Lecture 4

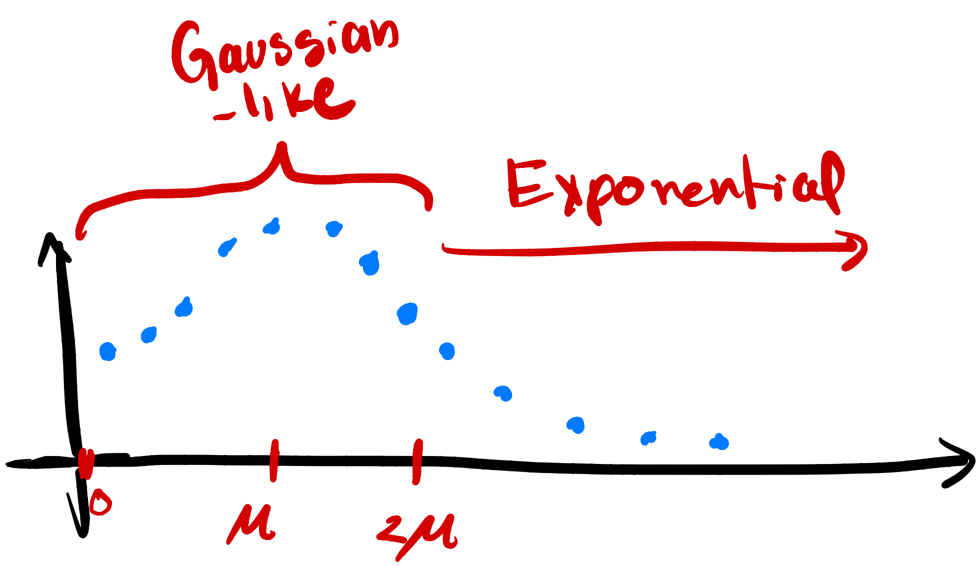
Last time
- Robust mean estimation
- Chernoff's inequality

Today
- Degree of random graphs.
- Subexponentials

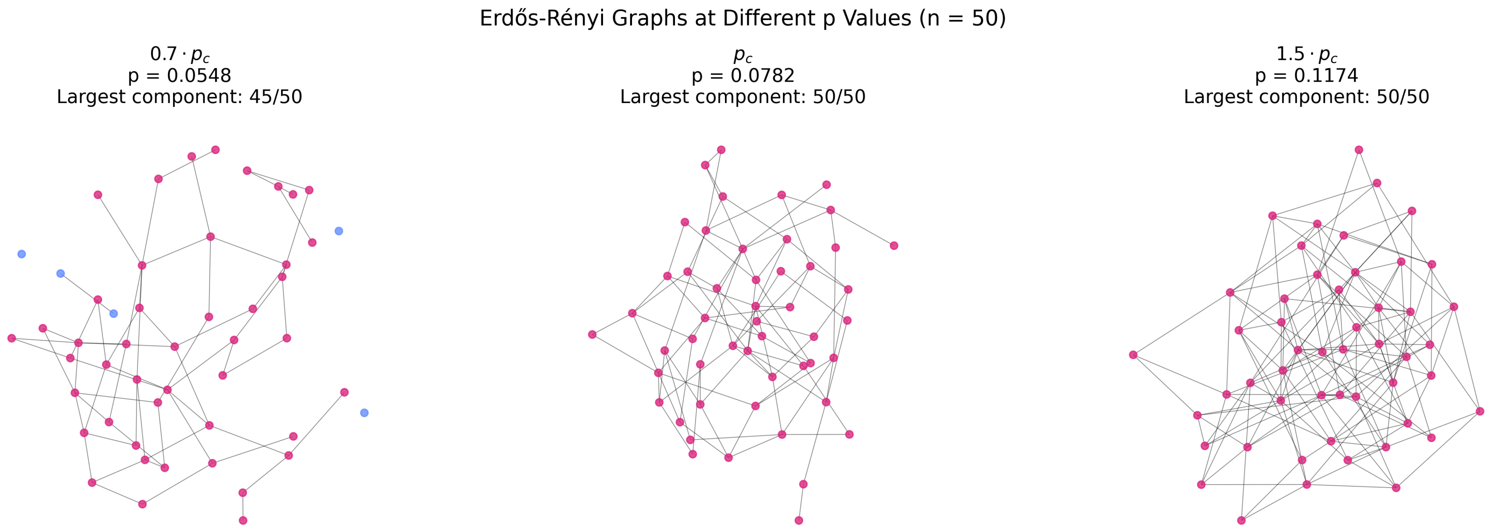
Erdös-Renyi Graphs

Last time we finished with:

Corollary: Take $\delta \in [0,1]$, then
$$\mathbb{P}(|S_n - \mu| \geq \delta\mu) \leq \exp\left(-\frac{\delta^2 \mu}{6}\right).$$



Def G(n,p): Fix a set of n nodes connect each pair independently with probability p. ⊣



The degree deg(i) of node i is equal to the number of edges connected to i.

Then,

$$\deg(i) = S_{n-1} = (n-1)\,p =: d$$

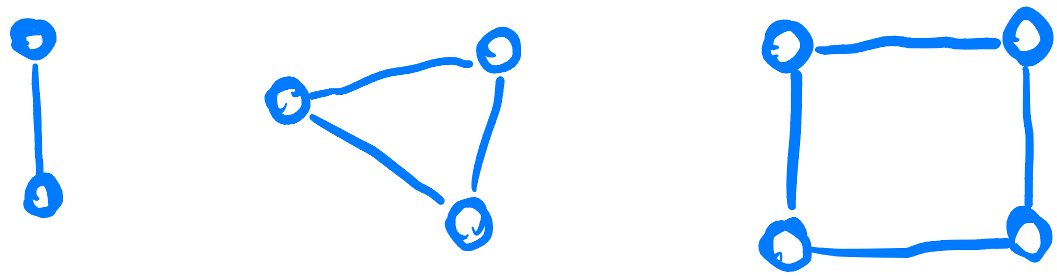
There is an interesting phase transition at $p = \log n / n$

$p \geq (1+\varepsilon) \log n / n \to$ Large component.

$p \leq (1-\varepsilon) \log n / n \to$ Disconnected.

Something similar happens with their degree.

Def: A graph is d-regular if $\deg(i) = d$ for all nodes.



Proposition: Take $\delta > 0$, and $(n-1)p \geq \frac{8}{\delta^2}\log(2n)$. Then, $G \sim G(n,p)$ is almost $\underbrace{(n-1)p}_{d}$-regular, i.e.,

$$\mathbb{P}\left(\forall i \in [n],\ (1-\delta)d \leq \deg(i) \leq (1+\delta)d\right) \leq 1 - \frac{1}{2n}.$$

Proof: For each i, we have

$$\deg(i) = \sum \mathbb{1}_{ij \in E} \leftarrow \text{Random edges of } G.$$

where $\mathbb{E}\deg(i) = (n-1)p$ and Chernoff's inequality yields

$$\mathbb{P}\big(\underbrace{|\deg(i) - d| \geq \delta d}_{\mathcal{E}_i^c}\big) \leq 2\exp(-\delta^2 d/4).$$

Taking union bound

$$\mathbb{P}\left(\cap \mathcal{E}_i\right) = 1 - \mathbb{P}\left(\cup \mathcal{E}_i^c\right)$$

$$\geq 1 - \sum \mathbb{P}(\mathcal{E}_i^c)$$
$$\geq 1 - 2n\exp(-\delta^2 d/4)$$
$$= 1 - \exp(\log(2n) - \delta^2 d/4)$$
$$= 1 - \frac{1}{2n}.$$

$\square$

Sub-exponential

Not all random variables are sub-Gaussian, indeed one often encounters squared Gaussians (χ^2). Let's see what the MGF looks in that case. If $Z \sim N(0,1)$,

$$\mathbb{E}[e^{\lambda(Z^2-1)}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\lambda(t^2-1)} e^{-t^2/2}\, dt$$
$$= \frac{e^{-\lambda}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(1-2\lambda)t^2/2}\, dt$$

If $|\lambda| \geq \frac{1}{2}$, the integrant diverges.

