Exploring Cooperation in Deterministic and Stochastic Spatial Prisoner's Dilemma

David Mateo Carpio R.[1]

[1] *Institute for Cross-Disciplinary Physics and Complex Systems IFISC (UIB-CSIC)*
Campus Universitat de les Illes Balears, E-07122 Palma de Mallorca, Spain

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Cooperative interactions are required for many levels of biological organization, ranging from single cells to groups of animals. The iterative Prisoner's Dilemma helps us understand the emergence of cooperators from an evolutionary game theory approach. To investigate how cooperative states emerge in spatial populations, we implement two variants of the game in a spatially structured population. We consider a deterministic model in which players update their strategies by imitating the most successful neighbor after each round of interactions, and a stochastic model in which players update their strategies probabilistically according to the Fermi distribution. The results show that in the deterministic version, the system's behavior is highly sensitive to the temptation to defect b. Small changes in b can lead to different dynamics, including static configurations, tiny periodic oscillations, or even chaotic patterns. Instead, the stochastic model shows a monotonic transition in the frequency of cooperators, from a complete cooperative state to an extinction of cooperation occurring at lower values of b than in the deterministic case. Finally, we identify that there is an optimal value of K that is capable of enhancing cooperative behaviour.

I. INTRODUCTION

Evolutionary Game Theory is a framework that models the dynamics of populations in which the fitness of individuals depends on their interactions with others. Unlike classical game theory, evolutionary game theory does not rely on rationality. Instead, it considers a population of players interacting in a game where the individuals have fixed strategies. The players interact with other individuals, and the payoffs of all these encounters are added up. Payoffs are seen as fitness indicators, while achieving success in the game is interpreted as reproductive success [1].

A central question in evolutionary biology is how cooperation can emerge and persist despite the evolutionary advantage often held by selfish behaviour. Cooperative interactions are essential at multiple levels of biological organization, from single-celled organisms to groups of animals[2]. Therefore, understanding the mechanisms that enable cooperation to arise and remain stable is a fundamental problem in evolutionary theory.

One of the most widely studied models in this context is the Prisoner's Dilemma, which captures the conflict between individual interests and cooperative behaviour. In this project, we implement two variants of the spatial Prisoner's Dilemma. First, we consider a deterministic model in which players imitate the most successful neighbor. Then, we explore a stochastic model in which strategy updates occur probabilistically based on payoff differences. We investigate how cooperative states emerge and persist in the spatial Prisoner's Dilemma under deterministic and stochastic strategy update rules.

II. ITERATIVE PRISONERS' DILEMMA

A. Deterministic Model

The standard Prisoners' Dilemma is a two-player game in which each participant can choose to cooperate, C, or defect, D, in any encounter. The outcome of the interaction depends on the combination of their choices. If both players choose C, both receive a payoff of magnitude R. If one player defects while the other cooperates, D receives the largest payoff, T, while C receives the smallest payoff, S. If both defects, both get P. The game is a Prisoner's Dilemma if the temptation to defect, T, exceeds the reward for mutual cooperation, R, which is greater than the punishment, P, for mutual defection, which is greater than the worst sucker's payoff, S. In other words, if $T > R > P > S$. The payoff matrix is represented as:

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix} \tag{1}$$

We chose the pay-offs of the Dilemma's matrix to have the values $R = 1$, $T = b$ (with $b > 1$), and $S = P = 0$. Then, we can represent the payoff matrix by:

