

Lecture 2

Last time

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- ▷ Motivation
- ▷ Overview

Today

- ▷ The setting
- ▷ convex sets
- ▷ Separation

The setting

We will work on a real Euclidean space \mathbb{E} . We (Finite dimensional Hilbert space) denote its inner product by $\langle \cdot, \cdot \rangle$.

Examples

- ▷ Standard space: \mathbb{R}^d with $x^T y$.
- ▷ Symmetric matrices:

$S^n = \{ \text{"symmetric } n \times n \text{ matrices"} \}$
with $\langle x, y \rangle = \text{tr}(x^T y)$.

We consider problems of the form

$$\left\{ \begin{array}{l} \min f(x) \\ \text{s.t. } x \in C \end{array} \right. \text{ or } \left\{ \begin{array}{l} \min f(x) \\ \text{s.t. } g_i(x) \leq 0 \quad \forall i \in [m] \\ [m] := \{1, \dots, m\} \end{array} \right.$$

for some functions $f, g_i : E \rightarrow \mathbb{R} \cup \{\pm\infty\}$
and $C \subseteq E$.

They can take
infinite values.

In the next few lectures we will focus on potential assumptions for C and f, g_i .

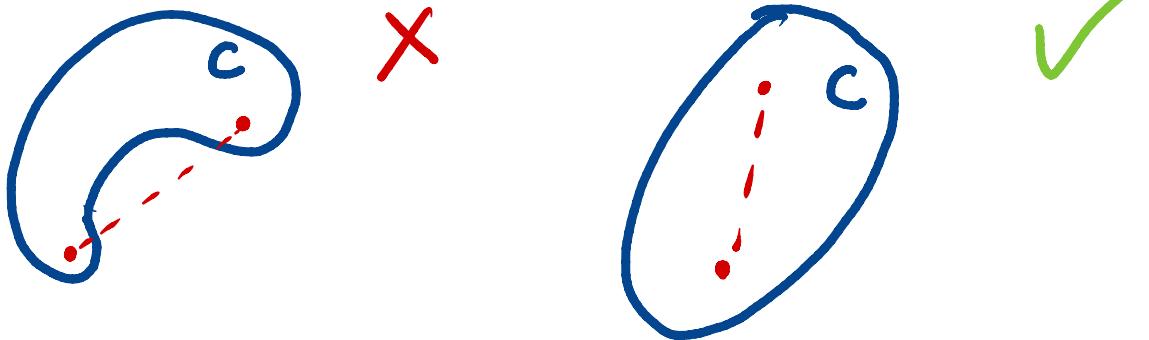
Convex sets

Def: A set $C \subseteq E$ is **convex** if for all $x, y \in C$ and $\lambda \in [0, 1]$ we have

$$\lambda x + (1-\lambda)y \in C.$$

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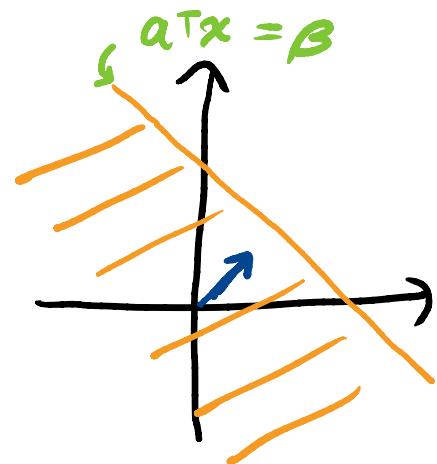


Examples

Two simple examples

▷ Half-spaces

$H := \{x \mid a^T x \leq \beta\}$
 for some $a \in E^*$ and $\beta \in \mathbb{R}$.
 \uparrow
 \uparrow
 $a \in E^*$



▷ Unit ball

Define $\|x\| := \sqrt{\langle x, x \rangle}$, and
 $B = \{x \in E \mid \|x\| \leq 1\}.$

The next result gives an easy way to identify convex sets.

Proposition: Arbitrary intersections of convex sets are convex.

Proof: Exercise. □

Example

▷ Polyhedra

Any set of the form

$$P = \{x \in E \mid \langle a_i, x \rangle \leq \beta_i \quad \forall i \in [m]\}$$

For some $a_i \in E$ and $\beta_i \in \mathbb{R}$.