If $|\lambda| < \frac{1}{2}$, this is a Gaussian with different variance.

$$\mathbb{E}\, e^{\lambda(Z^2-1)} = \begin{cases} \frac{e^{-\lambda}}{\sqrt{1-2\lambda}} & \text{if } |\lambda| < \frac{1}{2}, \\ +\infty & \text{if } |\lambda| \geq \frac{1}{2}. \end{cases}$$

We can further upper bound

$$\mathbb{E}\, e^{\lambda(Z^2-1)} \leq \frac{e^{-\lambda}}{\sqrt{1-2\lambda}} \leq e^{4\lambda^2/2} \quad \forall |\lambda| < 1/4$$

We follow the same strategy as with sub-Gaussians.

Def (sub-exponential): A r.v. X (σ, α)-subexponential if

$$\mathbb{E}\, e^{\lambda(X-\mu)} \leq e^{\sigma^2\lambda^2/2} \quad \forall |\lambda| \leq 1/\alpha.$$

Thus, if $Z \sim N(0,1)$, then, Z^2 is $(2,4)$-subexponential.

Theorem: Let X be a (σ, α)-sub-exponential r.v. with $\mathbb{E}X = \mu$. Then,

$$\mathbb{P}(X - \mu \geq t) \leq \begin{cases} e^{-t^2/2\sigma^2} & \text{if } |t| \leq \frac{\sigma^2}{\alpha} \\ e^{-\frac{t}{\alpha} + \frac{\sigma^2}{2\alpha^2}} & \text{otherwise.} \end{cases}$$

Proof: Applying the Chenoff bound†

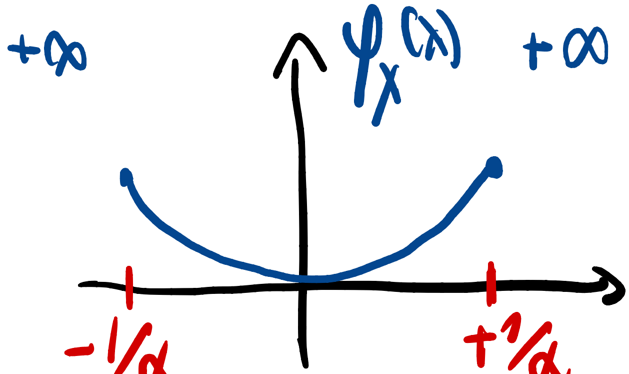
$$\log \mathbb{P}(X - \mu \geq t) \leq -\varphi_X^*(t)$$

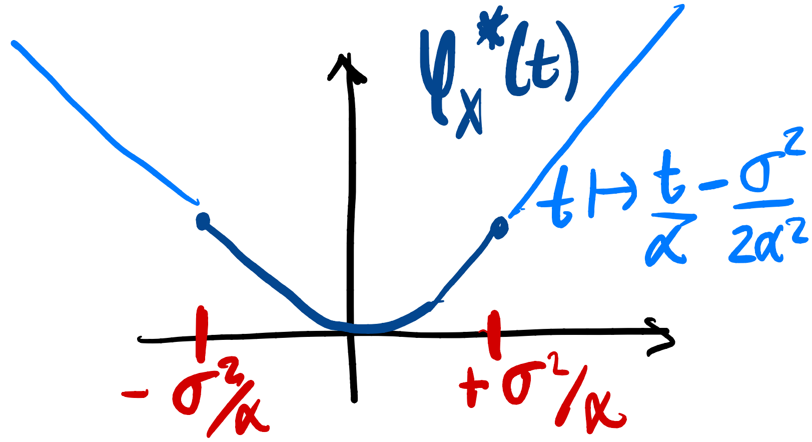
where

$$\varphi_X(\lambda) = \log \mathbb{E} e^{\lambda(X-\mu)} \leq \begin{cases} \sigma^2 \lambda^2 / 2 & \text{if } |\lambda| \leq \frac{1}{\alpha} \\ +\infty & \text{otherwise.} \end{cases}$$