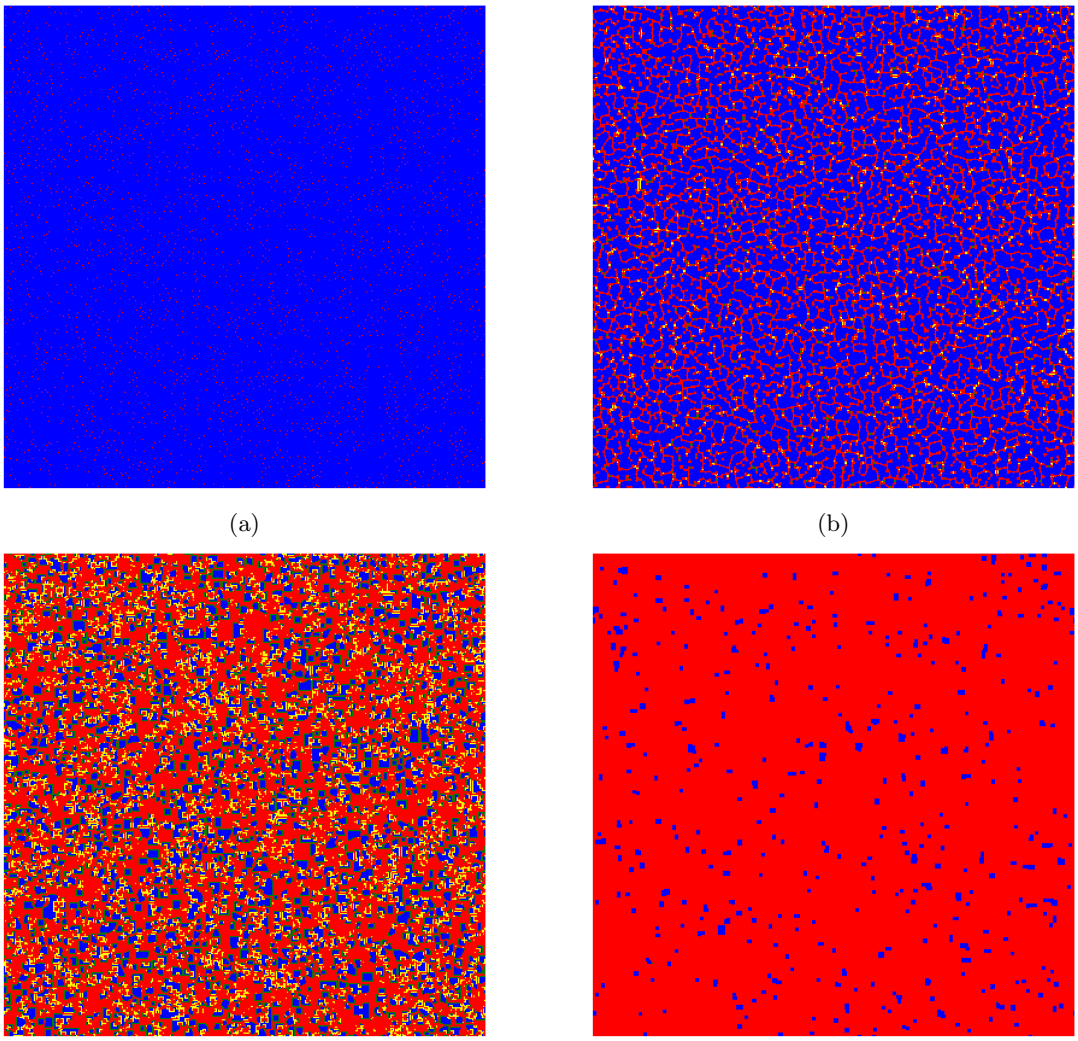
$$\begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \tag{2}$$

With this simplification, the model has only one parameter, b, representing the temptation to defect. If we set $P = \epsilon$ with $0 < \epsilon \ll 1$, the findings are qualitatively not altered [3]

We initialize the system on a two-dimensional square lattice of size $L \times L$, where each site hosts a player. Each player adopts one of the two strategies: cooperation (C) or defection (D). Then, at each time step:

1. Each player plays the game with their immediate neighbors and with themselves.
2. The score will be the sum of the payoffs in those encounters
3. For the next generation, each lattice site is occupied by the player with the highest score among the owner of the site and its immediate neighbors.

We consider the Moore neighborhood, where each site interacts with its eight nearest neighbors, those defined by a king's move on a chessboard. We consider periodic boundary conditions, so the edges of the square are wrapped around to generate a torus. This geometry has the advantage that all points on the grid are equivalent. We corroborate that the results are qualitatively unaltered if the lattice has fixed boundary conditions [3]. We characterize the system's state by the relative frequencies of the strategies. Since there are only cooperators and defectors, it suffices to consider the frequency of one of them. We denote by ρ the fraction of cooperators in the population. Then, $1 - \rho$ represents the fraction of defectors in the lattice.



(a) (b) (c) (d)

Figure 1: Asymptotic configuration in the deterministic model for different b values (a) $b = 1.05$, (b) $b = 1.78$, (c) $b = 1.9$, (d) $b = 2.1$ considering the initial condition $\rho(t_0) = 0.9$ of cooperators. Results computed in a square grid of lattice size $L = 400$ with periodic boundary conditions and considering a Moore neighborhood. Colors indicate strategy transitions of the individuals: blue ($C \to C$), red ($D \to D$), green ($D \to C$), yellow ($C \to D$).

