HOMEWORK 5

① Let V be a real vector space with an inner product f and a linear transformation $\tau : V \to V$ such that

$$f(\tau(v), w) = -f(v, \tau(w)) \quad \text{for all } v, w \in V.$$

Show that τ has all real eigenvalues equal to 0.

Since $\tau(v)$ is a linear transformation, we can write it as $\boxed{\tau(v) = \overset{*}{\lambda \cdot v}}$ for eigenvalue λ of τ with corresponding (non-zero) eigenvector v.

Since we know that the inner product f is PD, we know that $f(v,v) = 0$ only if $v = 0$

$$\left.\begin{aligned} f(\tau(v), v) &\overset{*}{=} f(\lambda v, v) \overset{\text{lin.}}{=} \underline{\lambda f(v,v)} \\ f(\tau(v), v) &\overset{\text{DEF}}{=} -f(v, \tau(v)) \overset{*}{=} -f(v, \lambda v) \overset{\text{LIN.}}{=} -\lambda f(v,v) \end{aligned}\right\} \begin{aligned} \lambda f(v,v) &= -\lambda f(v,v) \\ 0 &= 2\lambda f(v,v) \end{aligned}$$

$$\Downarrow$$

We know that $f(v,v) \neq 0$, since eigenvectors are not 0. This means that $\underline{\underline{\lambda = 0}}$.

② Find an inner product g on $\mathbb{R}[x]_2$ for which basis $\mathcal{B} = \{1, 1+x, 1+x^2\}$ is reciprocal to itself. Write an explicit formula for g.

~~We are looking for an orthonormal basis with respect~~

~~We are looking for g where $\mathcal{B}$ is an ortonormal basis.~~

$$\cancel{[v]_{\mathcal{B}}^T \cdot G_{\mathcal{B}} \cdot [v]_{\mathcal{B}} = \mathbb{I} \, G_g}$$

~~We know $G_{\mathcal{B}} = I$, since it is orthonormal basis.~~

$$\cancel{[v]_g^T \cdot G_g \cdot [v]_g = \mathbb{I} \, G_{\mathcal{B}}} \Rightarrow \cancel{[v]_{\mathcal{B}}^T \cdot G_g \cdot [v]_{\mathcal{B}} = \mathbb{I}} \quad G_g = [v]_{\mathcal{B}}^T$$

We are looking for g where $\mathcal{B}$ is an orthonormal basis.

$$G_g = \Lambda^T \cdot G_{\mathcal{B}} \cdot \Lambda = \Lambda^T \cdot \Lambda$$

orthonormal $G_{\mathcal{B}} = I$

$$L_{\mathcal{B}g} = \begin{array}{cc} \begin{array}{ccc} 1 & 1+x & 1+x^2 \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$$

$$\Lambda = L^{-1} = \underline{\underline{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

$$G_g = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}}}$$

$$g(p,q) = [p]_g^T G_g \cdot [q]_g = (a_0 \; a_1 \; a_2) \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = (a_0 \; a_1 \; a_2) \begin{bmatrix} b_0 - b_1 - b_2 \\ -b_0 + 2b_1 + b_2 \\ -b_0 + b_1 + 2b_2 \end{bmatrix} =$$

$$= \underline{\underline{a_0 b_0 - a_0 b_1 - a_0 b_2 - a_1 b_0 + 2 a_1 b_1 + a_1 b_2 - a_2 b_0 + a_2 b_1 + 2 a_2 b_2}}$$

③ In HW4 we proved $\alpha(p) = \int_0^1 p(x)\,dx$ is a linear form on $\mathbb{R}_2[x]$.

For $p, q \in \mathbb{R}_2[x]$ let

$$h(p,q) = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

be an inner product on $\mathbb{R}_2[x]$. Find a polynomial $r \in \mathbb{R}_2[x]$ such that $\alpha(p) = h(p,r)$ for all $p \in \mathbb{R}_2[x]$.

$$\int_0^1 (a + bx + cx^2)\,dx = ax + b\frac{x^2}{2} + c\frac{x^3}{3}\Big|_0^1 = \underline{a + \tfrac{1}{2}b + \tfrac{1}{3}c}$$

$$p(-1)q(-1) + p(0)q(0) + p(1)q(1) = (a-b+c)(d-e+f) + ad + (a+b+c)(d+e+f)$$

$$= ad - \cancel{ae} + af - \cancel{bd} + be - \cancel{bf} + cd - \cancel{ce} + cf +$$

$$+ ad +$$

$$+ ad + \cancel{ae} + af + \cancel{bd} + be + \cancel{bf} + cd + \cancel{ce} + cf =$$

$$= \underline{a(3d + 2f) + b(2e) + c(2d + 2f)}.$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1/2 \\ 2 & 0 & 2 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1/4 \\ 1 & 0 & 1 & 1/6 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \Rightarrow \underline{\underline{r(x) = \tfrac{2}{3} + \tfrac{1}{4}x - \tfrac{1}{2}x^2}}$$