Homework 6

(1) Show that every vector space V is isomorphic to $(V^*)^*$ by explicitly constructing the isomorphism.

We construct a map: $\varphi: V \to (V^*)^*$ $\quad \varphi(v): V^* \to \mathbb{R}, \ v \in V$

For any vector $v \in V$, $\varphi(v)$ is a linear form on V^* that takes a linear form f and evaluates at v.

For any $f \in V^*$: $\varphi(v)(f) = f(v) \in \mathbb{R}$

Linearity:

I) We know our map φ is linear since f is linear.

II) Bijectivity:

a) Surjectivity: We know $\dim(V) = \dim(V^*) = \dim((V^*)^*)$ this shows surjectivity.

b) Injectivity: By rank-nullity theorem we know that $\dim(N(\varphi)) + \dim(\mathrm{Im}(\varphi)) = \dim(\varphi)$, and we know $\mathrm{Im}(\varphi) = \dim(\varphi)$, we know it's injective.

Linear + Bijective = ISOMORPHISM. We proved that V is isomorphic to $(V^*)^*$.

(2) Find an inner product g on $\mathbb{R}^3$ for which basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is reciprocal to itself. Write an explicit formula for g.

We are looking for g where $\mathcal{B}$ is an orthonormal basis.

$$G_g = \Lambda^T \cdot G_{\mathcal{B}} \cdot \Lambda = \Lambda^T \cdot \Lambda$$

$G_{\mathcal{B}} = I$, since orthonormal

$$L_{\mathcal{B}\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda = L^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_g = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$ → computed in HW5

$$g(a,b) = (a_0\ a_1\ a_2) \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = a_0b_0 - a_0b_1 - a_0b_2 - a_1b_0 + 2a_1b_1 + a_1b_2 -$$
$$- a_2b_0 + a_2b_1 + 2a_2b_2$$