The Chaotic Pendulum

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Abstract

The behavior of a damped driven pendulum was simulated for different values of different adjustable parameters. Maintaining the damping strength and the driving force amplitude constant, the driving frequency was varied, thus obtaining the values for which the pendulum exhibited chaotic motion. A similar simulation was done for a varying driving force amplitude. The results were obtained graphically, and can be observed in the bifurcation diagrams below.

Introduction

In this project I demonstrate the chaotic behavior exhibited by a damped, periodically driven pendulum for different values of the different adjustable parameters. The equation of motion of the damped driven pendulum is:

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mgd\sin\theta = \Upsilon\cos(\omega_F t) \tag{1}$$

where I is the moment of inertia of the pendulum, b is the damping parameter, m is the mass, g is the acceleration due to gravity, d is the distance form the pivot to the center of mass, Υ is the amplitude, and ω_F is the angular driving frequency.

For a pendulum with small oscillations that has no driving force, its frequency of oscillation is the *natural frequency of oscillation of the pendulum*. This frequency is given by:

$$\omega_0 = \sqrt{\frac{mgd}{I}}. \tag{2}$$

Using Eq. (2), we can define the next dimensionless quantity:

$$t' = \omega_0 t. \tag{3}$$

Plugging in Eq. (3) into Eq. (1) we obtain the following relations:

$$\begin{bmatrix} A = \frac{\Upsilon}{(\omega_0)^2 I} \\ B = \frac{\omega_0 I}{b} \\ \omega_D = \frac{\omega_F}{\omega_0} \end{bmatrix} \tag{4}$$

and Eq. (1) can be rewritten as:

$$\frac{d^2\theta}{dt^2} + \frac{1}{B}\frac{d\theta}{dt} + \sin\theta = A\cos(\omega_D t) \tag{5}$$

where t' was replaced by t for aesthetic reasons. We can also see that in Eq. (5) there are only three adjustable parameters: the amplitude of the driving force A, the inverse of the damping strength B, and the ratio of the drive frequency to the natural frequency ω_D.

We can rewrite Eq. (5) as three first order differential equations, which can be written in matrix form as:

$$\frac{d}{dt}\begin{pmatrix}\theta \\ \omega \\ \phi\end{pmatrix} = \begin{pmatrix}\omega \\ -\frac{1}{B}\,\omega - \sin\theta + A\cos\phi \\ \omega_D\end{pmatrix} \tag{6}$$

Since Eq. (5) is nonlinear and can be written as three first-order differential equations, the underdamped pendulum exhibits chaotic behavior for certain values of the parameters A, B, and ω_D.

When the period of oscillation of the pendulum corresponds to the period of oscillation of the driving force, the pendulum has a *period one* oscillation. If the period of the pendulum is twice as long as the period of the driving force, then the pendulum has a *period two* oscillation. We can observe this evolution in the bifurcation diagrams below. Every bifurcation corresponds to a period doubling. Eventually, the period of the pendulum becomes infinite, i.e., the pendulum exhibits chaotic behavior.

References

[1] G. Baker, J. Gollub, *Chaotic Dynamics: An Introduction* (Cambridge University Press, Cambridge, 1990).