B. Deterministic Results

On the square lattice with Moore neighborhood and self-interactions, the possible payoffs for a cooperator range from $\{1, 2, 3, \ldots, 9\}$, while the payoffs for a defector take values from the set $\{0, b, 2b, \ldots, 8b\}$. The discrete nature of the possible payoff values means that there are only discrete transition points of b that can influence the dynamics. These breakpoints are given by: $b_{bp} = k_2/k_1$, where $k_1 = 2, 3, \ldots b$ and $k_2 = 3, 4, \ldots, b + 1$.

Also, the invasion dynamics between cooperators and defectors is strongly influenced by the value of the temptation parameter b (see [1] for further details). As b varies, the expansion behavior of cooperative and defective clusters exhibits distinct qualitative regimes:

- For the cooperators, when $b < 2$, a 2×2 cluster of C grows and invade neighboring defectors, while for $b > 2$, these clusters are not able to grow.
- For the defectors, when $b > 1.8$, a large cluster of D can grow, while for $b < 1.8$, the D cluster shrinks.
- An interesting intermediate regime occurs for $1.8 < b < 2.0$ where both C and D clusters can grow, leading to complex spatiotemporal dynamics.

Figure 1 shows the asymptotic spatial configurations for representative values of b with an initial population of cooperators given by $\rho(t_0) = 0.9$. We represent with blue color a cooperator that was a cooperator ($C \to C$), with red a defector that remains as a defector ($D \to D$), with green a cooperator that previously was a defector ($D \to C$), and with yellow a defector that in the previous steps was a cooperator ($C \to D$).

Figure 1 (a) shows the asymptotic pattern for $b = 1.05$, a regime where cooperators dominate the system, although a few isolated defectors survive. We obtain these static configurations for $8b < 9$ where, in each round, a solitary defector accumulates a payoff of $8b$, while the surrounded cooperators obtain a payoff of 8. However, defectors can not reproduce since the neighbors of the surrounding cooperators can accumulate a payoff of 9.

For higher values of b, defectors' income becomes sufficiently high to be followed by some neighboring cooperators. However, in the subsequent steps, cooperators strike back, leading to local oscillations. One example is illustrated in Figure 1 (b) for $b = 1.78$, corresponding to one of the periodic regimes delimited by $1.75 < b < 1.8$. This configuration is characterized by interconnected lines of resistant defectors in a background of cooperators.

Figure 1 (c) shows the results for $b = 1.9$, which lie in the dynamic coexistence regime $1.8 < b < 2$, where both defectors and cooperators can grow. As observed, both strategies can expand, resulting in a highly active pattern with many switches of strategy represented by the yellow and green sites. Figure 1 (d) shows the results for $b = 2.1$, corresponding to a region $b > 2$. This configuration is highly dominated by defectors, with some cooperators forming rectangular blocks resistant to defectors' invasions.

Figure 2 shows how the frequency of cooperators ρ evolves in time for the four mentioned examples. We corroborate that as we increase b, the frequency of cooperators decreases. For the case of $b = 1.9$, which belongs to $b \in [1.8, 2.0]$, we have fluctuations of ρ, where individuals oscillate irregularly between being a cooperator and being a defector. Then, the overall dynamics of the system is chaotic. For the case of $b = 1.78$, we find that the system exhibits tiny oscillations due to the periodic behaviour. On the other hand, for the cases of $b = 1.05$ and $b = 2.1$, we have that ρ converges to a fixed value, meaning that the individuals keep their strategies over time, and the system reaches a frozen state with asymmetric coexistence of strategies.

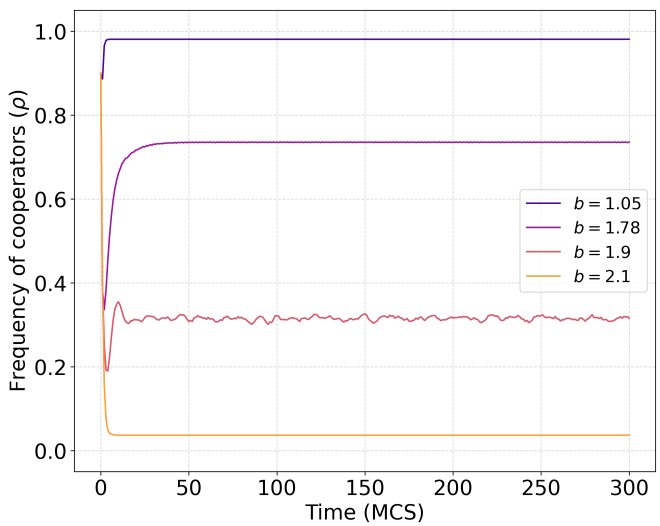


Figure 2: Time evolution of the cooperator frequency ρ for different values of b in the classical model. Results computed in a square lattice of size $L = 400$ with periodic boundary conditions, and starting from $\rho(t_0) = 0.9$.

