Final Presentation

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Complex Systems Modelling in Economics

May, 2025

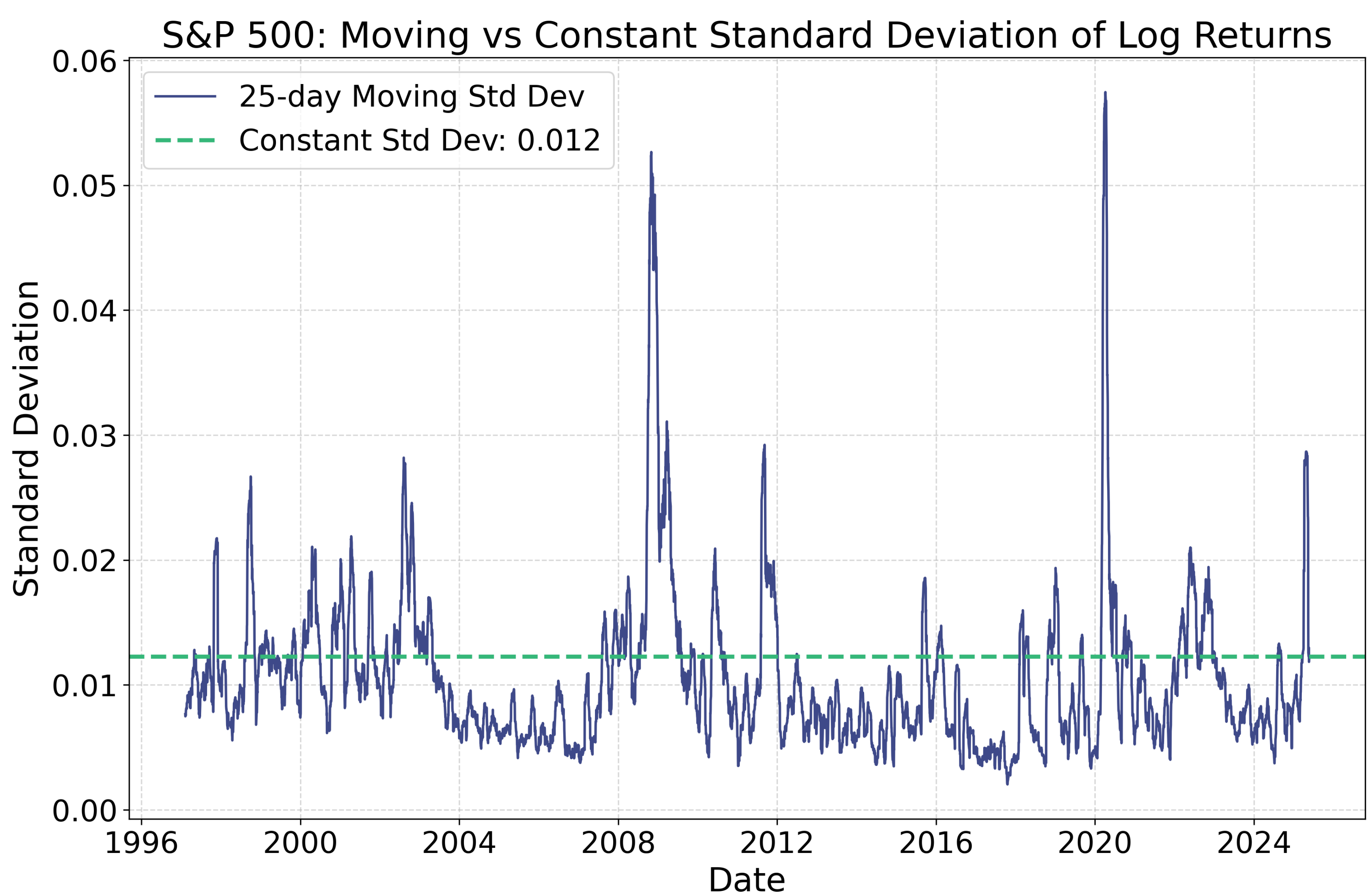
Assigment One:

ARCH and GARCH processes

Exercise 1. Analysis of the S&P 500 volatility time series:

Download daily price data for the S&P500 from 01/01/1997 to today and analyze the moving standard deviation of log returns with a window size of 25 days.

1. Is the standard deviation constant in time? Compare the moving standard deviation with the assumption of a constant σ in the whole period (where the value σ is computed from the data).



Results considering data on 25/05/2025.

Some features

- σ is not constant in time; instead, it fluctuates with remarkable peaks
- Maximum peak: 2020-04-06 with $\sigma \approx 0.057$.
- Another higher peak in the 2008 financial crisis.

Exercise 1. Analysis of the S&P 500 volatility time series:

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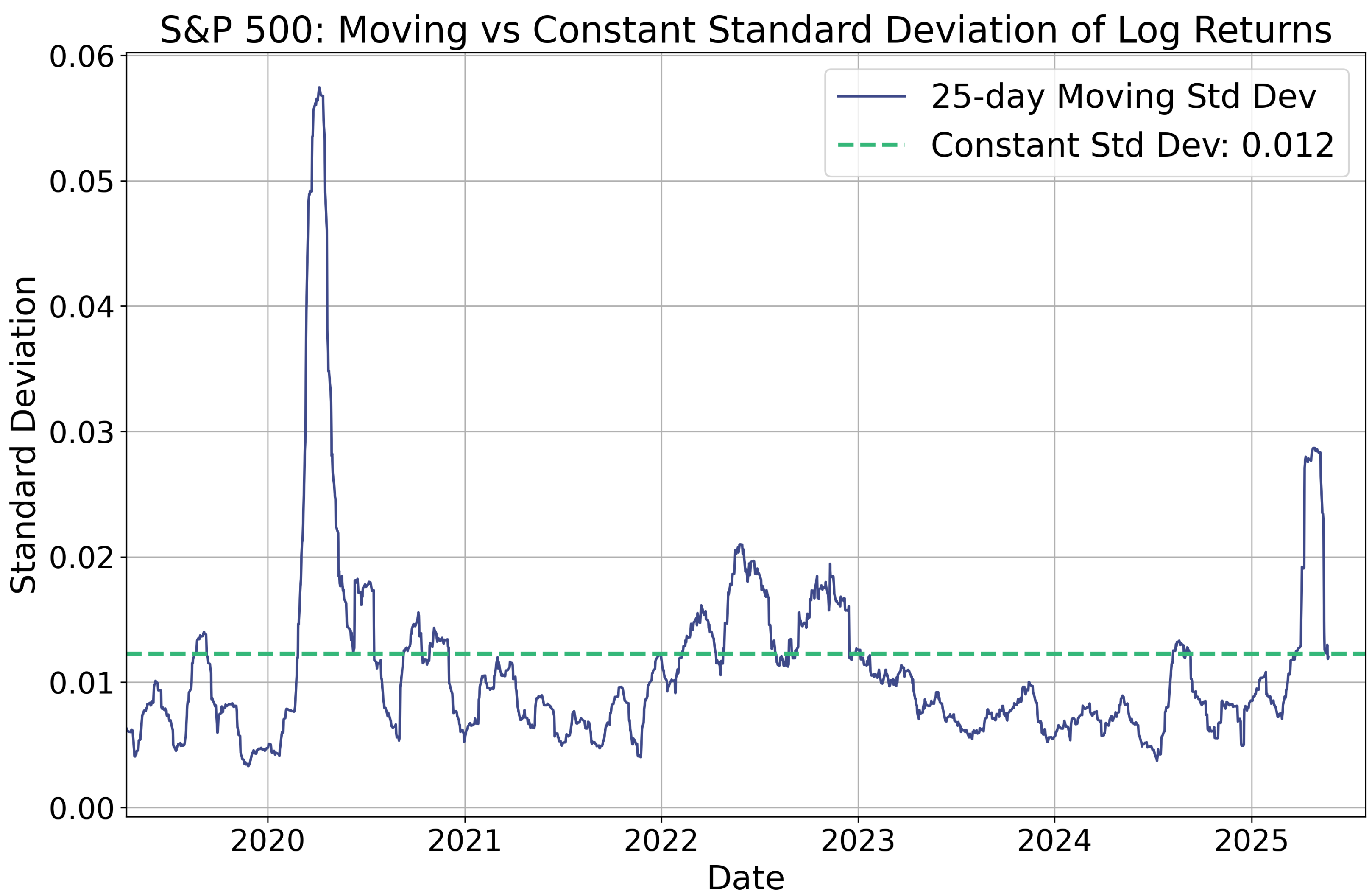


Results considering data on 25/05/2025.

Some features

- April 2025: highest peak since the pandemic.

Exercise 1. Analysis of the S&P 500 volatility time series:

2. Perform Engle's Test for Autoregressive Conditional Heteroscedasticity (ARCH). Summarise the idea behind the test and explain the results you obtain.

The Autoregressive Conditional Heteroscedasticity (ARCH) test is a statistical test used to detect whether the variance of a time series changes over time. If past squared returns help predict future variance, there's heteroscedasticity. If not, the series is homoscedastic.

- Null hypothesis H_0: No ARCH effects (variance is constant over time).

- Alternative hypothesis H_1: ARCH effects are present (variance depends on past stocks).

Statistic	**Value**
LM Statistic	1814.25
LM p-value	0.00
F Statistic	243.93
F p-value	0.00

p-value < 0.05, we reject the null hypothesis, and we accept the alternative hypothesis. This confirms the presence of time-varying volatility in the returns of the S&P 500.

Therefore, volatility is not constant.

Exercise 2. Volatility forecast with an ARCH model.

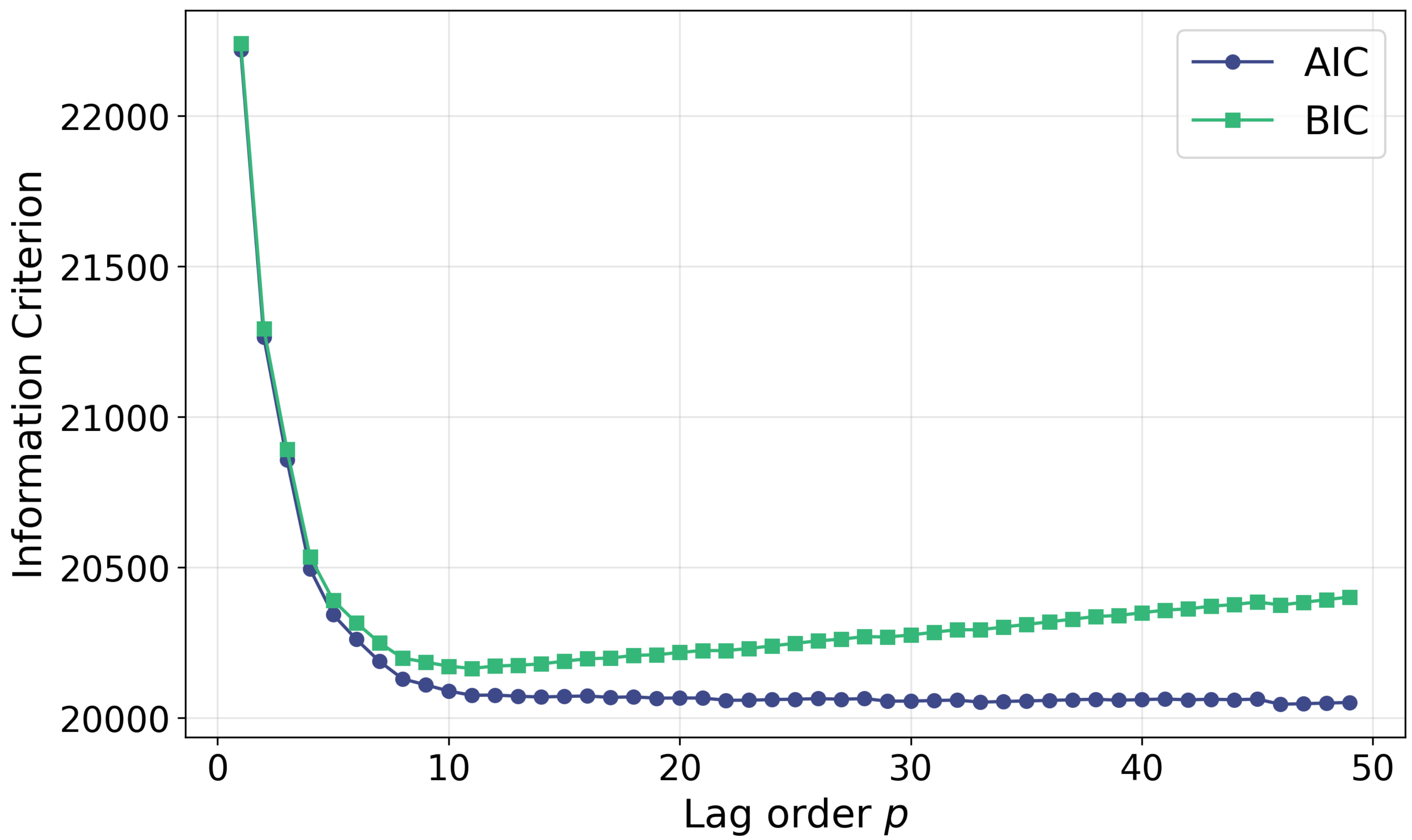
1. Fit an ARCH model. Choose a particular value p to fit an ARCH model to the data. Analyze the output carefully to understand how the library works.

```
                     Constant Mean - GARCH Model Results
==============================================================================
Dep. Variable:                 LogRet   R-squared:                       0.000
Mean Model:             Constant Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:               -10023.0
Distribution:                  Normal   AIC:                           20053.9
Method:            Maximum Likelihood   BIC:                           20081.4
                                        No. Observations:                 7143
Date:                Tue, May 27 2025   Df Residuals:                     7142
Time:                        12:21:24   Df Model:                            1
                                 Mean Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
mu             0.0649  9.952e-03      6.521  6.977e-11 [4.539e-02,8.441e-02]
                              Volatility Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
omega          0.0244  4.913e-03      4.957  7.155e-07 [1.473e-02,3.399e-02]
alpha[1]       0.1219  1.260e-02      9.678  3.738e-22   [9.723e-02,  0.147]
beta[1]        0.8626  1.282e-02     67.297      0.000     [  0.838,  0.888]
============================================================================
```

Example for $p = 5$.

Exercise 2. Volatility forecast with an ARCH model.

2. Determine the optimal value for p. Fit the model using different values of p (e.g. 1 to 25) and compute their AIC and BIC values to determine the optimal value for p.



BIC

Optimal value $p = 11$ (BIC = 20164)

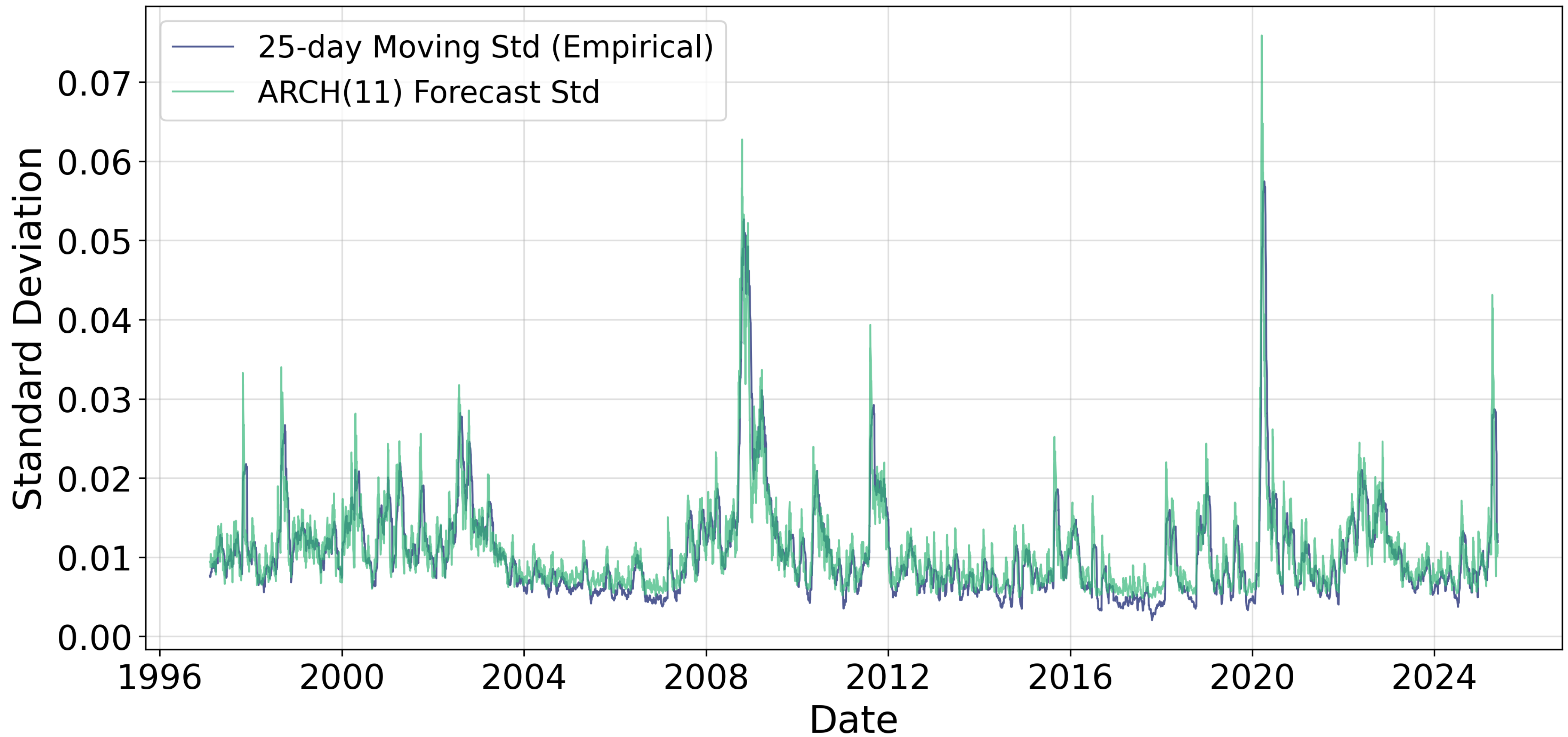
AIC

Optimal value $p = 46$ (AIC = 20044)

Exercise 2. Volatility forecast with an ARCH model.

3. Forecasts using the optimal model. Plot the predictions along with the moving volatility previously predicted. Do you think the model is performing well?

The prediction follows the overall behavior. However, we can notice that the peaks are overestimated.



Forecast using the ARCH(11) model.

Exercise 2. Volatility forecast with an ARCH model.

3. Forecasts using the optimal model. Plot the predictions along with the moving volatility previously predicted. Do you think the model is performing well?

