Jerk, Acceleration, and Speed on Inclined Surfaces

Mateo Patino* and Clinton Nash†

Thornton Academy, Saco, ME 04072, USA

(Dated: January 7, 2025)

Abstract

An algorithm is developed to generate the equations defining inclined surfaces such that objects placed on them experience constant kinematic jerk. Two different types of constant jerk are explored. The first is simply the tangential component of the total jerk and the second, the scalar jerk, is defined as the time rate of change of the tangential component of acceleration. The latter corresponds most directly to the notion of changes in the rate of change of speed. The surfaces so developed are almost parabolic.

I. INTRODUCTION

Galileo's study of the motion of marbles rolling down inclined planes is as good a place as any to mark the beginning of modern physics and even today inclines such as his form the foundation of much introductory instruction in kinematics and dynamics.[1] Among dynamical systems only free-fall is simpler, extensions to non-planar surfaces can be found in the literature and university-level textbooks as can related problems such as the classic brachistochrone.[2] After more than 400 years it might seem unlikely that there is any new insight to be gained from exploring these kinds of problems but this may not be true. Such became clear when a precocious student asked an insightful question; given that a planar surface results in a constant (tangential) acceleration, how would an inclined surface have to be shaped in order to produce a constant tangential jerk? Her wording was imprecise. It eventually became clear that what she meant to understand was a slightly different problem that concerned the rate at which an object would increase or decrease its rate of change of speed as it traveled down an incline, in other words its rate of change of tangential acceleration. Even after clarification, the answers to these questions are not obvious and, as far as we can tell, not to be found explicitly within the canon of elementary mechanics.[3]

If present at all, discussion of jerk in physics textbooks rarely ventures much beyond its amusing name.[4] This is perhaps surprising since, as the time derivative of acceleration, it represents a straightforward extension of standard kinematics.[5] Moreover, jerk is arguably the most easily perceptible kinematic quantity, and its management in a wide variety of scenarios is the focus of much engineering design work, especially in transportation and robotics.[6] Variably inclined surfaces also provide the simplest examples of jerk systems. The topic has most often been approached by assuming an object travels along a defined curve or trajectory and then computing the jerk and/or its components as it does so.[7] Our approach is different. First, we seek the vertical plane curve, $y(x)$, upon which a sliding object experiences a constant value of the tangential component of jerk. It is this component (as opposed to the total jerk which cannot be constant non-zero) that is most directly related to the tangential component of acceleration and therefore the rate of change of the speed along a surface. Second, we seek a different vertical plane curve defined by the property that a sliding object experiences a tangential component of acceleration that changes linearly with time. It is tempting to identify this quantity as the tangential jerk, but it is not the same thing. Because of this we find it convenient to define the term scalar jerk, j_s, to mean the time rate of change of the magnitude of tangential acceleration, a_T. It is this quantity that relates directly to the common sense notion of change in the rate of change of speed, $v = \frac{1}{2}j_s t^2 + a_0 t + v_0$ and the distance an object travels along the curve, $s = \frac{1}{6}j_s t^3 + \frac{1}{2}a_0 t^2 + v_0 t$, as functions of time. Our purpose here is to define and distinguish the relationship between these jerks (or jerk components) by determining the characteristics of two surfaces, the first corresponding to a constant tangential jerk and the second to a constant scalar jerk.

II. THEORY

A. Components of the Jerk Vector

We consider an object of constant mass m subject only to gravity and a normal force that slides without friction down an inclined surface, $y(x)$, having a non-constant slope (Figure 1). The vector sum of these comprises the net force according to Newton's Second Law and so the instantaneous acceleration of the object can be expressed as,

$$\vec{\boldsymbol{a}} = \frac{1}{m}\vec{\boldsymbol{F}}_{\text{net}} = \frac{1}{m}\left(\vec{\boldsymbol{F}}_N + m\vec{\boldsymbol{g}}\right) = \frac{1}{m}\vec{\boldsymbol{F}}_N + \vec{\boldsymbol{g}}. \tag{1}$$

By definition, Jerk is the time derivative of eq. (1) which, because of constant gravity, is proportional solely to the time rate of change of the normal force, F_N

$$\vec{\boldsymbol{j}} = \frac{d\vec{\boldsymbol{a}}}{dt} = \frac{d}{dt}\left(\frac{1}{m}\vec{\boldsymbol{F}}_N\right) + 0 \tag{2}$$

The normal force vector itself can be expressed as

$$\vec{\boldsymbol{F}}_N = F_N\hat{\boldsymbol{N}} = \left(mg\cos\alpha + \frac{mv^2}{R}\right)\hat{\boldsymbol{N}} \tag{3}$$

where the first term in parentheses is the magnitude of the normal component of gravitational force acting on the object at a particular point on $y(x)$ and the second term is a speed-dependent centripetal force directed toward the center of the osculating circle at that point. In this formulation, R is the signed osculating radius of the surface at the point of contact, v is the instantaneous (tangential) speed, and $\hat{\boldsymbol{N}}$ is the normal unit vector. The argument α is the local angle of incline for the object on $y(x)$ which is related to the geometry of $y(x)$ through

$$\alpha = \arctan\left(-y'(x)\right) \qquad \cos\alpha = \frac{1}{\sqrt{1+y'(x)^2}} \qquad \sin\alpha = \frac{-y'(x)}{\sqrt{1+y'(x)^2}} \tag{4}$$

