Isomorphism testing for embeddable graphs through definability *

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ABSTRACT

The k-dimensional Weisfeiler-Leman algorithm, for $k \geq 1$, is a natural and simple combinatorial algorithm attempting to decide whether two given graphs are isomorphic.

In this paper, we show that for every surface S (orientable or non-orientable) there is a $k \geq 1$ such that the k-dimensional WL-algorithm succeeds to decide isomorphism of graphs embeddable in S. To prove this, we use a close connection between the WL-algorithm and definability in certain finite variable logics that has been established by Cai, Fürer, and Immerman [7].

1. INTRODUCTION

The graph isomorphism problem asks whether two given graphs are isomorphic. While complexity theoretic results indicate that the isomorphism problem is not NP-complete (if it was, the polynomial hierarchy would collapse to its second level [6; 29]), no polynomial time algorithm for the general problem is known.

However, there is a number of important classes of graphs on which the isomorphism problem is known to be solvable in polynomial time. Building on work of Hopcroft and Tarjan [18; 19], Hopcroft and Wong [17] proved in 1974 that isomorphism of planar graphs can be decided in linear time. In 1980, Filotti, Mayer [12] and Miller [26] showed that for every orientable surface S there is polynomial time isomorphism test for graphs embeddable in S. Lichtenstein [22] showed the analogous result for graphs embeddable in the projective plane. Around the same time, Luks [23] gave his well-known group theoretic polynomial time algorithm testing isomorphism of graphs of bounded valence. Similar group theoretic algorithms have been used by Babai and others to give polynomial time isomorphism tests for a number of further classes of graphs (see, for example, [1; 2; 16; 24; 25; 28]). In 1990, Bodlaender [5] proved that isomorphism of graphs of bounded tree-width is decidable in polynomial time.

These algorithms are often divided into *combinatorial* algorithms, such as those for planar graphs or graphs of bounded tree-width, and *group theoretic* algorithms (cf. [16]). Whereas it is quite obvious that all the group theoretic algorithms are very similar and based on common ideas, this is not clear at all for the combinatorial algorithms. For example, Hopcroft and Wong's algorithm for planar graphs does not seem to have much in common with Bodlaender's algorithm for graphs of bounded tree-width. It is one of the contributions of this paper to give a uniform explanation for all the "combinatorial" classes

The prototype of a combinatorial isomorphism algorithm is the following *color refinement algorithm*, which attempts to decide whether two given graphs G and H are isomorphic:

> First, each vertex of G and H is colored by its valence. In each iteration step, the current color of each vertex is extended by the multiset of colors of its neighbors. The iteration step is repeated until it no longer leads to a refinement of the equivalence relation induced by the coloring. The algorithm answers "yes" if the multisets of colors occurring in G and H after the iteration has stopped are the same for both graphs; otherwise it answers "no".

To keep the colors short, between two iteration steps all occurring colors are ordered lexicographically, and every color is replaced by its order number. The algorithm can be implemented to run in time $O((m+n)\log(n))$, where m denotes the size of the edge set of the input graphs and n the size of the vertex sets [18; 21].

Obviously, the algorithm always recognizes isomorphic input graphs as isomorphic, but it does not always succeed in recognizing non-isomorphic input graphs as such. For example, it is not able to distinguish between two regular graphs of the same valence and order. Nevertheless, due to its efficiency and simplicity, color refinement is successfully used as the basis of almost all practical implementations of isomorphism tests. This can be explained by results of Babai, Erdös, Selkow [3] and Babai, Kučcera [4] showing that it succeeds "almost always".

A more powerful variant of the algorithm colors k-tuples according to their isomorphism type and iteratively refines this coloring according to the colors occurring among the neighbors of a tuple. This method is called the k-dimensional Weisfeiler-Leman (WL) algorithm. Based on the work of

*To appear in *Proceedings of the 32nd ACM Symposium on Theory of Computing*, 2000

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Weisfeiler and Leman, it has been developed in the late seventies and early eighties by many researchers. I refer the reader to [7] for a history of the algorithm (also see [11; 10]).

In 1992, Cai, Fürer and Immerman [7] observed a close connection between the WL-algorithm and definability in certain *finite variable logics* that have been intensely studied in descriptive complexity theory. Employing techniques from this area, Cai, Fürer and Immerman [7] could show the limits of the WL-algorithm. They proved that for every k there is a pair of non-isomorphic graphs (of size linear in k and valence 3) that cannot be distinguished by the k-dimensional WL-algorithm.

This logical approach can also be used to show, conversely, that on many important classes of graphs the WL-algorithm does decide isomorphism in polynomial time. It reduces the algorithmic question to a definability question. This method has successfully been applied to planar graphs [14] and graphs of bounded tree-width [15]. In this paper, we prove that for every (orientable or non-orientable) surface S there is a $k \geq 1$ (linear in the genus of S) such that every graph embeddable in S can be characterized up to isomorphism by a sentence of the logic $\mathrm{C}^k_{\infty\omega}$, answering a question of Immerman and Lander [21]. This result implies that the $(k-1)$-dimensional WL-algorithm decides isomorphism of graphs embeddable in S in time $O(n^k \log(n))$, where n is the number of vertices of the input graphs.

Thus we obtain a very simple and natural isomorphism algorithm for graphs embeddable in a fixed surface; the difficulty is only in proving that the algorithm works correctly. This gives our approach a clear advantage over the very complicated algorithms of Filotti, Mayer [12] and Miller [26]. Let me add that the running time of both our and their algorithms is $n^{O(g)}$, where g is the genus of the surface. Immerman and Lander [21] show how our result can be used to compute canonical forms for graphs embeddable in a fixed surface within the same time-bounds. Maybe most importantly, together with the results in [14; 15], our result shows that on all those classes where the isomorphism problem is known to be solvable in polynomial time by a "combinatorial" algorithm, it can actually be solved by the WL-algorithm. So we have a nice dichotomy for the classes of graphs where the isomorphism problem is solvable in polynomial time: Either, there is a $k \geq 1$ such that the isomorphism problem can be solved by the k-dimensional WL-algorithm, or the group theoretic approach is required.

The original motivation for this line of research was not to attack the graph isomorphism problem, but a question from database theory due to Chandra and Harel [8]. They asked whether there is a query language that *captures* polynomial time, that is, a language in which precisely the polynomial time computable (relational database) queries can be expressed. Using techniques from [14; 15], the results of this paper can easily be extended to show that for every surface S, *fixed-point logic with counting* captures polynomial time on the class of all databases whose underlying graph is embeddable in S.

2. NOTATION

To distinguish multisets from sets, they are put in double brackets, such as $\{\!\{1, 1, 2\}\!\}$. $\bar{x}$ always denotes a tuple $(x_1, \ldots, x_l)$, for some $l \geq 1$.

In this paper, graphs are always finite, undirected, and simple. Occasionally, we also consider *multigraphs*, which are allowed to have multiple edges and loops. The vertex set of a graph (or multigraph) G is denoted by V^G, the edge set by E^G.

The *valence* of a vertex $v \in V^G$, denoted by $\mathrm{val}(v)$, is the number of edges incident with v. A vertex of valence at least 3 is called a *branching vertex.*

Path and *cycles* in a multigraph are defined as usual. Depending on the context we may both consider them as sequences of vertices and edges, such as

$$v_1 e_1 v_2 e_2 \ldots e_{m-1} v_m,$$

or as subgraphs. A *chain* in a graph G is a path $v_1 e_1 \ldots v_m$, where $m \geq 2$, such that $\mathrm{val}(v_i) = 2$ for $2 \leq i \leq m-1$.

A *colored graph* is a pair

$$G = \big((V_i^G)_{i \geq 1}, (E_i^G)_{i \geq 1}\big),$$

where $(V_i^G)_{i \geq 1}$ is a sequence of finite sets and $(E_i^G)_{i \geq 1}$ is a sequence of anti-reflexive and symmetric binary relations on $V^G := \bigcup_{i \geq 1} V_i^G$. Furthermore, only finitely many of the V_i^G and E_i^G are non-empty. The *vocabulary* of G is defined to be the set

$$\tau(G) := \{V_i \mid V_i^G \neq \emptyset\} \cup \{E_i \mid E_i^G \neq \emptyset\}.$$

The *underlying graph* of G has vertex set V^G and edge set $E^G := \bigcup_{i \geq 1} E_i^G$. When talking about graph theoretic notions such as connectivity in colored graphs, we always refer to the underlying graph.

Colored graphs do not only generalize graphs, but also multigraphs, because multiple edges can be coded into the edge coloring and loops into the vertex coloring.

