline{\mathfrak{b}}$

- if $[X_\alpha, Y] = 0$, then $X_\alpha \in \mathfrak{g}^Y$, i.e. $\alpha(H) = 0$

- if $[X_\alpha, Y] \neq 0$, then the ad_H-eigenvalue of $[X_\alpha, Y] = \alpha(H) - 2$ therefore $\alpha(H) - 2 \in -\mathbb{N}$. This in total implies that $\alpha(H) = \{0, 1, 2\}$.

If we label every ndoe of the Dynkin diagram of $\mathfrak{g}$ with the eigenvalue $\alpha(H)$ on the corresponding root space $\mathfrak{g}_\alpha$, we obtain at most $3^{rank \mathfrak{g}}$ different Dynkin diagrams, called the **weighed Dynkin diagrams** of $\mathcal{O}_X$ denoted by $\Delta(\mathcal{O}_X)$.

Theorem 4.3. *(Kostant) The weighted Dynkin diagrams of a nilpotent orbit is an invariant, i.e.* $\Delta(\mathcal{O}') = \Delta(\mathcal{O}) \iff \mathcal{O} = \mathcal{O}'$ *for any two nilpotent* $\mathcal{O}, \mathcal{O}'$

Proof. " $\Leftarrow$ " is evident
" $\Rightarrow$ if $\Delta(\mathcal{O}') = \Delta(\mathcal{O})$, then there exist two standard triples $\{H, X, Y\}$ and $\{H', X', Y'\}$ with nilpotent X, X'. The neutral elements can be chosen to be identical, thus by Mal'cev's theorem we can choose Y and Y' to be conjugate. $\square$

Remark 4.4. *While the above chapters indeed give us an upper bound for the number of nilpotent orbits, it is very inaccurate and not easy to work with. For example the number of nilpotent orbits in* E_8 *is 70, but* 3^8 *is a much bigger number.*

5 Examples

In this section, we will focus on $\mathfrak{sl}_n$. While the example does not follow the theory explained above, it gives us the exact number of nilpotent orbits in $\mathfrak{sl}_n$ and is a very enlightening example.

5.1 Counting Orbits in $\mathfrak{sl}_n$

Definition 5.1. *let* $n \in \mathbb{N}$*, a* ***partition*** *of* n *is a tuple* $[d_1, ..., d_k]$ *such that*

- $d_1 \geq d_2 \geq .... \geq d_k \geq 0$
- $\sum d_i = n$

We say two partitions $[d_1, .., d_k]$ *and* $[p_1, .., p_n]$ *are equivalent if and only if* $p_i = d_i$ *for all* $p_i, d_i > 0$

Definition 5.2. *the* ***set of all partitions of*** n *is denoted by* $\mathcal{P}(n)$

We recall the definition of a Jordan block of type i from basic linear algebra: Construct the $i \times i$ matrix of the form

$$J_{d_i} = \begin{pmatrix} 0 & 1 & ... & \\ 0 & \ddots & 1 & ... \\ \vdots & & \ddots & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

Let $[d_1, \dots, d_k] \in \mathcal{P}(n)$ such that $d_k > 0$, we can construct the Jordan normalform associated to the partition via:

$$X_{[d_1,\dots,d_k]} = \begin{pmatrix} J_{d_1} & 0 & \dots \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_{d_k} \end{pmatrix}$$

This matrix is both nilpotent and has trace zero, therefore it is a nilpotent element of $\mathfrak{sl}_n$. Notice by the uniqueness of the Jordan normalform that any two distinct partitions of n define disjoint nilpotent orbits. Thus we can conclude

$$|\mathcal{P}(n)| \leq |\{\text{nilpotent orbits in } \mathfrak{sl}_n\}|.$$

To show the other inequality, $\geq$, it suffices to recall that any element X can be brought to it's unqiue Jordan normalform. If X is chosen to be nilpotent, then the eigenvalues of X are all 0, however the eigenvalues of an uppertriangular matrix are the elements on the diagonal. This concludes the statement.

5.2 $\mathfrak{sl}_4$ explicitly

We count how many different partitions the number 4 has, giving us

$$\mathcal{P}(4) = \{[4], [3, 1], [2, 2], [2, 1, 1], [1, 1, 1, 1]\} \tag{5}$$

This gives us 5 distinct Nilpotent orbits, the most obvious representatives being

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The next question we could try answering is how big is the dimension of each of those orbits? While the dimension corresponding to the orbit of $[1, 1, 1, 1]$ is evidently zero, it is unclear which (if any) of the non-trivial Orbits are maximal/minimal and what their dimensions might be.
It can be shown that there exists a non-trivial nilpotent orbit of minimal dimension and that it's dimension is $1+|\{\text{positive roots not orthogonal to the highesr root}\}|$ [1], applied to our case of $\mathfrak{sl}_4$, we obtain that the dimension is 6.

References

[1] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[2] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.