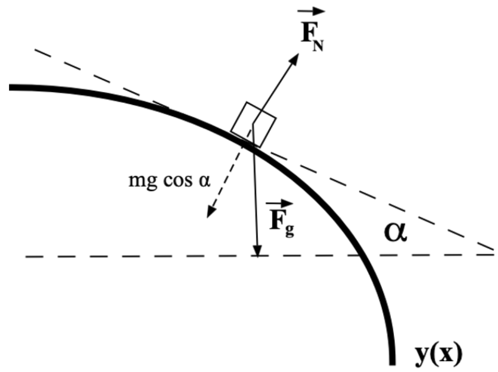


FIG. 1: Variable acceleration scenario

Substituting eq. (4) into the expression for normal force we obtain,

$$\frac{1}{m}\vec{\boldsymbol{F}}_N = \left(\frac{g}{\sqrt{1+y'(x)^2}} + \frac{v^2}{R}\right)\hat{\boldsymbol{N}} \tag{5}$$

and therefore for the jerk:

$$\vec{\boldsymbol{j}} = \hat{\boldsymbol{N}}\frac{d}{dt}\left(\frac{g}{\sqrt{1+y'(x)^2}} + \frac{v^2}{R}\right) + \left(\frac{g}{\sqrt{1+y'(x)^2}} + \frac{v^2}{R}\right)\frac{d\hat{\boldsymbol{N}}}{dt} \tag{6}$$

In this representation, the normal and tangent unit vectors are respectively:

$$\hat{\boldsymbol{N}} = \frac{-y'(x)\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{1+y'(x)^2}} \qquad \hat{\boldsymbol{T}} = \frac{\hat{\boldsymbol{\imath}} + y'(x)\hat{\boldsymbol{\jmath}}}{\sqrt{1+y'(x)^2}} \tag{7}$$

Making use of the implicit differentiation of $y(x)$,

$$\frac{dy(x)}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}v_x = \frac{y'(x)v}{\sqrt{1+y'(x)^2}}$$

it can easily be seen that

$$\frac{d\hat{\boldsymbol{N}}}{dt} = \frac{y''(x)v}{\left(1+y'(x)^2\right)^{\frac{3}{2}}}\left(-\hat{\boldsymbol{T}}\right)$$

Therefore, eq. (6) resolves into components

$$\vec{\boldsymbol{j}} = j_T\hat{\boldsymbol{T}} + j_N\hat{\boldsymbol{N}} \tag{8}$$

where

$$j_T = -\left(\frac{g}{\sqrt{1+y'(x)^2}} + \frac{v^2}{R}\right)\frac{y''(x)v}{\left(1+y'(x)^2\right)^{\frac{3}{2}}} \tag{9}$$

and

$$j_N = \frac{d}{dt}\left(\frac{g}{\sqrt{1+y'(x)^2}} + \frac{v^2}{R}\right) \tag{10}$$

To simplify both eq. (9) and eq. (10), it is necessary to introduce the explicit formula for the (reciprocal of) osculating radius R

$$\frac{1}{R} = \frac{f}{|v|}\left|\frac{d\hat{\boldsymbol{T}}}{dt}\right| = f\left|\frac{y''(x)}{\left(1+y'(x)^2\right)^{\frac{3}{2}}}\right| \tag{11}$$

where factor $f = \pm 1$ relates to the sign of R according to conventions summarized in Table 1.[8]

Using this in eq. (9) we obtain the following expression for the tangential component of jerk

$$j_T = \frac{-gy''(x)v}{\left(1+y'(x)^2\right)^2} - f\frac{y''(x)^2v^3}{\left(1+y'(x)^2\right)^3} \tag{12}$$

The derivation of the normal component of jerk, eq. (13), is considerably more involved as among other things it includes the derivative of an absolute value function and a third derivative of $y(x)$. For the sake of brevity and because it is not our primary focus we simply report the result here:

$$j_N = \frac{fv^3\sqrt{y''(x)^2}}{\left(1+y'(x)^2\right)^2}\left(\frac{-2gy'(x)}{v^2} + \frac{y'''(x)+y'''(x)y'(x)^2-3y'(x)y''(x)^2}{y''(x)\left(1+y'(x)^2\right)}\right) - \frac{gy'(x)y''(x)v}{\left(1+y'(x)^2\right)^2} \tag{13}$$

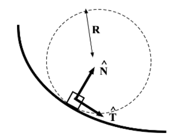
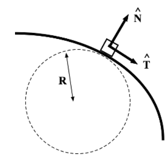
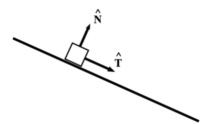
$\boldsymbol{R}$	$\boldsymbol{f}$	**Related Quantities**	**Scenario**
$R > 0$	1	$y''(x) > 0$, $j_T < 0$ $\lvert a_T \rvert$ decreases	
$R < 0$	-1	$y''(x) < 0$, $j_T > 0$ $\lvert a_T \rvert$ increases	
$R \to \infty$	1	$y''(x) = 0$, $j_T = 0$ No jerk, $\lvert a_T \rvert$ constant	

TABLE I: Definition of the sign of the osculating radius R, jerk factor f, and related quantities.

B. Scalar Jerk

In contrast to eq. (12) and eq. (13), the derivation of the equation for what we have defined as scalar jerk is straightforward and starts with the well-known expression for the tangential acceleration on an inclined surface,

$$a_T = g \sin \alpha = \frac{d^2 s}{dt^2} = \frac{-g y'(x)}{\sqrt{1 + y'(x)^2}} \tag{14}$$

The time derivative of eq. (14), after again taking advantage of implicit differentiation and simplification using eq. (4), is

$$j_s \equiv \frac{da_T}{dt} = \frac{d^3 s}{dt^3} = \frac{-g y''(x) v}{\left(1 + y'(x)^2\right)^2} \tag{15}$$

This expression for scalar jerk is clearly just the first term in the expression for tangential jerk, eq.(12), a fact that lays bare the close relationship between the two jerks while demonstrating conclusively that they are not one and the same. It can be shown that the second term in eq. (12), which accounts for the difference between j_T and j_s, is the tangential component of the time derivative of the normal (radial) component of acceleration $a_N \hat{\boldsymbol{N}}$:

$$\frac{d}{dt}\left(a_N\hat{\boldsymbol{N}}\right)=\frac{da_N}{dt}\hat{\boldsymbol{N}}+a_N\frac{d\hat{\boldsymbol{N}}}{dt}=\frac{da_N}{dt}\hat{\boldsymbol{N}}-a_N\frac{y''(x)v}{\left(1+y'(x)^2\right)^{\frac{3}{2}}}\hat{\boldsymbol{T}}$$

