Lecture 7 — Mon Feb/12/2024

Last time
- Continuation of the proof
- Properties
- Regular Conditional probability

Today
- Filtrations
- Martingales
- Stopping times

Filtrations

We now consider $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$, where
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a prob. space.
- $\{\mathcal{F}_n \mid n \geq 0\}$ is a *filtration*, i.e., an increasing family of sub-σ-algebras:

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}.$$

We define

$$\mathcal{F}_\infty := \sigma\left(\bigcup_n \mathcal{F}_n \right) \subseteq \mathcal{F}$$

Intuition

The information about $\omega \in \Omega$ available to us at "time" n consists in all values of $Z(\omega)$ for all $Z \in m\mathcal{F}_n$.

$\mathcal{F}_n$-measurable

Usually, $\{\mathcal{F}_n\}$ comes from a (stochastic) process $(W_n)_{n\in\mathbb{N}}$ via

$$\mathcal{F}_n = \sigma(W_0, W_1, \dots, W_n).$$

The information we have at time n is

$$W_0(\omega), \dots, W_n(\omega)$$

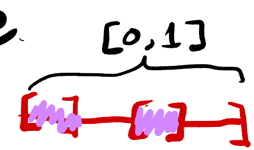
Example: Take $\Omega = [0,1]$, $\mathcal{F} = \overset{\text{Borel } \sigma\text{-algebra}}{\mathcal{B}}$, $\mathbb{P}$ to be the Lebesgue measure.

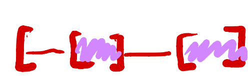
$$\mathcal{F}_0 = \sigma([\qquad]), \quad \mathcal{F}_1 = \sigma([\quad][\quad])$$

$$\mathcal{F}_2 = \sigma([\ ][\ ][\ ][\ ]), \dots, \quad \mathcal{F}_n = \sigma\left(\text{Grid with intervals of size } \frac{1}{2^n}\right).$$

In this case,

$$W_1 = \begin{cases} 0 & \text{if } \omega \leq \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$$

$W_2 = 0$ if $\omega \in$ 

$W_2 = 1$ if $\omega \in$ 

And the n'th random variable $W_n \in \{0,1\}$ tells you if you are on the left or the right of the interval at level $n-1$. $\dashv$

A process (X_n) is adapted to $(\mathcal{F}_n)$ if $X_n \in m\mathcal{F}_n \quad \forall n$.

Martingales

A process (X_n) is called a martingale (relative to $(\mathcal{F}_n)$) if

1) (X_n) is adapted,
2) $\mathbb{E}(|X_n|) < \infty \quad \forall n$,
3) $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$ a.s. $\forall n$.

A supermartingale is defined similarly but with 3) replaced by

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \leq X_{n-1} \quad \text{a.s.} \quad \forall n,$$

and submartingale

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \geq X_{n-1} \quad \text{a.s.} \quad \forall n.$$

Some remarks

1) Supermartingales decrease on average, submartigales increase on average.
There is nothing "super" about super

martingales, the name comes from notions in Harmonic Analysis.

2) Notice that if (X_n) is a $\binom{\text{sub}}{\text{super}}$ martingale, then so is $(X_n - X_0)$, so we can assume $X_0 = 0$.

3) The Tower Law implies that for supermartingales we have $\forall\, m < n$:

$$\begin{aligned}
\mathbb{E}[X_n \mid \mathcal{F}_m] &= \mathbb{E}\left[\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \mid \mathcal{F}_m\right] \\
&\leq \mathbb{E}[X_{n-1} \mid \mathcal{F}_m] \\
&\ \ \vdots \\
&\leq X_m .
\end{aligned}$$

Question: Why should we care about martingales?

Many stochastic processes are martingales and they have just enough structure to be amenable for analysis.

Examples

- Sums of independent zero-mean r.v.s.

Let $X_1, X_2, \dots$ be ind. r.v.s with

$\mathbb{E}(|X_n|) < \infty$ and $\mathbb{E}X_n = 0$ $\forall n$. Define the process (S_n) as

$$S_0 = 0$$
$$S_n = \sum_{k=1}^{n} X_k \qquad \forall n.$$

and the filtrations

(:)
$$\mathcal{F}_0 = \{\emptyset, \Omega\}$$
$$\mathcal{F}_n = \sigma(X_1, \dots, X_n) \qquad \forall n.$$

Then $\forall n \geq 1$

$$\mathbb{E}[S_n \mid \mathcal{F}_{n-1}] = \mathbb{E}[\underbrace{S_{n-1}}_{\in m\mathcal{F}_{n-1}} \mid \mathcal{F}_{n-1}] + \mathbb{E}[\underbrace{X_n | \mathcal{F}_n}_{\text{independent}}]$$
$$= S_{n-1} + \mathbb{E}[X_n]$$
$$= S_{n-1}.$$

Question for the future: When do we have that $\lim_{n\to\infty} S_n$ exists a.s.?

