① $\int f' \cdot f^{k} = \begin{cases} \dfrac{f^{k+1}(x)}{k+1} + C, & k \neq -1 \\ \int \dfrac{f'}{f} = \ln|f(x)| + C \end{cases}$

① $\int \sin^3 x \cdot \cos^4 x \, dx = \int \underbrace{\sin x}_{-(\cos x)'} \cdot \underbrace{\sin^2 x}_{(1-\cos^2 x)} \cdot \cos^4 x \, dx = -\int (\cos x)' \cdot \cos^4 x \, dx + \int (\cos x)' \cdot \cos^6 x \, dx$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C \in \mathbb{R}$$

② $\int \sin^4 x \cdot \cos^5 x \, dx \quad (x \in \mathbb{R}) = ?$

③ $\int \cos^4 x \, dx = \int (\cos^2)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int \left(1 + 2\cos(2x) + \cos^2(2x)\right)$

$$= \frac{1}{4}x + \frac{1}{2} \cdot \frac{\sin(2x)}{2} + \frac{1}{4}\int \frac{1+\cos 4x}{2} dx =$$

$$= \frac{1}{4}x + \frac{1}{4} \sin(2x) + \frac{1}{8} \cdot \frac{\sin(4x)}{4} + C \in \mathbb{R}$$

④ $\int \frac{1}{\cos^2 x \cdot \sqrt[3]{tg^2 x}} dx \quad x \in (0, \pi/2)$

$$= \int (tg\, x)' \cdot (tg\, x)^{-\frac{2}{3}} dx = \frac{(tg\, x)^{-\frac{2}{3}+1}}{-2/3+1} + C = 3 \cdot \sqrt[3]{tg\, x} + C$$

HF $\int \frac{1}{\cos^4 x} dx$

Parciális integrálás

$$\int (f\cdot g)' = \int f'\cdot g + \int f\cdot g'$$

$$f\cdot g = \int f'g + \int f\cdot g'$$

$$\Rightarrow \boxed{\int f\cdot g' = f\cdot g - \int f'\cdot g}$$

(A) $\int (x^2+2x-1)\cdot e^{-2x} \quad (x\in\mathbb{R})$ polinom

$$= \int (x^2+2x-1)\left(\frac{e^{-2x}}{-2}\right)' dx = \text{p.i.} =$$

$$= (x+2x-1)\cdot\left(\frac{e^{-2x}}{-2}\right) - \int (x^2+2x-1)'\left(\frac{e^{-2x}}{-2}\right) dx$$

$$= -\frac{1}{2}\cdot(x^2+2x-1)\cdot e^{-2x} + \frac{1}{2}\cdot\int (2x+2)\cdot e^{-2x}\, dx$$

$$= -\frac{1}{2}\cdot(x^2+2x-1)e^{-2x} + \underbrace{\int (x+1)\cdot\left(\frac{e^{-2x}}{-2}\right)' dx}_{\text{p.i.}}$$

$$(x+1)\left(\frac{e^{-2x}}{-2}\right) - \int (x+1)'\cdot\frac{e^{-2x}}{-2}$$

$$-\frac{1}{2}(x+1)e^{-2x} + \frac{1}{2}\int e^{-2x} =$$

$$-\frac{1}{2}\cdot(x^2+2x-1)e^{-2x} - \frac{1}{2}(x+1)e^{-2x} + \frac{1}{2}\cdot\frac{e^{-2x}}{-2} + C =$$

$$\frac{1}{2}\cdot e^{-2x}\left(-x^2-2x+1-x-1-\frac{1}{2}\right) + C = \underline{\frac{1}{2}e^{-2x}\left(-x^2-3x-\frac{1}{2}\right) + C}$$

HF. $\int x^2 \cdot \sin(3x)\,dx$; $\int x \cdot \cos^2 x\,dx$ $(x \in \mathbb{R})$

(3) <u>Inverz függvények tartalmazó integrálok</u>

a) $\int x^2 \cdot \ln x\,dx =$ nem inverz a derivált

$$\underset{g'}{x^2} \quad \underset{f}{\ln x}$$

$$= \int \left(\frac{x^3}{3}\right)' \cdot \ln x\,dx = \left(\frac{x^3}{3}\right) \cdot \ln x - \int \frac{x^3}{3} \cdot (\ln x)'\,dx =$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3} \cdot \int x^3 \cdot \frac{1}{x}\,dx =$$

$$\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9} \cdot x^3 + C^{\in\mathbb{R}}$$

b) $\int \operatorname{arctg}(3x)\,dx = \int \underbrace{1}_{(x)'} \cdot \operatorname{arctg}(3x)\,dx$

$$= \text{p.i.} = x \cdot \operatorname{arctg}(3x) - \int x (\operatorname{arctg} 3x)'\,dx$$

$$= x \cdot \operatorname{arctg}(3x) - \int x \cdot \frac{1}{1+9x^2} \cdot 3\,dx =$$