Using the reciprocal of the osculating radius, eq. (11), the tangential component becomes

$$\frac{-v^2}{R}\frac{y''(x)v}{\left(1+y'(x)^2\right)^{\frac{3}{2}}}\hat{\boldsymbol{T}}=-\left|\frac{y''(x)}{\left(1+y'(x)^2\right)^{\frac{3}{2}}}\right|\frac{y''(x)v^3}{\left(1+y'(x)^2\right)^{\frac{3}{2}}}\hat{\boldsymbol{T}}=-f\frac{y''(x)v^3}{\left(1+y'(x)^2\right)^{3}}\hat{\boldsymbol{T}}$$

which is evidently the second term in the tangential jerk.

C. Numerical Solutions

The surfaces we seek are solutions to eq. (12) for a constant value of j_T or solutions to eq. (15) for a constant value of j_s. Although the prospect of finding analytical solutions for either of them seems remote, both equations are particularly suited to solution by numerical methods. In particular, eq. (12) can be rearranged to a more manageable quadratic form

$$\left(\frac{fv^3}{\left(1+y'(x)^2\right)^3}\right)y''(x)^2+\left(\frac{gv}{\left(1+y'(x)^2\right)^2}\right)y''(x)+j_T=0 \tag{16}$$

which provides an expression for $y''(x)$ through the quadratic formula. After some simplification, $y''(x)$ becomes

$$y''(x)=\frac{1+y'(x)^2}{2fv^2}\left(-g+\sqrt{g^2-4fvj_T\left(1+y'(x)^2\right)}\right) \tag{17}$$

On the other hand, eq. (15) can be rearranged to give

$$y''(x)=\frac{-j_s}{gv}\left(1+y'(x)^2\right)^2 \tag{18}$$

Both eq. (17) and eq. (18) can be recast as a set of coupled differential equations

$$u(x)=y'(x) \qquad\qquad u'(x)=y''(x) \tag{19}$$

Because of the y-dependence of speed, v, under conservation of energy,

$$v=\sqrt{{v_0}^2+2g\left(y_0-y(x)\right)} \tag{20}$$

each set is autonomous, so its solution requires no explicit reference to time. Selected values of jerk, j_T or j_s, initial acceleration, a_0 (corresponding to an initial choice of $y'(x)$), initial speed, v_0, initial vertical position, y_0, and maximum horizontal run, $x_{\max}$, parameterize and define these curves. (The choice of y_0 is essentially arbitrary, it just serves as an initial reference point for vertical drop). We have encoded this procedure in a Python software package called *Jcurve*, which solves eq. (17) or eq. (18) using a 4th-order Runge-Kutta algorithm and offers multiple tools to analyze the resulting constant-jerk surfaces.[9] This software also computes and plots the instantaneous jerk and acceleration vectors appropriate to each solution, as well as their tangential and normal components.[10]

III. RESULTS

It won't come to anyone's surprise that surfaces exhibiting constant nonzero values of either tangential jerk or scalar jerk are curved; in fact for both jerks they are approximately, but only approximately, parabolic. Here we report results from the application of *Jcurve* to generate and characterize surfaces of constant jerk for two cases. Case A typifies a positive jerk and case B a negative jerk, as summarized in Table 2.

Parameters	**Case A**	**Case B**
j_T or j_s (m/s^3)	2	-4
a_0 (m/s^2)	0	7
v_0 (m/s)	1	0.001

TABLE II: Parameters used in example calculations using *Jcurve*.

Both eq. (17) and eq. (18) are singular at speed $v = 0$. The reason for this is simple: if an object is stationary on a surface ($v = 0$) the normal force is constant and eq. (2) is undefined for any non-zero value of jerk. We have found that a minimum speed $v = 0.001$ m/s is usually sufficient to avoid numerical instability in the solution and that is why this was the initial speed chosen in case B rather than $v = 0$. Figure 2 and Figure 5 respectively show the surface profiles for Case A and Case B solutions of the jerk equations eq. (12) and eq. (15).

In cases of positive jerk (tangential or scalar), if the selected horizontal run ($x_{\max}$) extends far enough the acceleration vector will eventually become vertical with a value of g. This occurs at the point on the surface where the normal component of gravity equals the centripetal acceleration. Beyond this, the object loses contact with the surface and is in free-fall. *Jcurve* will identify this divergence point and although it will generate surface points beyond this, these are physically meaningless. As shown for Case A in Figure 3, the divergence point for the curve associated with $j_T = 2$ occurs at $x_{\text{div}} = 34.52$ m whereas this point occurs at $x_{\text{div}} = 19.50$ m with $j_s = 2$. The large difference in divergence values belies how close the curves track each other for the first 10 m or so as shown in the last panel of Figure 2. Only when the curve becomes steep, and the normal component of acceleration changes quickly, do they differ significantly. Cases of positive jerk correspond to a slow clockwise rotation of both the jerk and acceleration vectors as seen in Figure 4 though the details of this evolution is subtly different for the two constant jerk curves.