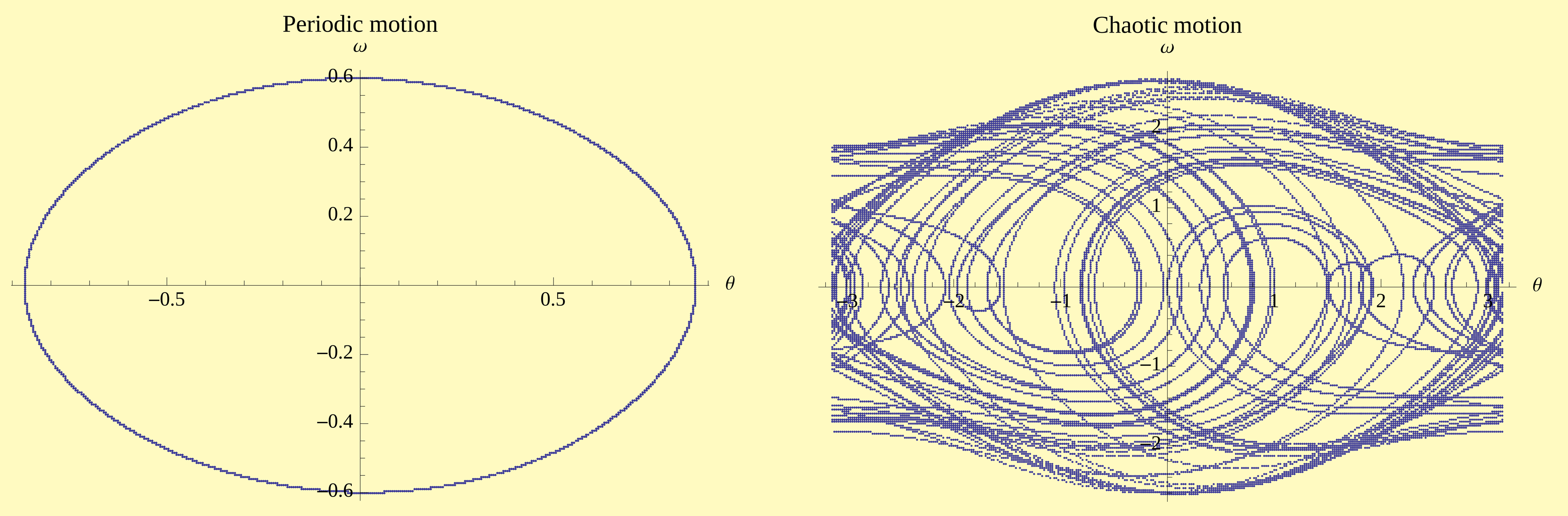
[2] J. Franklin, *Computational Methods For Physics* (Cambridge University Press, New York, 2013).

[3] G. Baker, J. Blackburn, *The Pendulum: A Case Study in Physics* (Oxford University Press, Oxford, 2005).

Method & Results

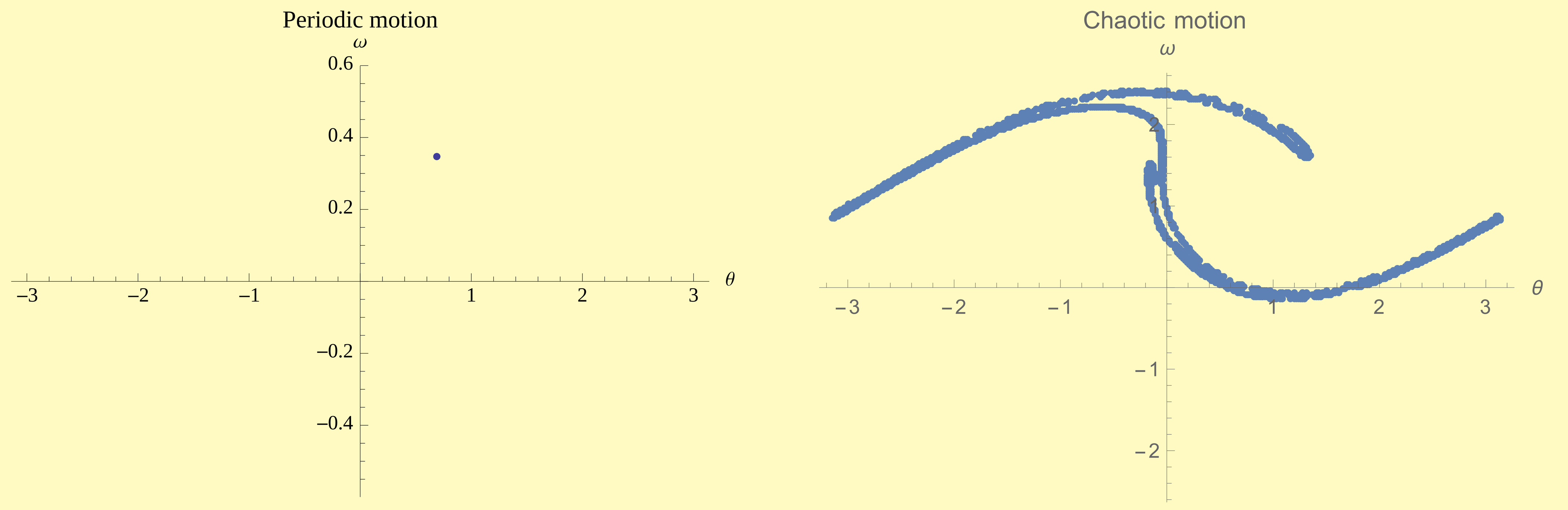
Using the Runge-Kutta method RK4, I solved Eq. (4) using a grid of $\text{n} = 2^{16}$ steps and $\Delta t = \frac{\omega_d}{64}$ spacing between steps. This method gives the value of the angle θ, the angular velocity ω, and the phase ϕ of the driving force frequency ω_D for every value of time $t = n\Delta t$.