If G is a colored graph and $U \subseteq V^G$, the colored subgraph induced by G on U, denoted by $\langle U \rangle^G$, is defined by $V_i^{\langle U \rangle^G} := U \cap V_i^G$ and $E_i^{\langle U \rangle^G} := U^2 \cap E_i^G$ (for $i \geq 1$). Furthermore, $G \setminus U := \langle V^G \setminus U \rangle^G$. For a colored subgraph H of G we let $G \setminus H := G \setminus V^H$. The union and intersection of two colored graphs G, H is defined in the obvious way; for example, we let $G \cup H$ be the colored graph with $V_i^{G \cup H} := V_i^G \cup V_i^H$ and $E_i^{G \cup H} := E_i^G \cup E_i^H$ (for $i \geq 1$).

3. THE WL-ALGORITHM

Let G be a colored graph. The *type* of a vertex $v \in V^G$ is the set

$$\mathrm{tp}(v) := \{l \mid l \geq 1, v \in V_l^G\}.$$

We also need to define types of tuples of elements. For a tuple $\bar{v} \in (V^G)^k$ we let

$$\begin{aligned}\mathrm{tp}(\bar{v}) :=& \big\{(l, i) \;\big|\; l \geq 1, 1 \leq i \leq k, v_i \in V_l^G\big\}\\ &\cup \big\{(0, i, j) \;\big|\; 1 \leq i, j \leq k, v_i = v_j\big\}\\ &\cup \big\{(m, i, j) \;\big|\; m \geq 1, 1 \leq i, j \leq k, (v_i, v_j) \in E_m^G\big\}.\end{aligned}$$

Observe that for two tuples $\bar{v}, \bar{w} \in (V^G)^k$ we have $\mathrm{tp}(\bar{v}) = \mathrm{tp}(\bar{w})$ if, and only if, the mapping $v_i \mapsto w_i$ is an isomorphism between the induced colored subgraphs $\langle \{v_1, \ldots, v_k\} \rangle^G$ and $\langle \{w_1, \ldots, w_k\} \rangle^G$.

We now reformulate the color-refinement algorithm of the introduction in the context of colored graphs: We inductively define labelings γ_i^1, for $i \geq 1$, of the vertex set of a colored graph G.

Initialization: For all $v \in V^G$ we let $\gamma_1^1(v) := \mathrm{tp}(v)$.

Refinement Step: For $i \geq 1$ and $v \in V^G$ we let

$$\gamma_{i+1}^1(v) := \Big\{\!\!\Big\{ \big(\mathrm{tp}(v,w), \gamma_i^1(w)\big) \;\Big|\; w \in V^G \Big\}\!\!\Big\}.$$

Let $n := |V^G|$. Note that for all $v, w \in V^G$ and $m > n$ we have

$$\gamma_m^1(v) = \gamma_m^1(w) \iff \gamma_n^1(v) = \gamma_n^1(w).$$

We let $\gamma^1(G) := \{\!\{\gamma_n^1(v) \mid v \in V\}\!\}$. It is not hard to see that if G is a plain (uncolored) graph, for all $i \geq 2$ and $v, w \in V^G$ we have $\gamma_i^1(v) = \gamma_i^1(w)$ if, and only if, v and w get the same label in the $(i-1)$th step of the color-refinement algorithm described in the introduction.

The *k-dimensional Weisfeiler-Leman algorithm*, for a $k \geq 2$, is a generalization of the color refinement algorithm that iteratively refines a coloring of k-tuples of vertices instead of just single vertices.

Let G be a colored graph and $k \geq 1$. We inductively define labelings γ_i^k of $(V^G)^k$, for $i \geq 1$.

Initialization: For all $\bar{v} \in (V^G)^k$ we let $\gamma_1^k(\bar{v}) := \mathrm{tp}(\bar{v})$.

Refinement Step: For $i \geq 1$ and $\bar{v} \in (V^G)^k$ we let

$$\begin{aligned}\gamma_{i+1}^1(\bar{v}) := \Big\{\!\!\Big\{ \big(\mathrm{tp}(v_1, \dots, v_k, w), &\gamma_i^k(v_1, \dots, v_{k-1}, w),\\ &\gamma_i^k(v_1, \dots, v_{k-2}, w, v_k),\\ &\vdots\\ &\gamma_i^k(w, v_2, \dots, v_k)\big) \;\Big|\; w \in V^G \Big\}\!\!\Big\}.\end{aligned}$$

Let $n := |V^G|$. For all $\bar{v}, \bar{w} \in (V^G)^k$ and $m > n^k$ we have

$$\gamma_m^k(\bar{v}) = \gamma_m^k(\bar{w}) \iff \gamma_{n^k}^k(\bar{v}) = \gamma_{n^k}^k(\bar{w}).$$

We let $\gamma^k(G) := \{\!\{\gamma_{n^k}^k(\bar{v}) \mid \bar{v} \in (V^G)^k\}\!\}$.

It is straightforward to turn the definition of $(\gamma_i^k)_{i \geq 1}$ into a polynomial time algorithm that decides whether two given graphs G, H satisfy $\gamma^k(G) = \gamma^k(H)$. With a little effort the following can be achieved:

THEOREM 1 (IMMERMAN AND LANDER [21]). *Let $k \geq 1$. Then there is an algorithm that decides whether two given colored graphs G, H satisfy $\gamma^k(G) = \gamma^k(H)$ in time $O\big(t \cdot n^{k+1} + k \cdot n^{k+1} \cdot log(n)\big)$, where $n := |V^G|$ and $t := |\tau(G)|$.*

4. FINITE-VARIABLE LOGICS

We have an infinite supply of variables, which we denote by x, y, z and variants such as x_1, x'. *Atomic formulas* are of the form $(x = y)$, $E(x,y)$, $E_s(x,y)$, $V_s(x)$, where x, y are variables and $s \geq 1$.

The class of $\mathrm{C}_{\infty\omega}$*-formulas* is defined inductively as follows:

- Atomic formulas are $\mathrm{C}_{\infty\omega}$-formulas.
- If φ is a $\mathrm{C}_{\infty\omega}$-formula, then so is $\neg\varphi$.
- If $(\varphi_i)_{i \in I}$ are $\mathrm{C}_{\infty\omega}$-formulas, for an arbitrary index set I, then $\bigwedge_{i \in I} \varphi_i$ is a $\mathrm{C}_{\infty\omega}$-formula.
- If φ is a $\mathrm{C}_{\infty\omega}$-formula, x a variable, and $m \geq 1$, then $\exists^{\geq m} x\varphi$ is a $\mathrm{C}_{\infty\omega}$-formula.

We also use notations like $\bigvee_{i \in I} \varphi_i$, $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$, $(\varphi_1 \to \varphi_2)$, $\exists x\varphi$, $\forall x\varphi$, and $\exists^{=m} x\varphi$ (as abbreviations of $\mathrm{C}_{\infty\omega}$-formulas) in a standard way.

The semantics of the logic $\mathrm{C}_{\infty\omega}$ is defined inductively in a straightforward manner. The meaning of the atomic formulas is obvious, $\neg\varphi$ means "not φ", $\bigwedge_{i \in I} \varphi_i$ means "φ_i holds for every $i \in I$", and $\exists^{\geq m} x\varphi$ means "there exist at least m vertices x such that φ holds".

For example, a colored graph G satisfies the $\mathrm{C}_{\infty\omega}$-sentence

$$\forall x \forall y \Big(\big(V_1(x) \wedge V_2(y)\big) \to \bigwedge_{i \geq 2} \neg E_i(x,y) \Big)$$

if $(V_1^G \times V_2^G) \cap E_i^G = \emptyset$ for all $i \geq 2$. *Sentences* are formulas without free variables. We write $G \models \varphi$ to denote that a colored graph G satisfies a sentence φ. We write $\varphi(x_1, \dots, x_l)$ to indicate that all free variables of φ are among $x_1, \dots, x_l$. For a colored graph G and vertices $v_1, \dots, v_l$ we write $G \models \varphi(v_1, \dots, v_l)$ to denote that G satisfies the formula $\varphi(x_1, \dots, x_l)$ if the free variables $x_1, \dots, x_l$ are interpreted by $v_1, \dots, v_l$, respectively.

If $\varphi(x_1, \dots, x_l)$ is a $\mathrm{C}_{\infty\omega}$-formula and $y_1, \dots, y_l$ are variables not occurring in φ, then $\varphi(y_1, \dots, y_l)$ is the result of simultaneously substituting $x_1, \dots, x_l$ by $y_1, \dots, y_l$, respectively, in φ. Often, however, we would like to substitute the x_i by variables that do also occur in φ. Furthermore, since we are going to treat the number of variables appearing in a formula as an important parameter, we would like to do this in such a way that we do not need to introduce new variables. This can always be achieved by permuting the variables in a suitable way. Instead of giving a formal definition, let us just illustrate it with an example: Let $\varphi(x,y) = \exists z(E(x,z) \wedge \exists x(E(z,x) \wedge E(x,y))$. Then we let $\varphi(z,x) := \exists y(E(z,y) \wedge \exists z(E(y,z) \wedge E(z,x))$.