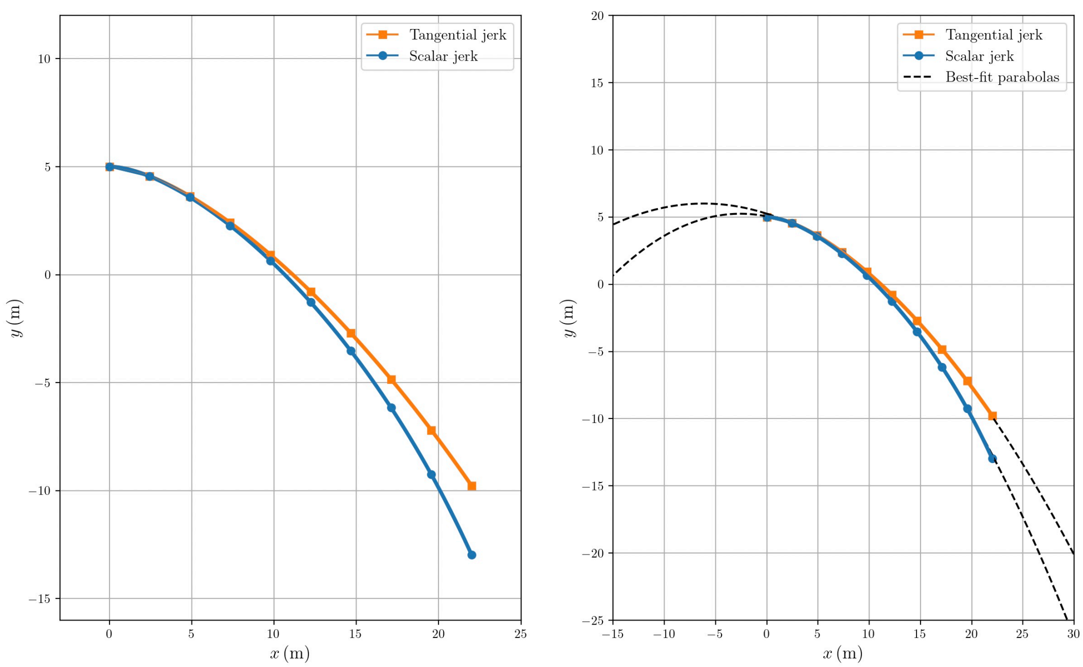
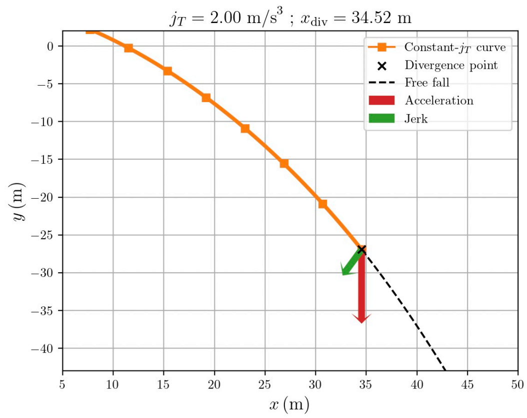
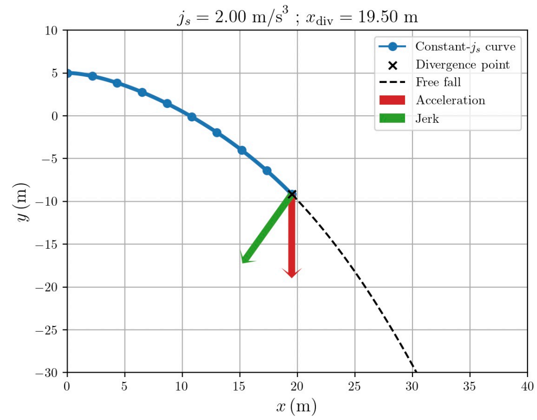


FIG. 2: Surface profiles of Case A ($j_T = 2$ m/s^3 and $j_s = 2$ m/s^3, $a_0 = 0$ m/s^2) curves for $x_{\max} = 22$ m. The tangential jerk curve has a length of $s = 26.98$ m and the scalar jerk curve of $s = 29.26$ m.

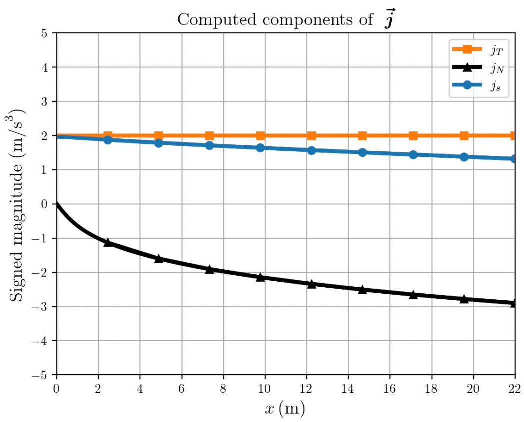
Negative jerk curves such as Case B are limited to values of $y(x)$ such that $y(x) \leq y_0 + \frac{{v_0}^2}{2g}$ which ensure positive values of speed. Owing to the high initial acceleration in B and the correspondingly rapid change in the normal component of acceleration, the j_s and j_T curves



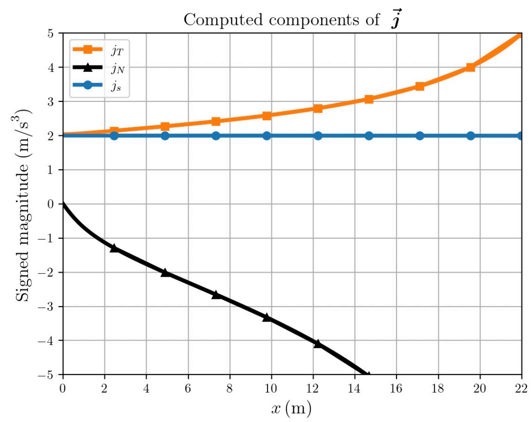
(a) $\vec{a}$ and $\vec{j}$ at divergence point for constant j_T



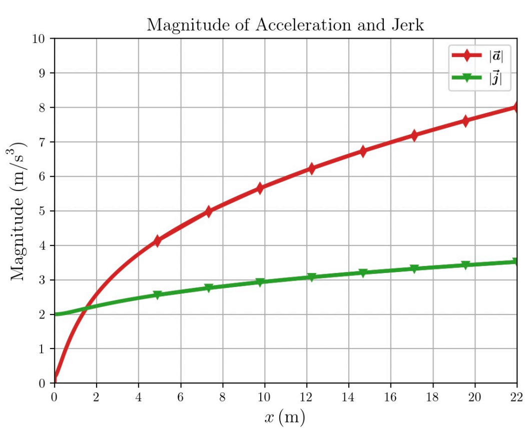
(b) $\vec{a}$ and $\vec{j}$ at divergence point for constant j_s



