Mathematics 2
Homework 1

The homework consists of six theoretical and practical problems. The solutions are to be submitted as **one .zip file** to the appropriate mailbox on ucilnica. The solutions should contain a .pdf file containing a clear and well described procedure, a code, explanation of choices of parameters, numerical results, etc.

1 Theoretical problems

1. Prove equivalence 1. $\Leftrightarrow$ 4. of Proposition 2.3 in notes.

2. Let $f(x,y) = x^2 + e^x + y^2 - xy$. Function f restricted to $K = [-2,2] \times [-2,2]$ is Lipschitz, smooth and strongly convex. Find some corresponding (preferably optimal) constants L, α and β on K. Furthermore, prove f is convex.

3. Find formulas for projections $\mathbb{R}^2 \to K$ to the closest point for the following convex sets: $x^2 + y^2 \leq 1.5; [-1,1] \times [-1,1]$; and the triangle with vertices $(-1,-1), (1.5,-1), (-1,1.5)$.

4. Let $f(x,y) = x^2 + 2y^2$. Starting with $x_1 = (1,1)$:

   (a) What is the minimal function value that can be achieved with one step of the gradient descend, i.e., find the minimum of $f(x_2)$.

   (b) How close to the actual minumum x^* of function f can we get with one step of the gradient descend, i.e., find the minimum of the distance from x^* to x_2.

2 Programming problems

For the following problems you need to submit a code and a short report on results.

5. Combining the function of problem 2. and results of problems 2. and 3. use PGD with:

   - initial point $x_1 = (-1,1)$,
   - three different learning rates from Theorem 3.3,
   - domain of f restricted to each of the three convex sets of problem 3.

   Together that gives 9 combinations of domains and learning rates.

   Perform 10 steps of the gradient descend. Compare the obtained result with theoretical guarantees of Theorem 3.3. Which one performs best on this example?

6. Find the global minumum of the following function on $[0,1]^3$ using GD or PGD.

$$f(z_1, z_2, z_3) = -\sum_{i=1}^{4} c_i \, e^{\left(-\sum_{j=1}^{3} a_{i,j}\,(z_j - p_{i,j})^2\right)}$$

i	$a_{i,j}, j = 1,2,3$			c_j	$p_{i,j}, j = 1,2,3$		
1	3.0	10	30	1.0	0.36890	0.1170	0.2673
2	0.1	10	35	1.2	0.46990	0.4387	0.7470
3	3.0	10	30	3.0	0.10910	0.8732	0.5547
4	0.1	10	35	3.2	0.03815	0.5743	0.8828

The minimal value of f is approximately -3.86278214782076. How close to that value can you get? How many steps does it take?