Lecture 8 Wed Feb/14/2024

Last time
- Filtrations
- Martingales
- Stopping times

Today
- Doob's optimal-stopping Theorem.
- Doob's convergence Theorem.

Last time we ended with

Theorem (⋈) Assume that X_n is a (super) martingale and let T be a stopping time. Then $X_{T\wedge n}$ is a (super) martingale. $\dashv$

Warning: $X_{T\wedge n}$ is very different from X_T. Let X_n be a simple $\pm$ random walk in $\mathbb{Z}$. Let

$$T := \inf\{n : X_n = 1\}.$$

We will eventually see that $\mathbb{P}(T < \infty) = 1$. However, even though

$$E(X_{T\wedge n}) = E(X_0) \quad \forall n.$$

we have

$$1 = \mathbb{E}[X_T] \neq \mathbb{E}[X_0] = 0.$$

Natural question: When can we say that
$$\mathbb{E}[X_T] = \mathbb{E}[X_0]?$$

Doob's Optional-Stopping Theorem

Theorem: Let T a stopping time. Let X be a (super)martingale. Then X_T is integrable and
$$\mathbb{E}[X_T] \overset{(\leq)}{=} \mathbb{E}[X_0]$$
in each one of the following situations:

a) There exists $N > 0$ s.t. $T(\omega) \leq N$ $\forall \omega$;

b) There exists $K > 0$ s.t. $|X_n(\omega)| \leq K$ $\forall \omega \forall n$ and T is a.s. finite;

c) $\mathbb{E}[T] < \infty$ and for some $K > 0$,
$$|X_n(\omega) - X_{n-1}(\omega)| \leq K \quad \forall n \, \forall \omega.$$

Proof: By Theorem (∞) we have that $X_{T \wedge n}$ is integrable and

(*) $$\mathbb{E}(X_{T \wedge n} - X_0) = 0 \quad \forall n.$$

For a) it suffices to use $n = N$.

For b), notice that if $P(T<\infty)=1$, then

$$X_{n\wedge T}(\omega) \to X_T(\omega) \quad \text{a.s.}$$

Further, since $|X_{n\wedge T}| < K$, then by BCT we have that $\lim \mathbb{E} X_{n\wedge T} = \mathbb{E} X_T$, then by (*) we have that $\mathbb{E} X_T = \mathbb{E} X_0$.

For (c) notice that

$$|X_{T\wedge n} - X_0| \leq \sum_{k=1}^{T\wedge n} |X_k - X_{k-1}| = KT$$

Then, since $\mathbb{E}T<\infty$, we can apply DCT to conclude that $X_{T\wedge n} \to X_T$ and thus $\mathbb{E}X_T = \mathbb{E}X_0$ from (*). $\square$

Doob's Convergence Theorem

stochastic

Super/Sub martingales are in a sense analogous to nonincreasing/nondecreasing deterministic sequences.