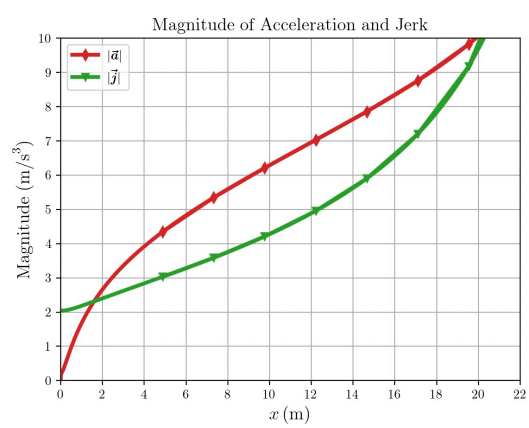
(c) Magnitudes of j_T, j_N, and j_s for constant j_T



(d) Magnitudes of j_T, j_N, and j_s for constant j_s



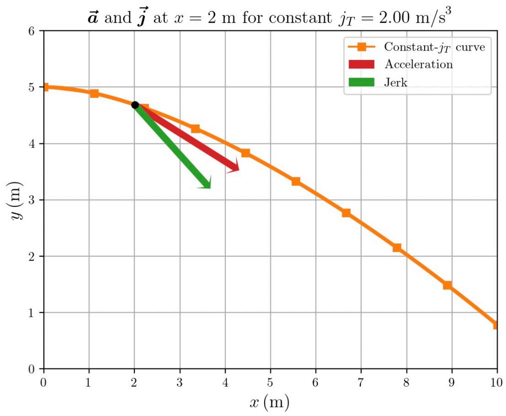
(e) $|\vec{a}|$ and $|\vec{j}|$ for constant j_T



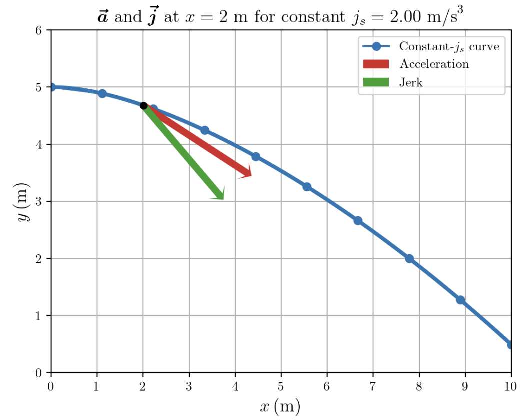
(f) $|\vec{a}|$ and $|\vec{j}|$ for constant j_s

FIG. 3: Critical behavior and dynamical data of Case A.

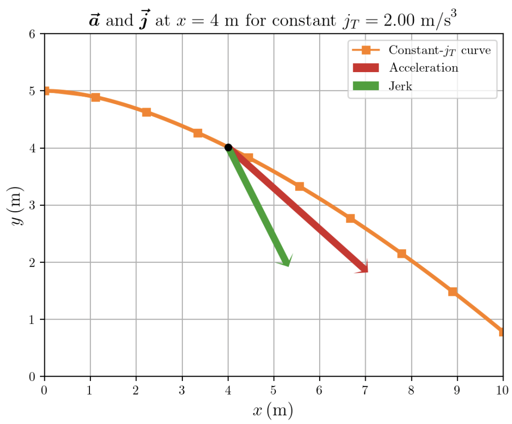
diverge almost immediately. Therefore, the constant-j_T curve in Case B reaches a maximum



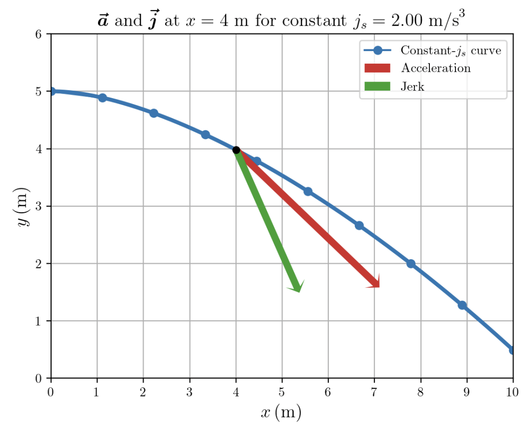
(a) $|\vec{a}| = 2.56$ m/s^2, $|\vec{j}| = 2.24$ m/s^3



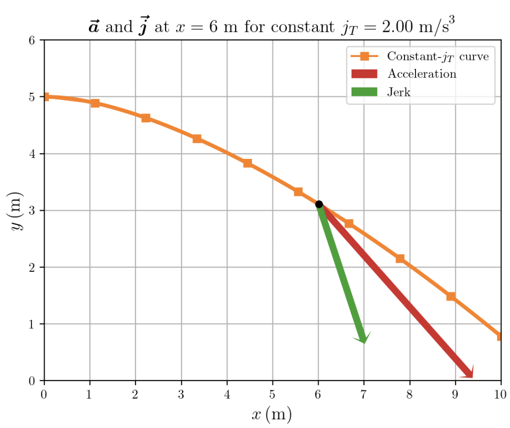
(b) $|\vec{a}| = 2.63$ m/s^2, $|\vec{j}| = 2.40$ m/s^3



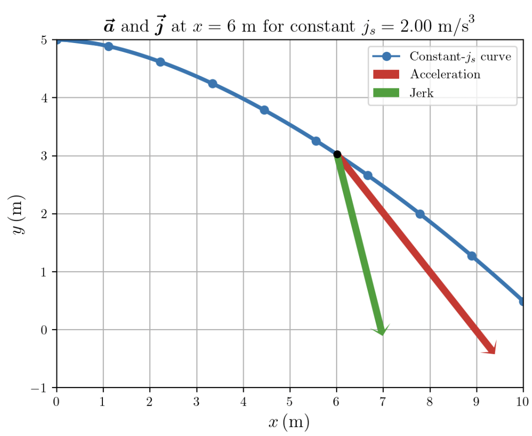
(c) $|\vec{a}| = 3.74$ m/s^2, $|\vec{j}| = 2.47$ m/s^3



