Applications and in-depth cases

Up until now, Malliavin calculus has been mostly a game or procedure we apply to stochastic processes. We will dive deeper and reach some interesting results.

Malliavin derivative of the Ornstein-Uhlenbeck process

Let's consider a random process N_t that fluctuates around a value μ and corrects itself more strongly the farther it is to μ, plus some noise. So, we want something something like this:

$$dN_t = \theta(\mu - N_t)\,dt + \sigma\,dW_t$$

For example, let's say that a lake can accept around $\mu = 1000$ trouts. If there are more than that, some will starve and die off, and if there are more, then they will reproduce. Finally, θ controls how strong is the correction. Here are some examples:

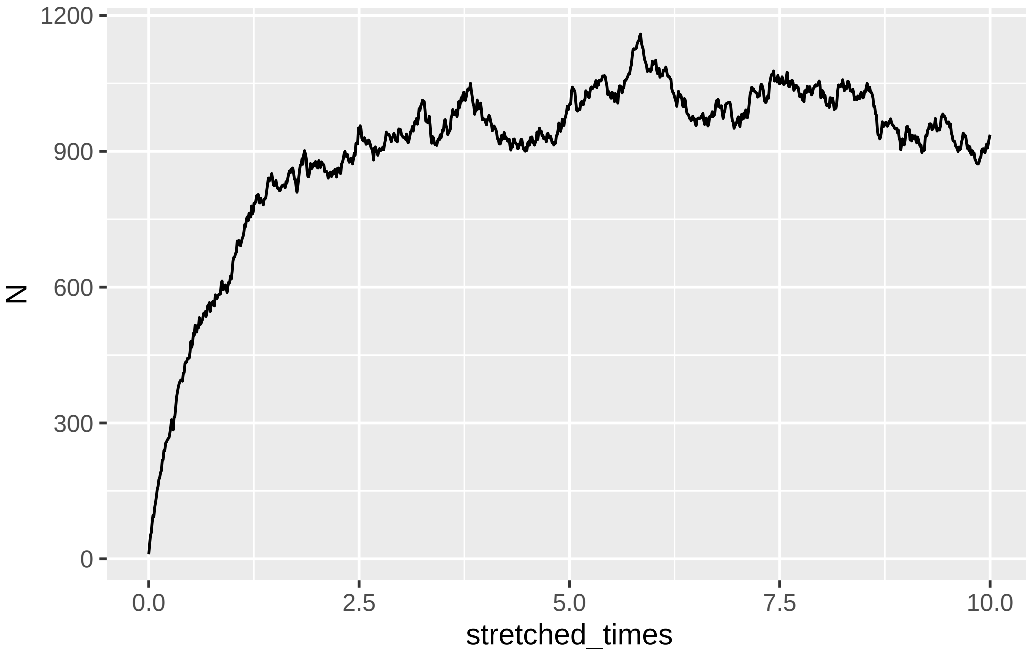
```r
library(ggplot2)

# Right
dt <- 0.01
total_time <- 10
N_0 <- 10
mu_ou <- 1000.0
sigma_ou <- 100
tetha_ou <- 1.0

steps = total_time / dt
times <- seq(from = 0, to = total_time, length.out=steps)
stretched_times <- times * tetha_ou
dBt <- rnorm(steps,mean = 0, sd = sqrt(dt))
N <- rep_len(x = N_0, length.out = length(times))

for (i in (2:length(times))) {
  N[i] <- N[i-1] + tetha_ou * (mu_ou - N[i-1]) * dt + sigma_ou * dBt[i]
}
```

```r
ggplot(mapping = aes(x = stretched_times, y = N)) + geom_line()
```



