HOMEWORK 4

① Let $\mathcal{B} = \{\vec{g}_1, \dots, \vec{g}_n\}$ be a basis for $\mathbb{R}^n$ and $\mathcal{B}^* = \{\beta^1, \dots, \beta^n\}$ its dual basis.

a) Show that $\beta^i(\vec{x}) = \vec{a}_i^T \vec{x}$ for some $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$

$\beta^i(g_j) = \delta^i_j$

We can rewrite $\vec{x}$ and $\vec{a}_i$ with our basis vectors.

$\vec{x} = x^1 \vec{g}_1 + \dots + x^n \vec{g}_n = x^i \vec{g}_i \Rightarrow \beta^i(\vec{x}) = x^1 \beta^i(\vec{g}_1) + \dots + x^n \beta^i(\vec{g}_n) = \underline{\underline{x^i}}$

$\vec{a}_i = a_i^1 \vec{g}_1 + \dots a_i^n \vec{g}_n = a_i^j \vec{g}_j$ $\qquad \beta^i(\vec{x}) = \cancel{x^i} \, x^i = \vec{a}_i^T \vec{x}$

$\vec{a}_i^T \cdot \vec{x} = a_i^j \vec{g}_j^T \cdot x^i \vec{g}_i = \sum_k^n \sum_j^n \underbrace{(a_i^j \cdot \vec{g}_j^T \cdot \vec{g}_k)}_{\text{when } k=i \text{ this is } 1 \text{, else } 0} x^k = x^i$

So: $\underbrace{[a_i^1, \dots a_i^n]}_{a_i^T} \cdot \underbrace{\begin{bmatrix} -g_1- \\ \vdots \\ -g_n- \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{bmatrix}}_{Q} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ith} = e_i$

$\Rightarrow \underline{\underline{a_i^T = e_i \cdot (Q^T \cdot Q)^{-1}}}$, so $\beta^i(\vec{x}) = \vec{a}_i^T \vec{x}$ for some $\underline{\underline{a_i^T = e_i (Q^T \cdot Q)^{-1}}}$

b) If $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^n$, show that $\vec{a}_i = \vec{g}_i$ for $i = 1, \dots, n$

Basis is orthonormal if $\vec{g}_i \cdot \vec{g}_j = \delta_i^j \quad \forall i,j = 1, \dots, n$

From part a) we know: $a_i^T \vec{x} = x^i$.

$g_i^T \vec{x} = g_i^T \cdot (x^i g_i) = g_i^T (x^1 \vec{g}_1 + \dots + x^n \vec{g}_n) = x^1 \underbrace{\vec{g}_i^T \vec{g}_1}_{0} + \dots x^i \underbrace{\vec{g}_i^T \vec{g}_i}_{1} + \dots + x^n \underbrace{\vec{g}_i^T \vec{g}_n}_{0} =$

$= \underline{x^i}$

Which means $\vec{a}_i = \vec{g}_i$ if $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^n$.

② Let us define:

$$f_1(p) = p(0) + p(1)$$
$$f_2(p) = \int_{-1}^{1} p(x)\, dx$$
$$f_3(p) = \int_{0}^{1} p(x)\, dx$$

and $g : R_2[x] \times R_2[x] \to \mathbb{R}$ as $g(p,q) = \int_0^1 p(x)\, q(x)\, dx$

a) Prove that $\{f_1, f_2, f_3\}$ form a basis for $(R_2[x])^*$.

$\{f_1, f_2, f_3\}$ is a basis if f_1, f_2 and f_3 are linearly independent (I) and span (II) $(R_2[x])^*$.

I) They are independent if: $\alpha_1 \cdot f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$

$$f_1(p) = \underline{a + b + 2c}$$
$$f_2(p) = \int_{-1}^{1} p(x)\, dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx \Big|_{-1}^{1} = \underline{\frac{2a}{3} + 2c}$$
$$f_3(p) = \int_{0}^{1} p(x)\, dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx \Big|_{0}^{1} = \underline{\frac{a}{3} + \frac{b}{2} + c}$$

$$\alpha_1 (a + b + 2c) + \alpha_2 \left(\frac{2a}{3} + 2c\right) + \alpha_3 \left(\frac{a}{3} + \frac{b}{2} + c\right) = 0$$

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & 1 & 2 \\ 2/3 & 0 & 2 \\ 1/3 & 1/2 & 1 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$$