(d) $|\vec{a}| = 3.90$ m/s^2, $|\vec{j}| = 2.83$ m/s^3



(e) $|\vec{a}| = 4.54$ m/s^2, $|\vec{j}| = 2.66$ m/s^3



(f) $|\vec{a}| = 4.82$ m/s^2, $|\vec{j}| = 3.28$ m/s^3

FIG. 4: The evolution of $\vec{a}$ and $\vec{j}$ along 10 m of Case A curves.

horizontal distance of 18.5 m, while the constant-j_s curve reaches a maximum of 13.5 m.

Although no divergence point exists for such cases, they do have critical minima where their graphs bottom out and reverse themselves; we see in Figure 6 that, for the j_s curve, these occur at lower x values than those for j_T. Negative jerk produces a counterclockwise rotation of the jerk and acceleration vectors. Figure 7 shows the evolution of the acceleration and the jerk vectors as an object moves along the curves in both cases, A and B, and for each type of constant jerk, j_T or j_s, while Figure 4 displays dynamical data pertinent to each.

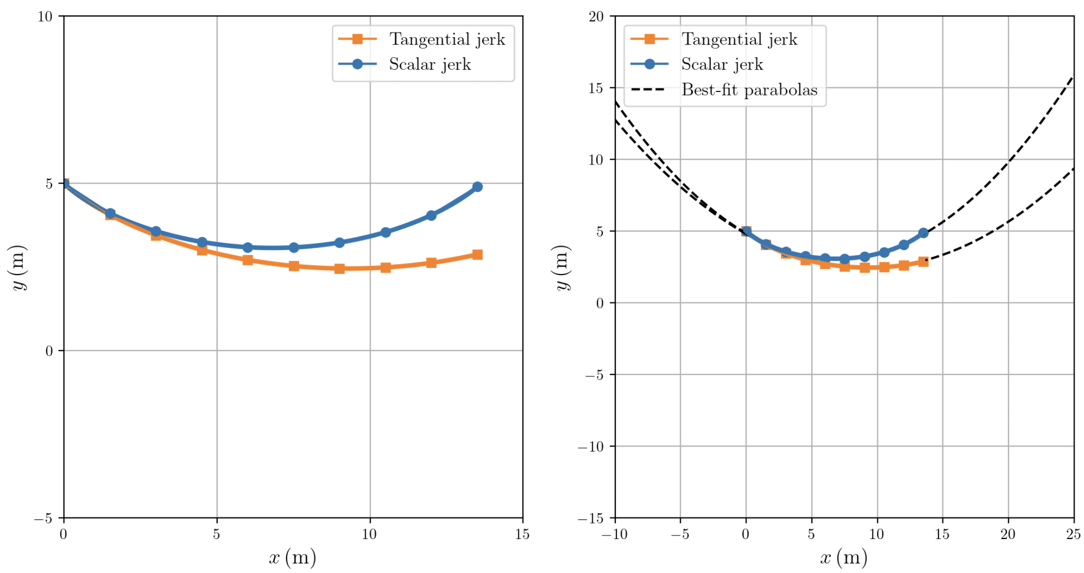
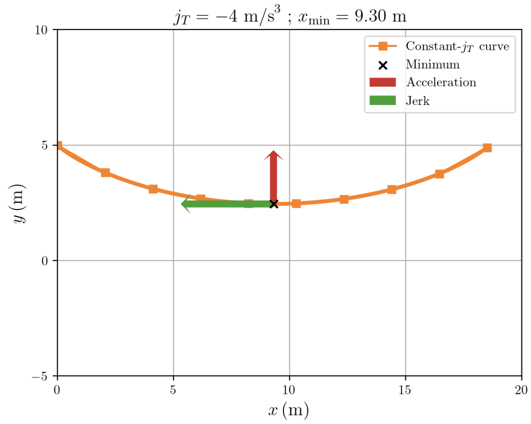
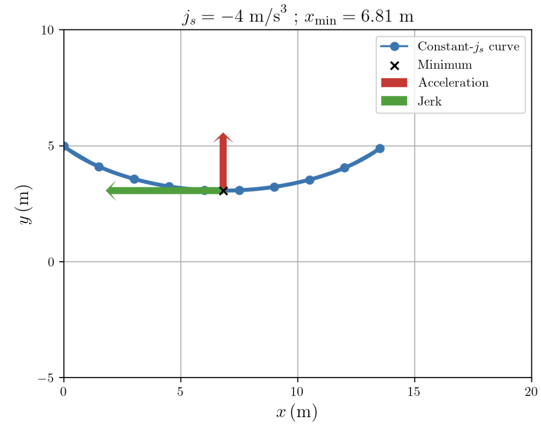


FIG. 5: Surface profiles of Case B ($j_T = -4$ m/s^3 and $j_s = -4$ m/s^3, $a_0 = 7$ m/s^2) curves for $x_{\max} = 13.5$ m. The lengths of the curves are 14.21 m and 14.04 m for j_s and j_T, respectively.

