

Nonlinear Optimization 2, Spring 2025 - Homework 2

Due at 11:49PM on Friday 2/28 (Gradescope)

Your submitted solutions to assignments should be your own work. You are allowed to discuss homework problems with other students, but should carry out the execution of any thoughts/directions discussed independently, on your own. Acknowledge any source you consult. **Do not use any type of Large Language Model, e.g., ChatGPT, to blindly answer this assignment. If you do, your submission will be voided and you will get zero as a grade.**

Problem 1 - Directional derivative formulae

- (a) Let $f: \mathbf{E} \rightarrow \mathbf{R}$ be continuous and directionally differentiable at zero such that there exists $g \in \mathbf{E}$ satisfying $f'(x; v) = \langle g, v \rangle$ for all $v \in \mathbf{E}$. Find a counterexample to show that this does not necessarily imply that f is differentiable at zero.
- (b) Let $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{+\infty\}$ be a proper, closed, convex function. Let $x \in \text{int dom } f$, show that for any v , the directional derivative $f'(x, v)$ exists and, moreover, it is equal to

$$f'(x; v) = \sup_{g \in \partial f(x)} \langle g, v \rangle \quad \text{for all } v \in \mathbf{E}.$$

- (c) Let $f_1, \dots, f_k: \mathbf{E} \rightarrow \mathbf{R}$ be differentiable functions and define $h(x) = \max_{j \in [k]} f_j(x)$. Prove that h is directionally differentiable for any $v \in \mathbf{E}$. Further, prove that

$$h'(x; v) = \max_{j \in M(x)} \langle \nabla f_j(x), v \rangle \quad \text{for all } x, v \in \mathbf{E},$$

where $M(x) = \{i \in [k] \mid f_i(x) = h(x)\}$.

Problem 2 - Arguments that we missed

Show the following two things we did not prove in class.

- (a) Let $f: \mathbf{E} \rightarrow \mathbf{R} \cup \{+\infty\}$ and $g: \mathbf{E} \rightarrow \mathbf{R} \cup \{+\infty\}$ be convex functions, and $A: \mathbf{E} \rightarrow \mathbf{Y}$ be a linear map. Define the value function $\nu: \mathbf{Y} \rightarrow \mathbf{R} \cup \{\pm\infty\}$ given by $\nu(z) = \inf_{x \in \mathbf{E}} f(x) + g(Ax + z)$.
- (1) Show that ν is a convex function.
 - (2) In Lecture 6, we concluded that if $\nu(0)$ is finite and $0 \in \text{int}\{\text{dom}(g) - A \text{dom}(f)\}$, then there exists a $y \in \partial \nu(0)$. To do this we used the "Existence of subgradients" Theorem from Lecture 4. However, this theorem only applies to functions whose image land in $\mathbf{R} \cup \{+\infty\}$, why can we apply it to ν ? (Recall that we saw an example where $\nu(1) = -\infty$ in Lecture 7).
- (b) Let $a_1, \dots, a_m \in \mathbf{E}$ be an arbitrary collection of points. Consider the following three statements.

- (1) The function $f(x) = \log \left(\sum_{i \in [m]} \exp(\langle a_i, x \rangle) \right)$ is bounded below.

- (2) There exists a vector $\lambda \in \mathbf{R}_+^m$ such that $\sum_{i \in [m]} \lambda_i = 1$ and $\sum_{i \in [m]} \lambda_i a_i = 0$.
(3) There is no vector $x \in \mathbf{E}$ such that $\langle a_i, x \rangle < 0$ for all $i \in [m]$.

In class we proved that (1) implies (2). Show that (2) implies (3) and (3) implies (1).

Problem 3 - Duality with cones

- (a) (**Krein-Rutman Theorem**) Consider a linear map $A: \mathbf{E} \rightarrow \mathbf{Y}$, and the indicator functions of convex cones $K \subseteq \mathbf{E}$ and $H \subseteq \mathbf{Y}$. Compute $\partial \iota_K(0)$ and use subdifferential calculus to find conditions guaranteeing $(K \cap A^{-1}H)^+ = K^+ + A^*H^+$ where $A^{-1}H = \{x \in \mathbf{E} \mid Ax \in H\}$.
(b) Given a nonempty set $K \subseteq \mathbf{E}$, by considering ι_K^{**} , prove $K = K^{++}$ if, and only if, K is a closed convex cone.
(c) Suppose that the closed convex cones $K \subseteq \mathbf{E}$ and $H \subseteq \mathbf{E}$ satisfy the condition $K^+ \cap \text{int } H^+ \neq \emptyset$. Prove $K + H$ is closed.
(d) Prove the sum of the closed convex cones in $\mathbf{R}^2 \times \mathbf{R}$

$$\{(x, r) \mid \|x\|_2 \leq r\} \quad \text{and} \quad \{(x, r) \mid x_1 = 0, r = x_2\}$$

is not closed. **Hint:** consider the point $-(1, 1, 1)$.

Problem 4 - Von Neumann Minimax Theorem

Suppose the sets $C \subseteq \mathbf{E}$ and $D \subseteq \mathbf{Y}$ are nonempty and convex, with D closed.

- (a) By considering the Fenchel problem

$$\inf_x \{\iota_C(x) + \iota_D^*(Ax)\}$$

prove that if either of the next conditions hold

- (i) D is bounded,
(ii) A is surjective and $0 \in \text{int } C$,

then

$$\inf_{x \in C} \sup_{y \in D} \langle Ax, y \rangle = \sup_{y \in D} \inf_{x \in C} \langle Ax, y \rangle$$

where the supremum on the right is attained whenever finite.

- (b) If both C and D are compact, prove

$$\min_{x \in C} \max_{y \in D} \langle Ax, y \rangle = \max_{y \in D} \min_{x \in C} \langle Ax, y \rangle$$

where all the maxima and minima are attained.