Vertices of matching polytope

Let Ψ be a convex set in $\mathbb{R}^d$. We call $x \in \Psi$ a vertex (also an extremal point) of Ψ if x cannot be expressed as a proper convex combination of $x_1, x_2 \in \Psi \backslash \{x\}$. Equivalently, x is a vertex of Ψ if $\Psi \backslash \{x\}$ is convex.

Let $G = (V, E)$ be a simple (undirected, finite) graph. A matching in G is a set of edges M which have no vertices in common. The maximal matching problem is an optimization problem looking for a matching of maximal cardinality.

The maximal matching problem can be described as a constraint satisfaction problem in the following way:

$$\max e^T x$$
$$\text{where for all } v \in V \text{ we have } \sum_{i \in E(v)} x_i \leq 1, \text{and}$$
$$x \in \{0, 1\}^E.$$

In the above e stands for the all 1s vector and $E(v)$ is the set of edges incident with a vertex v. The maximal matching problem can be linearly relaxed to

$$\max e^T x$$
$$\text{where for all } v \in V \text{ we have } \sum_{i \in E(v)} x_i \leq 1, \text{ and}$$
$$x \in [0, 1]^E,$$

allowing nonintegral coordinates in x. The feasible set in (MM-LR) is called a matching polytope.

9. Show that matchings are extremal points of the matching polytope.

We know that a non-empty matching has at least one coordinate equal to 1 in the matching polytope. Let's call it coordinate i. Let's say that a matching M is not an extremal point, which means it can be expressed as a proper convex combination of some x and y $\in \Psi \backslash \{x\}$. This means that M can be written as: $M = \lambda x + (1 - \lambda) y$, where $\lambda \in (0, 1)$. No matter which x and y we chooose, we cannot express coordinate i of M with only x and y, since values lie in [0,1]. We arrive at a contradiction, which means that M cannot be expressed as a proper convex combination and is therefore an extremal point.

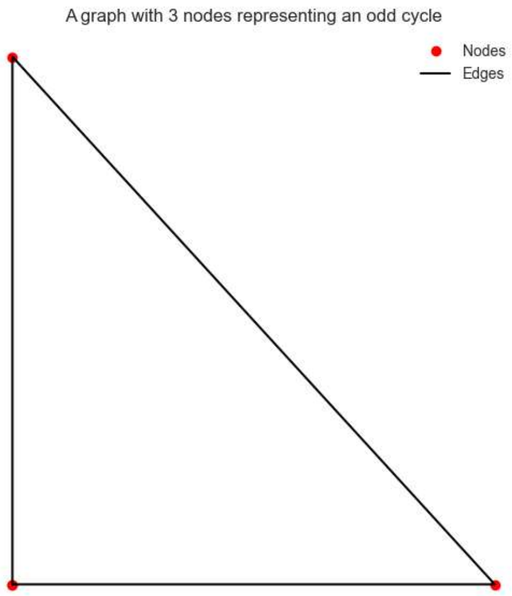
Note: For the empty-matching we arrive to the same conclusion, but using the fact that all coordinates are equal to 0.

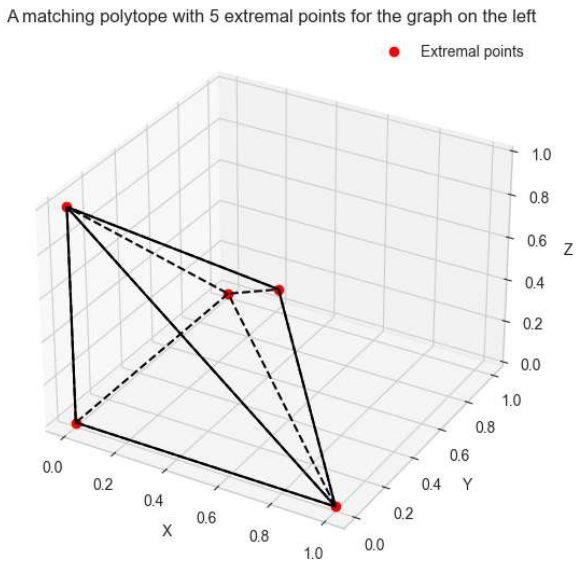
10. Show that there may exist extremal points of the matching polytope which are not matchings (try an odd cycle).

In [ ]:
```
import matplotlib.pyplot as plt
plt.style.use('seaborn-whitegrid')

fig = plt.figure(figsize=(14, 7))
ax1 = fig.add_subplot(1, 2, 1)
```

```
ax2 = fig.add_subplot(122,projection='3d')

x = [0, 1, 0, 0]
y = [0, 0, 1, 0]

ax1.plot(x, y, 'ro', label='Nodes')
ax1.plot(x, y, 'k', label='Edges')

ax1.axis('off')
ax1.set_title("A graph with 3 nodes representing an odd cycle")

ax1.legend()

x = [0, 1, 0, 0, 1/2]
y = [0, 0, 1, 0, 1/2]
z = [0, 0, 0, 1, 1/2]

xl = [0, 1, 0, 0, 0, 1, 1/2, 0, 0, 1/2]
yl = [0, 0, 0, 0, 1, 0, 1/2, 1, 0, 1/2]
zl = [0, 0, 1, 0, 0, 0, 1/2, 0, 1, 1/2]

ax2.plot(x, y, z, 'ro', label='Extremal points')
ax2.plot(xl, yl, zl, "black", linestyle='dashed')

xl = [1, 0, 0, 1, 1/2, 0]
yl = [0, 0, 0, 0, 1/2, 0]
zl = [0, 1, 0, 0, 1/2, 1]

ax2.plot(xl, yl, zl, 'black')

ax2.set_xlabel('X')
ax2.set_ylabel('Y')
ax2.set_zlabel('Z')

ax2.set_title("A matching polytope with 5 extremal points for the graph on the left"
ax2.legend()
```

Out[ ]: <matplotlib.legend.Legend at 0x1df02b036d0>





In this plot we show that for a fully connected graph with 3 nodes we obtain 5 extremal points.

- (0,0,0)
- (1,0,0)

- (0,1,0)
- (0,0,1)
- (0.5, 0.5, 0.5)

The fifth extremal point is NOT a matching, since it contains fractions in all dimensions.