This is a very common process, so much it has a name: the Ornstein-Uhlenbeck process. Now, let's calculate DN. We have $a(X) = \theta(\mu - X)$ and $b(x) = \sigma$, so it's not a simple Ito process. We better start from those before tackling the Ornstein-Uhlenbeck process. If we remember that we can split the integrals as $\int_0^t ...dW_t = \sum_{i=1}^n ...(W_{t_i} - W_{t_{i-1}})$, then we can transform a classical Ito process X_t into something we can easily apply chain and product rules:

$$
\begin{aligned}
D_r X_t &= \sum_{i=1}^n \mu_{t_i}(t_i - t_{i-1}) + \sum_{i=1}^n \sigma_{t_i}(W_{t_i} - W_{t_{i-1}}) \\
&= \sum_{i=1}^n D_r \mu_{t_i}(t_i - t_{i-1}) + \sum_{i=1}^n D_r[\sigma_{t_i} \cdot (W_{t_i} - W_{t_{i-1}})] \\
&= \sum_{i=1}^n D_r \mu_{t_i}(t_i - t_{i-1}) + \sum_{i=1}^n \sigma_{t_i} \cdot 1_{[t_{i-1}, t_i]}(r) + \sum_{i=1}^n D_r(\sigma_{t_i}) \cdot (W_{t_i} - W_{t_{i-1}})] \\
&= \int_0^t D_r \mu_s \, ds + \sigma_r 1_{[0,t]}(r) + \int_0^t D_r \sigma_s \, dW_s
\end{aligned}
$$

It looks messy, but things are what we expect: t is the terminal value of both integrals, s is the variable we use to integrate from 0 to t, and r is the variable we introduced with the Malliavin

Derivative. See that it makes sense: the influence of σ in the Ito process is the same along the whole Ito process path.

Now, we can use a similar argument for the number of trouts in the lake, which follows an Ornstein-Uhlenbeck process. The formula in general for a process that motivated Ito's Lemma is like so:

$$D_r X_t = \int_r^t \frac{\partial a}{\partial x}(s, X_s)\, D_r X_s\, ds + b(r, X_r) 1_{[0,t]}(r) + \int_r^t \frac{\partial b}{\partial x}(s, X_s)\, D_r X_s\, dW_s$$

Now, we know from above that $\frac{\partial a}{\partial x} = -\theta$ and $\frac{\partial b}{\partial x} = 0$, so we get:

$$\begin{aligned} D_r N_t &= \int_r^t (-\theta)\, D_r N_s\, ds + \sigma 1_{[0,t]}(r) \\ &= -\theta \int_r^t D_r N_s\, ds + \sigma 1_{[0,t]}(r) \\ &= \sigma\, e^{-\theta(t-r)} \end{aligned}$$

In the case above, we arrive at a neat expression that tell us something interesting: the effect of a perturbation on the number of trouts is exponentially smaller if $r << t$. This makes sense: the population of trouts N wants to be close to μ and the population from a distant past is irrelevant. It also tells us that there's no randomness in the fluctuation.

Finally, we lucked out above. The expression for $D_r N_t$ is simple, so we could write an analytic expression for it. If it were more convoluted, we would need to estimate a solution. Alos (2021) does a great job showcasing the above.

Clark-Ocone-Haussman Formula

Let's see a relatively simple application with profound implications. Let's consider this function:

$$F(t) = e^{\int_0^t h\, dB - \frac{1}{2}\int_0^t h^2 d\lambda}$$

This function is an exponential martingale, that is, $\mathbb{E}_s[F(t)] = F(s)$. We now take $F(1) = \mathcal{E}(h)$ and we now calculate the derivative as mentioned above, to obtain DF as such:

$$\begin{aligned}
D_t F &= e^{-\frac{1}{2}\int_0^1 h^2 d\lambda} \cdot D_t(e^{\int_0^1 h\, dB}) \\
&= \underbrace{e^{-\frac{1}{2}\int_0^1 h^2 d\lambda} \cdot e^{\int_0^1 h\, dB}}_{F} \cdot h(t) \\
&= F h(t)
\end{aligned}$$

Now, this is only valid at $t = 1$, but we use the expectation operator and martingale property to get the values at a previous time. We will call $\mathcal{F}_s$ the filtration up to time s, and it's the way we call the "history" or "information" known up to time s, and then:

$$\begin{aligned}
\mathbb{E}[D_t F | \mathcal{F}_t] &= \mathbb{E}[F(1) h(t) | \mathcal{F}_t] \\
&= h(t) \mathbb{E}[F(1) | \mathcal{F}_t] \\
&= h(t) F(t)
\end{aligned}$$