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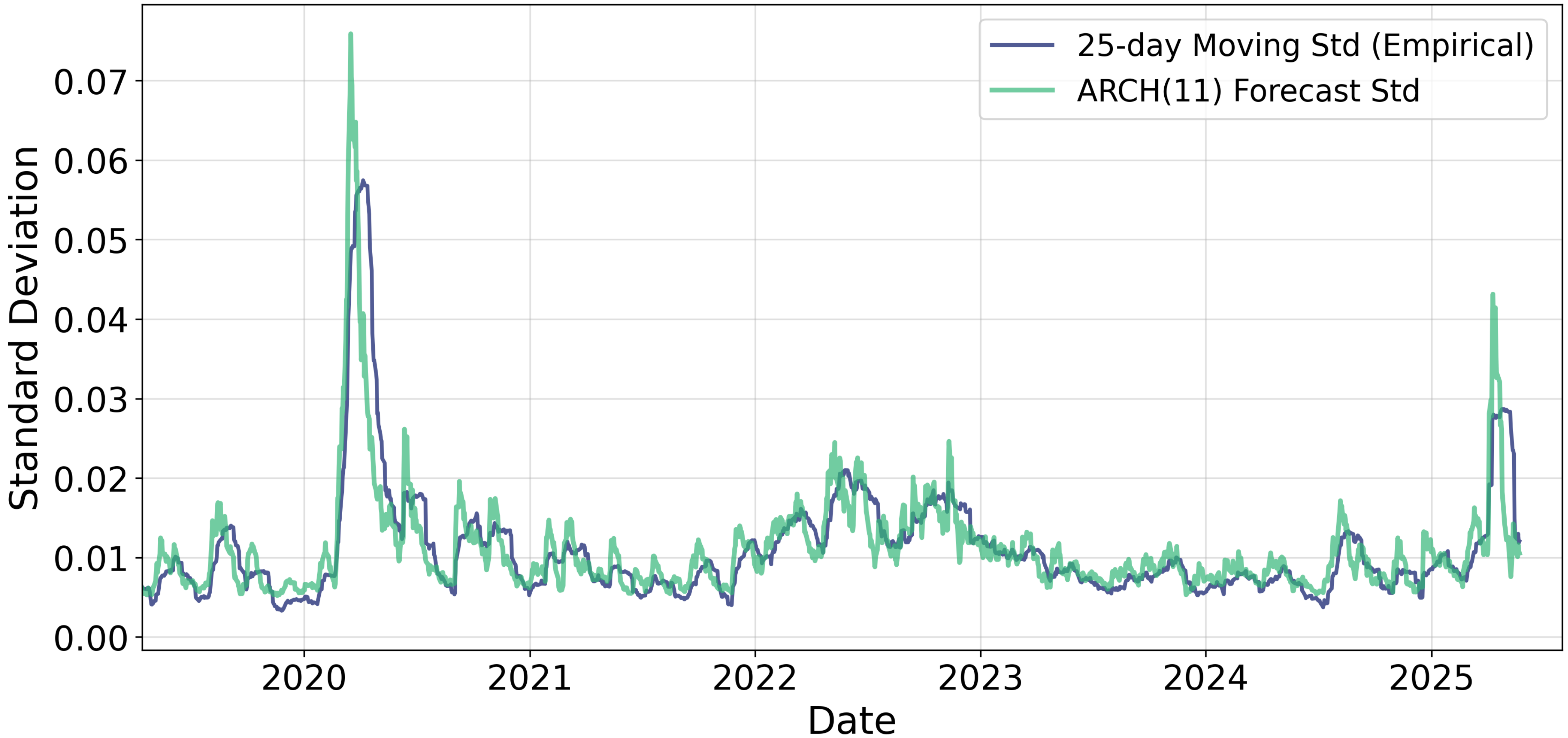


Forecast using the ARCH(11) model.

Exercise 3. Volatility forecast with GARCH models.

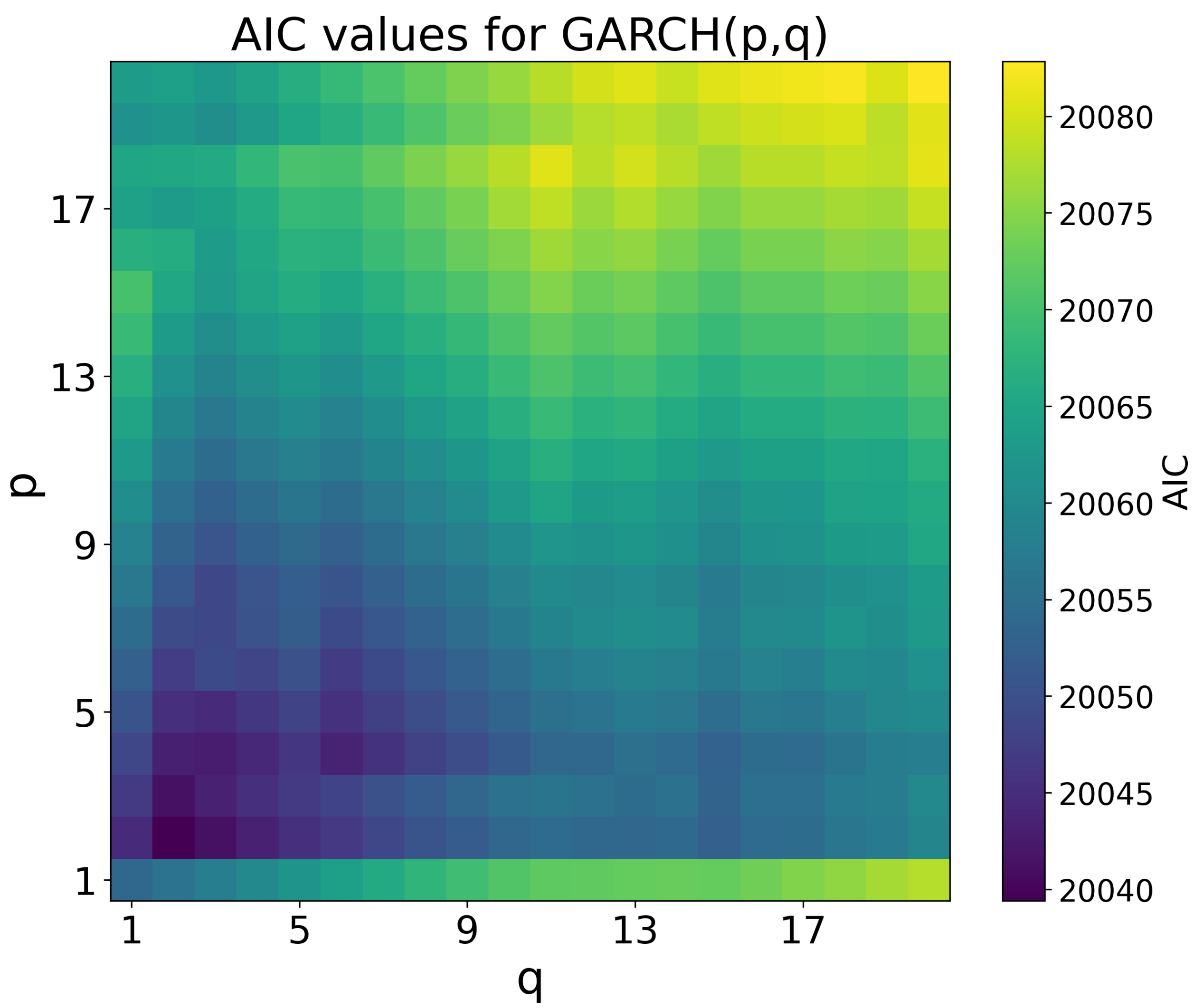
1. Fit a GARCH(1,1) model.

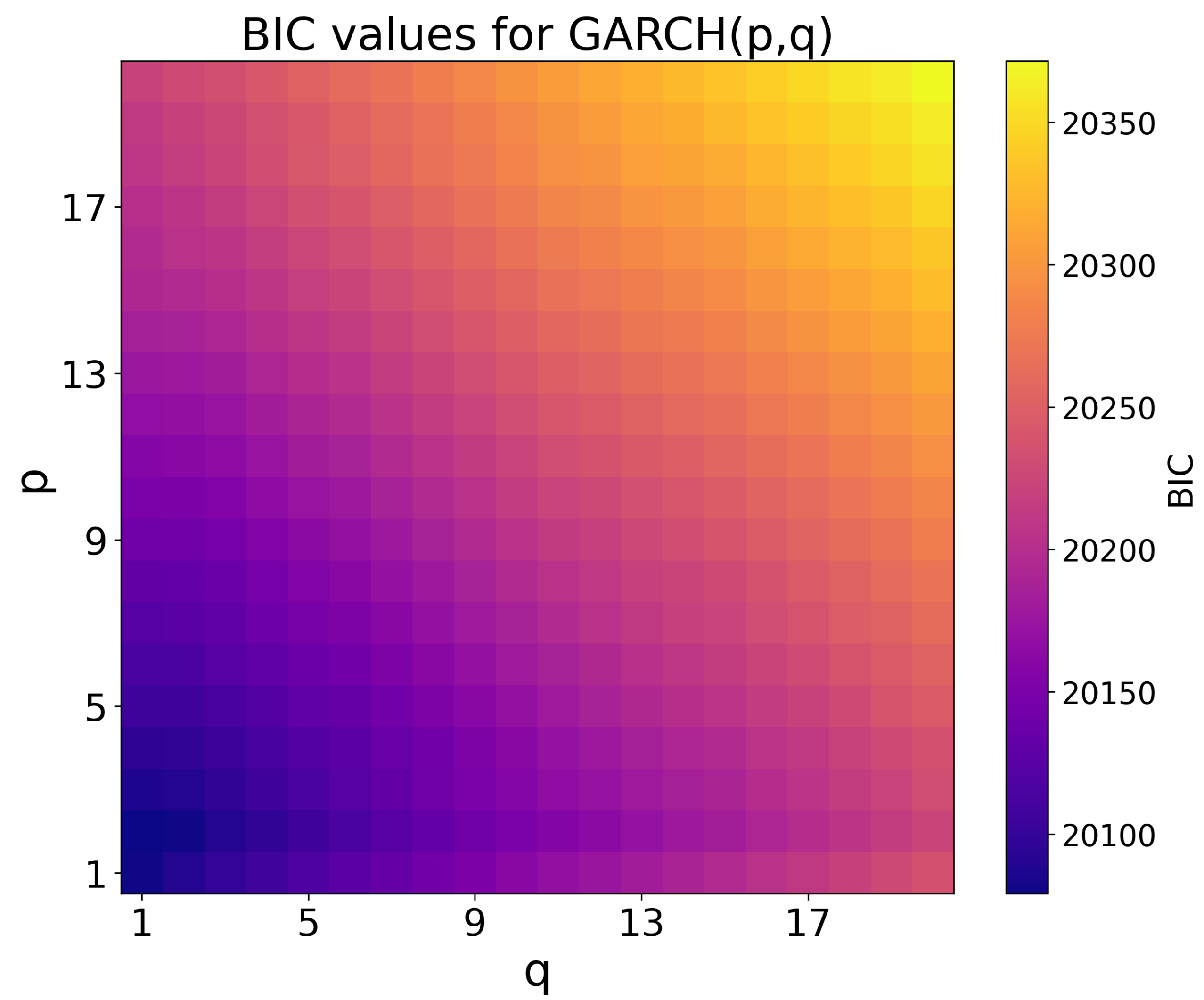
```
                      Constant Mean - ARCH Model Results
==============================================================================
Dep. Variable:                 LogRet   R-squared:                       0.000
Mean Model:             Constant Mean   Adj. R-squared:                  0.000
Vol Model:                       ARCH   Log-Likelihood:               -10164.2
Distribution:                  Normal   AIC:                           20342.5
Method:            Maximum Likelihood   BIC:                           20390.6
                                        No. Observations:                 7143
Date:                Tue, May 27 2025   Df Residuals:                     7142
Time:                        11:58:49   Df Model:                            1
                                 Mean Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
mu             0.0693  1.000e-02      6.930  4.205e-12 [4.971e-02,8.891e-02]
                              Volatility Model
==========================================================================
                 coef    std err          t      P>|t|    95.0% Conf. Int.
--------------------------------------------------------------------------
omega          0.3206  2.359e-02     13.590  4.613e-42   [  0.274,  0.367]
alpha[1]       0.1134  2.053e-02      5.527  3.259e-08 [7.321e-02,  0.154]
alpha[2]       0.2053  2.238e-02      9.173  4.613e-20   [  0.161,  0.249]
alpha[3]       0.1624  2.049e-02      7.929  2.211e-15   [  0.122,  0.203]
alpha[4]       0.1953  2.360e-02      8.279  1.247e-16   [  0.149,  0.242]
alpha[5]       0.1375  1.968e-02      6.986  2.838e-12 [9.890e-02,  0.176]
==========================================================================
```

Example for $p = 1$, $q = 1$.

Exercise 3. Volatility forecast with GARCH models

2. Determine the optimal values for p and q.





AIC

Optimal values $p = 2, q = 2$

AIC = 20039.

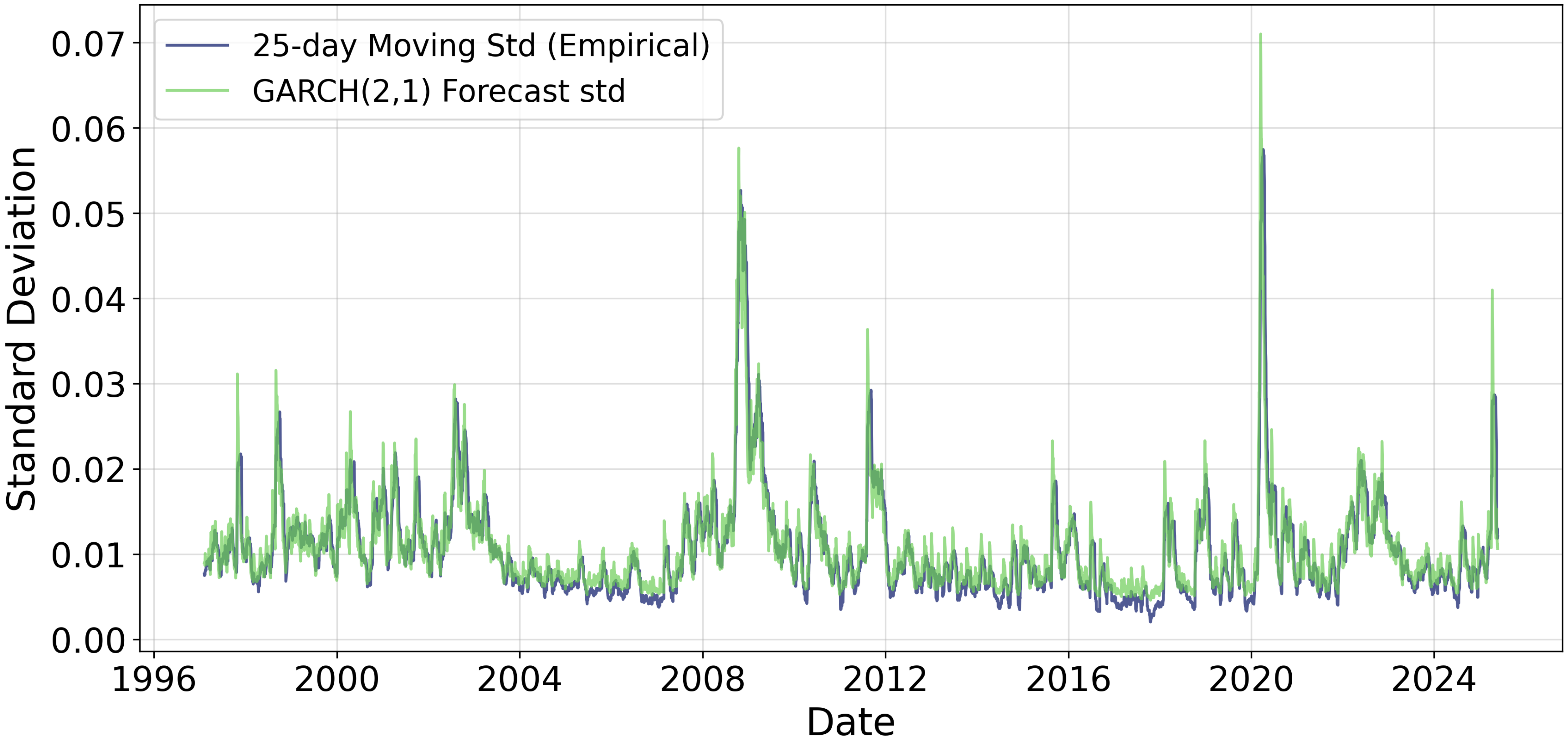
BIC

Optimal values $p = 2, q = 1$

AIC = 20079.

Exercise 3. Volatility forecast with GARCH models.

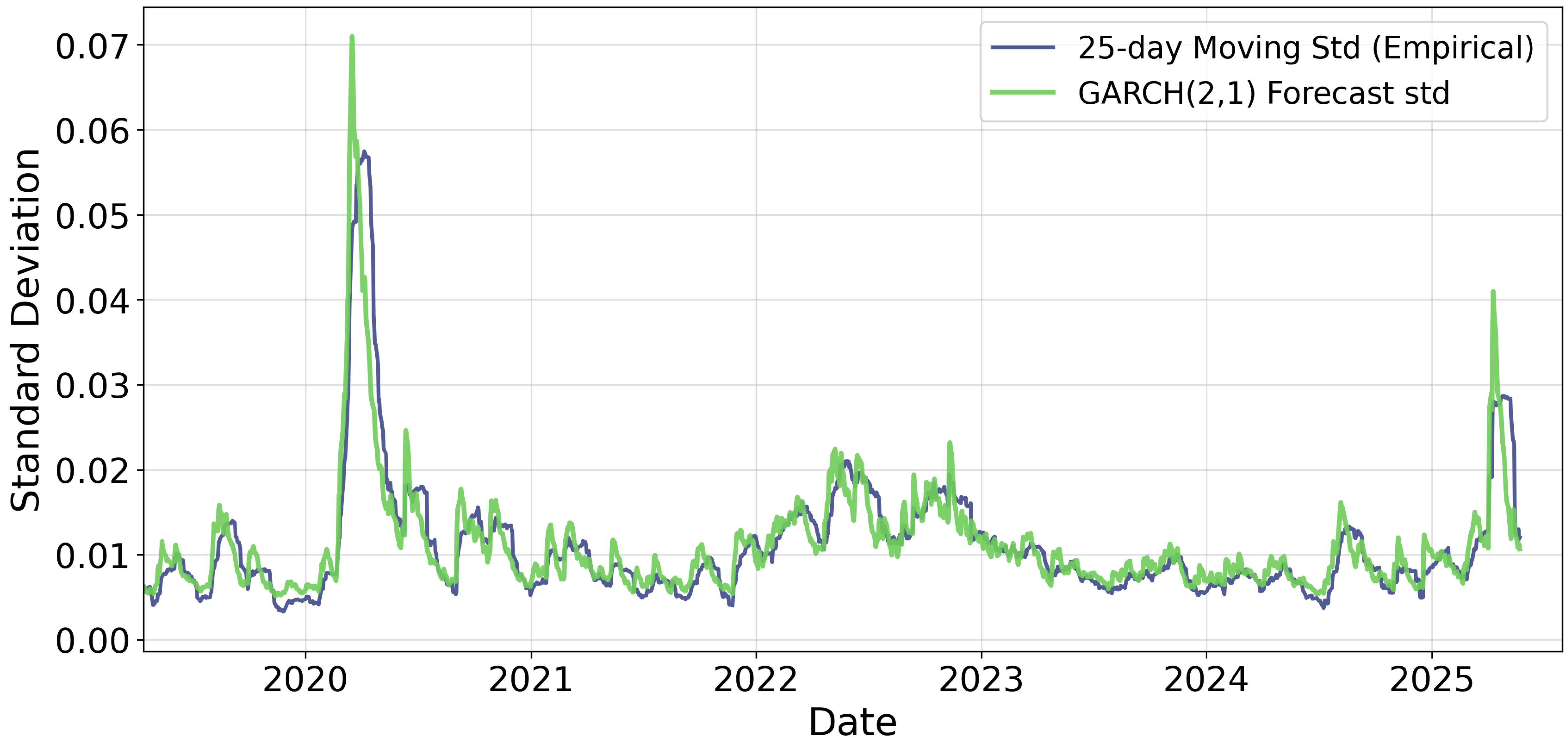
3. Forecasts.



Forecast using GARCH(2,1) model.

Exercise 3. Volatility forecast with GARCH models.

3. Forecasts.



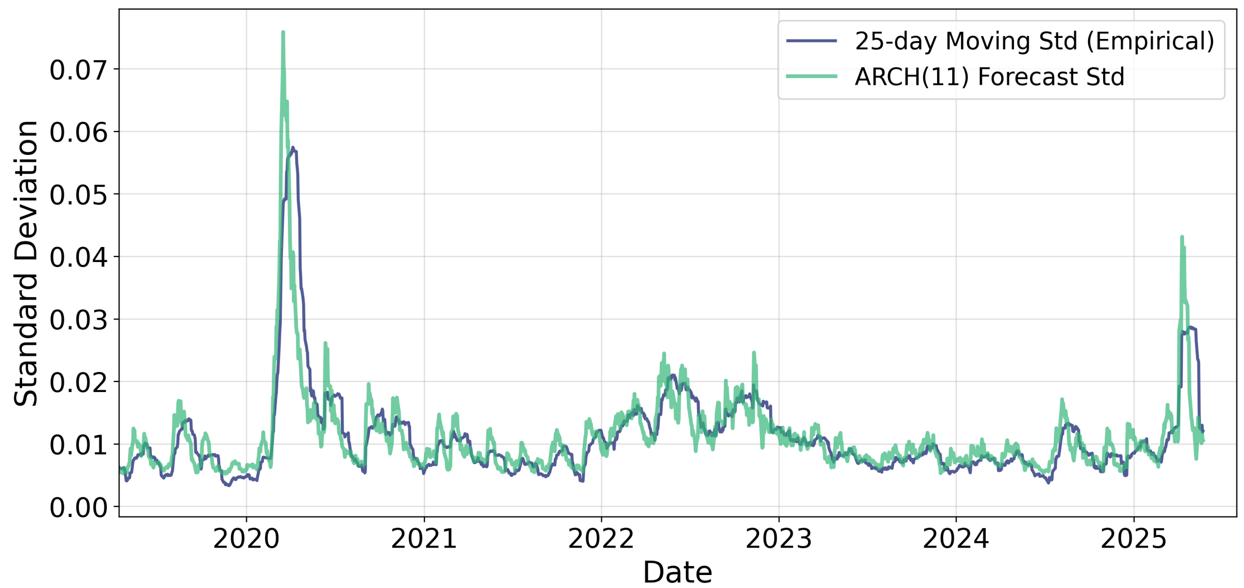
Forecast using GARCH(2,1) model.