The next values were used to obtain data: for periodic: $A = 0.5$, $B = 2$, $\omega_D = \frac{2}{3}$; for chaotic: $A = 1.5$, $B = 2$, $\omega_D = \frac{2}{3}$.



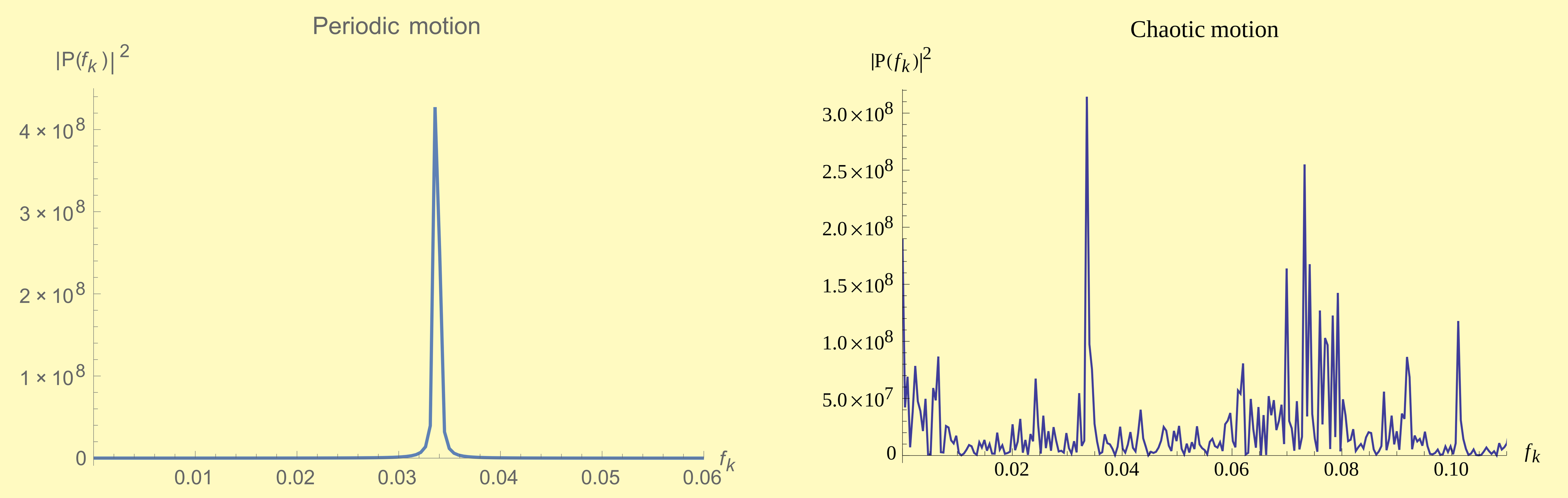
a) Phase Space

For the Poincaré sections, I obtained the value of θ and ω for every value of $\phi = \text{k}\,2\pi$, where k is an integer. The value of 2π is chosen because that is the period of the force.



