Mathematics 2, Part 3	**2022/23**

Homework 4

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The homework consists of two problems. The solutions are to be submitted to the appropriate mailbox on Učilnica before the exam, but preferably in a week. The solutions should contain a clear and well explained proofs, procedures, explanations, etc.

(1) Let $\mathcal{B} = \{\vec{q}_1, \dots, \vec{q}_n\}$ be a basis for $\mathbb{R}^n$ and $\mathcal{B}^* = \{\beta^1, \dots, \beta^n\}$ its dual basis.
  (a) Show that $\beta^i(\vec{x}) = \vec{a}_i^\top \vec{x}$ for some $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$.
  (b) If $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^n$, show that $\vec{a}_i = \vec{q}_i$ for $i = 1, \dots, n$.

(2) Let us define $f_1(p) = p(0) + p(1)$, $f_2(p) = \int_{-1}^1 p(x)\,dx$, $f_3(p) = \int_0^1 p(x)\,dx$ and $\varphi\colon \mathbb{R}_2[x] \times \mathbb{R}_2[x] \to \mathbb{R}$ as $\varphi(p,q) = \int_0^1 p(x)q(x)\,dx$.
  (a) Prove that $\{f_1, f_2, f_3\}$ form a basis for $(\mathbb{R}_2[x])^*$.
  (b) Find the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ for $\mathbb{R}_2[x]$ such that $\mathcal{B}^* = \{f_1, f_2, f_3\}$.
  (c) Show that φ is bilinear form on $\mathbb{R}_2[x]$ and express φ as a linear combination of $\{f_i \otimes f_j\colon\ i,j = 1,2,3\}$.
  (Hint: you can do it directly. Alternatively, prove that

$$\varphi = \sum_{i=1}^{3} \sum_{j=1}^{3} \varphi(b_i, b_j) f_i \otimes f_j.)$$