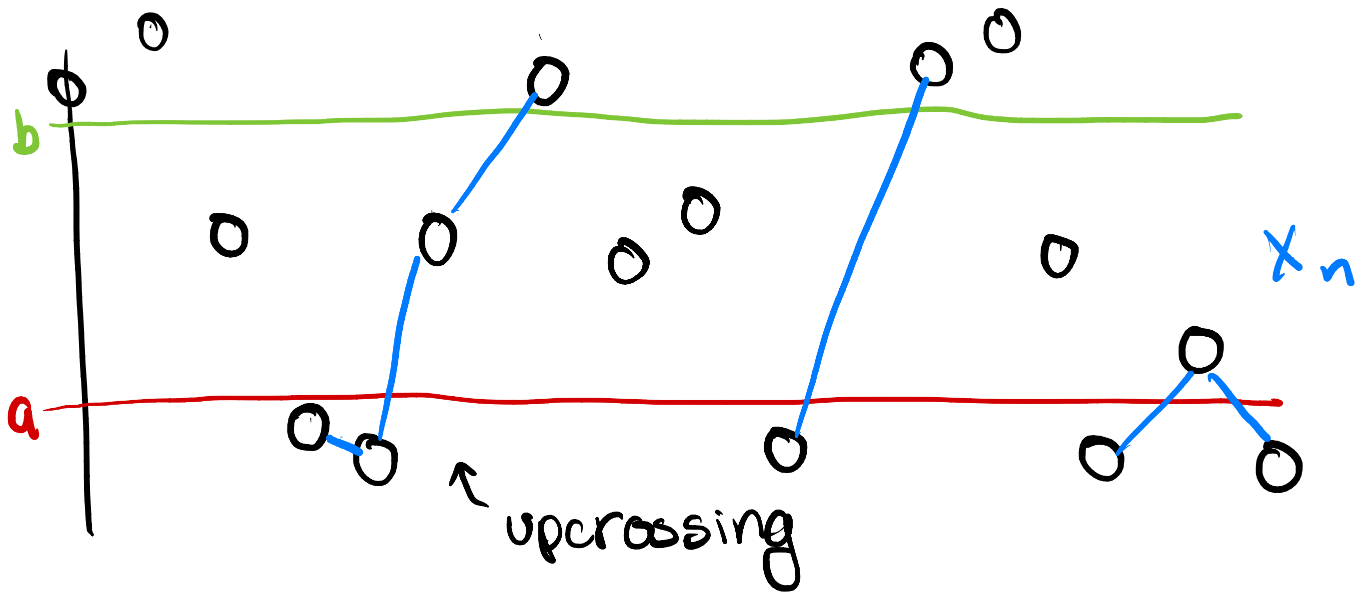
Theorem: Let (X_n) be a supermartingale such that $\sup_n \mathbb{E}|X_n| < \infty$. Then, almost surely, $X_\infty = \lim X_n$ exists and is finite.

$\dashv$

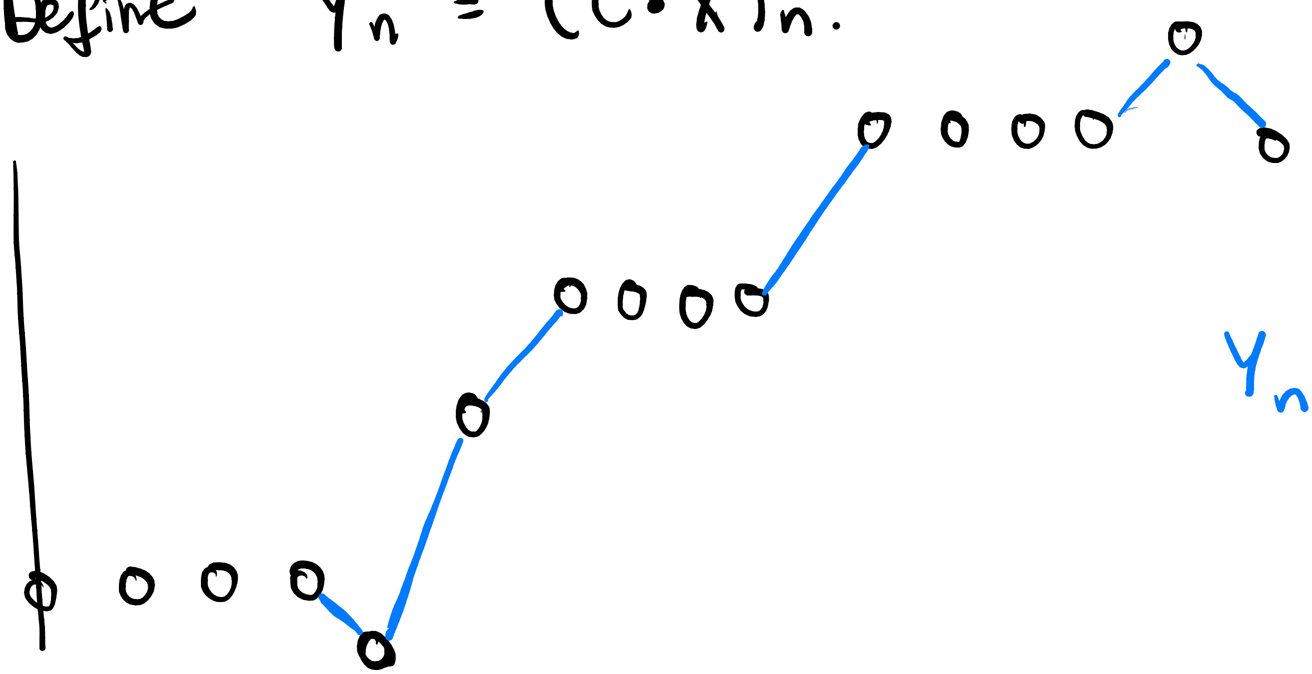
To formalize this we consider the following betting strategy $C_1 = \mathbb{1}_{\{X_0 < a\}}$ and $n \geq 2$