$$= x \operatorname{arctg}(3x) - \int \frac{3x}{1+9x^2}\,dx$$

$$(1+9x^2)' = 18x$$

$$= x \cdot \operatorname{arctg} 3x - \frac{1}{6}\int \underbrace{\frac{18x}{1+9x^2}}\,dx =$$

$$\int \frac{(1+9x^2)'}{1+9x^2}\,dx$$

$$= x\cdot \text{arctg}\,3x - \frac{1}{6}\cdot \ln|\underbrace{1+9x^2}_{+}| + C = \underline{x\cdot \text{arctg}\,3x - \frac{1}{6}\cdot \ln(9x^2+1) + C}$$

Hf. $\int \arcsin(2x)\,dx$

$\int \frac{\text{arctg}^2 x}{1+x^2}$

$\boxed{C}$ Egyenlettel megoldható integrálok

a) $\int \underset{f}{e^{2x}} \cdot \underset{g'}{\sin x}\,dx = \int e^{2x}\cdot(-\cos x)'\,dx =$

$$= e^{2x}\cdot(-\cos x) - \int (e^{2x})'\cdot(-\cos x)\,dx = -\cos x\cdot e^{2x} + 2\int e^{2x}\cdot \underbrace{\cos x}_{(\sin x)'}\,dx$$

$$-\cos x\cdot e^{2x} + 2\cdot\left(\sin x\cdot e^{2x} - \int (e^{2x})'\cdot \sin x\,dx\right) =$$

$$I(x) := \int e^{2x}\cdot \sin x \Rightarrow I(x) = -\cos x\cdot e^{2x} + 2\sin x\cdot e^{2x} - 4\int e^{2x}\cdot \sin x\,dx$$

$I(x) := e^{2x}\cdot(2\sin x - \cos x) - 4I(x) \Rightarrow$ egyenlet I-re

$$I(x) = \frac{1}{5}\cdot e^{2x}(2\sin x - \cos x) + C \quad (C\in\mathbb{R})$$

b, $\int \sqrt{1-x^2}\, dx = \int 1 \cdot \sqrt{1-x^2}\, dx = \int (x)' \cdot \sqrt{1-x^2} =$ pi

$= x \cdot \sqrt{1-x^2} - \int x \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot (-2x)\, dx =$

$= x \cdot \sqrt{1-x^2} + \int \frac{x^2 - 1 + 1}{\sqrt{1-x^2}}\, dx = x \cdot \sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}\, dx + \int \frac{1}{\sqrt{1-x^2}}\, dx =$

$= x \cdot \sqrt{1-x^2} + \arcsin x - \int \sqrt{1-x^2}\, dx$ | egyenlet átrendezve

$\Rightarrow \quad 2 \int \sqrt{1-x^2}\, dx = x \cdot \sqrt{1-x^2} + \arcsin x$

$\Rightarrow \int \sqrt{1-x^2} = \frac{1}{2} x \cdot \sqrt{1-x^2} + \frac{1}{2} \arcsin x + C$

a) $\int \frac{2x}{x^2+3}\,dx \quad (x \in \mathbb{R})$ $\qquad \frac{f'}{f}$

$= \int \frac{(x^2+3)'}{x^2+3}\,dx = \ln(x^2+3)+c \qquad f^{r} \cdot f'$

b) $\int \sin^3 x \cdot \cos x\,dx = \int \sin^3 x \cdot (\sin x)' = \frac{\sin^4 x}{4}$

c) $\int \frac{dx}{x \cdot \ln x} = \int \frac{1}{x \cdot \ln x}\,dx = \int \frac{\frac{1}{x}}{\ln x}\,dx = \int \frac{(\ln x)'}{\ln x}\,dx =$

$\ln(\ln x)+c$

d) $\int \sin^3 x \cdot \cos^4 x\,dx = \int \sin x \cdot \sin^2 x \cdot \cos^4 x\,dx =$

$\int \sin x \cdot (1-\cos^2 x) \cdot \cos^4 x\,dx = \int \sin x \cdot (\cos^4 x - \cos^6 x)\,dx =$

$\int \sin x \cdot \cos^4 x - \sin x \cdot \cos^6 x\,dx = -\int (\cos x)' \cdot \cos^4 x + \int (\cos x)' \cdot \cos^6 x$

$-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}$

c) $\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx = \int \frac{1}{2} - \frac{\cos 2x}{2} dx =$

$\frac{1}{2} \int 1 - \cos 2x = \frac{1}{2} \cdot x - \frac{\sin 2x}{2}$

d) $\int \frac{1}{\cos^2 x \cdot \sqrt{tg^3 x}} dx = \int (tg\, x)' \cdot tg^{-\frac{3}{2}} dx =$

$\frac{tg^{-\frac{1}{2}} x}{-\frac{1}{2}} + C$