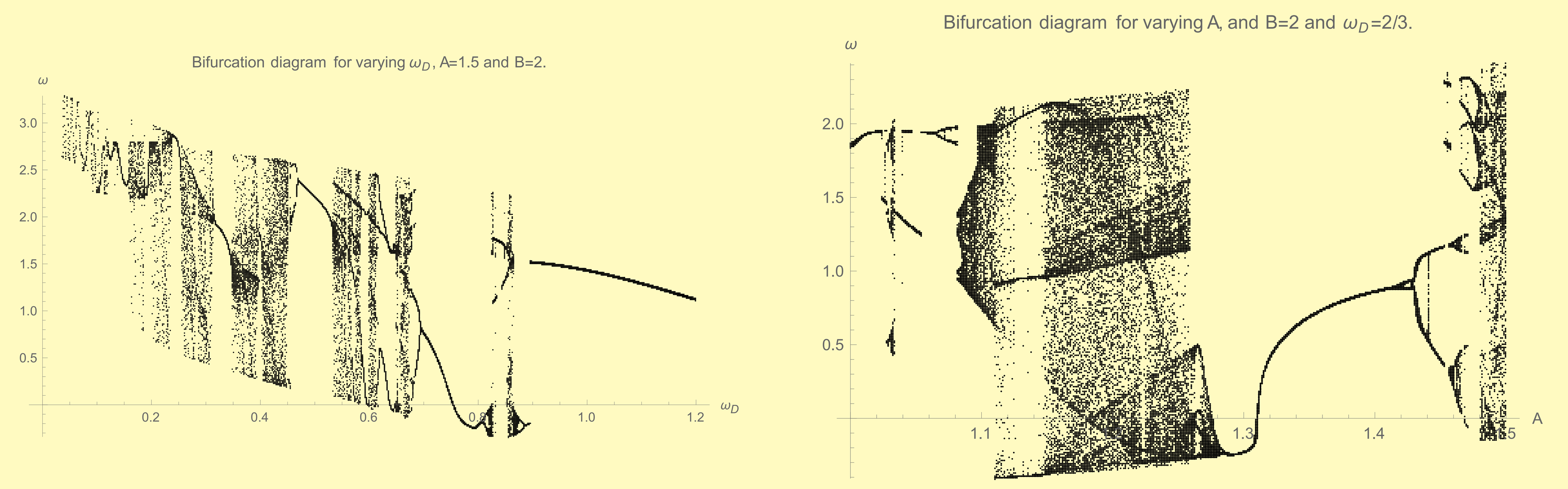
a) Poincaré Sections

The power spectrum was obtained by applying the Fast Fourier transform on the data obtained using RK4.



a) Power Spectrum

For the frequencies that correspond to the relevant terms that describe the motion, only frequencies whose amplitudes in the power spectrum are bigger than 10^8 are taken into account.



a) Bifurcation Diagrams

Bifurcation diagrams for varying ω_D and A.

Conclusion

As opposed to the pendulum with small oscillations, the nonlinear damped driven pendulum exhibits chaotic behavior. As it can be seen from the bifurcation diagrams, the regions where behavior is chaotic do not extend to infinity. Periodic motion can occur after chaotic motion as the values of the parameters are increased. From the power spectrum we can see that periodic motion has only one main frequency, whereas chaotic motion has several main frequencies. This indicates that, as opposed to periodic motion, chaotic motion has to be described by several sinusoidal functions, each one with a different frequency.