The logic $\mathrm{C}_{\infty\omega}$ itself is far too expressive to be useful; it is not hard to show that for every class $\mathcal{C}$ of graphs there is a $\mathrm{C}_{\infty\omega}$-sentence that is satisfied precisely by the graphs in $\mathcal{C}$. We consider the *finite variable fragments* of $\mathrm{C}_{\infty\omega}$: For $k \geq 1$, we let $\mathrm{C}_{\infty\omega}^k$ be the set of all $\mathrm{C}_{\infty\omega}$-formulas that contain at most k variables. Finite variable logics play an important role in descriptive complexity theory; for background material on such logics I refer the reader to the monographs [9; 20] and the survey [13].

We give three typical examples to illustrate the expressive power of our logics:

Example 1. For all $r \geq 1$, the $\mathrm{C}_{\infty\omega}^2$-sentence

$$\forall x \exists^{=r} y\, E(x,y)$$

says that a graph is r-regular. Similarly, for all $r, s, t \in \mathbb{N}$ there is a $\mathrm{C}_{\infty\omega}^3$-sentence saying that a graph is strongly regular with parameters r, s, t.

The next example illustrates a fundamental trick in the theory of finite variable logics: *re-using variables.*

Example 2. The $\mathrm{C}_{\infty\omega}^3$-formula

$$\begin{aligned}\varphi_9(x,y) :=\, &\exists z(E(x,z) \wedge \exists x(E(z,x) \wedge\\ &\quad\exists z(E(x,z) \wedge \exists x(E(z,x) \wedge\\ &\qquad\exists z(E(x,z) \wedge \exists x(E(z,x) \wedge\\ &\qquad\quad\exists z(E(x,z) \wedge \exists x(E(z,x) \wedge E(x,y) \cdots)\end{aligned}$$

says that there is a walk of length 9 from x to y. (A *walk* in a graph is a "path" on which repeated vertices and edges are allowed.) Similarly, for every $m \geq 0$ there is a $C^3_{\infty\omega}$-formula $\varphi_m(x,y)$ saying that there is a walk of length m from x to y. Then the $C^3_{\infty\omega}$-sentence $\forall x \forall y \bigvee_{i \geq 0} \varphi_i(x,y)$ says that a graph is connected.

We say that a sentence φ *characterizes* a colored graph G *up to isomorphism* if for every colored graph H we have $H \models \varphi$ if, and only if, H is isomorphic to G.

Example 3. For every colored graph G there exists a $k \leq |V^G|$ and a sentence $\varphi \in C^k_{\infty\omega}$ that characterizes G up to isomorphism. I leave it to the reader to find such a φ.

We say that two colored graphs G, H are $C^k_{\infty\omega}$*-equivalent* if they satisfy the same $C^k_{\infty\omega}$-sentences. Remember that $\gamma^k(G)$ denotes the multiset of colors of G computed by the k-dimensional WL-algorithm.

LEMMA 2 (CAI, FÜRER, AND IMMERMAN [7]). *For all $k \geq 1$, two colored graphs G, H are $C^{k+1}_{\infty\omega}$-equivalent if, and only if, $\gamma^k(G) = \gamma^k(H)$.*

We can now use techniques from logic, in particular so-called *Ehrenfeucht-Fraïssé games*, to derive results about the Weisfeiler-Leman algorithm. Using such games, it is easy to prove that any two r-regular graphs of the same order are $C^2_{\infty\omega}$-equivalent. Furthermore, any two graphs of the same order that are strongly regular with the same parameters are $C^3_{\infty\omega}$-equivalent. Since it is known that not all such graphs are isomorphic, we obtain non-isomorphic graphs that cannot be distinguished by the 2-dimensional Weisfeiler-Leman algorithm. I do not know any natural examples of graphs that are $C^4_{\infty\omega}$-equivalent, but not isomorphic.
However, Cai, Fürer and Immerman proved the following strong theorem:

THEOREM 3 (CAI, FÜRER, AND IMMERMAN [7]). *For all $k \geq 1$ there are graphs G_k, H_k that are $C^k_{\infty\omega}$-equivalent, but not isomorphic. Furthermore, the graphs can be chosen to be of order $O(k)$ and 3-regular.*

Thus Cai, Fürer, and Immerman used Lemma 2 to find non-isomorphic graphs that cannot be distinguished by the Weisfeiler-Leman algorithm. We use the same lemma to prove that many interesting graphs can be distinguished by the algorithm.

Definition 1. Let $k \geq 1$ and $\mathcal{C}$ a class of graphs. We say that $\mathcal{C}$ is $C^k_{\infty\omega}$*-characterizable* if for every colored graph G whose underlying graph is in $\mathcal{C}$ there is a $C^k_{\infty\omega}$-sentence that characterizes G up to isomorphism.

It follows from Lemma 2 that if a class $\mathcal{C}$ of graphs is $C^k_{\infty\omega}$-characterizable, then two graphs $G, H \in \mathcal{C}$ are isomorphic if, and only if, $\gamma^{k-1}(G) = \gamma^{k-1}(H)$. Thus the $(k-1)$-dimensional Weisfeiler-Leman algorithm gives rise to a polynomial time isomorphism algorithm for graphs in $\mathcal{C}$. The interesting, and somewhat surprising fact behind this approach is that when characterizing a graph we can use the full expressive power of the infinitary logic $C^k_{\infty\omega}$, without having to worry about conciseness and efficiency. The logical machinery behind Lemma 2 takes care of transforming our sentence into an efficient algorithm.

THEOREM 4. (1) (IMMERMAN AND LANDER [21]) *The class of trees is $C^2_{\infty\omega}$-characterizable.*
(2) (GROHE [14]) *There is a $k \geq 1$ such that the class of planar graphs is $C^k_{\infty\omega}$-characterizable.*
(3) (GROHE AND MARIÑO [15]) *For every $k \geq 1$, the class of graphs of tree-width at most k is $C^{k+2}_{\infty\omega}$-characterizable.*

For later reference, let us state the following lemma that is proved in [14]:

LEMMA 5. *Let $k \geq 3$ and $\mathcal{C}$ a class of graphs such that the class $\{G \in \mathcal{C} \mid G$ 3-connected$\}$ is $C^k_{\infty\omega}$-characterizable. Then $\mathcal{C}$ is $C^{k+2}_{\infty\omega}$-characterizable.*

5. EMBEDDINGS OF GRAPHS

A *surface* is a compact connected 2-manifold. Recall that for each orientable surface S there is a $g \geq 0$ such that S is homeomorphic to the surface S_g obtained by attaching g handles to the sphere. For each non-orientable surface S there is an $h \geq 1$ such that S is homeomorphic to the surface N_h obtained by attaching h crosscaps to the sphere. The number g or h, respectively, is called the *genus* of S. The *Euler characteristic* of a surface S, denoted by $\chi(S)$, is $2-2g$ if S is orientable of genus g, and $2-h$ if if S is non-orientable of genus h.
A simple closed curve γ in S is called *contractible* if it can be contracted to a point, or more formally, if it is homotopic to a constant map. Every simple closed curve on the 2-sphere is contractible, so let us assume that $\chi(S) < 2$. Consider the space $S \setminus \gamma$; it has either one or two connected components. If γ is contractible, then $S \setminus \gamma$ has two connected components, precisely one of which is homeomorphic to an open disk. We call this component the *interior* of γ.
Let $S/_\gamma$ be the space obtained from $S \setminus \gamma$ by sewing a disc on the one or two boundary cycles that arise from cutting out γ. $S/_\gamma$ is either a surface or the disjoint union of two surfaces.
The inductive proof of our main theorem is based on the following well-known lemma:

LEMMA 6. *Let S be a surface and γ a non-contractible simple closed curve in S. Then each component of $S/_\gamma$ is a surface of strictly greater characteristic.*

An *embedding* of a multigraph G into a surface S is a mapping Π that associates distinct points of S with the vertices of G and internally disjoint simple curves in S with the edges of G in such a way that a vertex v is incident with an edge e if, and only if, $\Pi(v)$ is an endpoint of $\Pi(e)$. We let $\Pi(G) = \{\Pi(v) \mid v \in V^G\} \cup \bigcup_{e \in E^G} \Pi(e)$. The *faces* of Π are the connected components of $S \setminus \Pi(G)$. The embedding Π is *2-cell* if all faces are homeomorphic to open disks (topological 2-cells). The set of all faces of Π is denoted by F^Π.
The *Euler formula* says that for every 2-cell embedding Π of a connected multigraph G into a surface S we have $|V^G| - |E^G| + |F^\Pi| = \chi(S)$. If Π is not 2-cell, we still have $|V^G| - |E^G| + |F^\Pi| \geq \chi(S)$.
The boundary of a face f in S is the image of a subgraph of G under Π. We call every $B \subseteq G$ such that $\Pi(B)$ is the boundary of some face $f \in F^\Pi$ a Π*-boundary*. If a Π-boundary is a cycle, we also call it a Π-facial cycle.

