Lecture 9 Mon Feb/19/2024

Last time

- Doob's optimal-stopping Theorem.
- Doob's convergence Theorem.

Today

- A word of warning
- Symmetric random walk
- Branching process.

A word of Warning

The last result from the previous lecture gives:

Corollary (♡) If $X_n \geq 0$ is a supermartingale, then $X_\infty = \lim_{n\to\infty} X_n < \infty$ exists almost surely.

Warning We cannot conclude that X_n converges to X_∞ in $\mathcal{L}^1$. ← Will come back to $\mathcal{L}^1$ next class.

Example: Consider a simple random walk $S_n = \sum_{k=1}^{n} \xi_k$ with $S_0 = 1$, and $\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2$. Define

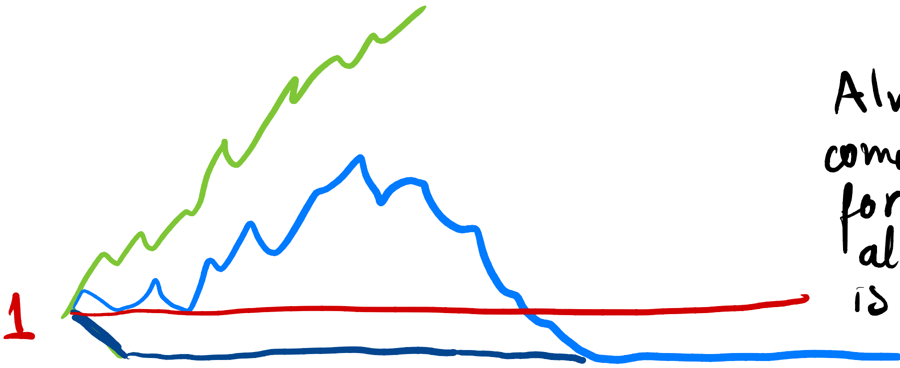
$$T = \inf\{ n \mid S_n = 0 \} \quad \text{and} \quad X_n = S_{T\wedge n}.$$

Since X_n is a martingale $\mathbb{E}X_n = \mathbb{E}X_0 = 1$.

By the corollary.

On the other hand, $X_n \to X_\infty$ a.s.
Notice that $X_\infty(\omega) = 0 \;\; \forall \omega$ since
if $X_n(\omega) = K > 0 \Rightarrow X_{n+1}(\omega) = K \pm 1$, and so
$\lim X_n \neq K$.

This example illustrates the warning as
$\mathbb{E}|X_n - X_\infty| \geq |\mathbb{E}X_n - \mathbb{E}X_\infty| = 1.$ $\dashv$



Almost every trajectory comes back to 0. But for any n there is always a portion that is far away.

Applications to Random walks

Consider $S_0 = 0$, $S_n = \sum_{k=1}^{n} \xi_k$ with iid r.v. ξ_k
s.t. $\mathbb{E}\xi_k = \mu$ and $\mathbb{E}\xi_k^2 = \sigma^2$.

Claim The following are martingales:

a) $M_n = S_n - \mu n$.

b) $M_n = S_n^2 - \sigma^2 n$.

c) $M_n = \exp(\theta S_n)/\phi(\theta)^n$ with $\phi(\theta) = \mathbb{E}\exp(\theta\xi_1) < \infty$.

Proof: a) $\mathbb{E}[S_n - \mu n \mid \mathcal{F}_{n-1}] = \mathbb{E}[S_{n-1} + \xi_n - \mu n \mid \mathcal{F}_n] = S_{n-1} - \mu(n-1)$

b) $\mathbb{E}[S_n^2 - \sigma^2 n \mid \mathcal{F}_{n-1}] = \mathbb{E}[(S_{n-1} + \xi_n)^2 - \sigma^2 n \mid \mathcal{F}_{n-1}]$

$= \mathbb{E}[S_{n-1}^2 + \cancel{2\xi_n S_{n-1}} + \xi_n^2 - \sigma^2 n \mid \mathcal{F}_{n-1}]$

0 because $\mathbb{E}\xi_n = 0$ and $\xi_n \perp \mathcal{F}_{n-1}$.

Useful to analyze MGF (we will not cover it)

$$= S_{n-1}^2 + \sigma^2 - \sigma^2 n$$

$$= S_{n-1}^2 - \sigma^2(n-1).$$

c) Follows from the fact that $X_k = \exp(\theta \xi_k)/\phi(\theta)$ are i.i.d. positive random variables with mean 1, and $M_n = \prod_{k=1}^{n} X_k$. (Recall the example from Lecture 7) $\square$

We will leverage these to obtain explicit formulas for probabilities and expectations of stopped random walks.

Theorem (Wald's equation) Consider a random walk $S_n = \sum_{k=1}^{n} \xi_k$ with $P(\xi_k = 1) = 1 - P(\xi_k = -1)$. Let N be a stopping time with $\mathbb{E}N < \infty$. Then,

$\mu = \mathbb{E}\xi_k$.

$$\mathbb{E}S_N = \mu \mathbb{E}N.$$

Proof: Optional stopping Thm applied to $S_n - \mu n$. $\square$

Theorem: Consider the symmetric random walk $S_n = \sum_{k=1}^{n} \xi_k$ with $\{\xi_k\}$ i.i.d. s.t. $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 1/2$.