Exercise 3. Volatility forecast with GARCH models.

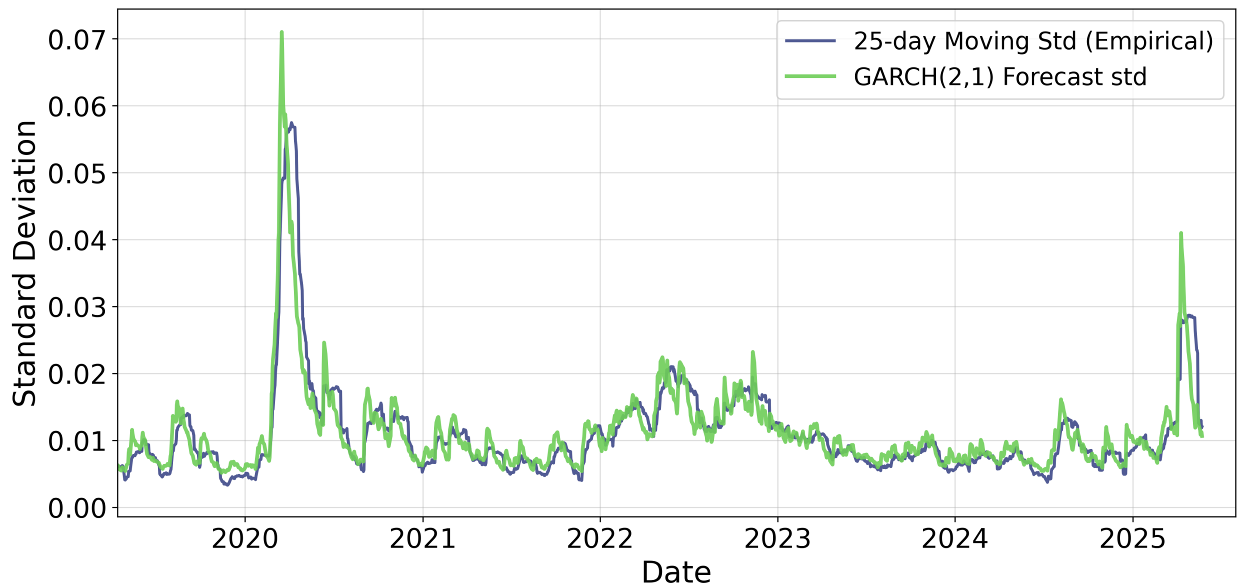
4. ARCH or GARCH?. Compare the best fit models $ARCH(p^*)$ and $GARCH(p^*, q^*)$ to determine which is the overall best one.

GARCH performs more accurately as it has lower AIC and BIC values:

Model	AIC	BIC
ARCH(11)	20074.52	20163.88
GARCH(2,1)	20044.75	20079.12



Forecast using the ARCH(11) model.

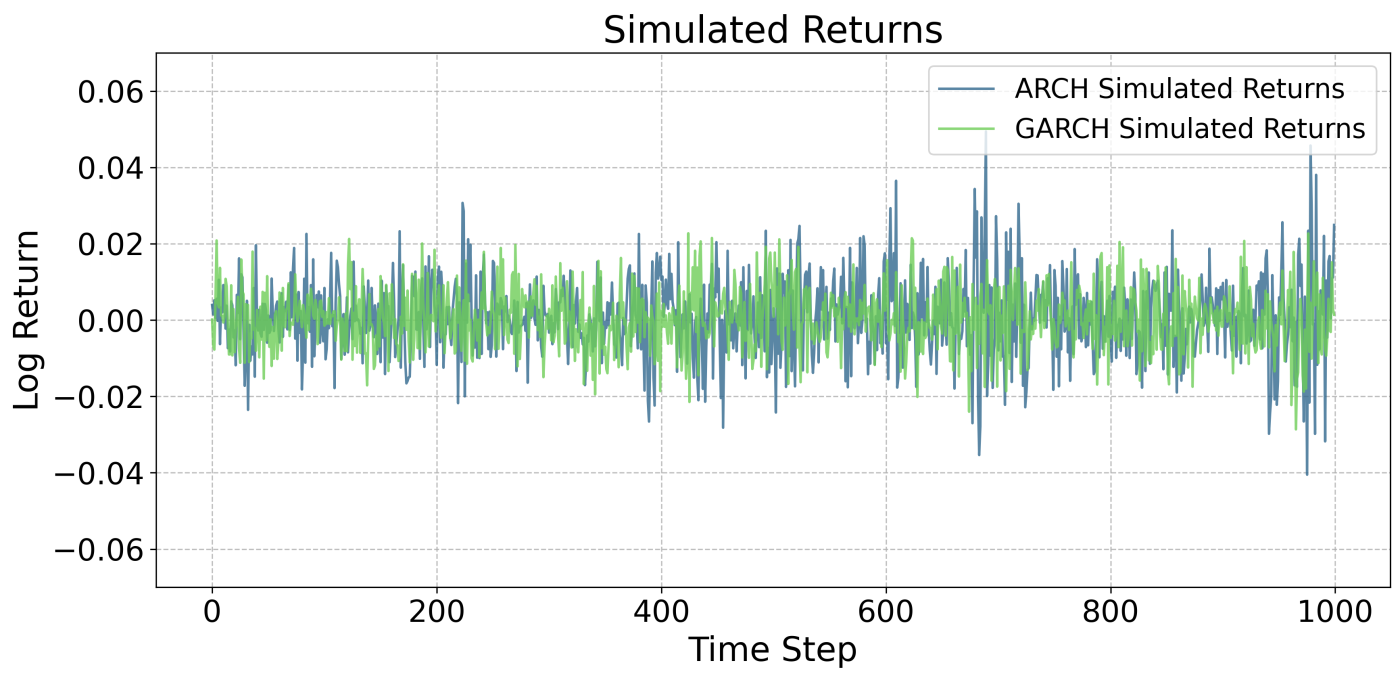


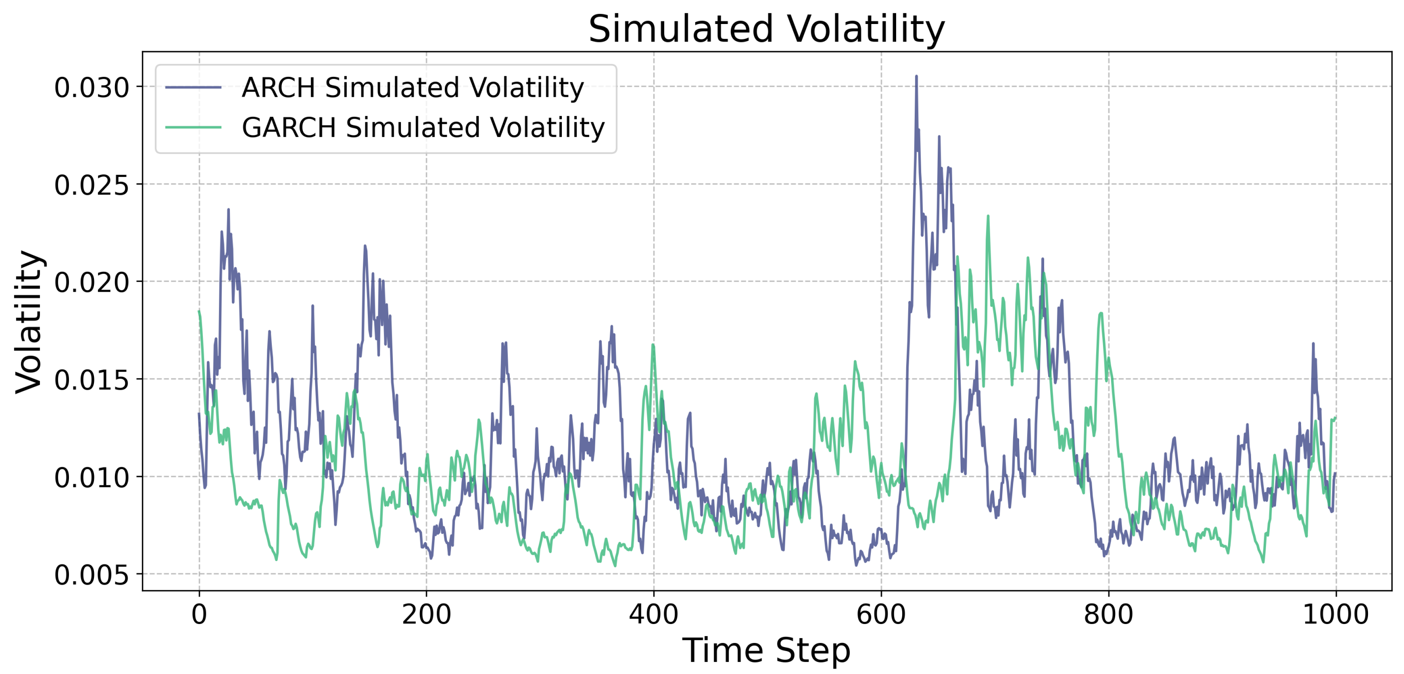
Forecast using GARCH(2,1) model.

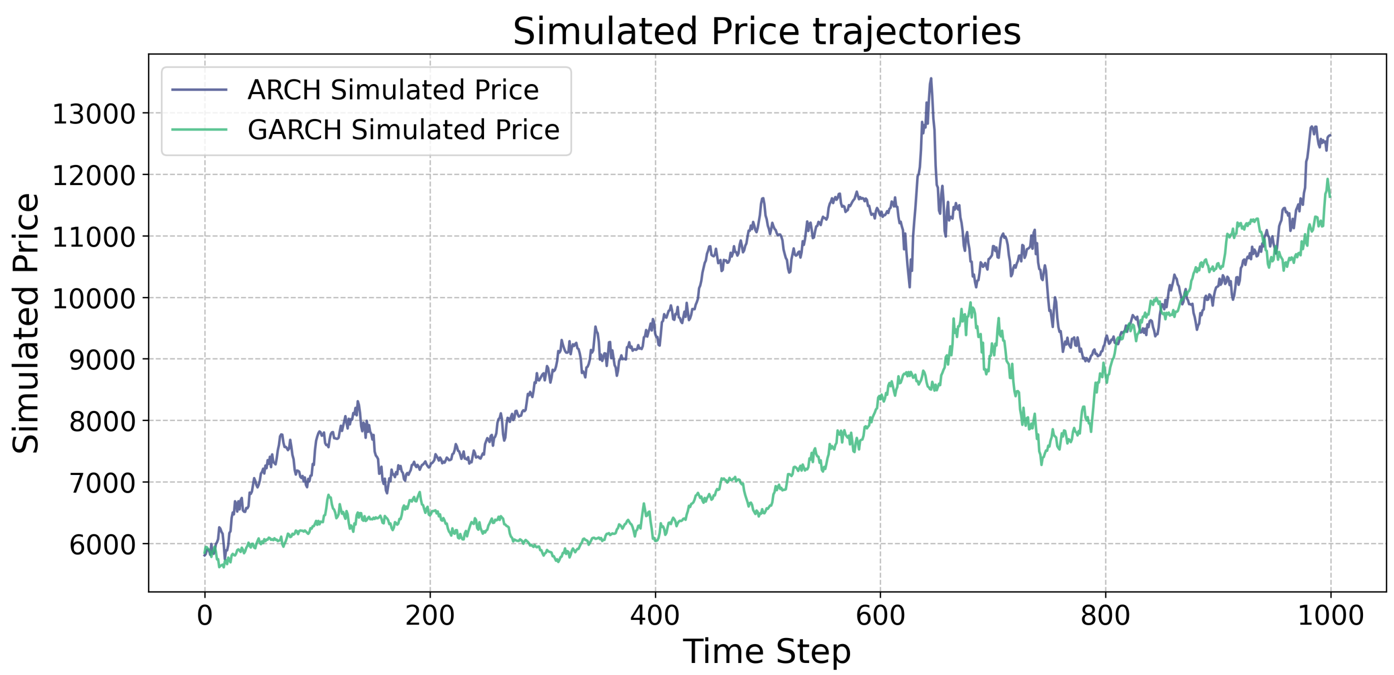
Exercise 4. Simulation of ARCH and GARCH processes.

1. Simulate price dynamics with an ARCH and GARCH model. Use the optimal ARCH(p^*) and GARCH(p^*,q^*) models, which you should have found previously.

2. Plot the simulated returns, volatility and prices.

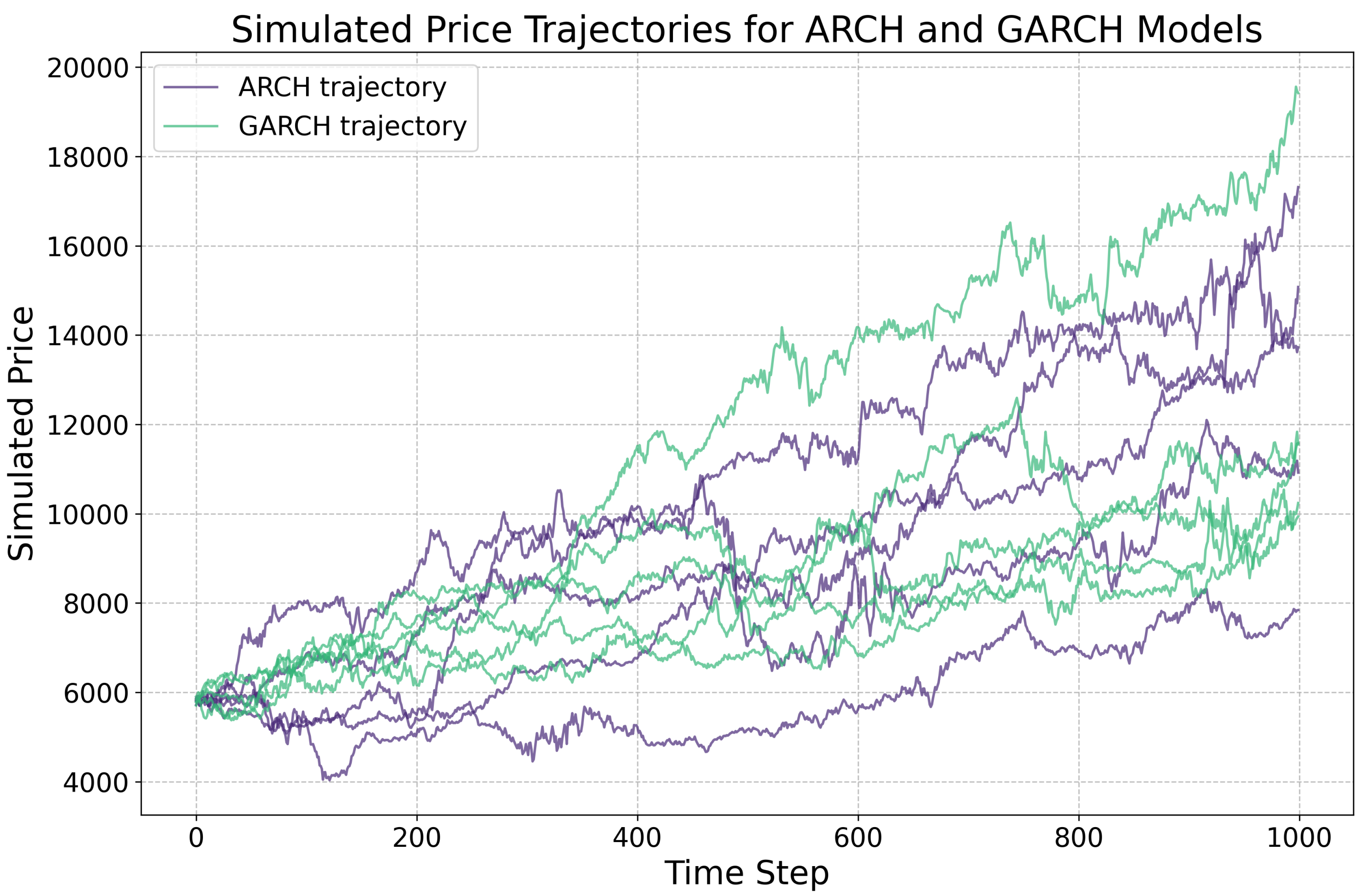




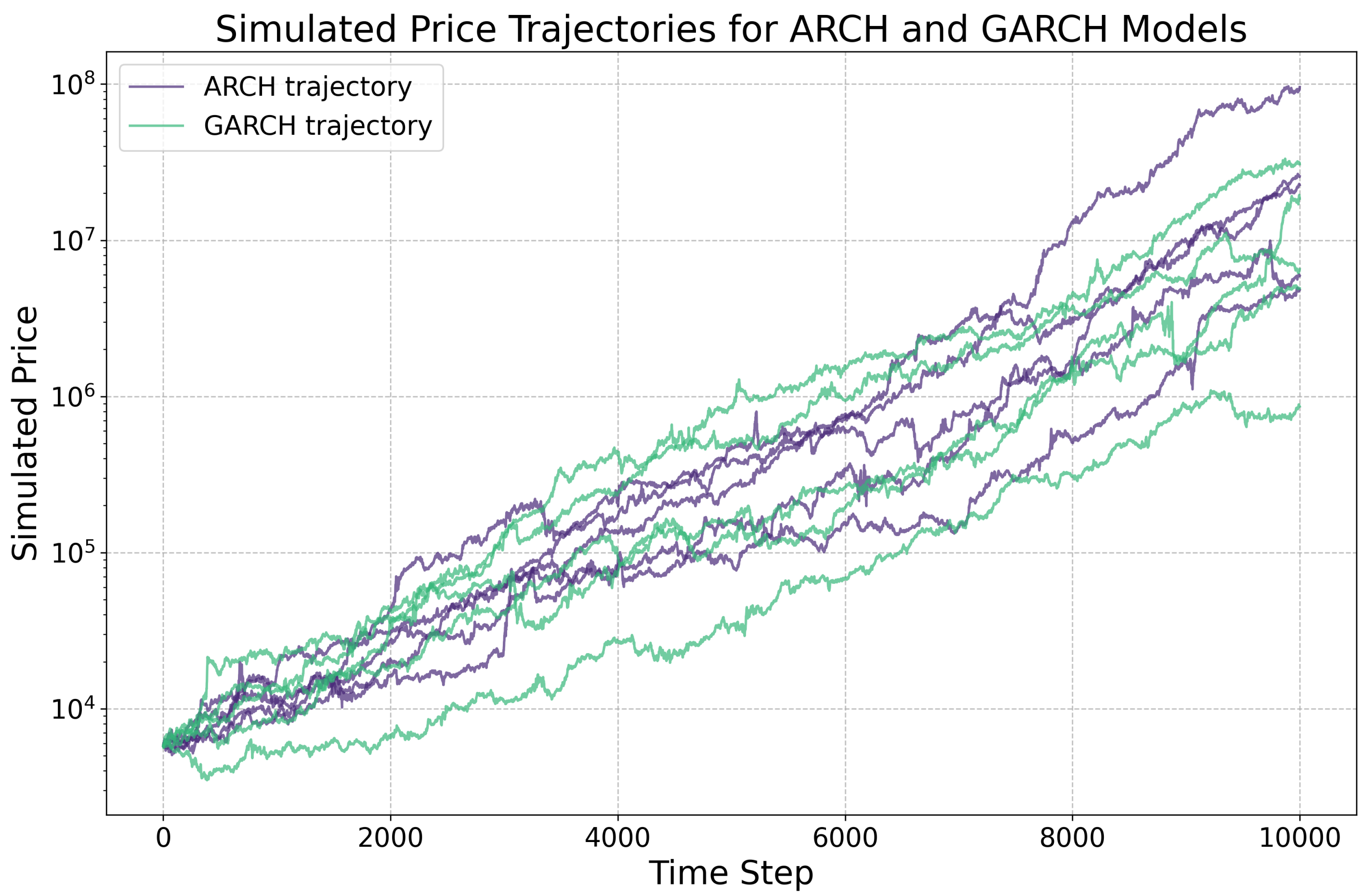


Exercise 4. Simulation of ARCH and GARCH processes.