▷ Products of nonnegative ind r.v.s

Let $X_1, X_2, \dots$ be ind r.v. such that

$$X_n \geq 0 \qquad \text{and} \qquad \mathbb{E}X_n = 1 \qquad \forall n.$$

Define $(\mathcal{F}_n)$ as in (:) and

$$M_0 = 1$$
$$M_n = \prod_{k=1}^{n} X_k.$$

Then, $\forall n \geq 1$ a.s.

$$\mathbb{E}[M_n | \mathcal{F}_{n-1}] = \mathbb{E}[M_{n-1} X_n | \mathcal{F}_{n-1}]$$
$$= \mathbb{E}[X_n] \, \mathbb{E}[M_{n-1} | \mathcal{F}_{n-1}]$$
$$= M_{n-1}.$$

Questions for the future: Does $M_\infty = \lim M_n$ exist a.s.? If so, do we have that $\mathbb{E}(M_\infty) = 1$?

▷ Accumulating data of a r.v.

Let $(\mathcal{F}_n)$ be filtration and $X \in \mathcal{L}^1$,

Define $M_n = \mathbb{E}[X | \mathcal{F}_n]$. By Tower Law

$$\mathbb{E}[M_n | \mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{E}[X | \mathcal{F}_n] | \mathcal{F}_{n-1}] = \mathbb{E}[X | \mathcal{F}_{n-1}]$$
$$= M_{n-1} \quad \text{a.s.}$$

Questions for the future: Does $M_\infty = \lim_{n \to \infty} M_n$ exist a.s.? When is it that $X = M_\infty$?

Martingales and Gambling

One can think of $X_n - X_{n-1}$ as net winnings per each dollar that you bet at time n.

For martingales, the bets are fair:

$$\mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] = 0.$$

For supermartigales the game is unfavorable to you

$$\mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}] \leq 0.$$

Now, if we let C_n be how much money you bet at time n, then

$$\int C\,dX \qquad Y_n = \sum_{k \in [n]} C_k (X_k - X_{k-1}) = (C \bullet X)_n$$

encodes your total winnings up to time n.

Def: A process $(C_n)_{n=1}^{\infty}$ is **previsable** if $C_n \in m\mathcal{F}_{n-1}$.

Suppose that you have that $\xi_n = X_n - X_{n-1}$ with $\mathbb{P}(\xi_n = 1) = p = 1 - \mathbb{P}(\xi_n = -1)$.

The infamous "martingale" gambling strategy is defined as

$$C_1 = 1 \qquad n = \begin{cases} 2C_{n-1} & \text{if } \xi_n = -1 \\ 1 & \text{if } \xi_n = 1. \end{cases}$$

Theorem (♡) (You cannot game the system)

Let (C_n) be a bounded, nonnegative, predictable process and (X_n) be a (super) martingale. Then, $(C \cdot X)_n$ is a (super) martingale.

Proof: Linearity of cond. expectation gives

$$\mathbb{E}((C \cdot X)_n \mid \mathcal{F}_{n-1}) = (C \cdot X)_{n-1} + \mathbb{E}[C_n (X_n - X_{n-1}) \mid \mathcal{F}_{n-1}]$$
$$= (C \cdot X)_{n-1} + C_n \mathbb{E}[X_n - X_{n-1} \mid \mathcal{F}_{n-1}]$$
$$= (C \cdot X)_{n-1}.$$

□

Stopping times

A natural idea to make money is to stop while you are winning.

Def: We say that a map $T: \Omega \to \{0, 1, \dots, \infty\}$ is a stopping time if

$$\{T \leq n\} \in \mathcal{F}_n \quad \forall n \leq \infty.$$

or equivalently

$$\{T = n\} \in \mathcal{F}_n \quad \forall n \leq \infty.$$

Why are these equiv?

⊣

Intuition:

If we are playing a game, T tell us whether to stop playing a time n with the information that we have at time n.

⊣

Let $T \wedge n = \min\{T, n\}$, note that $T \wedge n$ is $\mathcal{F}_n$-measurable.

Theorem Assume that X_n is a (super) martingale. Then $X_{T \wedge n}$ is a (super) martingale.

Exercise: Proof this Theorem using Theorem (♡).