Lemma 7. *Let G be a connected multigraph with $val(v) \geq 3$ for all $v \in V^G$. Let Π be an embedding of G into a surface S. Then either $|E^G| \leq -21\chi(S)$ or there exists a Π-boundary with at most 6 edges.*

The proof is a simple application of Euler's formula.

Let G be a 3-connected graph and Π a 2-cell embedding of G into a surface S. Furthermore, let $\mathcal{C}$ be a family of Π-facial cycles. Note that every edge of G is contained in at most 2 cycles in $\mathcal{C}$. Let us call two cycles in $\mathcal{C}$ *adjacent* if they are distinct and have an edge in common. A *region (of G with respect to $\mathcal{C}$)* is a subgraph of G that can be written as the union of all cycles in a connected component of $\mathcal{C}$ with respect to adjacency. More formally, R is a region if there exists an $\mathcal{R} \subseteq \mathcal{C}$ such that $\mathcal{R}$ is a connected component of the graph defined on $\mathcal{C}$ by the adjacency relation, and $R = \bigcup_{C \in \mathcal{R}} C$.

We define two subgraphs $I, X \subseteq G$, the *interior* and *exterior* with respect to $\mathcal{C}$ as follows: E^I is the set of all edges of G that are contained in at least one cycle in $\mathcal{C}$ and V^I the set of all vertices of G that are incident with an edge in E^I. Similarly, E^X is the set of all edges of G that are contained in at most one cycle in $\mathcal{C}$ and V^X the set of all vertices of G that are incident with an edge in E^X. Note that the graph I is the union of all regions.

Recall that a branching vertex is a vertex of valence at least 3.

Lemma 8. *Let G be a 3-connected graph and Π an embedding of G in a surface S of characteristic χ. Let $\mathcal{C}$ be a family of Π-facial cycles of G. Suppose that the graph X has more than -42χ branching vertices. Then there is a Π-boundary $B \subseteq X$ that contains at most 12 branching vertices of X.*

Proof. The idea of the proof is to contract all regions of G to single vertices, thus obtaining a new graph G'. The embedding Π gives rise to an embedding Π' of G', and by Lemma 7, there is a Π'-boundary with at most 6 edges. Going back to G, this Π'-boundary induces a Π-boundary with at most 12 branching vertices.

For every $C \in \mathcal{C}$, let f_C be a face whose Π-boundary is C and $\overline{f}_C := f \cup \Pi(C)$. For every region R of G with respect to $\mathcal{C}$, the set

$$\Pi(R) := \bigcup_{\substack{C \in \mathcal{C} \\ C \subseteq R}} \overline{f}_C$$

is a closed connected submanifold of S with boundary. Its boundary is a family of disjoint simple closed curves in S, each of which is the image of a cycle in $I \cap X$. We call these cycles the *boundary cycles of R.*

Let G' be the graph defined as follows: The vertex set $V^{G'}$ consists of all branching vertices of X together with a new vertex v_R for every region R. The vertex set $E^{G'}$ consists of all edges in $E^X \setminus E^I$ and a new edge between v_R and all vertices of G' on the boundary of R.

The graph G' is connected, and every vertex v_R, corresponding to a region R, has valence at least 3 (because G is 3-connected). All vertices of valence ≤ 2 in G' lie on the boundary of some region R. Let G'' be the multigraph whose vertices are all vertices of G' valence at least 3 and whose edges are all chains of G' (with the obvious incidence relation).

The embedding Π gives rise to an embedding Π' of G' into S obtained as follows: For every $v \in V^X$, let $\Pi'(v) = \Pi(v)$, and for every edge $e \in E^X \setminus E^I$, let $\Pi'(e) = \Pi(e)$. For every region R, let $\Pi(v_R)$ be an arbitrary point in the interior of $\Pi(R)$. Since the interior of $\Pi(R)$ is connected, there are internally disjoint simple curves in the interior of $\Pi(R)$ connecting $\Pi(v_R)$ with the images of its neighbors in G'. We choose such curves as the images of the edges incident with v_R. Π' canonically gives rise to an embedding Π'' of G'' into S.

I claim that G'' has more than -21χ edges. Let Y be the set of all branching vertices of X. We partition Y in the set $Y_{\geq 3}$ of all vertices that have valence at least 3 in G' (hence are also vertices of G'') and $Y_{=2} := Y \setminus Y_{\geq 3}$. Every chain K in G' is of one of the following three forms:

(1) $K = vew$, where $v \in Y_{\geq 3}$ and $w \in Y_{\geq 3}$ or $w = v_{R'}$ for a region R',

(2) $K = v_R eve'w$, where R is a region, $v \in Y_{=2}$, and $w \in Y_{\geq 3}$ or $w = v_{R'}$ for a region R',

(3) $K = v_R eve'v'e''v_{R'}$, where R, R' are regions and $v, v' \in Y_{=2}$.

This implies that a chain contains at most 2 branching vertices of X, and we have $2|E^{G''}| \geq |Y|$. Since $|Y| > -42\chi$, the claim follows.

Thus by Lemma 7, there is face f whose Π''-boundary B'' has at most 6 edges. Obviously, f is also of face of Π'; its Π'-boundary B' is obtained by replacing some edges in B'' by chains.

Consider the open set

$$g := f \setminus \bigcup_{R \text{ region}} \Pi(R).$$

It is not hard to see that $g \neq \emptyset$, because G' does not contain the edges that form the boundary of the regions. Let f_0 be a connected component of g. Then f_0 is a face of Π. Let B be the Π-boundary of f_0. We shall prove that B contains at most 12 branching vertices of X.

B consists of pieces of B' and pieces of boundary cycles of the regions. Note that whenever an interval $v_1e_1v_2e_2v_3$ that is part of a boundary cycle C of some region R is contained in B, then v_2 is not a branching vertex of X, because if it was, it would be incident with an edge e that does not belong to R. But with $\Pi(R)$ on one side and f_0 on the other, there would be no room for $\Pi(e)$ (see Figure 1).

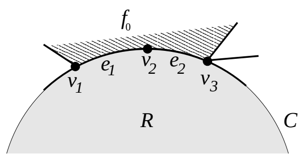

Figure 1.

Hence without loss of generality we can assume that B is not just a boundary cycle of some region, because if it was it would contain no branching vertices of X at all.

Suppose that B contains the interval $v_1e_1 \dots v_l$ of some boundary cycle C of a region R, and that v_1 and v_l are

branching vertices of X. Then G' has an edge e^1 from v_1 to v_R and an edge e^l from v_l to v_R. I claim that $v_l, v_p, v_R \in V^{B'}$ and $e^1, e^l \in E^{B'}$. This can be seen as follows: Let x be an interior point of $\Pi(e_1)$. Then $x \in f$, because e_1 is not an edge of G'. It is not hard to see that for every point $y \in \Pi'(e^1) \cup \Pi'(e^l)$ there is a simple curve in $\Pi(R)$ that connects x with y and does not intersect $\Pi'(G')$ otherwise. f contains the interior of this curve and has $\Pi(y)$ in its boundary. Thus the edges e^1, e^l and their endpoints v_l, v_p, and v_R belong to the boundary of f and hence to B'.
Furthermore, every vertex of B that does not belong to a boundary cycle of some region is also a vertex of B'. Recall that Y denotes the set of all branching vertices of X. We have just proved that $Y \cap V^B \subseteq Y \cap V^{B'}$. Since every chain in G' contains at most 2 branching vertices of X, we have $|V^{B'} \cap Y| \leq 2|E^{B''}| \leq 12$. □

It is well-known that all boundaries of a 2-connected graph embedded into the 2-sphere (or equivalently, into the plane) are cycles. This is no longer true for embeddings into higher surfaces. The following lemma serves as a substitute. A cycle C of G is Π-*contractible* if the simple closed curve $\Pi(C)$ in S is contractible, and Π-*non-contractible* otherwise.
The proof of the next lemma is a bit tedious, but straighforward.

LEMMA 9. *Let G be a 2-connected graph and Π an embedding of G into a surface S. Then for each Π-boundary B, either B is a cycle or there is a Π-non-contractible cycle $C \subseteq B$.*

A cycle C in a graph G is *chordless*, if it is an induced subgraph, and it is non-separating if $G \setminus C$ is connected. Tutte [30] proved that the boundaries of a 3-connected graph G embedded into the 2-sphere are precisely the non-separating chordless cycles of G. The following three lemmas serve as a higher order substitute of this fact.