We also study how the frequency of cooperators in the asymptotic state depends on the temptation to defect b. Figure 3 shows the results of ρ as a function of b computed in the interval $b \in [1.0, 2.4]$. To account for the chaotic and oscillatory behaviors, we average over the last 100 MCS after the dynamics stabilizes. To avoid bias in the distribution of the initial configuration, we average the results over 100 realizations. We consider an initial configuration $\rho(t_0) = 0.9$, represented by the blue solid line. We also compare the results with those obtained from a different initial condition $\rho(t_0) = 0.5$, shown as a dashed black line.

For $b < 1.8$, we obtain that the system converges to a system with most individuals being cooperators. Within this regime, a variety of dynamical behaviors emerge, such as a period 2 oscillator ($b = 1.21$), an equilibrium without oscillations ($b = 1.26$), or even period 6 and period 24 oscillators for $b = 1.79$ and 1.65, respectively. These complex dynamics are well-documented in [4].

We observe a sharp drop of ρ at $b = 1.8$, the limit where the defectors can invade cooperators. In the range $b \in [1.8, 2)$, the system stabilizes around $\rho \approx 0.32$. Notably, this value is independent of the initial population of cooperators in the system [5]. In contrast, in the non-chaotic regimes, cooperators' final density depends on the initial population, as evidenced by the black solid line computed for the initial condition $\rho(t_0) = 0.5$.

At $b = 2$, the population of cooperators considerably drops. A subsequent drop at $b = 2.24$ further reduces their population, almost extinguishing them. In this regime, cooperators have a fair chance of survival if they form compact colonies. Ultimately, for values beyond $b = 3$, these last survivors are extinguished as defectors can invade any block of cooperators.

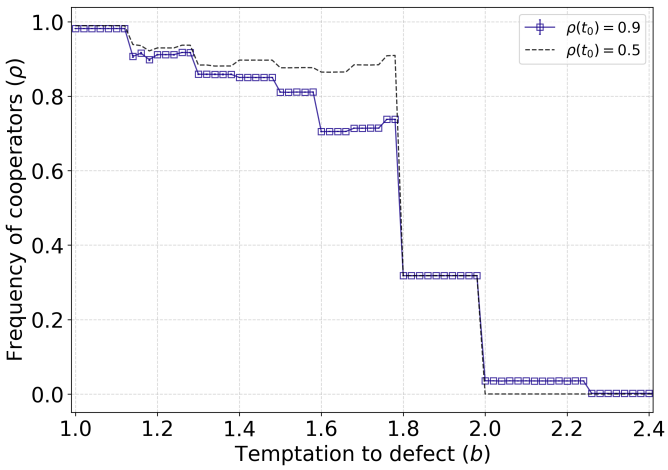


Figure 3: Frequency of the cooperators ρ for different values of b in the deterministic model. Results computed for a lattice size $L = 400$ with periodic boundary conditions, averaging over 100 MCS after reaching the asymptotic dynamics and over 100 realizations. The solid blue line corresponds to the initial condition $\rho(t_0) = 0.9$, and the dashed black line to $\rho(t_0) = 0.5$.

As evidence, the deterministic spatial Prisoner's Dilemma leads to several transitions slightly varying b. These abrupt changes, along with the sensitivity to initial conditions in certain parameter ranges, motivate the introduction of a stochastic model. By incorporating noise into the strategy update process, we aim to smooth out these irregularities and better approach the system's long-term behavior.