$$C_n = \mathbb{1}_{\{C_{n-1}=1\}} \mathbb{1}_{\{X_{n-1} \leq b\}} + \mathbb{1}_{\{C_{n-1}=0\}} \mathbb{1}_{\{X_{n-1} < a\}}.$$

which is easier to parse with a picture



Define $Y_n = (C \bullet X)_n$.



We define $U_N[a,b](\omega)$ to be the number of upcrossing of $[a,b]$ made by $X_n(\omega)$ by time N, i.e., the largest $k \in \mathbb{N}$ s.t. there exist

$$0 \leq s_1 < t_1 < \dots < s_k < t_k \leq N$$

with

$$X_{s_i}(\omega) < a \quad \text{and} \quad X_{t_i}(\omega) > b \quad \forall i \in [k].$$

From the picture we see

$$Y_N(\omega) \geq (b-a)\, U_N[a,b](\omega) - (X_N(\omega) - a)^-.$$

Lemma (Doob's Upcrossing lemma)
Let (X_n) be a supermartingale. Then

$$(\heartsuit) \qquad (b-a)\, \mathbb{E}\, U_N[a,b] \leq \mathbb{E}\,(X_N - a)^-$$

Proof: The process (C_n) is previsable, bounded and nonnegative $\Rightarrow Y_n = (C \bullet X)_n$ is a supermartingale and so $\mathbb{E}\, Y_n \leq 0$. The result follows from $(\heartsuit)$. $\square$

Proposition Assume X_n is a supermartingale such that $\sup_n \mathbb{E}|X_n| < \infty$.

Then, if we set $U_\infty[a,b] := \uparrow \lim_{N\to\infty} U_N[a,b]$,

$$(b-a)\,\mathbb{E}\,U_\infty[a,b] \le |a| + \sup_n \mathbb{E}|X_n| < \infty.$$

So, $\mathbb{P}(U_\infty[a,b] = \infty) = 0$.

Proof: By the Lemma above,

$$(b-a)\,\mathbb{E}\,U_N[a,b] \le |a| + \mathbb{E}|X_N|$$
$$\le |a| + \sup_n \mathbb{E}|X_n|$$

Using DCT, we can let $N \to \infty$ and the result follows. $\square$

Proof of Doob's Convergence Theorem

Define

Note the [ ]

$$\Lambda := \{\omega \mid X_n(\omega) \text{ does not converge in } [-\infty,\infty]\}$$
$$= \{\omega \mid \liminf_n X_n(\omega) < \limsup_n X_n(\omega)\}$$
$$= \bigcup_{\{a,b\in\mathbb{Q}\,:\,a<b\}} \underbrace{\{\omega \mid \liminf X_n(\omega) < a < b < \limsup X_n(\omega)\}}_{\Lambda_{a,b}.}$$

Notice that

$$\Lambda_{a,b} \subseteq \{\omega \mid U_\infty[a,b] = \infty\},$$

the by the Proposition above

$\mathbb{P}(\Lambda_{a,b}) = 0$ and $\mathbb{P}(\Lambda) = 0$. Hence,

$$X_\infty := \lim X_n \text{ exists a.s. in } [-\infty, \infty].$$

By Fatou's Lemma,

$$\begin{aligned} \mathbb{E}(|X_\infty|) &= \mathbb{E}(\liminf |X_n|) \\ &\le \liminf \mathbb{E}(|X_n|) \\ &\le \sup \mathbb{E}(|X_n|) < \infty. \end{aligned}$$

so that

$$\mathbb{P}(X_\infty \text{ is finite}) = 1. \qquad \square$$