LEMMA 10. *Let G be a connected graph and C a cycle of G that is non-separating and chordless. Let Π be an embedding of G into a surface S. Then C is either Π-non-contractible or Π-facial.*

PROOF. Suppose that C is Π-contractible. Then $S \setminus \Pi(C)$ has two components S_1 and S_2. Since C is non-separating and chordless, either $\Pi(G) \cap S_1$ or $\Pi(G) \cap S_2$ must be empty, thus C is Π-facial. □

An embedding Π of a graph G into a surface S is *minimal* if G cannot be embedded into a surface S' with $\chi(S') > \chi(S)$. A cycle C in a graph G is *reducing* if for every embedding Π of G into a surface S, every component of $G \setminus C$ can be be embedded into a surface S' with $\chi(S') > \chi(S)$. Of course, to show that a cycle is reducing, it suffices to consider minimal embeddings.
The following Lemma is an immediate consequence of Lemma 6.

LEMMA 11. *Let G be a graph, C a cycle of G, and Π a minimal embedding of G into a surface S such that C is Π-non-contractible. Then C is reducing.*

LEMMA 12. *Let G be a 3-connected graph and Π an embedding of G into a surface S. Let C be a non-reducing Π-facial cycle. Then C is non-separating and chordless.*

PROOF. Without loss of generality we can assume that S is not the sphere. By Lemma 11, C is Π-contractible, thus the interior I of $\Pi(C)$ is homeomorphic to an open disk.
Suppose for contradiction that C separates G, and let H be a connected component of $G \setminus C$ that cannot be embedded into a surface S' with $\chi(S') > \chi(S)$. Let w be a vertex of $G \setminus (C \cup H)$. Since G is 3-connected, by Menger's theorem there are three internally disjoint paths P_1, P_2, P_3 from w to a vertex of H. Since C separates w from H, these paths intersect C. For $1 \leq i \leq 3$, let v_i be the first vertex on P_i that belongs to C. Let P_i' be the initial segment $w \ldots v_i$ of P_i and P_i'' the end segment starting from v_i.
Let D be the cycle obtained by walking along P_1' from w to v_1, then along C from v_1 to v_2 in such a way that v_3 is not met, and then back to w along P_2'. Then D is Π-contractible, because $H \subseteq G \setminus D$. Let J be the interior of $\Pi(D)$.
Suppose first that $I \subseteq J$. Then v_3 is in the interior of D, H in its exterior, and the path P_3'' connects v_3 with H. This is impossible, as can be seen as follows (cf. Figure 2): We connect the points $\Pi(v_1)$ and $\Pi(v_2)$ by a simple

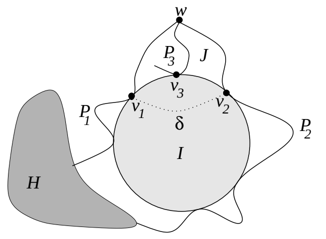

Figure 2.

curve δ through I. We obtain contractible simple closed curve γ in S consisting of $\Pi(P_1')$, δ, and $\Pi(P_2')$. Since the paths P_1, P_2, P_3 are internally disjoint and C is Π-facial, $\Pi(P_3'')$ does not intersect γ. But it connects the interior of γ with its exterior, which is a contradiction. Hence C is non-separating.
If $I \not\subseteq J$, then instead of D we consider the cycle D' obtained by walking along P_1' from w to v_1, then along C from v_1 to v_3 in such a way that v_2 is not met, and then back to w along P_2'. D' is also contractible, but now v_3 must be in its interior, and we can argue analogously.

It remains to prove that $C = v_1 e_1 \ldots v_n e_n v_1$ is chordless. Suppose for contradiction that e is an edge from v_i to v_j, where $1 \leq i < j \leq n$, $j \neq i+1$. Since C is non-reducing, the cycle $C' = v_i e_{i+1} \ldots e_j v_j e v_i$ is also non-reducing, thus by Lemma 11 it is Π-contractible. Arguing similarly as above, we see that the graph $G \setminus \{v_i, v_j\}$ is not connected, contradicting the fact that G is 3-connected. □

6. THE MAIN THEOREM

THEOREM 13. *For every surface S there is a $k \geq 1$ such that the class of all graphs embeddable into S is $\mathrm{C}^k_{\infty\omega}$-characterizable.*

PROOF. For $\chi \leq 2$, let $k(\chi)$ be the least natural number k such that the class of all graphs embeddable into a surface of characteristic χ is $\mathrm{C}^k_{\infty\omega}$-characterizable, or ∞ if no such k

exists. To keep things formally correct, we let $\mathrm{C}^{\infty}_{\infty\omega} := \mathrm{C}_{\infty\omega}$; recall from Example 3 that every colored graph can be characterized up to isomorphism by a $\mathrm{C}^{\infty}_{\infty\omega}$-sentence. Inductively, we show that $k(\chi)$ is finite for all $\chi \leq 2$. For $\chi = 2$, this is just Theorem 4(2), but our proof does not depend on this theorem.

It is easy to see that $k(\chi) \geq 3$ for all $\chi \leq 2$.

Let G be a colored graph that is embeddable into a surface of characteristic $\chi \leq 2$, but not into any surface of characteristic $> \chi$. Because of Lemma 5, we can assume without loss of generality that G is 3-connected.

Let $k, l \geq 1$ and $W \subseteq V^G$. We say that W is *k-distinguishable with l parameters* if there are $v_1, \ldots, v_l \in V^G$ such that for every $w \in W$ there is a $\mathrm{C}^k_{\infty\omega}$-formula $\varphi_w(x_1, \ldots, x_k, y)$ such that

$$\{v \in V^G \mid G \models \varphi_w(v_1, \ldots, v_k, v)\} = \{w\}.$$

We say that a subgraph of G is *k-distinguishable with l parameters* if the set of its vertices is.

LEMMA 14. *Let $l \geq 1$, $k \geq l + 2$ and suppose that G itself is k-distinguishable with l-parameters. Then G can be characterized up to isomorphism by a $\mathrm{C}^k_{\infty\omega}$-sentence.*

PROOF. Suppose that $V^G = \{v_1, \ldots, v_n\}$, where $v_i \neq v_j$ for $1 \leq i < j \leq n$.

Let $\bar{u} \in (V^G)^l$ and, for $1 \leq i \leq n$, let $\varphi_i(\bar{x}, y)$ be a $\mathrm{C}^k_{\infty\omega}$-formula such that $\{w \mid G \models \varphi_i(\bar{u}, w)\} = \{v_i\}$.

We let

$$\psi(\bar{x}) = \bigwedge_{i=1}^{n} \exists^{=1} y \varphi_i(\bar{x}, y) \wedge \forall y \bigvee_{i=1}^{n} \varphi_i(\bar{x}, y).$$

For $1 \leq i \leq n$, we let

$$\alpha_i(y) = \bigwedge_{\substack{j \geq 1 \\ v_i \in V_j^G}} V_j(y) \wedge \bigwedge_{\substack{j \geq 1 \\ v_i \notin V_j^G}} \neg V_j(y),$$

and for $1 \leq i < j \leq n$, we let

$$\beta_{ij}(y, z) = \bigwedge_{\substack{m \geq 1 \\ (v_i, v_j) \in E_m^G}} E_m(y, z) \wedge \bigwedge_{\substack{m \geq 1 \\ (v_i, v_j) \notin E_m^G}} \neg E_m(y, z).$$

Then

$$\exists \bar{x} \Big(\psi(\bar{x}) \wedge \bigwedge_{1 \leq i \leq m} \exists y \big(\varphi_i(\bar{x}, y) \wedge \alpha_i(y) \big) \wedge \bigwedge_{1 \leq i < j \leq m} \exists y \exists z \big(\varphi_i(\bar{x}, y) \wedge \varphi_j(\bar{x}, z) \wedge \beta_{ij}(y, z) \big) \Big)$$

characterizes G up to isomorphism. □

Recall the notion of a *reducing cycle* of G from Page 6.

LEMMA 15. *Let $l \geq 1$ and $k \geq l + 2$. Suppose that G has a reducing cycle C that is k-distinguishable with the l parameters. Then G can be characterized up to isomorphism by a $\mathrm{C}^{\max(k, k(\chi-1)+l+2)}_{\infty\omega}$-sentence.*

PROOF. Say, $C = v_1 e_1 \ldots v_m e_m v_1$. Choose an l-tuple $\bar{u} \in (V^G)^l$ and, for $1 \leq i \leq l$, a $\mathrm{C}^k_{\infty\omega}$-formula $\varphi_i(\bar{x}, y)$ such that $\{v \mid G \models \varphi_i(\bar{u}, v)\} = \{v_i\}$. Let $\nu(\bar{x}) := \bigwedge_{i=1}^m \exists^{=1} y \varphi_i(\bar{x}, y)$. Whenever we refer to a graph G' and a tuple $\bar{u}' \in (V^{G'})^l$ such that $G' \models \nu(\bar{u}')$ below, we let v_i' denote the unique vertex of G' such that $G' \models \varphi_i(\bar{u}', v_i')$.

Let $H_1, \ldots, H_h$ be the components of $G \setminus C$. Since C is reducing, every H_i is embeddable in a surface of characteristic $> \chi$.