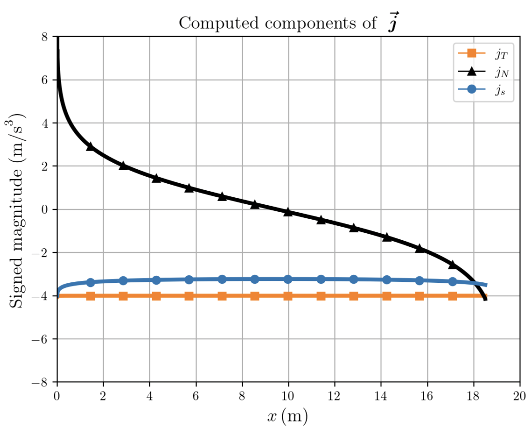
In cases of both positive and negative jerk, surfaces generated with constant j_T don't curve as much as those with the same value of constant j_s; they don't need to. For the former, the tangential jerk caused by the curvature is augmented by the tangential component of the derivative of the acceleration perpendicular to the curve. In Figure 4, the components of jerk calculated for each surface using eq. (12), eq. (13), or eq. (16) for j_T, j_N and j_s, respectively, are shown. Not surprisingly, surfaces defined by constant j_T are marked by varying j_s and vice versa. As noted earlier, the surfaces in both Case A and Case B, as well as all others we tested using *Jcurve*, are almost parabolic. In fact, perfectly parabolic surfaces will produce non-constant values of j_T, j_N, and j_s. Almost exact agreement can be reached between the numerical curve and polynomials as high as 5th order in all of the cases we have studied but of course that doesn't mean anything about the nature of constant

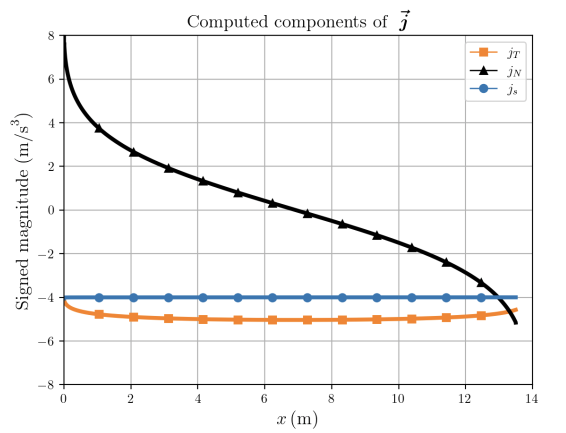




(a) $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{j}}$ at minimum for constant j_T

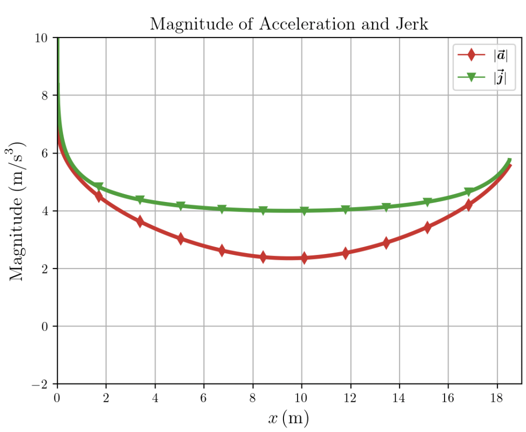
(b) $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{j}}$ at minimum for constant j_s

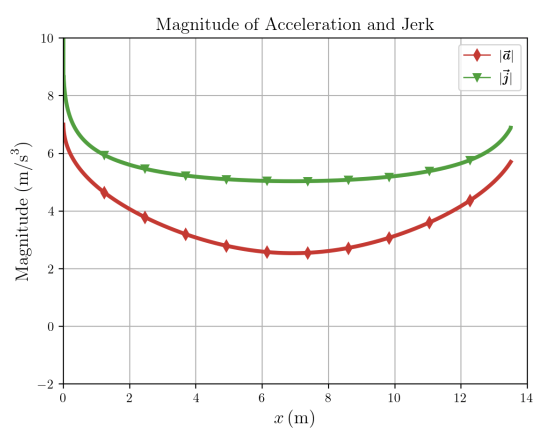




(c) Magnitudes of j_T, j_N, and j_s for constant j_T

(d) Magnitudes of j_T, j_N, and j_s for constant j_s



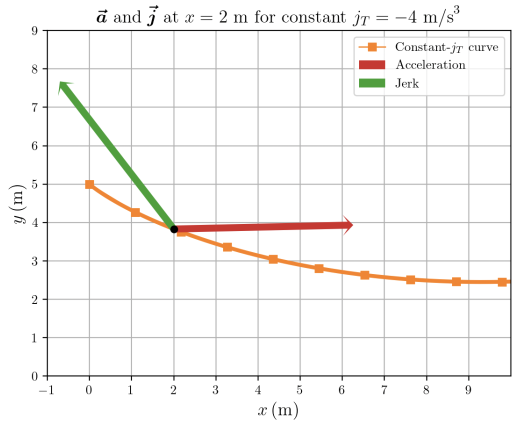


(e) $|\vec{\boldsymbol{a}}|$ and $|\vec{\boldsymbol{j}}|$ for constant j_T

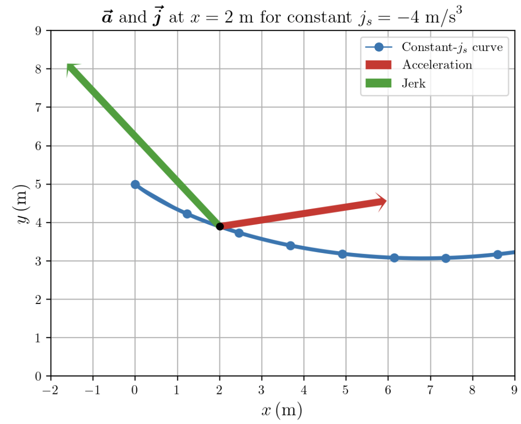
(f) $|\vec{\boldsymbol{a}}|$ and $|\vec{\boldsymbol{j}}|$ for constant j_s

FIG. 6: Critical behavior and dynamical data of Case B.