3. Plot different realisations of the future simulated prices.



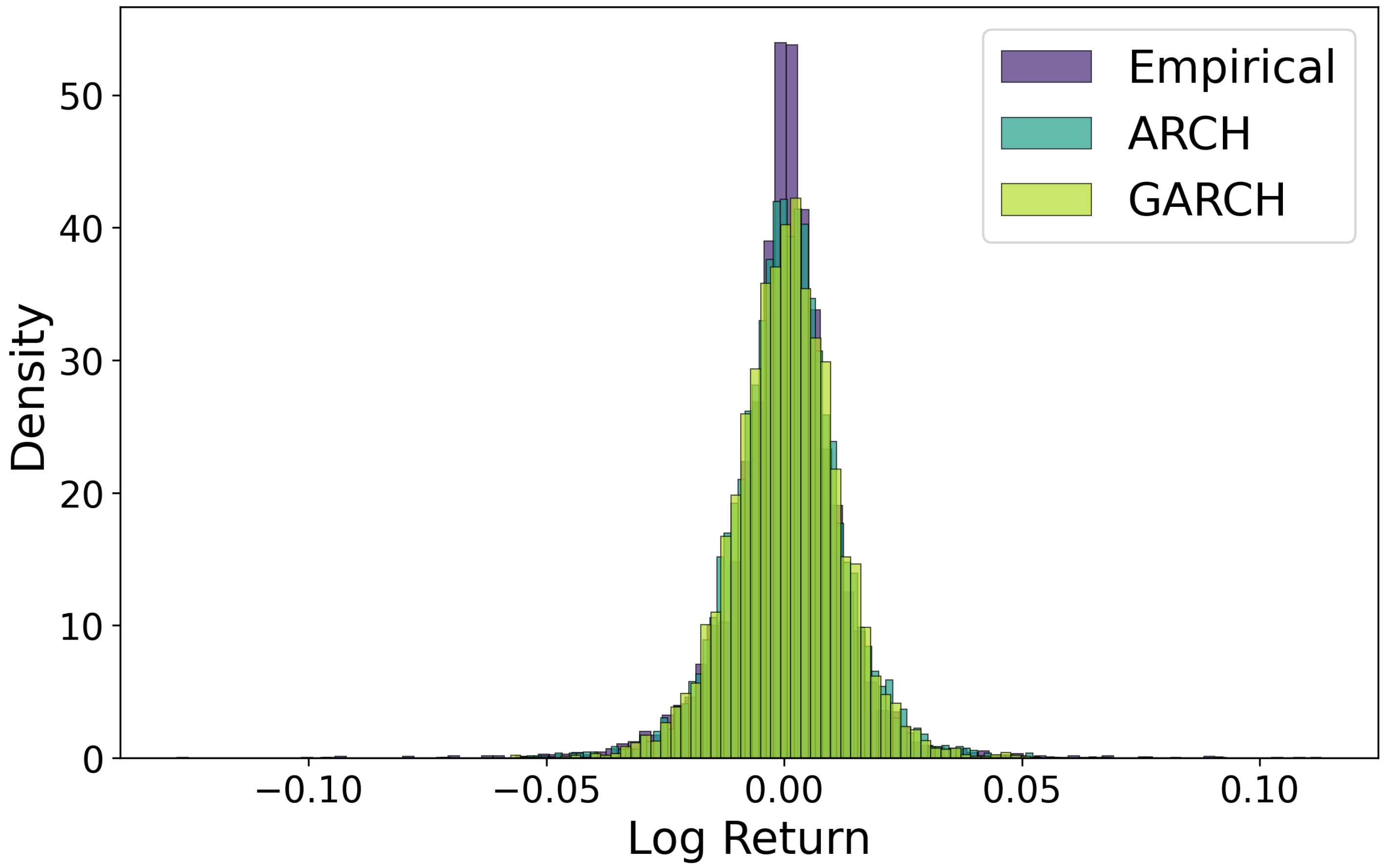
Results for $N = 10^3$ time steps.



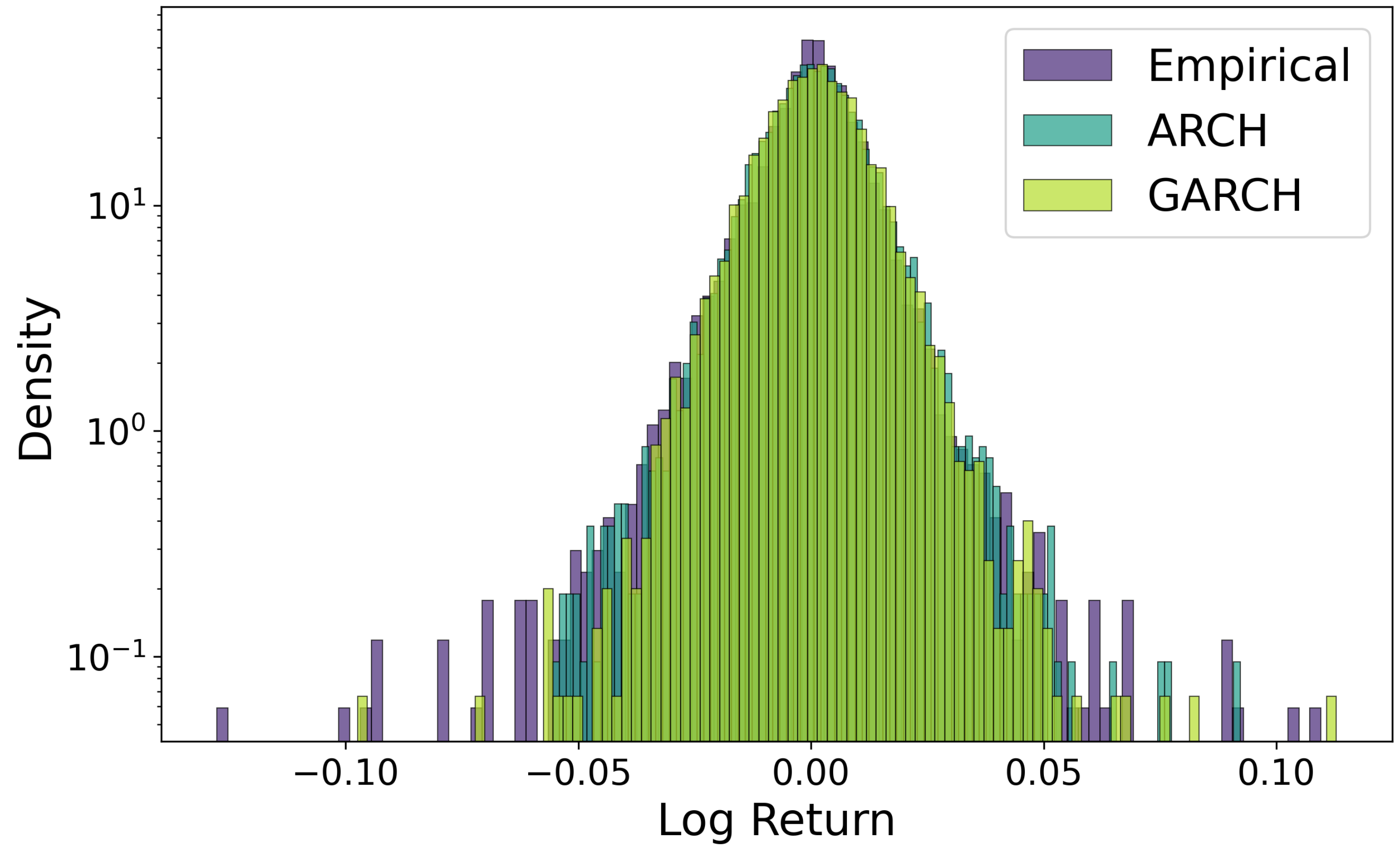
Results in semi-log scale for $N = 10^4$ time steps.

Exercise 4. Simulation of ARCH and GARCH processes.

4. Plot the return distribution of the data (S&P 500), and the ones simulated with the ARCH and GARCH models.



Results for $N = 7143$ time steps.



Results in semi-log scale for $N = 7143$ time steps.

Assigment Two:

Options

Exercise 1. Option Pricing using the binomial model.

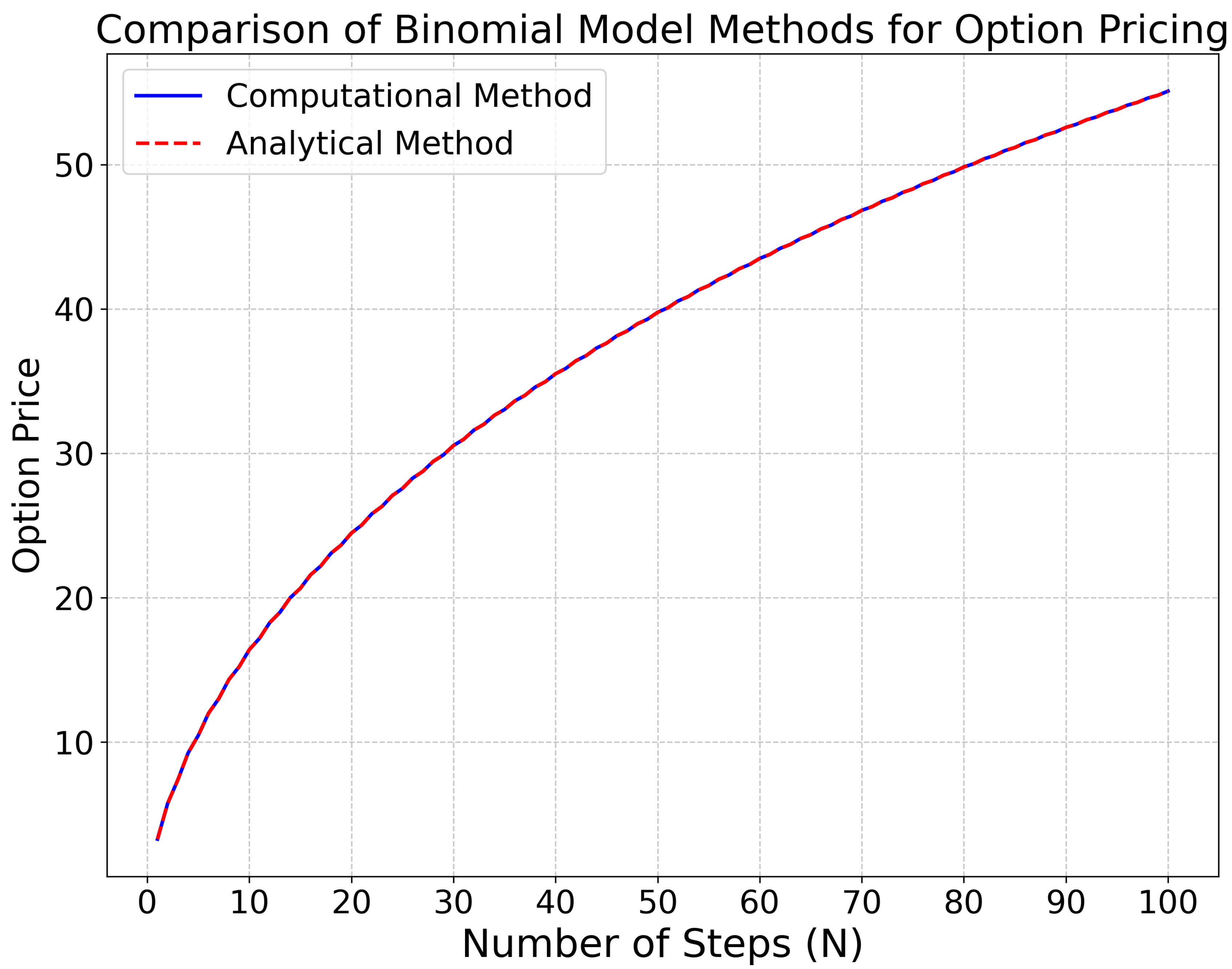
1. Complete the functions below to implement the binomial model for a call option.

```
def comb(n, i): #Returns the result of n combined with i

    return np.math.factorial(n) / (np.math.factorial(n-i)*np.math.factorial(i))

def call_price_binomial_computational(N, Y0, K, r, dt, u, d):

    pe = (np.exp(r*dt) - d) / (u - d)
    C0 = 0.0

    for j in range(N+1):
        payoff = max(u**j * d**(N - j) * Y0 - K, 0)
        probability = comb(N, j) * pe**j * (1 - pe)**(N - j)
        C0 += probability * payoff

    return np.exp(-r * N * dt) * C0



def call_price_binomial_analytical(N, Y0, K, r, dt, u, d):
    pe = (np.exp(r*dt)-d)/(u-d)
    qe = 1 - pe
    qe_prime = 1 - u * np.exp(-r*dt)*pe
    a = np.ceil(np.log(K/(Y0*d**N))/np.log(u/d))
    F_qe = binom.cdf(N - int(a), N, qe)
    F_qe_prime = binom.cdf(N-int(a), N, qe_prime)
    return Y0*F_qe_prime - np.exp(-r*N*dt)*K*F_qe
```

Function to implement the Binomial Model for a call option.

Exercise 1. Option Pricing using the binomial model.

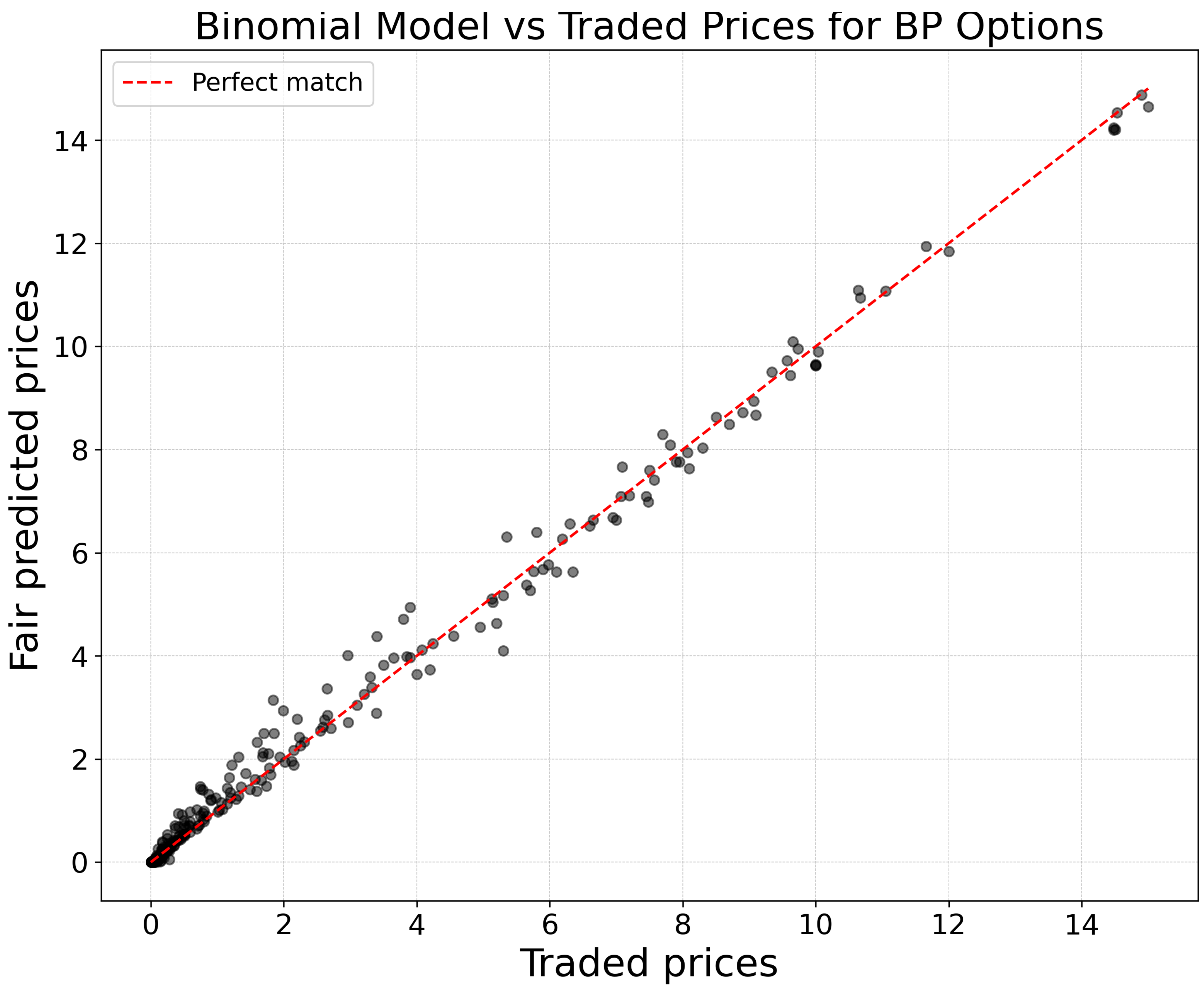
2. Compare computational and theoretical methods.



Parameters: $Y_0 = 100$, $K = 110$, $\sigma = 0.5$, $r = 0.0$, $dt = 0.1$.

Exercise 1. Option Pricing using the binomial model.

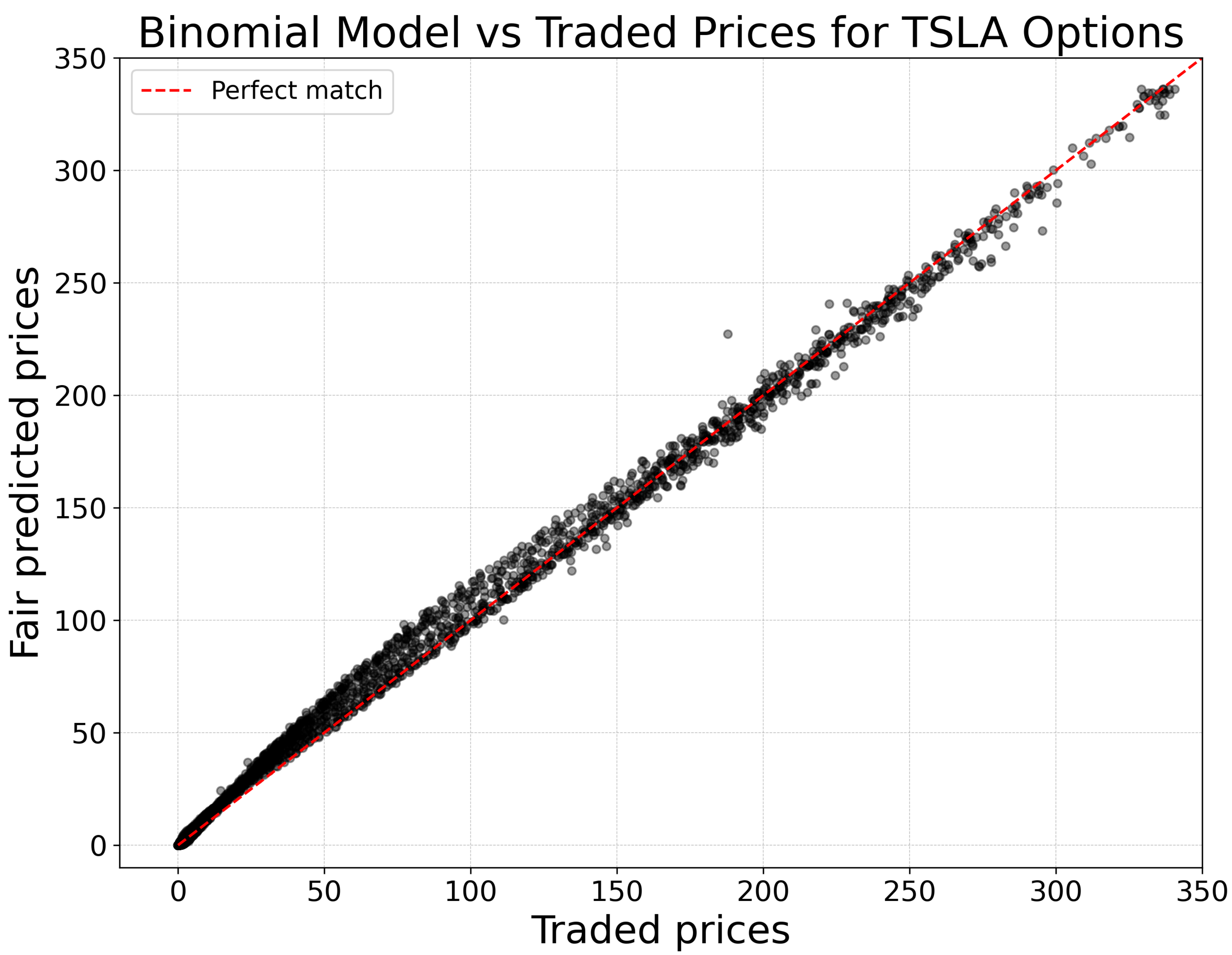
3. Price Real Options from BP. Price each Call option using the binomial model and compare the results with the real traded prices.