Without loss of generality, we can assume that $\tau(G) = \{V_1, \ldots, V_r, E_1, \ldots, E_s\}$. For $1 \leq q \leq s$ and $1 \leq i \leq m$, let $W_{qi} := V_{r+m(q-1)+i}$. For $1 \leq i \leq h$, let H_i^* be the colored graph of vocabulary $\tau(G) \cup \{V_{r+1}, \ldots, V_{r+ms}\}$ with $E_q^{H_i^*} := E_q^{H_i}$ for all $q \geq 1$, $V_p^{H_i^*} := V_p^{H_i}$ for all $p \notin \{r+1, \ldots, r+ms\}$, $W_{qj}^{H_i^*} := \{w \in V^{H_i} \mid E_q^G(v_j, w)\}$ for all q, j with $1 \leq q \leq s$, $1 \leq j \leq m$. Note that H_i^* has the same underlying graph as H_i. Let ψ_i be a $\mathrm{C}^{k(\chi-1)}_{\infty\omega}$-sentence that characterizes H_i^* up to isomorphism; such a sentence exists by induction hypothesis. Let $x_1, \ldots, x_l, y, z$ be variables not occurring in $\psi_1, \ldots, \psi_h$. We shall define a $\mathrm{C}^{\max(k, l+3)}_{\infty\omega}$-formula $\kappa(\bar{x}, z, z')$ such that for all graphs G' and $\bar{u}' \in (V^{G'})^l$ such that $G' \models \nu(\bar{u}')$, and for all $v, w \in V^{G'}$ we have $G' \models \kappa(\bar{u}', v, w)$ if, and only if v and w belong to the same connected component of $G' \setminus \{v_1', \ldots, v_m'\}$. (Recall that v_i' denotes the unique vertex of G' such that $G' \models \varphi_i(\bar{u}', v_i')$.)

We let $\kappa_0(\bar{x}, y, z) := \bigwedge_{i=1}^m \neg\varphi_i(\bar{x}, y) \wedge (y = z)$ and for $i \geq 0$

$$\kappa_{i+1}(\bar{x}, y, z) := \bigwedge_{i=1}^{m} \neg\varphi_i(\bar{x}, z) \wedge \exists z' \big(\kappa_i(\bar{x}, y, z') \wedge E(z', z) \big).$$

Then we let $\kappa(\bar{x}, y, z) := \bigvee_{i \geq 0} \kappa_i(\bar{x}, y, z)$.

In particular, for $v, w \in V^G$ we have $G \models \kappa(\bar{u}, v, w)$ if, and only if, v and w belong to the same connected component of $G \setminus C$.

For $1 \leq i \leq h$, let $\psi_i'(\bar{x}, y)$ be the formula obtained from ψ_i by *relativizing* all quantifiers to $\kappa(\bar{x}, y, \_)$, that is, by replacing each subformula of the form $\exists^{\geq t} \tilde{z} \xi$ by $\exists^{\geq t} \tilde{z} (\kappa(\bar{x}, y, \tilde{z}) \wedge \xi)$. Then for all colored graphs G' and $\bar{u}', v \in V^{G'}$ such that $G \models \nu(\bar{u}')$ we have $G' \models \psi_i'(\bar{u}', v)$ if, and only if, the connected component of $G' \setminus \{v_1', \ldots, v_m'\}$ that contains v is isomorphic to H_i^*. By suitably renaming the variables, we can achieve that $\psi_i \in \mathrm{C}^{\max(k, k(\chi-1)+l+1)}_{\infty\omega}$.

Now suppose that $H_1^*, \ldots, H_{h'}^*$ are pairwise non-isomorphic and for $h' + 1 \leq i \leq h$ there exists a $j, 1 \leq j \leq h'$ such that H_i^* is isomorphic to H_j^*. For $1 \leq i \leq h'$, let h_i be the number of j such that $1 \leq j \leq h$ and H_j^* is isomorphic to H_i^*. Let $h_i^\times := h_i \cdot |V^{H_i^*}|$. Then $\sum_{i=1}^{h'} h_i = h$ and $\sum_{i=1}^{h'} h_i^\times = |V^G| - m$. Let

$$\psi(\bar{x}) := \Big(\bigwedge_{i=1}^{h'} \exists^{=h_i^\times} y \, \psi_i'(\bar{x}, y) \wedge \forall y \big(\bigvee_{i=1}^{m} \varphi_i(\bar{x}, y) \vee \bigvee_{i=1}^{h'} \psi_i'(\bar{x}, y) \big) \Big).$$

For all colored graphs G' and $\bar{u}' \in (V^{G'})^l$ such that $G' \models \nu(\bar{u}')$, we have $G' \models \psi(\bar{u}')$ if, and only if, the graph $G' \setminus \{v_1', \ldots, v_m'\}$ is isomorphic to the union $H_1^* \cup \ldots \cup H_h^*$.

For $1 \leq i \leq m$, let $\psi'(\bar{x})$ be the formula obtained from $\psi(\bar{x})$ obtained by replacing each subformula of the form $W_{qi} z$ by $\exists y (\varphi_i(\bar{x}, y) \wedge E_q y z)$ (for $1 \leq q \leq s$, $1 \leq i \leq m$).

It remains to describe the subgraph induced on $\{v_1, \ldots, v_m\}$. It is easy to define a $\mathrm{C}^k_{\infty\omega}$-formula $\theta(\bar{x})$ such that for all colored graphs G' and $\bar{u}' \in (V^{G'})^l$ such that $G' \models \nu(\bar{u}')$, we have $G' \models \theta(\bar{u}')$ if, and only if, the induced subgraph

$\langle\{v'_1,\dots,v'_m\}\rangle^{G'}$ is isomorphic to $\langle\{v_1,\dots,v_m\}\rangle^{G}$ (this can be proved as Lemma 14).
Then the $\mathrm{C}_{\infty\omega}^{\max(k,k(\varphi)+l+2)}$-sentence $\varphi_G := \exists\bar{x}\big(\nu(\bar{x})\wedge\theta(\bar{x})\wedge\psi'(\bar{x})\big)$ characterizes G up to isomorphism. □

A *facial cycle* of G is a cycle that is Π-facial for every minimal embedding of G. Let $\mathcal{C}$ be a family of facial cycles of G. We can define the notion of a region of G with respect to $\mathcal{C}$ and the subgraphs I, X in the same way as we did it with respect to a family of Π-facial cycles on Page 5. But note that now these notions do not depend on any particular embedding.
We call a family $\mathcal{C}$ of cycles of G *k-definable* (for a $k\geq 1$) if there is a $\mathrm{C}^k_{\infty\omega}$-formula $\xi(z_1,\dots,z_4)$ such that for all $w_1,\dots,w_4\in V^G$ we have $G\models\xi(w_1,\dots,w_4)$ if, and only if, there is a cycle in $C\in\mathcal{C}$ on which $w_1,\dots,w_4$ appear in this order. (More precisely, this means that C can be written as

$$w_1e_1w_2e_2w_3e_3w_4e_4w_5\dots w_me_mw_1$$

for a suitable $m\geq 0$, $w_5,\dots,w_m\in V^G$, and $e_1,\dots,e_m\in E^G$. We allow the case that C is a triangle and $w_4=w_1$.)
This notion k-definability is motivated by the observation that two facial cycles in a 3-connected graph have at most two consecutive vertices in common. So by fixing three consecutive vertices, we completely fix a facial cycle; the fourth free variable of ξ can be used to step further through this cycle. These remarks will become clearer in the proof of the following lemma.

LEMMA 16. *Let $k\geq 1$ and $\mathcal{C}$ a k-definable family of facial cycles of G.*
Then every region of $\mathcal{C}$ is $(k+3)$-distinguishable with 3 parameters.