$S_0 = 0$

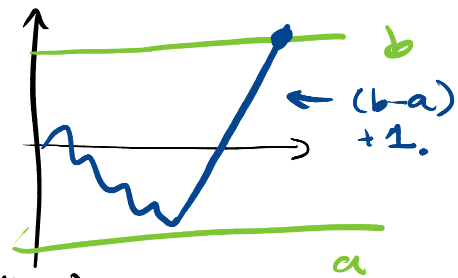
Let $N = \inf\{n \mid S_n \in (a,b)^c\}$. $a < 0 < b$. Then,

a) $\mathbb{P}(S_N = a) = \dfrac{b}{b-a}$ and $\mathbb{P}(S_N = b) = -\dfrac{a}{b-a}$.

b) $\mathbb{E}[N] = -ab.$

Proof: a) We want to apply Wald's and so we need

$\mathbb{E}N < \infty$. Notice that if we have a $(b-a)$ consecutives $+1$, then we stop.



Then, N is dominated by $(b-a)X$ where X is a geometric random variable with success probability $p = 2^{-(b-a)}$. Thus

$$\mathbb{E}N < (b-a)\,\mathbb{E}X = (b-a)\,2^{(b-a)} < \infty.$$

Thus, Wald's equation tell us

$$0 = \mathbb{E}S_N = a\,\mathbb{P}(S_N = a) + b\left(1 - \mathbb{P}(S_N = a)\right),$$

rearranging

$$\mathbb{P}(S_N = a) = \frac{b}{b-a}.$$

Analogously, we derive $\mathbb{P}(S_N = b) = \frac{-a}{b-a}$.

b) Using that $S^2_{N\wedge n} - N\wedge n$ is a martingales, we obtain using the Optional Stopping Theorem and the fact that

$$\begin{aligned} \left|S^2_{N\wedge n} - S^2_{N\wedge(n-1)}\right| &\le \left|(S_{N\wedge n-1} + \xi_n)^2 - S^2_{N\wedge n-1}\right| \\ &= \left|2S_{N\wedge n-1}\,\xi_n + 1\right| \\ &\le 2\max\{b, -a\} + 1, \end{aligned}$$

we conclude that

$$\mathbb{E}N = \mathbb{E}S_N^2 = a^2\frac{b}{b-a} - b^2\frac{a}{b-a} = -ab\left(\frac{b}{b-a} - \frac{a}{b-a}\right) = -ab.$$

$\square$

Branching Processes

Suppose we want to study the dynamics of a population throughout generations. A simple model considers iid nonnegative r.v. ξ_k^n = "number of children of kth individual at generation n."

Define $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \dots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0. \end{cases}$$

This is known as the Galton - Watson process, and $p_\ell = \mathbb{P}(\xi_k^n = \ell)$ is called offspring distribution.

Lemma: Let $\mathcal{F}_n = \sigma(\xi_k^m : 1 \le k, 1 \le m \le n)$ and $\mu = \mathbb{E}\xi_i^m \in (0, \infty)$. Then Z_n/μ^n is a martingale.

Proof: By construction $Z_n \in m\mathcal{F}_n$ and $\mathbb{E}|Z_n| < \infty$. Then,

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = \mathbb{E}[\xi_1^{n+1} + \dots + \xi_{Z_n}^{n+1} | \mathcal{F}_n]$$

$$= Z_n \mathbb{E}\xi_1^{n+1} \quad \leftarrow \text{independence.}$$

Proved in HW 3

$$= Z_n \mu.$$

Thus $\mathbb{E}[Z_n/\mu^n \mid \mathcal{F}_{n-1}] = Z_{n-1}/\mu^{n-1}.$ $\square$

Thus, by Corollary (♡) we have that $Z_n/\mu^n \to$ a limit a.s.

Theorem If $\mu < 1$, then $Z_n = 0$ for all large n, and so $Z_n/\mu^n \to 0$ a.s.

Proof: $\mathbb{E}[Z_n/\mu^n] = \mathbb{E}[Z_0] = 1 \Rightarrow \mathbb{E}Z_n = \mu^n$.
Now, by Markov's inequality
$$\mathbb{P}(Z_n > 0) = \mathbb{P}(Z_n \geq 1) \leq \mathbb{E}Z_n = \mu^n \to 0.$$
The result follows by Borell-Cantelli. $\square$

Theorem If $\mu = 1$ and $\mathbb{P}(\xi_k^m = 1) < 1$, then $Z_n = 0$ for all large n a.s.

Proof: When $\mu = 1$, Z_n is a nonnegative martingale, thus $Z_n \to Z_\infty$ a.s. Since Z_n is integer-valued $Z_n = Z_\infty$ for large enough n. If $\mathbb{P}(\xi_k^m = 1) < 1$, then
$$\mathbb{P}(Z_m = k \text{ for all } n \geq N)$$
$$= \mathbb{P}(Z_n = k, Z_{n+1} = k, \dots)$$
$$= \mathbb{P}(Z_n = k) \prod_{l=n}^{\infty} \mathbb{P}(Z_{l+1} = k \mid Z_l = k) \qquad (\infty)$$

Note that $\mathbb{P}(Z_{\ell+1} = k \mid Z_\ell = k) = \mathbb{P}(\xi_1^{\ell+1} = 1)^{k-1}\,\mathbb{P}(\xi_k^{\ell+1} \neq 1) < 1$. Thus the infinite product in $(\infty) \to 0$.

$\square$