Results considering data on 26/05/2025.

Exercise 1. Option Pricing using the binomial model.

4. Repeat the last step for options from Tesla.



Results considering data on 26/05/2025.

Exercise 2. Implement the Black-Scholes model.

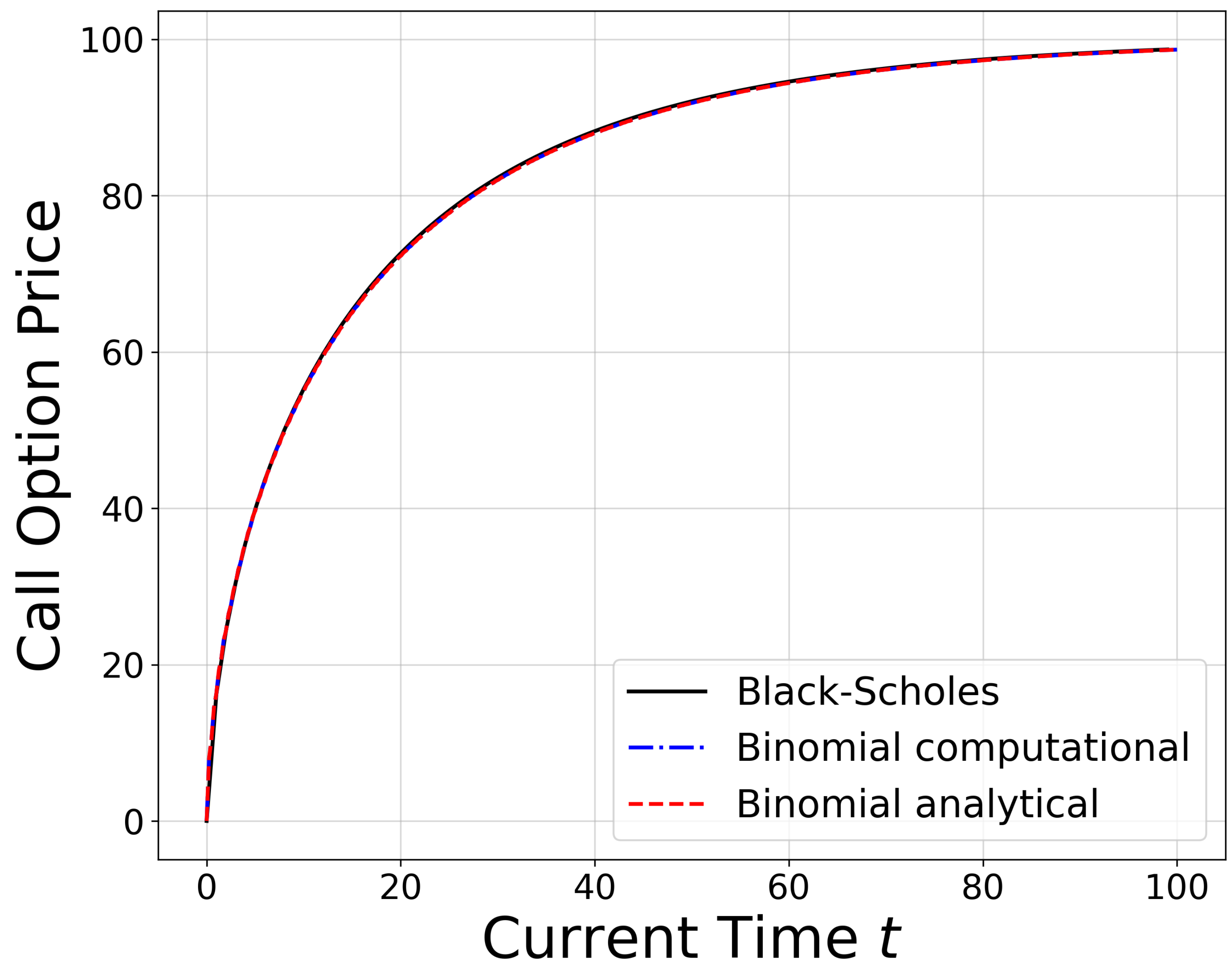
1. Implement the Black-Scholes model.

```
def black_scholes(t, S, K, T, r, sigma):

    d1 = (np.log(S/K) + (r + sigma**2/2)*(T - t)) / (sigma * np.sqrt(T - t))
    d2 = d1 - sigma*np.sqrt(T-t)

    call = S * norm.cdf(d1) - K * np.exp(-r*(T-t)) * norm.cdf(d2)

    return call
```
✓ 0.0s

Function to implement the Black-Scholes Model.

Exercise 2. Implement the Black-Scholes model

2. Compute the price of an option with the given parameters using the Black-Scholes model and the binomial model (with both analytical and computational methods). Plot the results for the values at different times. What do you observe?

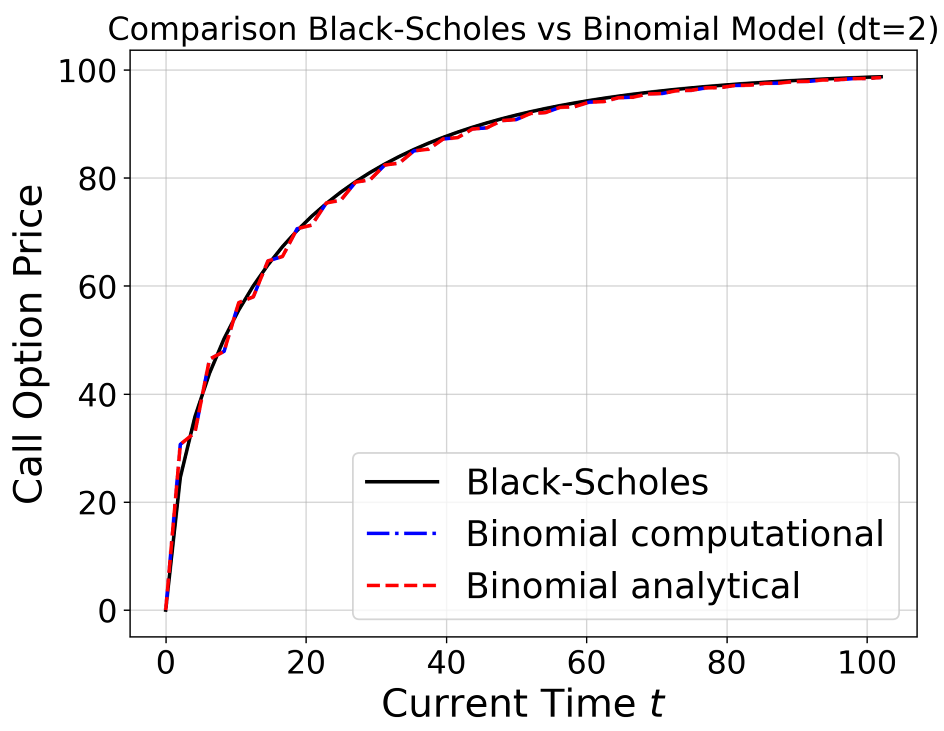


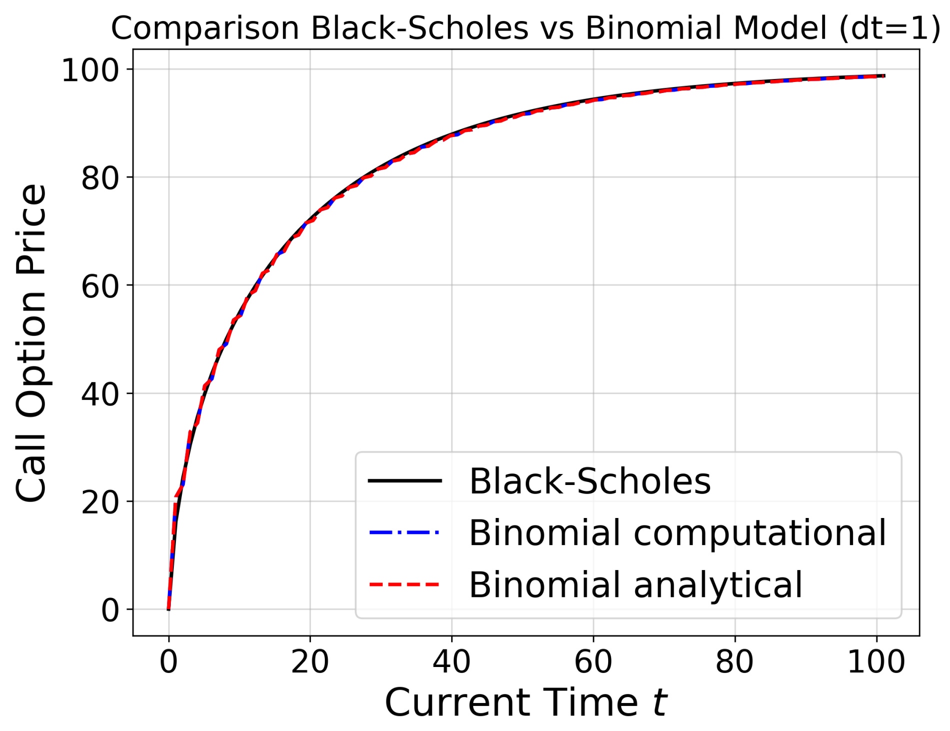
- Both models predict the same behavior for call option prices.
- The Binomial Model accurately approximates the Black-Scholes model.

Parameters: $Y_0 = 100$, $K = 110$, $\sigma = 0.5$, $r = 0.0$, $dt = 0.25$.

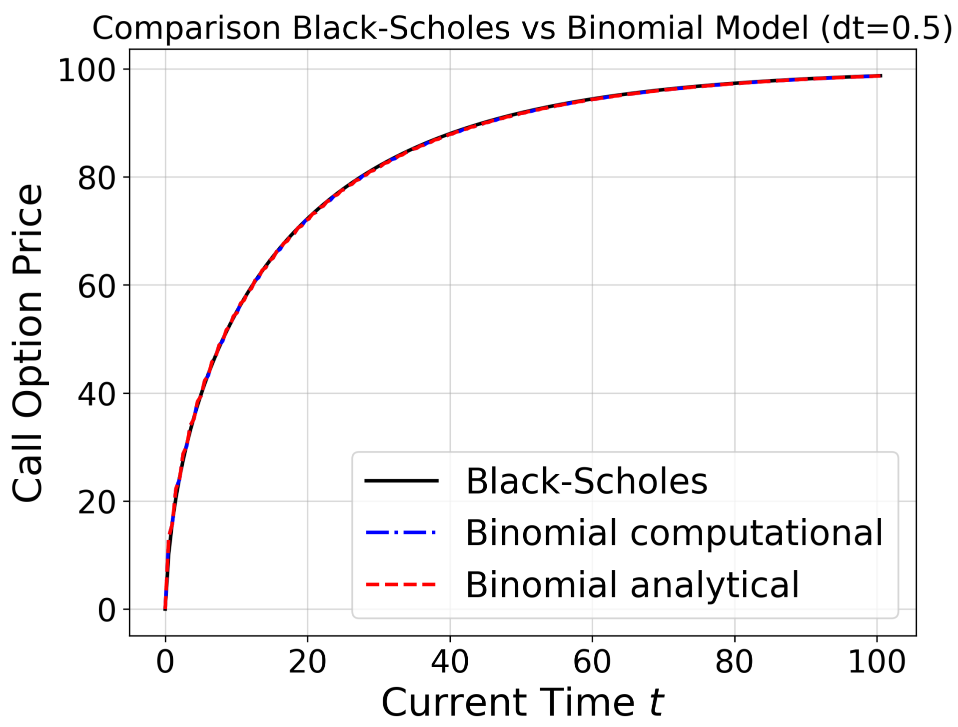
Exercise 2. Implement the Black-Scholes model.

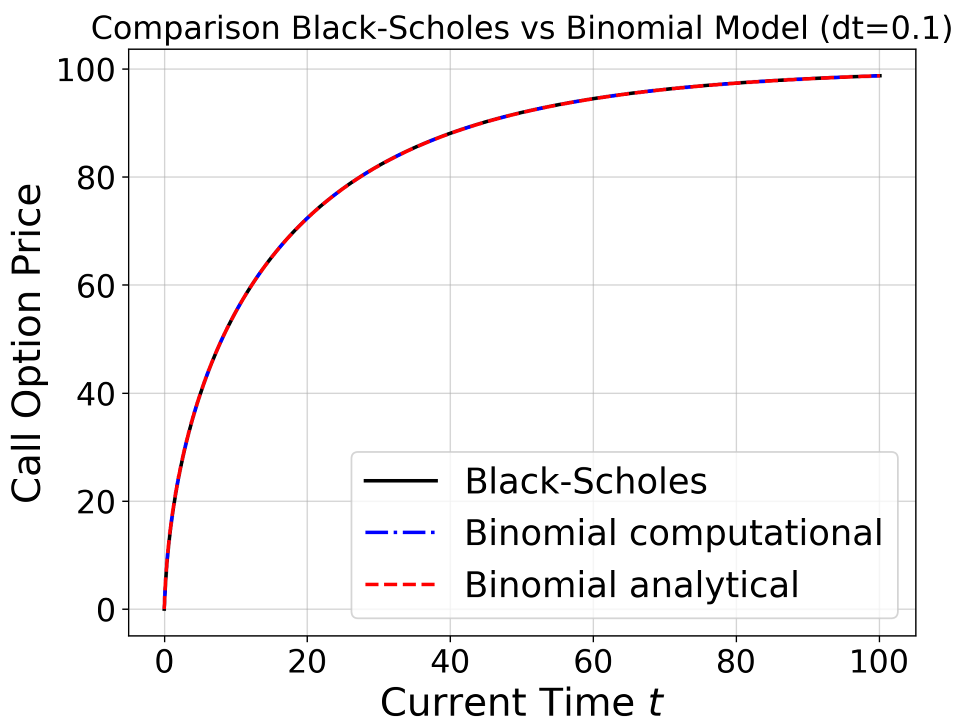
3. Repeat the previous step, decreasing the time step. Plot it for 4 different time steps, each one smaller than the previous one. What do you observe?





- The Binomial Model converges to the Black-Sholes Model as we decrease dt.





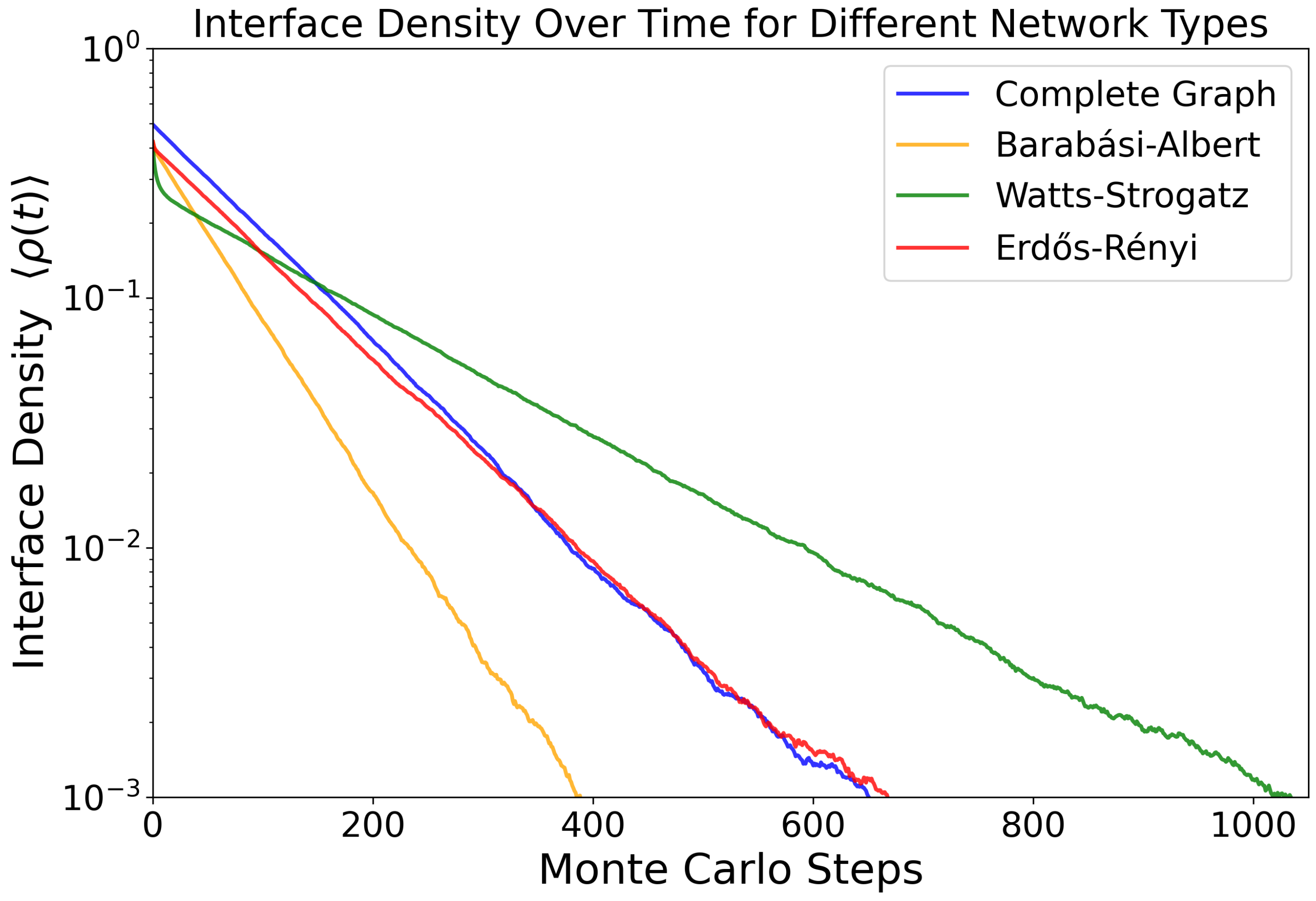
- For $dt = 0.1$, the models match properly.