PROOF. Let $C_1,\dots,C_m$ be the cycles in $\mathcal{C}$ forming region R, ordered in such a way that for $1\leq i\leq(m-1)$ there exists j, $1\leq j\leq i$ such that C_{i+1} and C_j have an edge in common. Such an ordering of the cycles exists by the definition of a region.
Say, $C_1=v_1e_1\dots v_re_rv_1$. For $1\leq i\leq 3$, let

$$\varphi_{v_i}(x_1,x_2,x_3,y)=(y=x_i).$$

For $4\leq i\leq r$ let

$$\begin{aligned}\varphi_{v_i}(x_1,x_2,x_3,y):=\exists z_1\exists z_2\exists z_3\exists z_4\Big(&\varphi_{v_{i-3}}(z_1)\\ &\wedge\varphi_{v_{i-2}}(z_2)\\ &\wedge\varphi_{v_{i-1}}(z_3)\\ &\wedge(z_4=y)\\ &\wedge\xi(z_1,z_2,z_3,z_4)\Big).\end{aligned}$$

Now suppose that, inductively, we have already defined formulas φ_v for all vertices v appearing on $C_1,\dots,C_i$. Choose $j\leq i$ such that C_{i+1} has an edge, say (w_1,w_2) in common with C_j. Let u be the vertex following w_1,w_2 on C_j. Suppose that $C_{i+1}=w_1e_1w_2e_2\dots w_se_sw_1$. Let

$$\begin{aligned}\varphi_{w_3}(x_1,x_2,x_3,y):=&\neg\varphi_u(x_1,x_2,x_3,y)\\ &\wedge\exists z_1\exists z_2\exists z_3\exists z_4\big(\ \varphi_{w_1}(x_1,x_2,x_3,z_1)\\ &\qquad\wedge\varphi_{w_2}(x_1,x_2,x_3,z_2)\\ &\qquad\wedge(z_3=y)\\ &\qquad\wedge\xi(z_1,z_2,z_3,z_4)\big).\end{aligned}$$

To see that this defines w_3, recall that C_j and C_{i+1} are the only cycles in $\mathcal{C}$ on which the edge (w_1,w_2) appears. Thus lines 2–4 of the formula says that y is the vertex following w_1,w_2 on either C_j or C_{i+1}. Line 1 rules out C_j. For $4\leq j\leq s$ we let

$$\begin{aligned}\varphi_{w_j}(x_1,x_2,x_3,y):=\exists z_1\exists z_2\exists z_3\exists z_4\big(&\varphi_{w_{i-3}}(x_1,x_2,x_3,z_1)\\ &\wedge\varphi_{w_{i-2}}(x_1,x_2,x_3,z_2)\\ &\wedge\varphi_{w_{i-1}}(x_1,x_2,x_3,z_3)\\ &\wedge(z_4=y)\\ &\wedge\xi(z_1,z_2,z_3,z_4)\big).\end{aligned}$$

□

LEMMA 17. *Let $l\geq 1$, $k\geq 2l+2$, and $\mathcal{C}$ a k-definable family of facial cycles of G. Then every cycle $C\subseteq X$ that contains at most l branching vertices is $\mathrm{C}^k_{\infty\omega}$-distinguishable with $2l$ parameters.*

PROOF. Let $C\subseteq X$ be a cycle which contains l branching vertices. We only consider the case $l\geq 1$; it will be obvious how to modify our proof for the case $l=0$. Let $v_{10},\dots,v_{l0}$ be the branching vertices on C, in the order in which they appear on C. To simplify the notation, we let $v_{(l+1)0}=v_{10}$. For $1\leq i\leq l$, let $v_{i0}e_{i0}v_{i1}e_{i1}\dots v_{im_i}e_{im_i}v_{im_i+1}=v_{(i+1)0}$, for an $m_i\geq 0$, be a path from v_{i0} to $v_{(i+1)0}$ in X such that for $1\leq j\leq m_i$, v_{ij} has valence 2 in X. We show how to distinguish C with the parameters $v_{10},\dots,v_{l0},v_{11},\dots,v_{l1}$.
Let $\xi(z_1,\dots,z_4)$ be a $\mathrm{C}^k_{\infty\omega}$-formula that defines $\mathcal{C}$. Then the $\mathrm{C}^k_{\infty\omega}$-formula

$$\epsilon_X(z_1,z_2):=E(z_1,z_2)\wedge\neg\exists^{\geq 2}z_3\exists z_4\xi(z_1,\dots,z_4)$$

defines the edge-relation of the graph X. Thus the $\mathrm{C}^k_{\infty\omega}$-sentence

$$\beta(z_1):=\exists^{\geq 3}z_2\epsilon_X(z_1,z_2)$$

defines the set of branching vertices of X and the formula

$$\alpha(z_1)=\exists z_2\epsilon_X(z_1,z_2)\wedge\neg\beta(z_2)$$

defines the set of vertices of valence at most 2 in X.
Let $\delta_0(x,y)=(y=x)\wedge\alpha(x)$ and, for $i\geq 0$,

$$\delta_{i+1}(x,y):=\alpha(y)\wedge\neg\delta_{i-1}(x,y)\wedge\exists z(\delta_i(x,z)\wedge\epsilon_X(z,y))$$

(for $i=0$ we omit the subformula $\neg\delta_{i-1}(x,y)$). Then for all $i\geq 1$ and $w,u\in V^G$ we have $G\models\delta_i(w,u)$ if, and only if, there is path of length i from w to u in X on which only vertices of valence 2 appear. If w is a vertex of valence 2 in X that is adjacent to a branching vertex, then there is at most one such path.
For $1\leq i\leq l$, let $\varphi_{i0}(x_1,\dots,x_l,y_1,\dots,y_l,z)=(z=x_i)$ and, for $1\leq j\leq m_i$,

$$\varphi_{ij}(x_1,\dots,x_l,y_1,\dots,y_l,z)=\delta_j(y_i,z).$$

Then clearly for $1\leq i\leq l$, $0\leq j\leq m_i$ we have

$$\{w\mid G\models\varphi_{ij}(v_{10},\dots,v_{l0},v_{11},\dots,v_{l1},w)\}=\{v_{ij}\}.$$

□

LEMMA 18. *Let $l\geq 1, k\geq 2l+3$ and $\mathcal{C}$ a k-definable family of facial cycles of G. Then the set of all chordless and non-separating cycles $C\subseteq X$ with at most l branching vertices is $\mathrm{C}^k_{\infty\omega}$-definable.*

PROOF. Using the $C^k_{\infty\omega}$-formulas $\epsilon_X(z_1, z_2)$, $\beta(z_1)$, and $\alpha(z_1)$ defined in the proof of the previous lemma, it is not hard to see that there is a $C^k_{\infty\omega}$-formula $\varphi(\bar{x}, \bar{y})$ such that for $\bar{v}, \bar{w} \in (V^G)^l$ we have

$$G \models \varphi(v_1, \ldots, v_l, w_1, \ldots, w_l)$$

if, and only if, there is a cycle $C \subseteq X$ such that

- $v_1, \ldots, v_l, w_1, \ldots, w_l \in V^C$,
- for $1 \leq i \leq l$, w_i is a neighbor of v_i on C,
- if we start walking around C starting with v_1, w_1, then the vertices appear in the order
$$v_1, w_1, v_2, w_2, \ldots, v_l, w_l,$$
- the branching vertices of C are among $v_1, \ldots, v_l$.

Arguing as in the proof of the previous lemma, we see that the choice of vertices $v_1, \ldots, v_l, w_1, \ldots, w_l$ uniquely determines C. It is not hard to write down a $C^{2l+1}_{\infty\omega}$-formula $\psi(\bar{x}, \bar{y}, z)$ such that for all $\bar{v}, \bar{w} \in (V^G)^l$ with $G \models \varphi(\bar{v}, \bar{w})$ we have

$$\{u \mid G \models \psi(\bar{v}, \bar{w}, u)\} = V^C,$$

where C is the cycle determined by $\bar{v}, \bar{w}$.
The cycle C determined by $\bar{v}, \bar{w}$ is chordless if, and only if, $G \models \zeta(\bar{v}, \bar{w})$, where

$$\zeta(\bar{x}, \bar{y}) := \forall z \Big(\psi(\bar{x}, \bar{y}, z) \to \exists^{=2} z' \big(\psi(\bar{x}, \bar{y}, z') \wedge E(z, z') \big) \Big).$$

To say that C is non-separating we define a $C^{2l+3}_{\infty\omega}$-formula $\kappa(\bar{x}, \bar{y}, z, z')$ saying that z, z' belong to the same component of $G \setminus C$; this can be achieved in a similar way as in the proof of Lemma 15. Then

$$\nu(\bar{x}, \bar{y}) := \forall z \forall z' \Big(\kappa(\bar{x}, \bar{y}, z, z') \vee \big(\psi(\bar{x}, \bar{y}, z) \wedge \psi(\bar{x}, \bar{y}, z') \big) \Big)$$

says that the cycle determined by $\bar{x}, \bar{y}$ is chordless.
The formula

$$\begin{aligned} \xi(x_1, y_1, z_3, z_4) := \\ \exists x_2 \ldots \exists x_l \exists y_2 \ldots \exists y_l \Big(& \varphi(\bar{x}, \bar{y}) \wedge \zeta(\bar{x}, \bar{y}) \wedge \nu(\bar{x}, \bar{y}) \\ & \wedge \psi(\bar{x}, \bar{y}, z_3) \wedge \psi(\bar{x}, \bar{y}, z_4) \\ & \wedge E(y_1, z_3) \wedge E(z_3, z_4) \Big) \end{aligned}$$

defines the desired class of cycles. □

Now we are ready to put things together and prove Theorem 13. Let

$$K := \max(-42\chi, 12).$$

We inductively define a sequence $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_m$ of families of facial cycles of G in such a way that every $\mathcal{C}_i$ is $(2K+3)$-definable. The number m will be determined later. Once $\mathcal{C}_m$ is defined, we use it to characterize G. We let $\mathcal{C}_0 := \emptyset$.
Suppose now that $\mathcal{C}_i$ is already defined for some $i \geq 0$.
Case 1: $X = \emptyset$ (here X is defined with respect to $\mathcal{C}_i$).
This means that there is just one region with respect to $\mathcal{C}_i$, G itself. Thus by Lemma 16, G is $(2K+6)$-distinguishable with 3 parameters. By Lemma 14, this means that G can be characterized up to isomorphism by a $C^{2K+6}_{\infty\omega}$-sentence. We let $m := i$ and we are done.