III. NOISY ITERATIVE PRISONERS' DILEMMA

A. Stochastic Model

To extend the classical spatial Prisoner's Dilemma model, we consider a stochastic model where strategies are updated via random sequential imitation [6]. In contrast to the deterministic case, where individuals imitate the most successful neighbor, the stochastic model introduces uncertainty in the decision process. Specifically, players adopt strategies probabilistically based on payoff differences, using the Fermi update rule. If an individual i with payoff Π_i compares its strategy s_i with a randomly selected neighbor j with payoff Π_j and strategy s_j, the probability that i adopts the strategy of j is given by the Fermi function:

$$W(s_i \leftarrow s_j) = \frac{1}{1 + \exp\left(\frac{\Pi_i - \Pi_j}{K}\right)}, \tag{3}$$

where K is the noise parameter, which controls the level of stochasticity in the imitation process. For $K \to 0$, the process recovers deterministic imitation, while for large K, strategy adoption becomes completely random [7].

The model's implementation proceeds as follows. The population is arranged on a square lattice of size $L \times L$ with periodic boundary conditions. Each individual occupies a site and plays either as a cooperator (C) or defector (D). In all our simulations, we consider an initial cooperator density of $\rho(t_0) = 0.5$. In contrast to the deterministic model, the stochastic model is independent of initial conditions, so the results remain consistent across different starting configurations. Interactions occur with the nearest, next-nearest neighbor, and themselves. The payoff is the same as in the deterministic case (equation 2), where we restrict to values $b > 1$. The simulation evolves through the following algorithm:

1. A random individual i is selected from the lattice.
2. One of its nearest neighbors j is selected uniformly at random.
3. Both players play the Prisoner's Dilemma game with all their immediate neighbors and themselves, and their total payoffs Π_i and Π_j are calculated.
4. Player i adopts the strategy of player j with probability $W(s_i \leftarrow s_j)$ as defined in equation 3.

One Monte Carlo Step (MCS) consists of $L \times L$ updates, so that on average each individual has one opportunity to revise its strategy. After a sufficient number of steps, the system reaches an asymptotic state. We measure the fraction of cooperators ρ over time and across multiple independent realizations to account for stochastic fluctuations.

B. Stochastic Results

Including noise has remarkable differences from the deterministic model. First, the frozen patterns with the coexistence of cooperators and defectors cannot be observed for this random model. Instead, the system always develops into one of the homogeneous absorbing states with $\rho = 0$ or 1, or into a two-strategy dynamical coexistence.

To illustrate those differences, Figure 4 shows the time evolution of the cooperator frequency for different b values and a fixed noisy value $K = 0.1$. At low values of the temptation to defect b, noisy dynamics eliminate defectors, and cooperators dominate the system. For $b = 1.12$, we observe how the system converges to a complete cooperative state with $\rho = 1$. The convergence to a state where all individuals are cooperators is stable if b does not exceed a threshold value b_1^c.

Another striking difference is that for noisy dynamics, cooperation can only be maintained until a significantly lower threshold value of b. The curve for $b = 1.4$ shows how the system reaches to extinction of cooperators with $\rho = 0$. In this case, the extinction of cooperators remains stable for $b \geq b_2^c$.

For values $b_1^c < b < b_2^c$, the system transitions to a mixed phase where cooperators and defectors coexist. We observe that the system evolves with a coexistence of strategies in the curves for $b = 1.2$ and $b = 1.35$. The fluctuations presented in these curves are not indicative of periodic or chaotic dynamics but are instead a natural consequence of the stochasticity introduced in the update process.

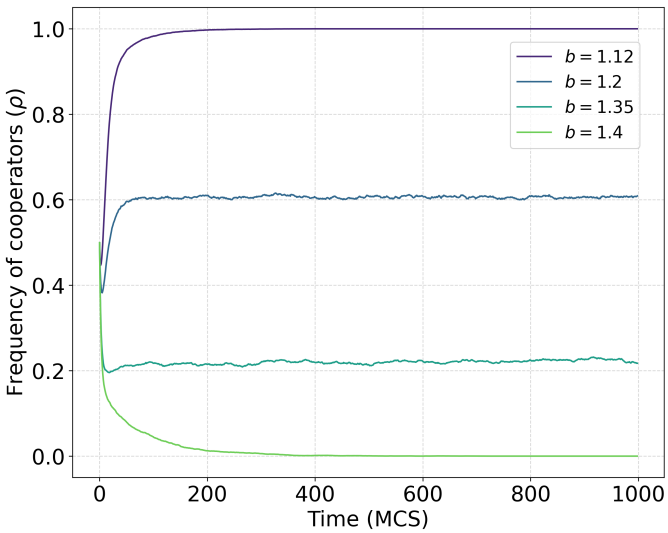


Figure 4: Temporal evolution of the cooperator frequency ρ for different values of b in the stochastic model. Results were computed for the noisy parameter $K = 0.1$ in a lattice size of $L = 400$, considering the Moore neighborhood with periodic boundary conditions.