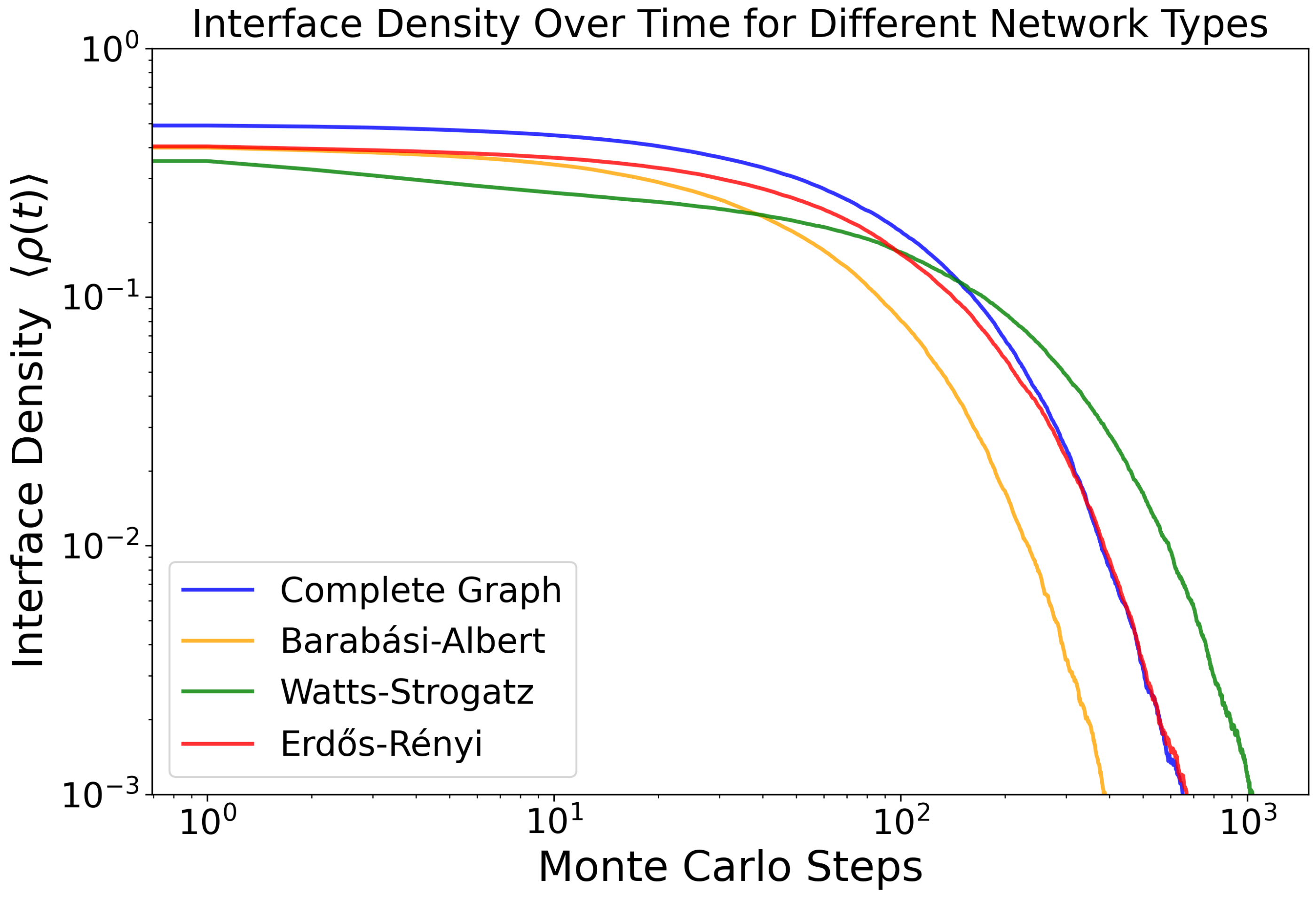
Assigment Three:

Agent Based Models

Exercise 1. The average interface density of the Voter Model.

1. Compute the average interface density of the voter model for the 4 networks Barabasi-Albert, Watts-Strogatz, Erdös-Rényi and complete graph. Comment the results that you obtain. **Note**: Use $N = 200$ nodes, $T = 100$ time steps and $M = 100$ number of realisations. It will take a while, but should be fine.

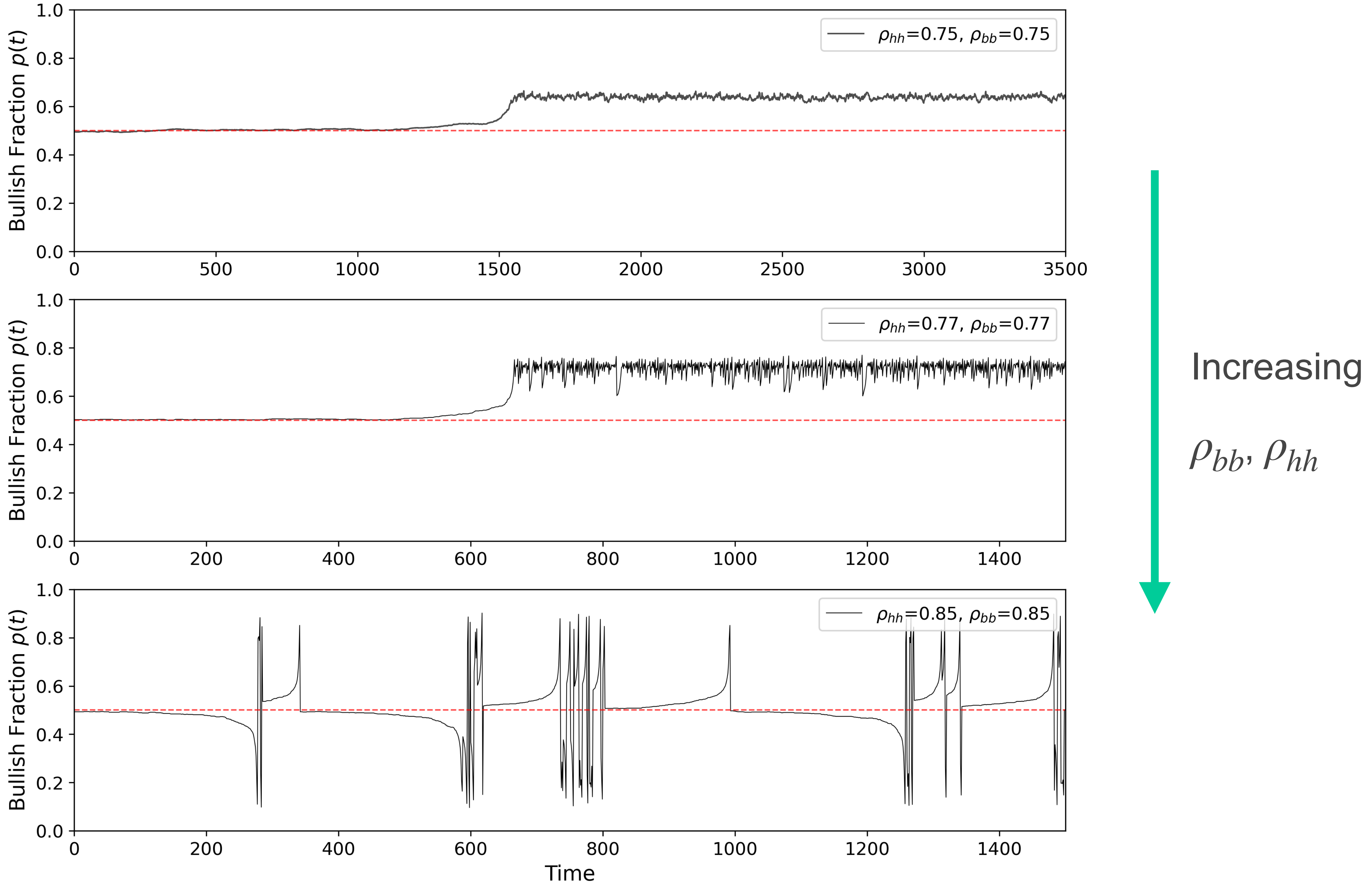




Results for $N = 200$ nodes and $M = 10^4$ realizations considering $\langle k \rangle = 6$ for BA, WS and ER networks.

Exercise 2: Implement the Imitation and Contrarian Behavior model.

Plot the time evolution of the number of bullish agents $p(t)$. Use the following parameters: $m = 60$, $\rho_{bb} = \rho_{hh} = 0.75$, $\rho_{hb} = \rho_{bh} = 0.72$. Then, repeat the simulation for $\rho_{bb} = \rho_{hh} = 0.77$ and $\rho_{bb} = \rho_{hh} = 0.85$. Use a population of $N = 1000$ agents.



Final project:

The Minority Game as a Model for Adaptive Competition and Market Dynamics

D. Challet and Y.-C. Zhang , Physica A 246, 407 (1997)

Problem:

Most of the economic theories are deductive in origin. However, most actions of real players are based on trial-and-error inductive thinking.

Proposal:

A simple model in which agents adapt by learning from past outcomes and make forecasts using only limited historical information.

Core idea:

1. If there are **more buyers** than sellers, prices rise → **sellers benefit** by selling at a higher price.
2. If there are **more sellers** than buyers, prices fall → **buyers benefit** by purchasing at a lower price.

→ Success comes from going against the crowd, **being in the minority.**

Explore the dynamics of the Minority Game by analyzing how volatility, predictability, and gains emerge from the interactions of agents.

Minority Game Model

Consider N agents that at each time step t_k decide whether to buy an asset $a_i(t_k) = +1$ or sell it $a_i(t_k) = -1$. Player used a finite set of ad hoc strategies to make their decision, based on the past record. Playes keeps a register with the accumulated virtual scores of the strategies $Q_{i,s}(t)$.

strategy

$$s_i(t) = \arg\max_s Q_{i,s}(t)$$

action

$$a_i \in \{-1,1\}$$

scores

$$Q_{i,s}(t+1) = Q_{i,s}(t) + g_{i,s}(t)$$

$$A(t) = \sum_{i=1}^{N} a_i(t)$$

attendance

At each time step:

1. Each agent i selects the strategy with the highest virtual score.
2. The agent follows the action suggested by the chosen strategy.
3. The global attendance is determined.
4. All strategies are scored based on their prediction, following the reward:

$$g_{i,s}(t) = -a_{i,s}(t) sign A(t)$$

Strategies S, histories μ and memory M.

- The input information of a strategy is: $\vec{\mu}(t_k) = [-signA(t_{k-1}), \ldots, -signA(t_{k-M})]$.
- We can represent the information of a history by an integer $\mu \in \{0,\ldots,2^M - 1\}$.
- A strategy is a map $\{-1,+1\}^M \to \{-1,+1\}$.

Number of history states combinations

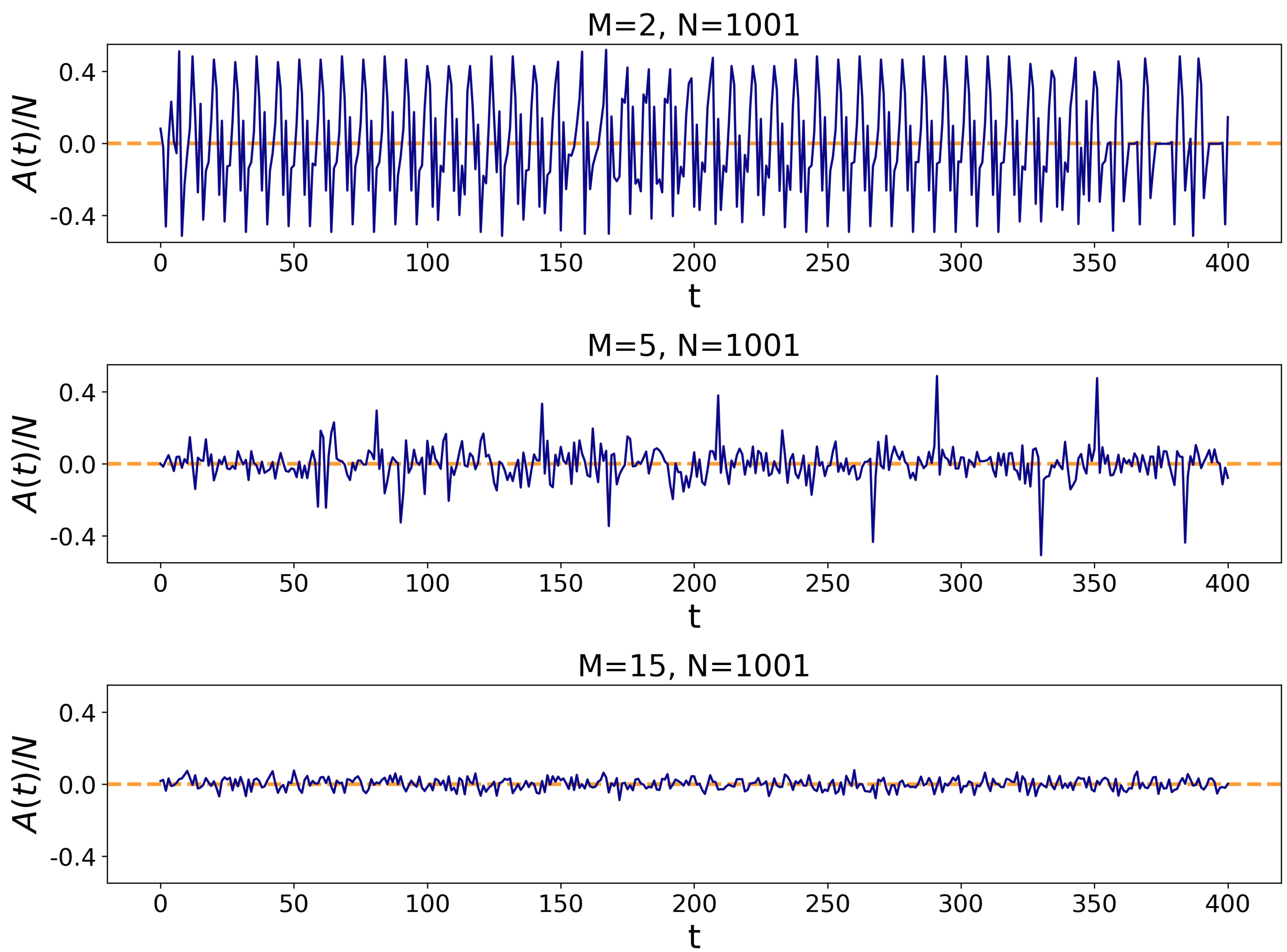
$P = 2^M$

History	Information	Prediction
$---$	0	Sell -1
$--+$	1	Buy $+1$
$-+-$	2	Sell -1
$-++$	3	Buy $+1$
$+--$	4	Buy $+1$
$+-+$	5	Sell -1
$++-$	6	Buy $+1$
$+++$	7	Sell -1

One of the possible strategies in the case of $M = 3$, the number of possible estrategies is $2^P = 2^{2^M}$.

Time evolution of the attendance.

$$A(t) = \sum_{i=1}^{N} a_i(t)$$

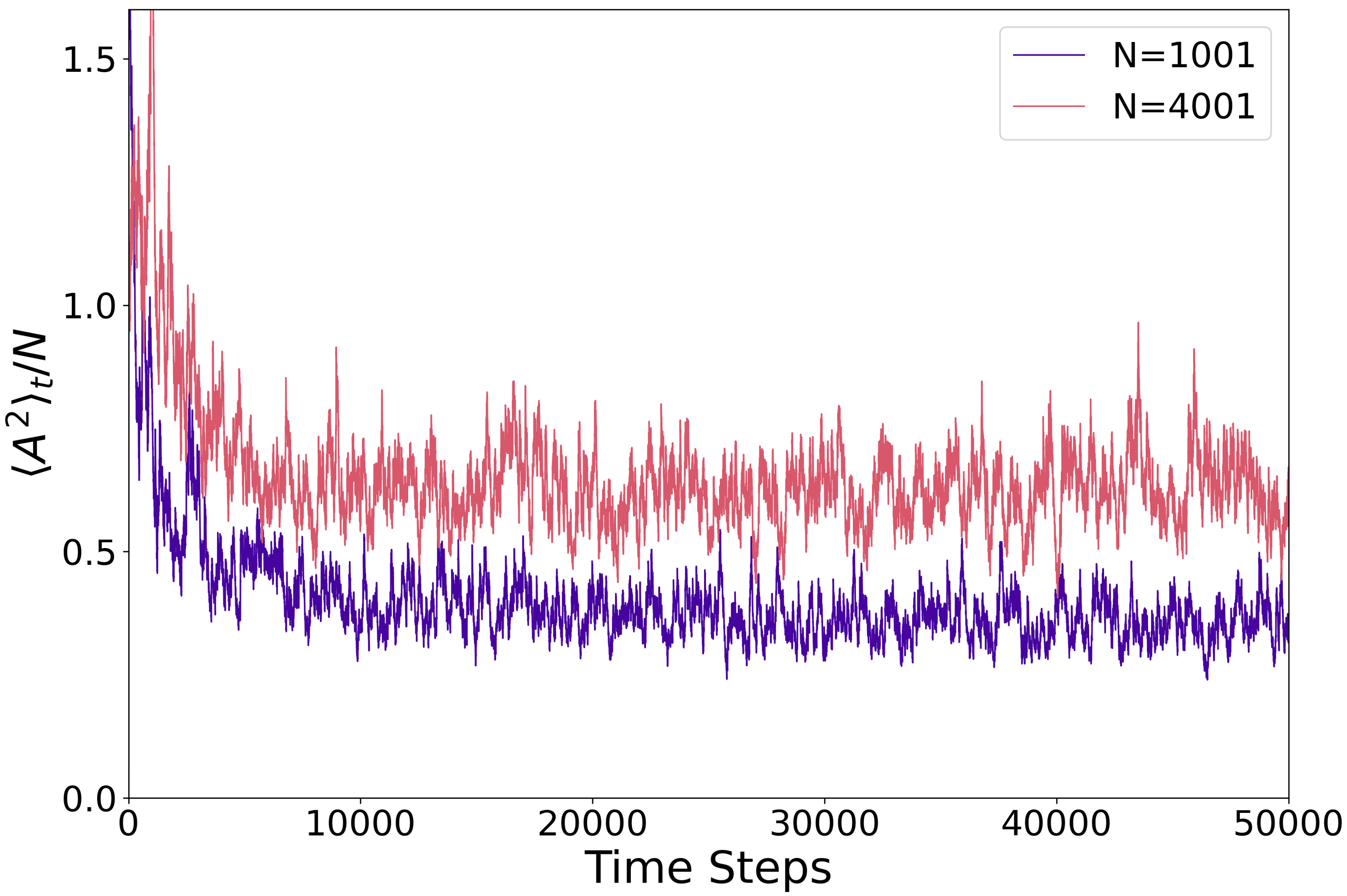


Fixed: $S = 2, N = 1001.$

- Low M: large variations, system in a quasi-periodic state.
- Intermediate M: the system follows a chaotic course.
- Large M, fluctuations get narrowed.

Time-dependent volatility: averaging the volatility over roughly $(1-\lambda)^{-1}$ last steps.

$$\boxed{\langle A^2\rangle_t = \lambda A^2(t) + (1-\lambda)\langle A^2\rangle_{t-1}}$$



$\langle A^2\rangle_t$ decreases from a high initial value until a certain value that depends on N and M.

Sign of self-organization, through adaptation.

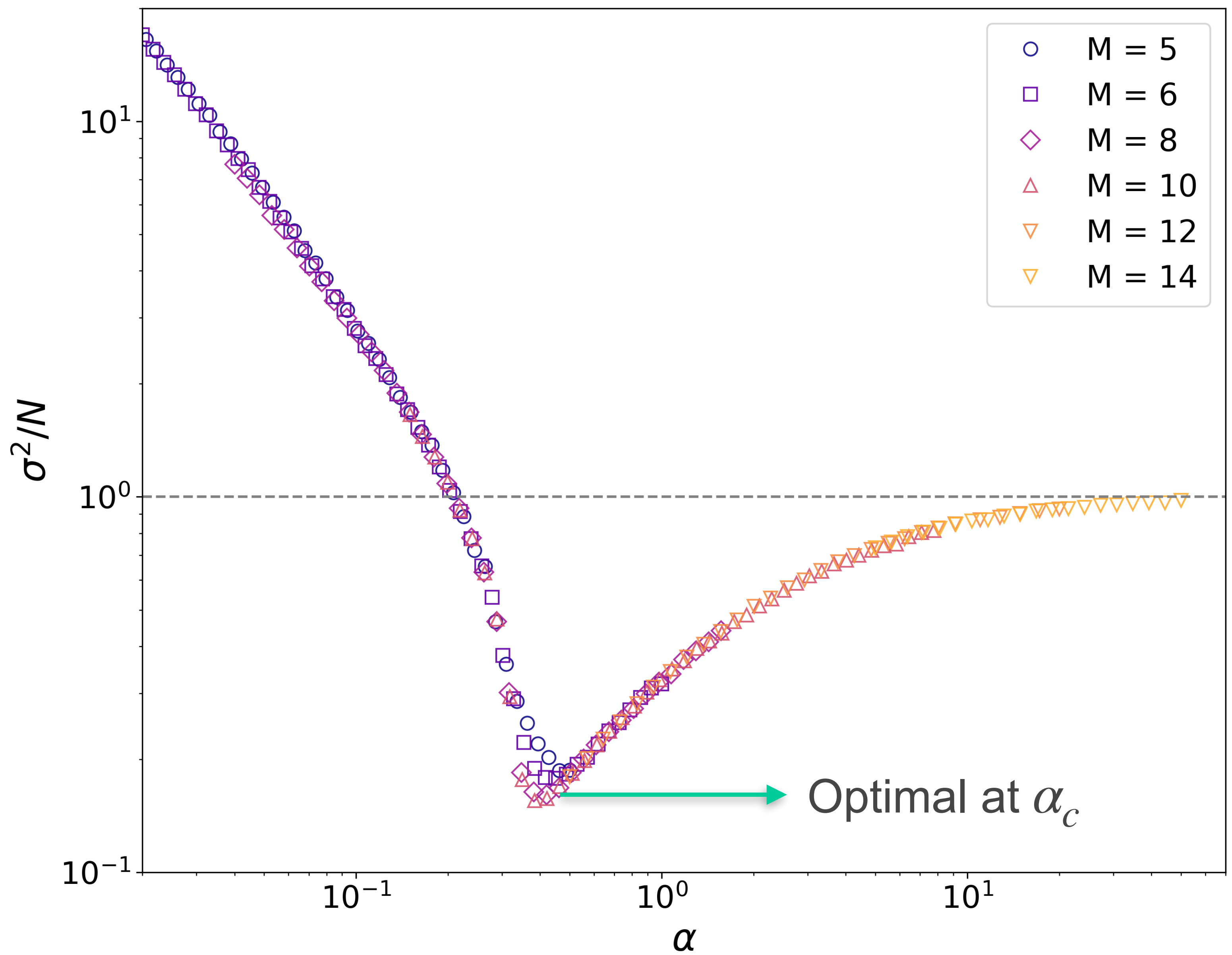
Fixed: $\lambda = 0.01$, $S = 2$, $M = 10$.