In the following, we assume that $X \neq \emptyset$. This implies that for every minimal embedding Π of G there is at least one Π-boundary $B \subseteq X$.

Case 2: G has a reducing cycle C that contains at most K branching vertices of X.
Then by Lemma 17, C is $C^{2K+2}_{\infty\omega}$-distinguishable with 2K parameters. Thus by Lemma 15, G can be characterized up to isomorphism by a $C^{\max(2K+2, k(\chi-1)+2K+2)}_{\infty\omega}$-sentence. Again we let $m := i$ and we are done.

Case 3: G has no reducing cycle that contains at most K branching vertices of X.
Let Π be a minimal embedding of G.
If X has more than -42χ branching vertices, let $C \subseteq X$ be a Π-boundary with at most 12 branching vertices; such a C exists by Lemma 8. If G has at most -42χ branching vertices, let $C \subseteq X$ be an arbitrary Π-boundary. Either way, C has at most K branching vertices. Since there are no reducing cycles with at most K branching vertices, by Lemma 9 C is actually a Π-facial cycle.
We let $\mathcal{C}_{i+1}$ be the union of $\mathcal{C}_i$ with the set of all chordless and non-separating cycles $C \subseteq X$ with at most K branching vertices. $\mathcal{C}_{i+1}$ is $C^{2K+3}_{\infty\omega}$-definable by Lemma 18.
By Lemma 11, G has no Π-non-contractible cycle $C \subseteq X$ with at most K branching vertices. Thus by Lemma 10, every cycle in $\mathcal{C}_{i+1}$ is Π-facial. Since the definition of $\mathcal{C}_{i+1}$ does not depend on the choice of Π, every cycle in $\mathcal{C}_{i+1}$ is facial. On the other hand, by Lemma 12, every Π-facial cycle $C \subseteq X$ with at most K branching vertices is contained in $\mathcal{C}_{i+1}$. Since such a cycle exists, this implies $\mathcal{C}_{i+1} \supset \mathcal{C}_i$.

Since we cannot increase the size of the $\mathcal{C}_i$ forever, we will eventually end up in either Case 1 or 2. □

7. FURTHER RESEARCH

Inspection of our proof shows that $k(\chi)$, the minimal number of variables needed to characterize all graphs embeddable in a surface of characteristic χ, is bounded by a linear function in χ; Theorem 3 also yields a linear lower bound (because every graph G can be embedded into the orientable surface of genus $|E|^G$). It would be interesting, though probably difficult, to determine the $k(\chi)$ exactly.
The fact that we have a linear lower bound for $k(\chi)$ does not mean that there cannot be a linear time algorithm to decide isomorphism of graphs embeddable in a surface S. It may actually be possible to combine the methods we developed here with the linear time embedding algorithm of Mohar [27] to obtain such an algorithm.
Finally, we may ask on which further classes of graphs the WL-algorithm can be used to decide isomorphism. We say that a class $\mathcal{C}$ of graphs has an *excluded minor* if there is at least one graph H that is not a minor of any graph in $\mathcal{C}$. I conjecture that for every class $\mathcal{C}$ with an excluded minor there is a k such that $\mathcal{C}$ is $C^k_{\infty\omega}$-characterizable.

8. REFERENCES

[1] L. Babai. Moderately exponential bound for graph isomorphism. In F. Gcseg, editor, *Fundamentals of Computation Theory, FCT'81*, volume 117 of *Lecture Notes in Computer Science*, pages 34–50. Springer-Verlag, 1981.

[2] L. Babai, D. Y. D.Y. Grigoryev, and D. Mount. Isomorphism of graphs with bounded eigenvalue multiplicity. In *Proceedings of the 14th ACM Symposium on Theory of Computing*, pages 310–324, 1982.

[3] L. Babai, P. Erdös, and S. Selkow. Random graph isomorphism. *SIAM Journal on Computing*, 9:628–635, 1980.

[4] L. Babai and L. Kučera. Canonical labelling of graphs in linear average time. In *Proceedings of the 20th Annual IEEE Symposium on Foundations of Computer Science*, pages 39–46, 1979.

[5] H. Bodlaender. Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees. *Journal of Algorithms*, 11:631–643, 1990.

[6] R. Boppana, J. Hastad, and S. Zachos. Does co-NP have short interactive proofs? *Information Processing Letters*, 25:127–132, 1987.

[7] J. Cai, M. Fürer, and N. Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica*, 12:389–410, 1992.

[8] A. Chandra and D. Harel. Structure and complexity of relational queries. *Journal of Computer and System Sciences*, 25:99–128, 1982.

[9] H.-D. Ebbinghaus and J. Flum. *Finite Model Theory.* Springer-Verlag, second edition, 1995.

[10] S. Evdokimov, M. Karpinski, and I. Ponomarenko. On a new high dimensional Weisfeiler-Lehman algorithm. *Journal of Algebraic Combinatorics*, 10:29–45, 1999.

[11] S. Evdokimov and I. Ponomarenko. On highly closed cellular algebras and highly closed isomorphism. *Electronic Journal of Combinatorics*, 6:#R18, 1999.

[12] I. S. Filotti and J. N. Mayer. A polynomial-time algorithm for determining the isomorphism of graphs of fixed genus. In *Proceedings of the 12th ACM Symposium on Theory of Computing*, pages 236–243, 1980.

[13] M. Grohe. Finite-variable logics in descriptive complexity theory. *Bulletin of Symbolic Logic*, 4:345–399, 1998.

[14] M. Grohe. Fixed-point logics on planar graphs. In *Proceedings of the 13th IEEE Symposium on Logic in Computer Science*, pages 6–15, 1998.

[15] M. Grohe and J. Mariño. Definability and descriptive complexity on databases of bounded tree-width. In C. Beeri and P. Buneman, editors, *Proceedings of the 7th International Conference on Database Theory*, volume 1540 of *Lecture Notes in Computer Science*, pages 70–82. Springer-Verlag, 1999.

[16] C. Hoffmann. *Group-theoretic algorithms and graph isomorphism*, volume 136 of *Lecture Notes in Computer Science.* Springer-Verlag, 1982.

[17] J. Hopcroft and J. Wong. Linear time algorithm for isomorphism of planar graphs. In *Proceedings of the 6th ACM Symposium on Theory of Computing*, pages 172–184, 1974.

[18] J. E. Hopcroft and R. Tarjan. Isomorphism of planar graphs (working paper). In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations.* Plenum Press, 1972.

[19] J. E. Hopcroft and R. Tarjan. A $v\log v$ algorithm for isomorphism of triconnected planar graphs. *Journal of Computer and System Sciences*, 7:323–331, 1973.

[20] N. Immerman. *Descriptive Complexity.* Springer-Verlag, 1999.

[21] N. Immerman and E. Lander. Describing graphs: A first-order approach to graph canonization. In A. Selman, editor, *Complexity theory retrospective*, pages 59–81. Springer-Verlag, 1990.

[22] D. Lichtenstein. Isomorphism for graphs embeddable on the projective plane. In *Proceedings of the 12th ACM Symposium on Theory of Computing*, pages 218–224, 1980.

[23] E. Luks. Isomorphism of graphs of bounded valance can be tested in polynomial time. *Journal of Computer and System Sciences*, 25:42–65, 1982.

[24] G. Miller. Isomorphism of graphs which are pairwise k-separable. *Information and Control*, 56:21–33, 1983.

[25] G. Miller. Isomorphism of k-contractible graphs. A generalization of bounded valence and bounded genus. *Information and Control*, 56:1–20, 1983.

[26] G. L. Miller. Isomorphism testing for graphs of bounded genus. In *Proceedings of the 12th ACM Symposium on Theory of Computing*, pages 225–235, 1980.

[27] B. Mohar. Embedding graphs in an arbitrary surface in linear time. In *Proceedings of the 28th ACM Symposium on Theory of Computing*, pages 392–397, 1996.

[28] I. N. Ponomarenko. The isomorphism problem for classes of graphs that are invariant with respect to contraction. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 174(Teor. Slozhn. Vychisl. 3):147–177, 182, 1988. In Russian.

[29] U. Schöning. Graph isomorphism is in the low hierarchy. *Journal of Computer and System Sciences*, 37:312–323, 1988.

[30] W. Tutte. How to draw a graph. *Proceedings of the London Mathematical Society*, 13:743–768, 1963.