Now, we pull everything together. This martingale $F(t)$ is the solution of the (stochastic) differential equation $dF(t) = h(t)F(t)dB_t$, when $F(0) = 1 = \mathbb{E}[F]$. Replacing with the above expression we get:

$$\begin{aligned}
F(t) &= \mathbb{E}[F] + \int_0^1 h(t) F(t) dB_t \\
&= \mathbb{E}[F] + \int_0^1 \mathbb{E}[D_t F | \mathcal{F}_t] dB_t
\end{aligned}$$

This is the Clark-Ocone formula, which is an explicit formula for the Martingale Representation Theorem. The Malliavin derivative is the key ingredient to have an analytic expression and it's the main reason one would care about it. What this is telling us is that any random variable indexed by time F_t can be split into a sum of a "deterministic" portion, the expected value, and a martingale that fluctuates around it.

We can apply this to the Ornstein-Uhlenbeck process for our trouts. We know from before that $D_r N_t = \sigma\, e^{-\theta(t-r)}$. Notice that there's no W_t, so the expected value is just the derivative. We could estimate the value of $\mathbb{E}[N]$, but the process already has a solution for it. Here's in all its splendor:

$$\begin{aligned}
N(t) &= \mathbb{E}[N] + \int_0^t \mathbb{E}_{\mathbb{r}}[D_r N_t] dW_r \\
&= \underbrace{X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})}_{Deterministic} + \underbrace{\int_0^t \sigma\, e^{-\theta(t-r)}\, dW_r}_{Martingale}
\end{aligned}$$

Why do we care about this? Well, for once, if we wanted to simulate different paths of our fish population N, we only need to simulate the martingale portion, and the deterministic portion is only calculated once. Secondly, we can use this to calculate a variance for N. Indeed, remember that $\mathbb{Var}[N] = \mathbb{E}[N - \mathbb{E}[N]]^2$, so then, using Ito's Isometry:

$$\begin{aligned}
\mathbb{Var}[N] &= \mathbb{E}\left[\left(\int_0^t \mathbb{E}_{\mathbb{r}}[D_r N_t] dW_r\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t \left(\sigma\, e^{-\theta(t-r)}\right)^2 dr\right] \\
&= \sigma^2 \mathbb{E}\left[\int_0^t e^{2\theta(r-t)} dr\right] \\
&= \frac{\sigma^2}{2\theta}\mathbb{E}\left[1 - e^{-2\theta t}\right] \\
&= \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)
\end{aligned}$$

Notice that this works for any variable: you only need the Malliavin Derivative to get the variance. We see above that: - Our trout population N in the long term ($t \to \infty$) will have a stable, constant variance around μ - Variance at the introduction of trouts ($t \approx 0$) is very small because the growth of N, trying to reach μ as soon as possible, dominates the noise - A large θ will not only make the long term variance arrive sooner, it will also prevent large deviations from μ

Is there anything else?

There's a lot left! But I haven't had time to round up the concepts. There are three things that I've excluded, this is long as it is.

First of all, Malliavin Derivatives make it simpler to work with stochastic variances. That is, σ is an Ito process. It's more tractable to use the techniques above than simulating and estimating the evolution of N.

Secondly, we can use an integration by parts to flip between a function derivative and the Skorohod integral. In the case of non-anticipatory processes, the formula is easy to compute because it's just an Ito integral. This makes it, in turn, easier to calculate/estimate derivatives of functions of processes against other processes. For example, the derivative of an insurance payoff for some asset against the price of the asset is called Δ. Switching to less risky assets to reduce this derivative is called delta-hedging. When this derivative is zero, changes on the asset's price don't affect the insurance payoff anymore and the insurance is considered delta-neutral.

A final point is that Skorohod integrals also work for anticipatory integrals. We haven't touched those here because they aren't proper Ito integrals and it was already too complicated, but this is also where it shines.

Alos, & Lorite, E. 2021. *Malliavin Calculus in Finance: Theory and Practice (1st Ed.)*. 1st ed. Financial Mathematics Series. Chapman; Hall/CRC. https://doi.org/10.1201/9781003018681.