Thus,

$$\varphi_X^*(t) = \sup_\lambda \{t\lambda - \varphi_X(\lambda)\}$$





Clearly, $|\lambda| \le 1/\alpha$ in the sup.
Thus, there are two cases:

Case 1: Suppose $\lambda \mapsto t\lambda - \sigma^2\lambda^2/2$ has a maximizer with $|\lambda| \le 1/\alpha$.
Taking derivatives equal zero

$\lambda = t/\sigma^2$ so $|t| \le \frac{\sigma^2}{\alpha}$ and

$$\psi_X^*(t) = t^2/2\sigma^2$$

Case 2: The maximizer λ^* of $\lambda \mapsto t\lambda - \sigma^2\lambda^2/2$ satisfying $|\lambda^*| > \frac{1}{\alpha}$ and so if $\lambda^* > 0$ we take the extreme $\lambda = 1/\alpha$

$$\psi_X^*(t) = t/\alpha - \frac{\sigma^2}{2\alpha^2}.$$

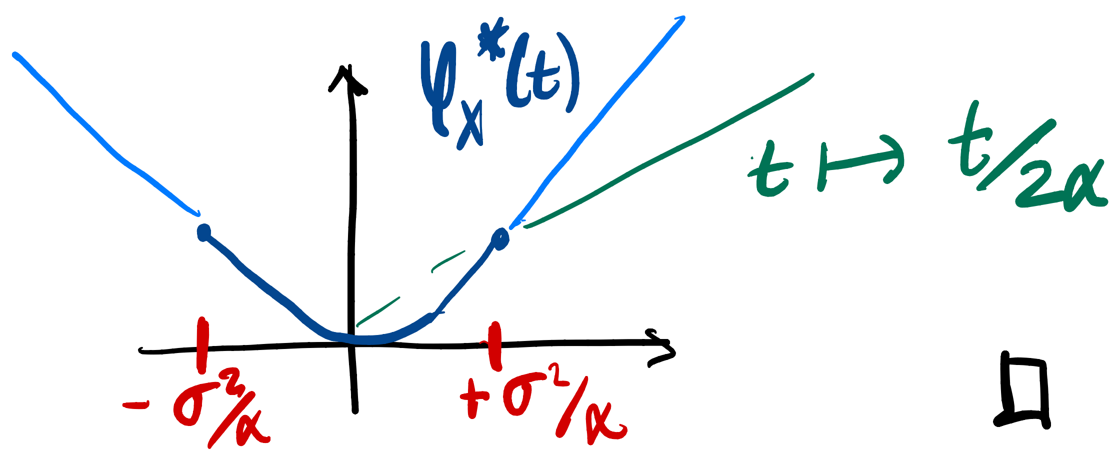
$\square$

Corollary :) : Let X be (σ, α)-subexponential r.v. with $\mathbb{E}X = \mu$. Then,

Minimum.

$$\mathbb{P}(|X-\mu| \geq t) \leq 2\exp\left(-\left(\frac{t^2}{2\sigma^2} \wedge \frac{t}{2\alpha}\right)\right)$$

Proof: Proof by picture:



$\square$

Lemma (sum rule): Let $X_1, \ldots, X_n$ be independent r.v. with X_i (σ_i, α_i)-subexponential. Then, $\sum_{i=1}^{n} X_i$ is $(\|\sigma\|_2, \|\alpha\|_\infty)$

$\max |\alpha_i|$

- subexponential.

Combining the sum rule with Corollary :) yields:

Theorem (Bernstein's Inequality):

Let $X_1, \dots, X_n$ be independent r.vs with X_i (σ_i, α_i) subexponential and $\mathbb{E}X_i = \mu$. Then,

$$\mathbb{P}\left(\left|\sum_{i=1}^{n}(X_i - \mu_i)\right| \geq t\right) \leq 2\exp\left(-\left(\frac{t^2}{2\|\sigma\|_2^2} \wedge \frac{t}{\|\alpha\|_\infty}\right)\right)$$