$N(A) = 0$, so there exist only trivial solutions

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

$\forall a, b, c \in \mathbb{R}$

II) We know that the dimension of basis $(R_2[x])^*$ is 3 and there are 3 independent elements in $\{f_1, f_2, f_3\}$, so it is spanning $(R_2[x])^*$

b) Find the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ for $\mathbb{R}_2[x]$ such that $\mathcal{B}^* = \{f_1, f_2, f_3\}$

$\mathbb{R}_2[x]$, $\mathcal{S} = \{1, x, x^2\}$ $\quad B = \Omega^{-1}$ $\quad \Omega = \begin{bmatrix} 1 & 1 & 2 \\ 2/3 & 0 & 2 \\ 1/3 & 1/2 & 1 \end{bmatrix}$

$$B = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 1 & 0 \\ 1/3 & 1/2 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 2/3 & 1 & 2 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} \cancel{1/3}\,1 & 0 & 0 & 3 & 0 & -6 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 2/3 & 0 & 2 & 0 & +1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -6 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1/2 & 2 \end{array}\right]$$

$2 \quad 0 \quad 6 \qquad +3$

$-6 \quad +3 \quad +12$

$-1 \quad +1/2 \quad +2$

$$\mathcal{B} = \{3x^2 - 1, -x + 1/2; 6x^2 + 2x + 2\}$$

c) Show that φ is bilinear form on $\mathbb{R}_2[x]$ (I) and express φ as linear combination (II) of $\{f_i \otimes f_j : i, j = 1, 2, 3\}$

I) $\varphi(p, g) = \int_0^1 p(x)\, g(x)\, dx$, $\quad p, s, g \in \mathbb{R}_2[x]$, $\alpha, \beta \in \mathbb{R}$

$$\varphi(\alpha p + \beta s, g) = \int_0^1 (\alpha p + \beta s)(x) \cdot g(x)\, dx = \int_0^1 (\alpha p(x) + \beta(x)) \cdot g(x)\, dx = \int_0^1 \alpha p(x) \cdot g(x) + \beta s(x) \cdot g(x)\, dx$$

$$= \int_0^1 \alpha p(x) \cdot g(x)\, dx + \int_0^1 \beta s(x) \cdot g(x)\, dx = \alpha \int_0^1 p(x) g(x)\, dx + \beta \int_0^1 s(x)\, g(x)\, dx$$

$$= \alpha \cdot \varphi(p, g) + \beta \cdot \varphi(s, g)$$

Similarly it holds for $\varphi(p, \alpha g + \beta s)$.

II) $g = \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{ij}\, f_i \otimes f_j = \sum_{i=1}^{3}\sum_{j=1}^{3} g(b_i, b_j)\, f_i \otimes f_j$.

$\mathcal{B} = \{\underbrace{3x^2-1}_{b_1}, \underbrace{-x+1/2}_{b_2}, \underbrace{-6x^2+2x+2}_{b_3}\}$, $g(p,q) = \int_0^1 p(x)\,q(x)\,dx$

$g(b_1,b_1) = \int_0^1 (3x^2-1)^2\,dx = \ldots = \frac{4}{5}$

$g(b_1,b_2) = \int_0^1 (3x^2-1)\cdot(-x+1/2)\,dx = \ldots = -1/4$

$g(b_1,b_3) = \int_0^1 (3x^2-1)\cdot(-6x^2+2x+2) = \ldots = \ldots = -11/10$

$g(b_2,b_1) = g(b_1,b_2) = -1/4$

$g(b_2,b_2) = \int_0^1 (-x+1/2)^2\,dx = \ldots = 1/12$

$g(b_2,b_3) = \int_0^1 (-x+1/2)\cdot(-6x^2+2x+2)\,dx = \ldots = 1/3$

$g(b_3,b_1) = g(b_1,b_3) = -11/10$

$g(b_3,b_2) = g(b_2,b_3) = 1/3$

$g(b_3,b_3) = \int_0^1 (-6x^2+2x+2)^2\,dx = \cancel{11/3}\ \frac{38}{15}$

$g = \frac{4}{5} f_1 \otimes f_1 - 1/4\, f_1 \otimes f_2 - 11/10\, f_1 \otimes f_3 -$
$- 1/4\, f_2 \otimes f_1 + 1/12\, f_2 \otimes f_2 + 1/3\, f_1 \otimes f_2 -$
$- 11/10\, f_3 \otimes f_1 + 1/3\, f_3 \otimes f_2 + \frac{38}{15} f_3 \otimes f_3$