▷ PSD matrices

$$S_+^n = \{X \in S^n \mid y^T X y \geq 0 \quad \forall y \in \mathbb{R}^d\}$$

is convex since

$$0 \leq y^T X y = \text{tr}(y^T X y) = \text{tr}(X y y^T) = \langle X y, y \rangle,$$

thus it is an infinite intersection of half spaces. \dashv

Denote $\text{int } C$ and $\text{cl } C$ the interior and closure of C . They'll play a key role later on and interact nicely with convex sets.

Proposition: Closures of convex sets are convex.

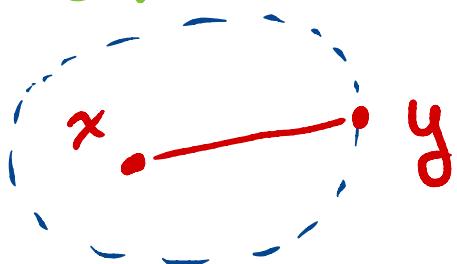
Proof: Exercise. \square

Lemma (\rightarrow): Suppose C is convex.

If $x \in \text{int } C$ and $y \in \text{cl } C$, then

$$(1-\lambda)x + \lambda y \in \text{int } C \quad \forall \lambda \in [0, 1)$$

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The segment $\bullet \cdots \bullet$ belongs to $\text{int } C$.

Proof: Assume first $y \in S$.

There is a $\delta > 0$ st $x + \delta B \subseteq S$.

By convexity, $\forall \lambda \in (0,1)$

$$(1-\lambda)(x + \delta B) + \lambda y \subseteq S$$

$$\Rightarrow (1-\lambda)x + \lambda y + (1-\lambda)\delta B \subseteq S,$$

which implies $(1-\lambda)x + \lambda y \in \text{int } S$.

Now suppose $y \in \text{cls } S$, then there is a sequence $\{y_k\} \subseteq S$ s.t $y_k \rightarrow y$.
For $\lambda \in (0,1)$, we can write

$$\begin{aligned} (1-\lambda)x + \lambda y \\ = (1-\lambda)x + \lambda y_k + \lambda(y - y_k) \\ = (1-\lambda) \left(x + \underbrace{\frac{\lambda}{(1-\lambda)}(y - y_k)}_{z_k} \right) + \lambda y_k. \end{aligned}$$

For large enough K , $z_K \in \text{int } S$. Then, our first argument shows the conclusion.

□

Corollary: The interior of convex sets is convex.

Proof: Take $x, y \in \text{int } C$. Then, $y \in \text{cl } C \Rightarrow (1-\lambda)x + \lambda y \in \text{int } C \quad \forall \lambda \in (0, 1)$ by Lemma (\rightarrow). \square

The next set of results will form the foundation of duality.

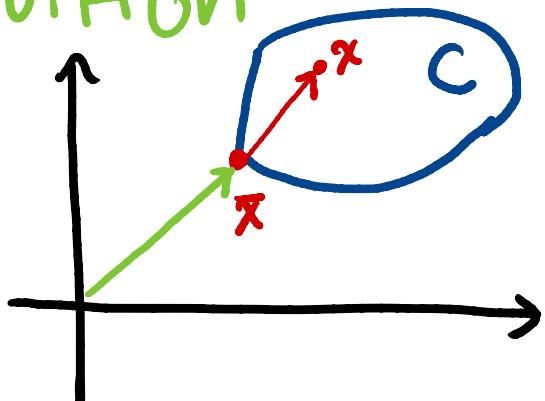
Theorem (Best approximation) Any nonempty closed convex set C has a unique shortest vector

$$\bar{x} = \underset{x \in C}{\operatorname{argmin}} \|x\|.$$

Moreover, it is characterized by
(\Leftarrow) $\langle \bar{x}, x - \bar{x} \rangle \geq 0 \quad \forall x \in S.$

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↑ and → are aligned.

Proof:

Existence. Choose any $\hat{x} \in C$, consider $C_1 = C \cap \|\hat{x}\|B$. Then

$$\min_{x \in S_1} \|x\| \leftarrow \begin{array}{l} \text{continuous} \\ \text{compact} \end{array}$$

achieves a minimizer x^* . Moreover, $\forall x \in C \setminus C_1$, we have $\|x^*\| \leq \|\hat{x}\| < \|x\|$.