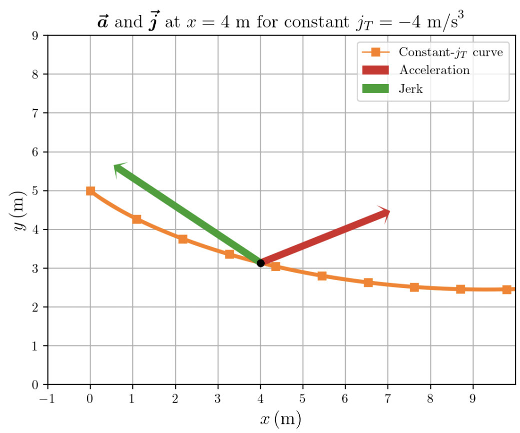
jerk curves except that they are well-behaved. There is no term for the shape of constant j



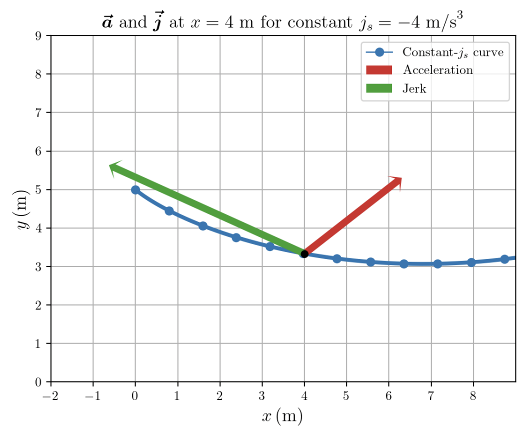
(a) $|\vec{a}| = 4.26$ m/s^2, $|\vec{j}| = 4.70$ m/s^3



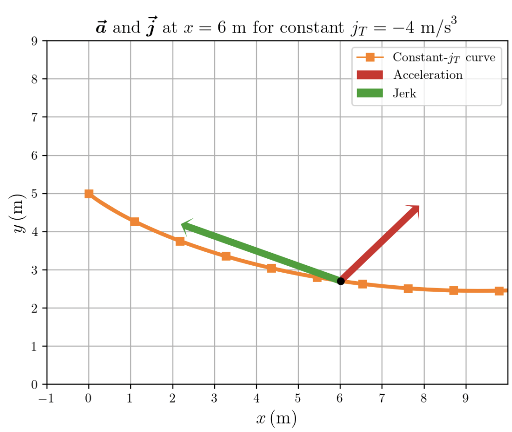
(b) $|\vec{a}| = 4.02$ m/s^2, $|\vec{j}| = 5.58$ m/s^3



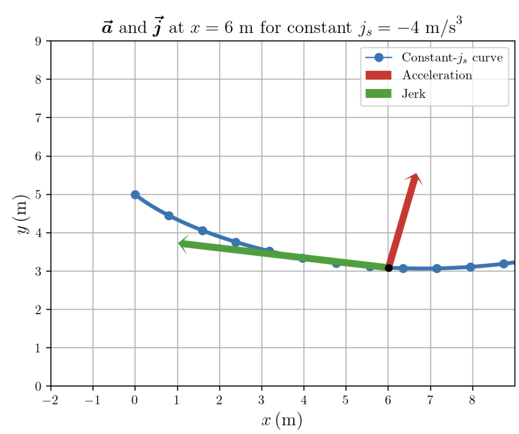
(c) $|\vec{a}| = 3.32$ m/s^2, $|\vec{j}| = 4.27$ m/s^3



(d) $|\vec{a}| = 3.03$ m/s^2, $|\vec{j}| = 5.17$ m/s^3



(e) $|\vec{a}| = 2.72$ m/s^2, $|\vec{j}| = 4.09$ m/s^3



(f) $|\vec{a}| = 2.55$ m/s^2, $|\vec{j}| = 5.03$ m/s^3

FIG. 7: The evolution of $\vec{a}$ and $\vec{j}$ along 10 m of Case B curves.

curves, they are unique to themselves.

IV. CONCLUSIONS

So our question has been answered; inclined surfaces producing constant jerk are curved in a way that is almost but not quite parabolic. There may be no simple explanation for this except to follow the mathematics, which is not particularly satisfying. A natural follow-up question might concern how the shape of a constant jerk surface is different if an object rolls without slipping rather than sliding down it. In fact, it is straightforward to move beyond a frictionless scenario to incorporate rolling into this kind of analysis, we have just decided to focus here on the ideal here. In any case, good questions are good questions and they deserve answers, especially after ripening for 400 years.

ACKNOWLEDGMENTS

We want to thank Mr. Matthew Amoroso and Mr. Brian Laich for their helpful comments about this work.

---

* mateo.patino25@thorntonacademy.org

† clinton.nash@thorntonacademy.org

[1] Frank S. Crawford. "Rolling and slipping down Galileo's inclined plane: Rhythms of the spheres," *American Journal of Physics*, 64, 441-446, 1996.

[2] Rod Cross, "Sliding and rolling along circular tracks in a vertical plane," *American Journal of Physics*, 91, 351-356, 2023.

[3] Hasti Hayati et al., "Jerk within the Context of Science and Engineering-A Systematic Review," *Vibration*, 3, 371-409, 2020.

[4] T.R. Sandlin, "The Jerk," *The Physics Teacher*, 28, 36-40, 1990.

[5] Steven Schot, "Jerk: The Time Rate of Change of Acceleration," *American Journal of Physics*, 46, 1090-1094, 1978.

[6] Ann-Marie Pendrill and David Eager, "Velocity, acceleration, jerk, snap and vibration: forces in our bodies during a roller coaster ride," *Physics Education*, 55, 1-13, 2020.

[7] Arjun Tan, Raja Surabhi, and Ashwith Chilvery. "Dynamics of a Particle Sliding Down a Smooth Exponential Incline," *Latin American Journal of Physics Education*, 7, 341-344, 2013.

[8] Carl E. Mungan. "A refresher on curvature for application to centripetal acceleration," *Latin American Journal of Physics Education*, 4, 27-31, 2010.

[9] J.C. Butcher. "A history of Runge-Kutta methods," *Applied Numerical Mathematics*, 20, 247-260, 1996.

[10] *Jcurve* is available on GitHub under a general MIT license at https://github.com/mateo-patino/jcurve.