Volatility: time-averaged square of the attendance.

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} A(t)^2$$

Scaling parameter

$$\alpha = \frac{2^M}{N}$$



Fixed: $S = 2$.

- Low α (crowded regime): large fluctuations, agents are uncoordinated.
- At Intermediate α values, there is an optimal value α_c where agents coordinate.
- High α: excess of information leads to confusion - system approach toss-coin limit.

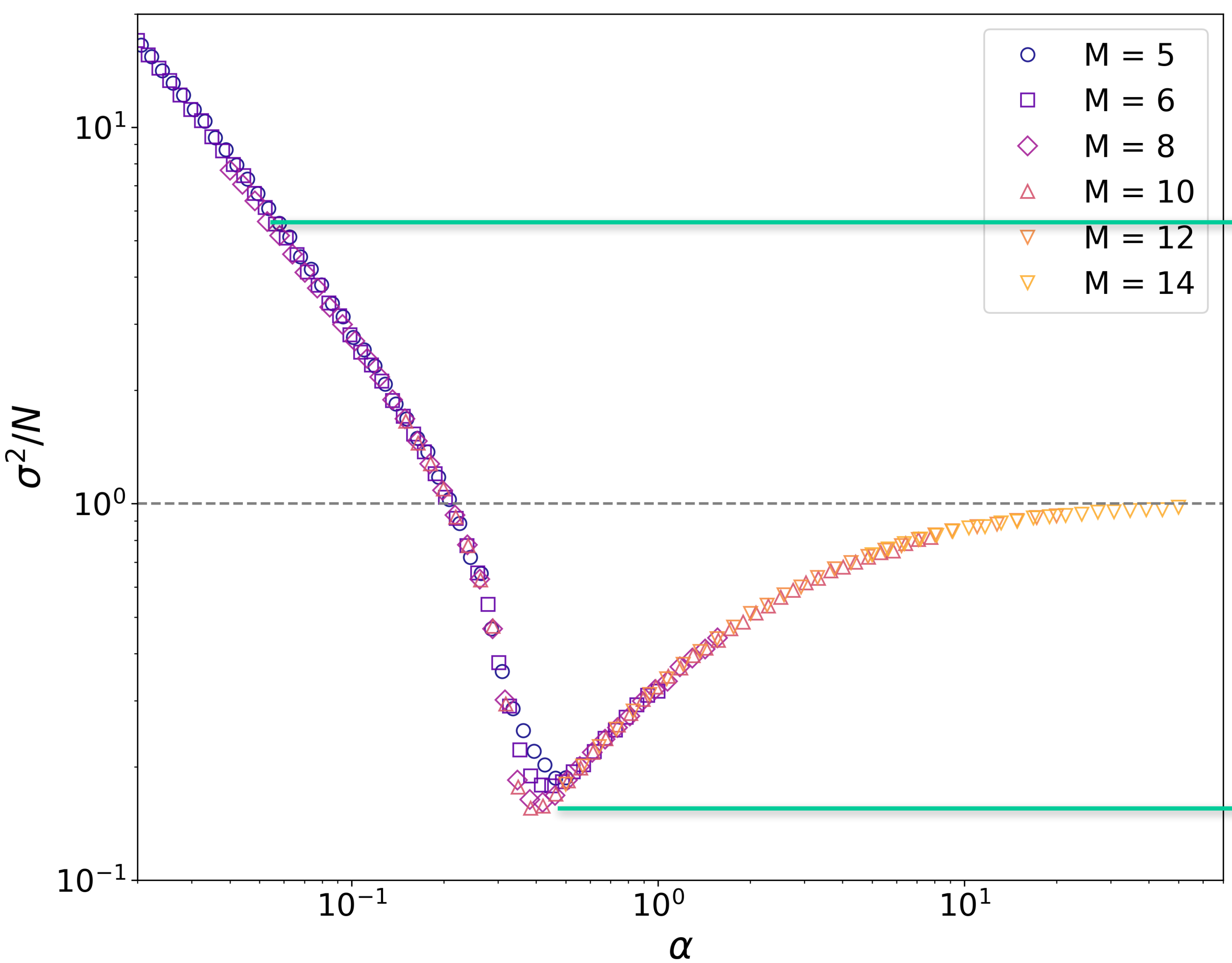
Optimal value for efficient use of shared information.

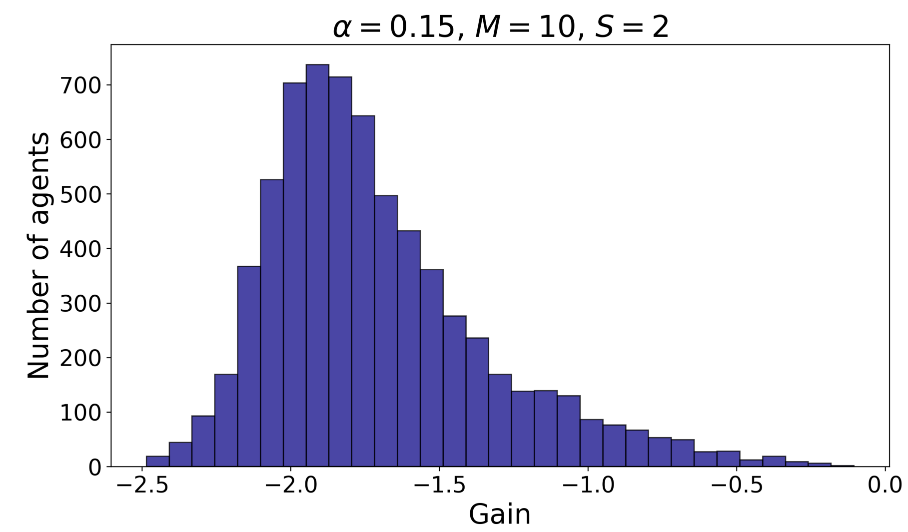
Volatility: time-averaged square of the attendance.

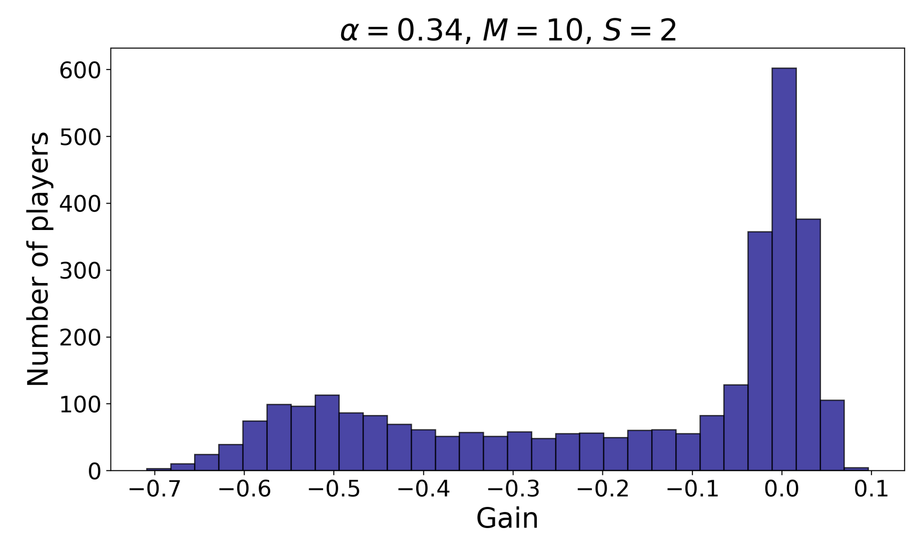
$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} A(t)^2$$

Average gain per agent

$$\langle G_i \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left(-a_i(t) \cdot A(t)\right)$$







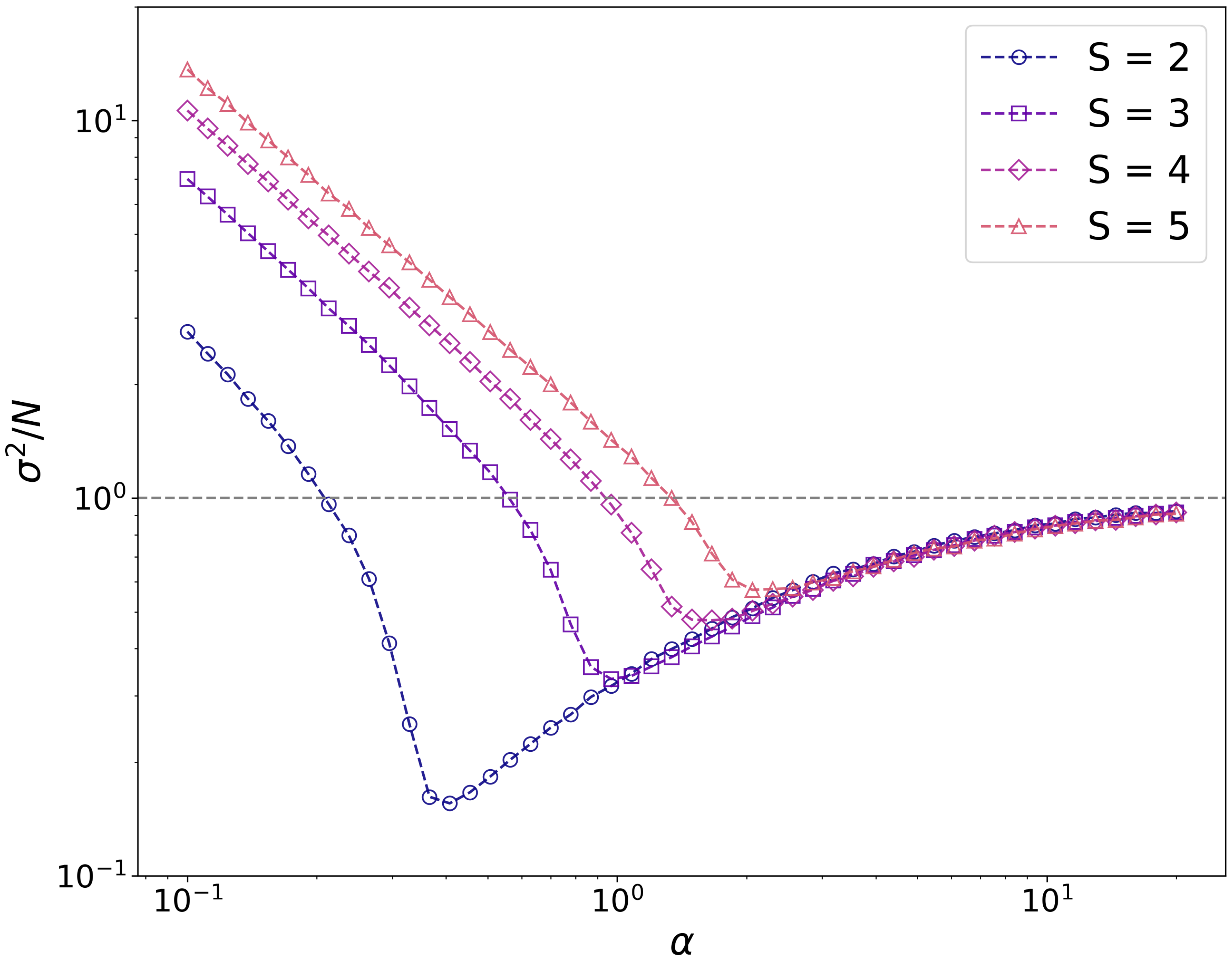
Fixed: $S = 2$.

Volatility for different number of strategies.

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} A(t)^2$$

Scaling parameter

$$\alpha = \frac{2^M}{N}$$



Fixed: $M = 10$ while varying N.

Effects of increasing S:

- Shifts the crowded region to the right.
- The minimum of the volatility increases.

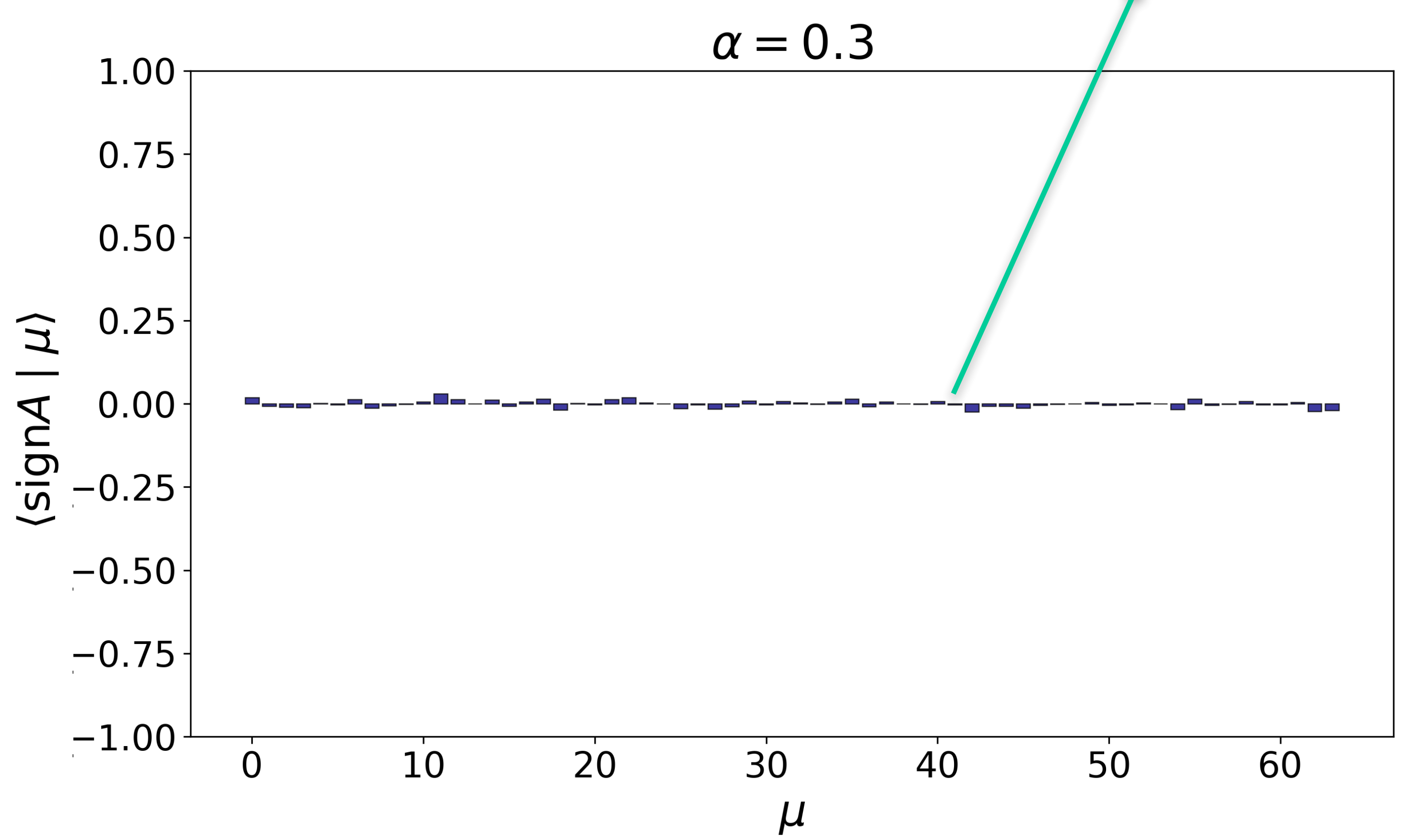
Harder for agents to coordinate effectively.

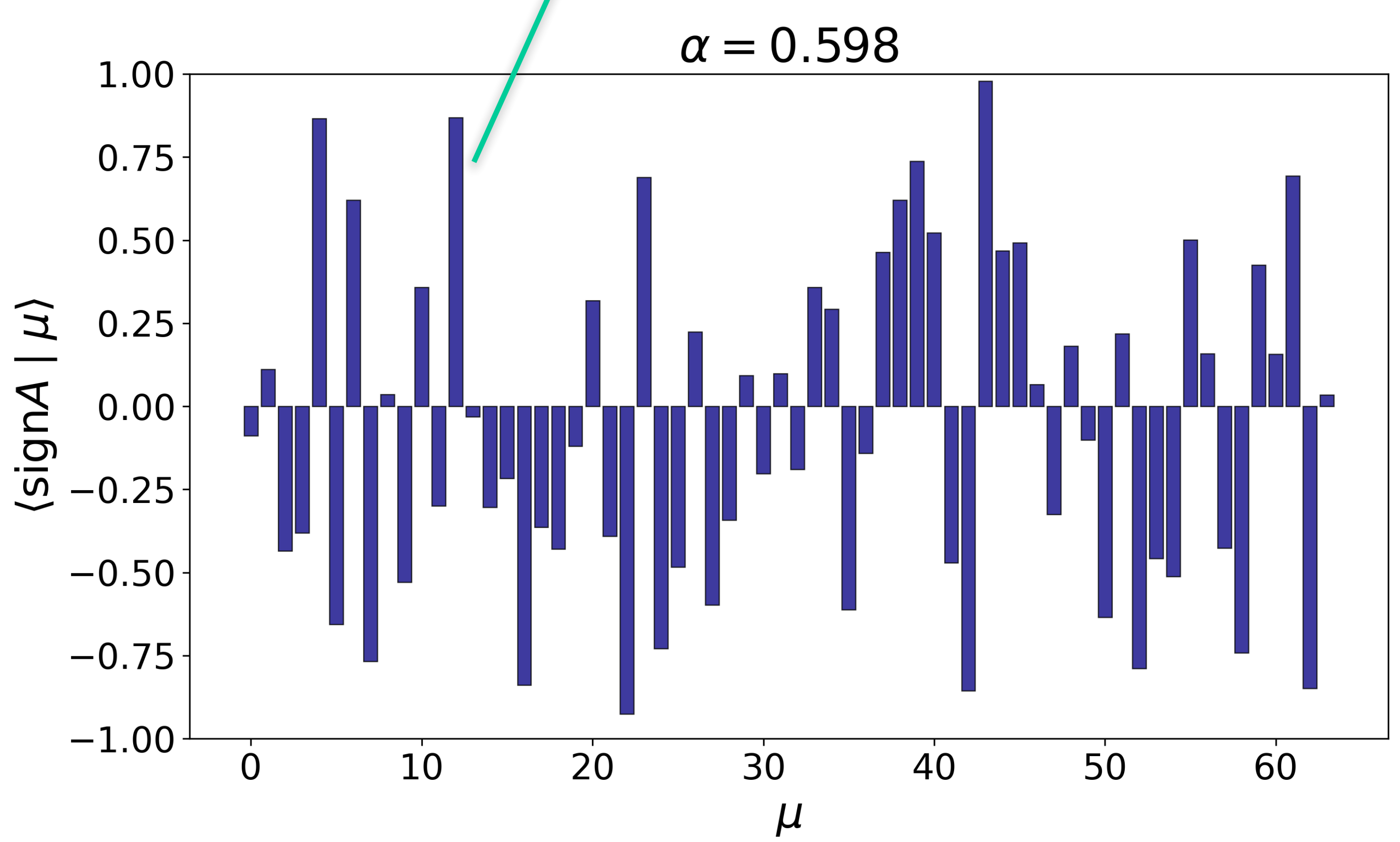
Average sign of the attendance given history μ.

$$\boxed{\left\langle \text{sign } A \mid \mu \right\rangle = \lim_{T\to\infty} \frac{\sum_{t=1}^{T} \delta_{\mu,\mu(t)} \text{ sign } A(t)}{\sum_{t=1}^{T} \delta_{\mu,\mu(t)}}}$$

Directional bias: predictable patterns.

Unbiased: no predictive information.