Characterization. Let $\bar{x} \in \operatorname{argmin}_{x \in C} \|x\|$,

$$\text{then } \|\bar{x}\|^2 \leq \|\bar{x} + \lambda(x - \bar{x})\|^2 \quad \forall x \in C.$$

Expanding

$$0 \leq \lambda \|x - \bar{x}\|^2 + 2 \langle \bar{x}, x - \bar{x} \rangle,$$

taking $\lambda \downarrow 0$ yields (\Leftarrow) . The other direction follows easily.

Uniqueness. Suppose \bar{x}_1, \bar{x}_2 satisfy (\Leftarrow) . Then

$$\begin{aligned} & \langle \bar{x}_1, \bar{x}_1 - \bar{x}_2 \rangle \leq 0 \\ & + \langle -\bar{x}_2, \bar{x}_1 - \bar{x}_2 \rangle \leq 0 \\ \hline & \|\bar{x}_1 - \bar{x}_2\|^2 \leq 0 \end{aligned}$$

Thus, $\bar{x}_1 = \bar{x}_2$. □

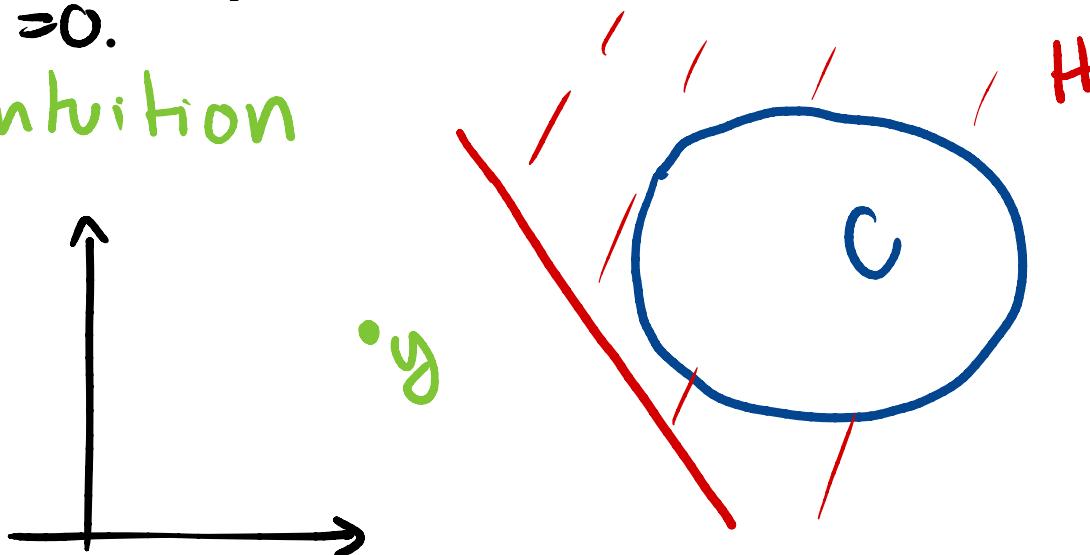
Theorem (Basic separation)

Suppose C is a nonempty closed convex set and $y \notin C$. Then there exists a half space H s.t.

$$C \subseteq H \quad \text{and} \quad y \notin H.$$

Proof: Apply previous result after a change of variables so that $y = 0$. □

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Theorem (Hahn-Banach)

Suppose C is convex and $\text{int } C \neq \emptyset$ and let $y \notin \text{int } C$. Then, there exists a half space H containing C

clC \ intC

with $y \in \text{bdC}$.

Proof: WLOG assume $0 \in \text{intC}$. For $n \in \mathbb{N}_+$, define

$$z_n = \left(1 + \frac{1}{n}\right)y.$$

Notice that $y = \frac{1}{n+1}0 + \frac{n}{n+1}z_n$,

then Lemma (\rightarrow) implies $z_n \notin \text{clC}$. Thus, by basic separation we obtain $\exists \{a_n\} \subseteq E^*$ s.t.

$$\langle a_n, z_n \rangle \geq \langle a_n, x \rangle \quad \forall x \in C.$$

WLOG we can take $\|a_n\|=1$, (why?) then by the Bolzano - Weierstrass theorem there exist a subsequence a_{n_k} s.t. $a_{n_k} \rightarrow a$. Therefore

$$\langle a, y \rangle \geq \langle a, x \rangle \quad \forall x \in C.$$

The result follows by taking

$$H = \{x \mid \langle a, x \rangle \leq \langle a, y \rangle\}. \quad \square$$