Figure 5 shows a snapshot of the spatial configuration at the steady-state distribution where cooperators and defectors coexist. For $b = 1.2$, the system displays a high density of cooperators (blue) scattered among defectors (red), forming a dynamic equilibrium that is sustained by stochastic interactions.

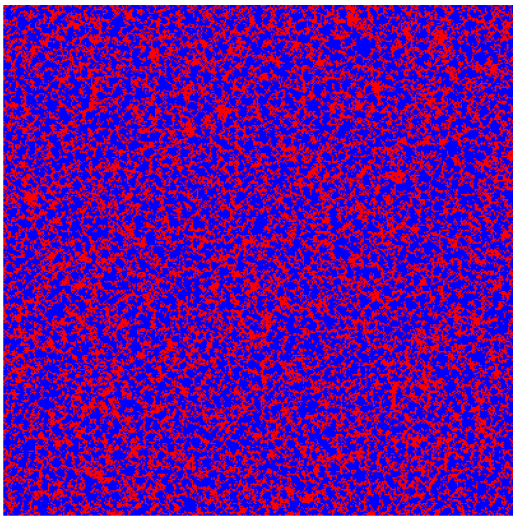


Figure 5: Asymptotic configuration for $b = 1.2$ in the stochastic model. Results computed for the noise value $K = 0.1$ in a grid of lattice size $L = 400$ with periodic boundary conditions. Cooperators are represented by blue, while defectors are represented by red.

As b increases within the coexistence regime, the overall density of cooperators decreases. For low values of ρ, the cooperators can survive if they form scattered colonies in the background of defectors, as illustrated in Figure 6. Different from the deterministic case, the colonies are not static, but can perform branching and coalescing random walks. Moreover, the structure of these colonies is irregular and constantly evolving, in contrast to the more compact and stable configurations observed in the deterministic case.

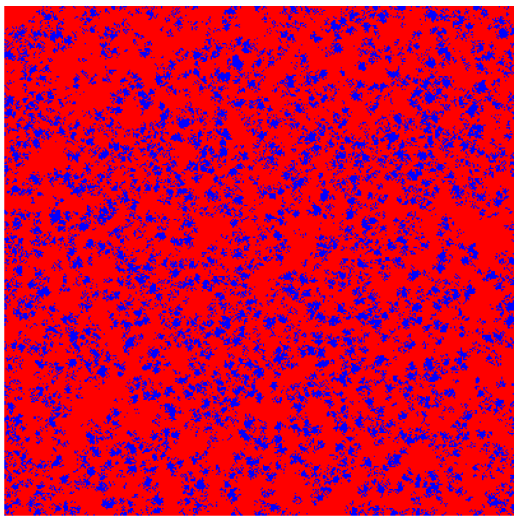


Figure 6: Asymptotic configuration for $b = 1.35$. Results computed for the noise value $K = 0.1$ in a grid of lattice size $L = 400$ with periodic boundary conditions. Cooperators are represented by blue, while defectors are represented by red.

To better capture how the temptation to defect b affects the asymptotic behaviour of the system, we compute the asymptotic density of cooperators over 100 Monte Carlo steps after reaching the asymptotic state across 100 iterations. Figure 7 shows the b dependence of the density of cooperators ρ in the coexistence region for $K = 0.1$. Unlike the deterministic model, ρ decreases monotonically with increasing b. This behavior differs significantly from the deterministic case, where the transition was irregular and marked by abrupt changes. In the stochastic model, we observe a smooth transition from complete cooperation to extinction of cooperators, occurring between the critical values b_1^c and b_2^c, respectively. For $K = 0.1$, we estimate these transitions to occur at approximately $b_1^c \approx 1.13$ and $b_2^c \approx 1.38$. Nevertheless, the transition regime will depend on the noise parameter K.