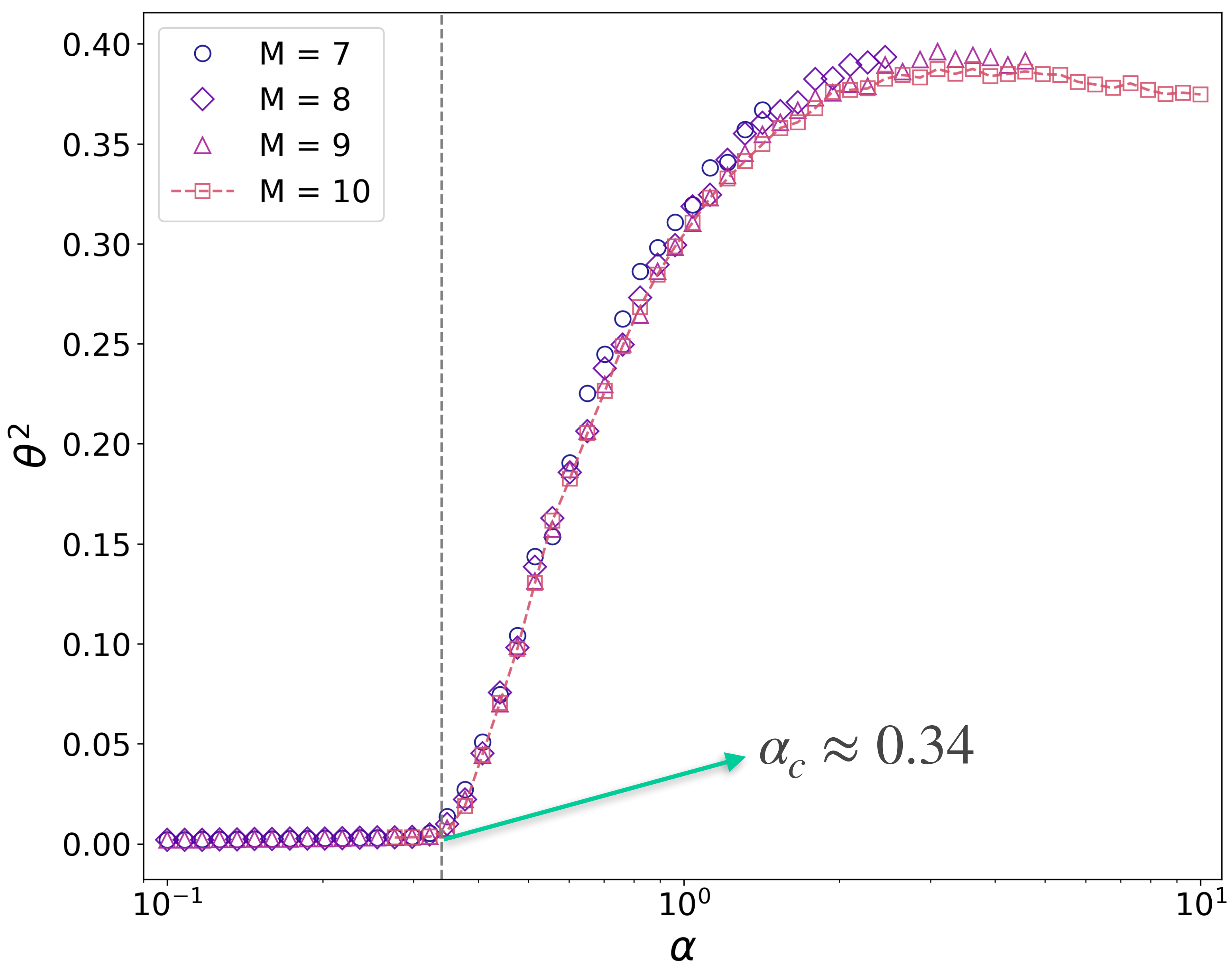
Fixed: $M = 6, N = 213, S = 2.$

Fixed: $M = 6, N = 107, S = 2.$

Predictability:

$$\theta^2 = \frac{1}{2^M} \sum_{\mu=0}^{2^M-1} \left(\langle \operatorname{sign} A \mid \mu \rangle \right)^2$$

Scaling parameter $\alpha = \dfrac{2^M}{N}$



Fixed: $S = 2$.

- Low α: Efficient phase - no directional signal $\theta \sim 0$.
- Phase transition at $\alpha_c \approx 0.34$.
- High α: Predictable phase - system develops memory, leading to dependent directional bias.

θ^2 captures the phase transition between efficient and inefficient regimes.

Speculators and producers:

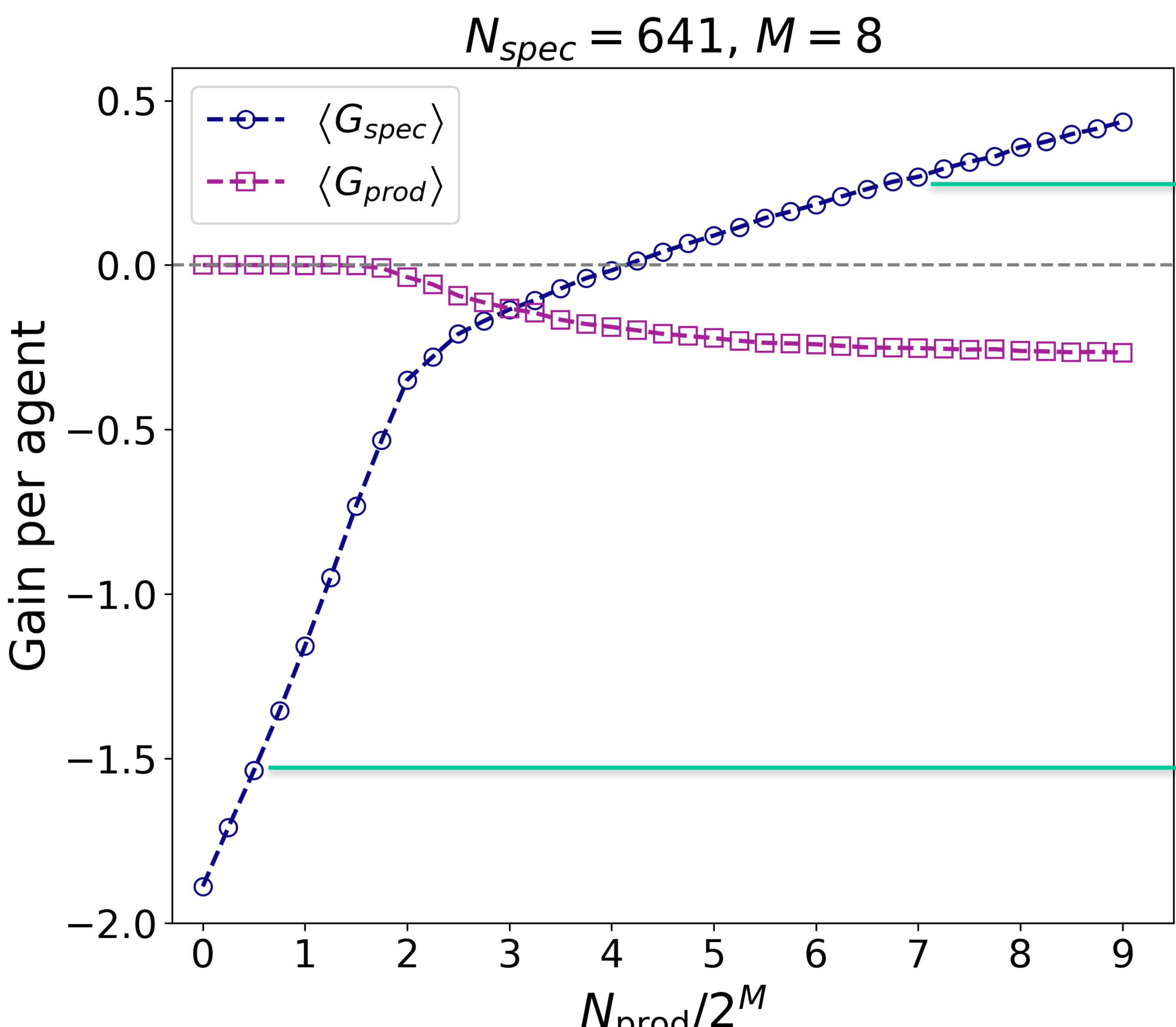
- **Speculators:** $S = 2$
- **Producers:** $S = 1$

Average gain per agent for:

- Speculators: $\langle G_{\text{spec}} \rangle$
- Producers: $\langle G_{\text{prod}} \rangle$

Average gain per agent

$$\langle G_i \rangle = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left(-a_i(t) \cdot A(t) \right)$$



Speculators exploit producers' information, improving their gains.

Speculators compete over a limited amount of information and perform poorly.

Speculators and producers:

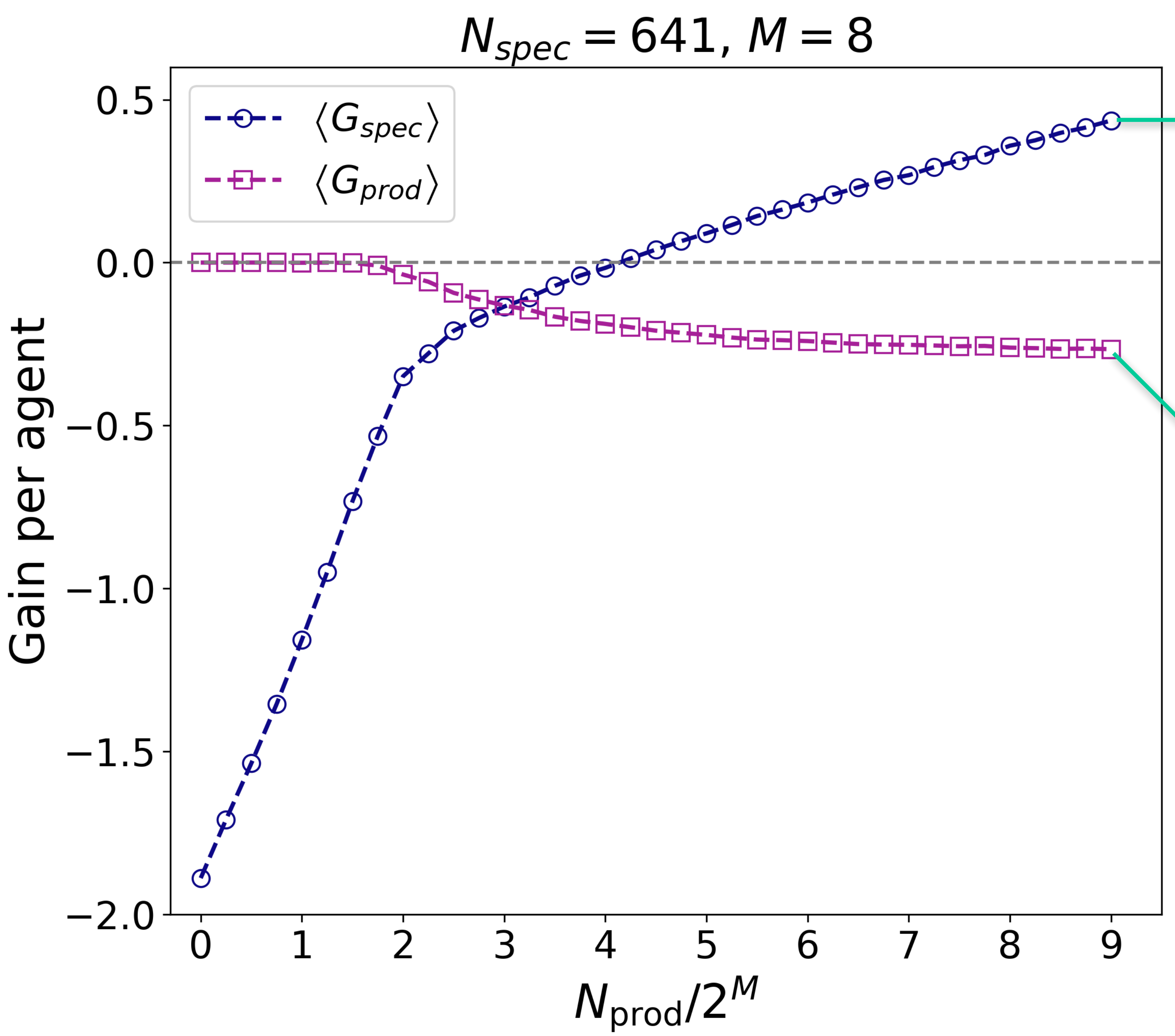
- **Speculators:** $S = 2$
- **Producers:** $S = 1$

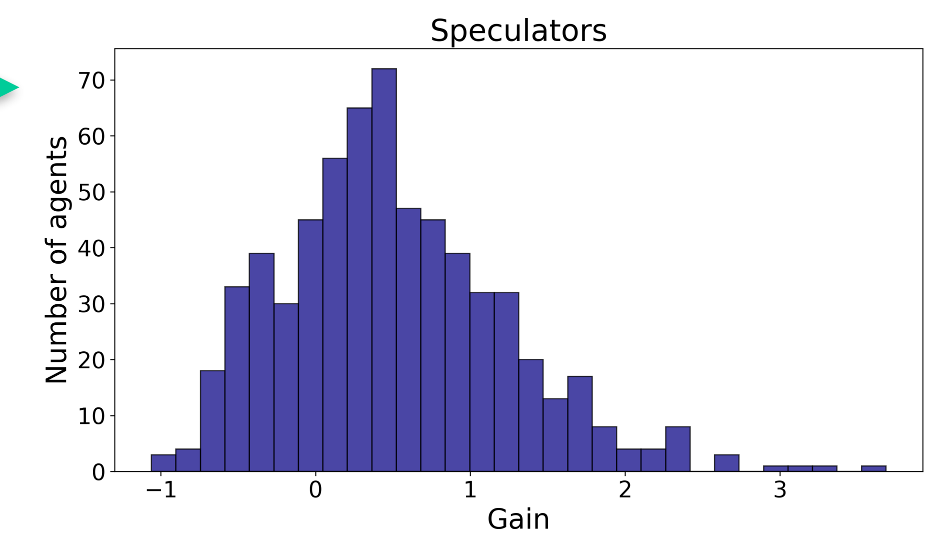
Average gain per agent for:

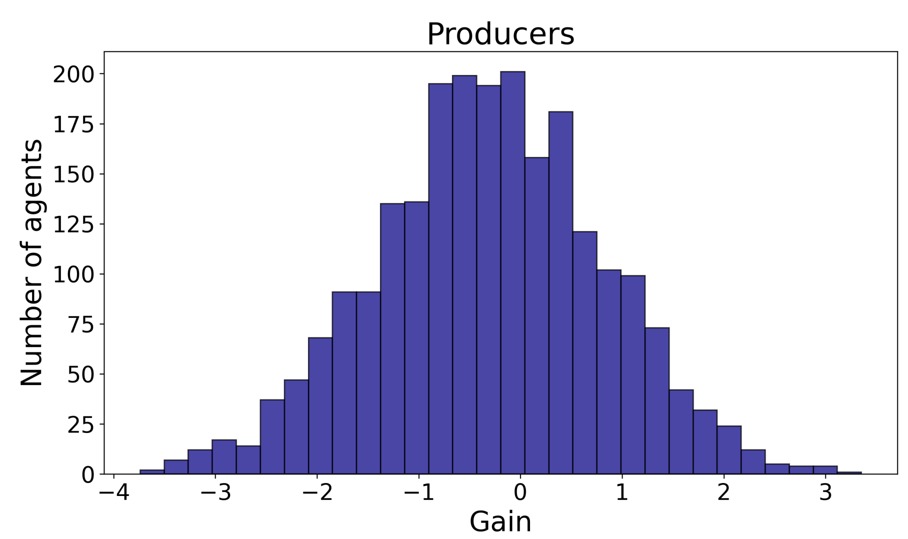
- Speculators: $\langle G_{\text{spec}} \rangle$
- Producers: $\langle G_{\text{prod}} \rangle$

Average gain per agent

$$\langle G_i \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left(-a_i(t) \cdot A(t)\right)$$







Summary & Conclusions

- We have implemented the **Minority Game** as a model for adaptive competition in financial markets.

- In the crowded regime, the system shows high **volatility (σ^2)** and a lack of coordination. Around the critical point α_c, agents self-organize optimally. For large α, the excess of information leads to uncoordinated and random behavior.

- **Predictability θ^2** serves as an **order parameter**, vanishing in the efficient regime and increasing beyond the critical point α_c, revealing predictability in the system.

- **Speculators** can **enhance** their **gains** by exploiting external information provided by **producers**.

- **Future Work:**
  - Consider noisy agents.
  - Communication between agents.

Main References

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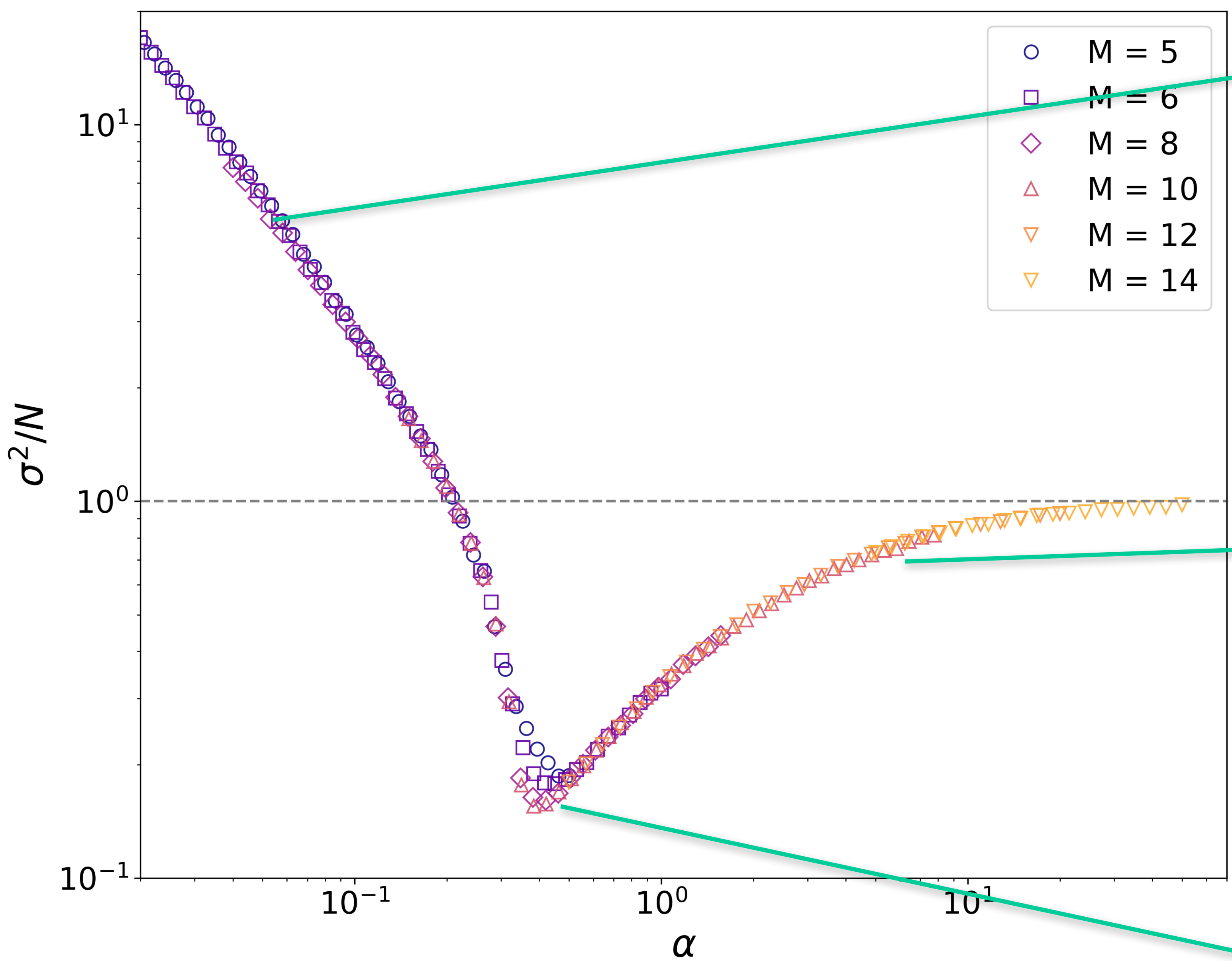
Extra Slide

Gain per agent

$$\langle G_i \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left(-a_i(t) \cdot A(t)\right)$$



Fixed: $S = 2$