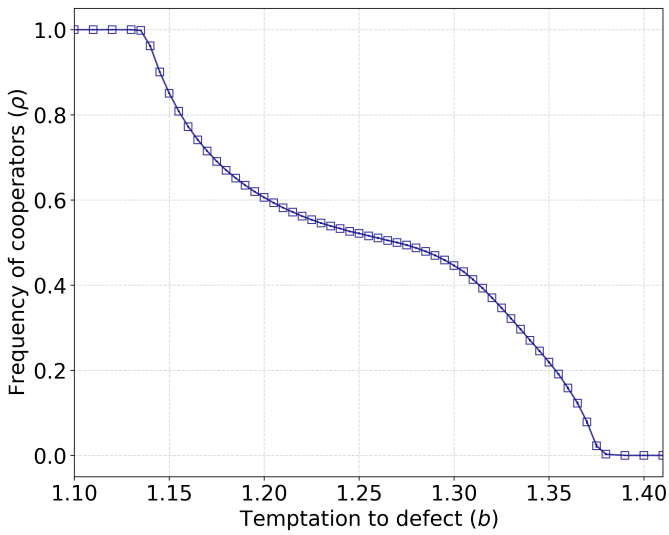


Figure 7: Asymptotic frequency of the cooperators ρ for different values of b in the stochastic model. Results computed for the noisy parameter $K = 0.1$ in a lattice of size $L = 200$, averaging over 100 MCS after reaching the asymptotic state and over 100 realizations.

Finally, we explore how the level of stochasticity influences the cooperative behavior of the system. Figure 8 shows the asymptotic frequency of cooperators as a function of the noise parameter K, for a fixed value of $b = 1.35$. We observe that the cooperator frequency initially increases with K, reaches a maximum, and then decreases for high K values. In other words, there exists an optimal level of noise that maximizes cooperation. This means that noise plays a crucial role in the emergence of cooperation.

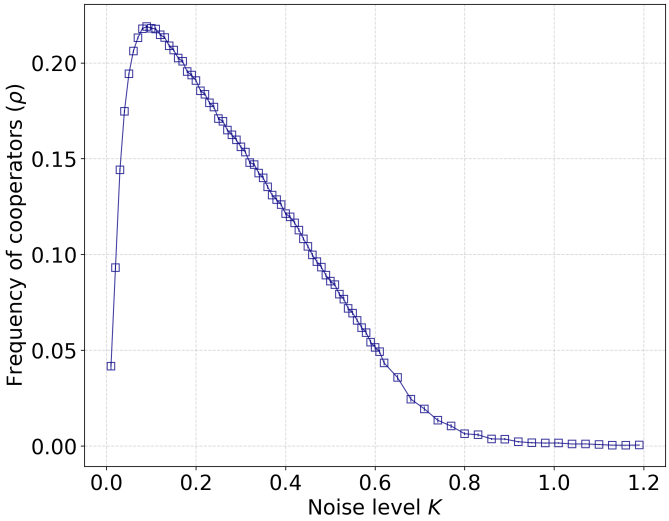


Figure 8: Asymptotic frequency of the cooperators ρ for different values of K for fixed $b = 1.35$ in the stochastic model. Results computed in a lattice of size $L = 200$, averaging over the 100 MCS after reaching the asymptotic state and over 100 realizations.

IV. CONCLUSIONS

We have implemented both deterministic and stochastic versions of the iterative Prisoner's Dilemma on a regular lattice to investigate the emergence and stability of cooperation under different update dynamics.

Our finding shows that in the classical and deterministic model, the temptation to defect b drives the system through a series of distinct dynamical regimes. For low values of b, the system reaches frozen configurations dominated by cooperators, with some solitary defectors resistant to extinction. As the temptation increases, defectors form interconnected lines, and the overall system can present oscillatory behaviour. We identify a dynamically active region where both cooperators and defectors can grow, leading to chaotic behaviour. Interestingly, even for relatively high values of b, cooperators can persist if they form regular clusters.

The stochastic model introduces significant changes to the dynamics of cooperation. Most notably, the threshold value of b at which cooperation goes extinct is reduced compared to the deterministic case. At high b values, we do not find the static clusters of cooperators, eventually eliminated by defectors. When b is sufficiently small, defectors are completely eliminated, leading to complete cooperation. In the transition regime, we observe the coexistence of cooperators and defectors. Unlike the deterministic model, this region does not exhibit chaotic or small oscillatory dynamics. Instead, fluctuations of the populations arise from the stochasticity of the model. Finally,

we observe that an optimal level of noise exists, characterized by the parameter K, that maximizes the prevalence of cooperation in the population.

Future work could explore the impact of varying network topologies [7] or incorporating inter-species interactions [8], which may further enrich the understanding of the evolutionary dynamics and stability of cooperation.

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