Lecture 12 Wed Feb/28/2024

Last time
- $\triangleright$ Uniform integrability
- $\triangleright$ $\mathcal{L}^1$ convergence

Today
- $\triangleright$ Optional stopping via UI
- $\triangleright$ Backwards Martingales

Optional stopping via UI

Today we add one more condition to the list of conditions in the optional stopping theorem. We will need the next result.

Theorem (:) If X_n is a UI (sub)martingale and T is a stopping time with $T<\infty$ a.s., then $X_{T\wedge n}$ is UI.

Proof: Since X_n is a martingale, by HW $|X_n|$ is a submartingale. Moreover $|X_{T\wedge n}|$ is a submartingale and so $\mathbb{E}|X_{T\wedge n}| \le \mathbb{E}|X_n|$. Thus,

$$\sup \mathbb{E}|X_{T\wedge n}| \le \sup \mathbb{E}|X_n| \underset{\text{UI}}{<} \infty.$$

Therefore, by Doob's convergence Thm, $X_{T\wedge n} \to X_T$ a.s. and $\mathbb{E}|X_T| < \infty$.

Then ← we use $T<\infty$.

$$\mathbb{E}\left[|X_{T\wedge n}| \mathbb{1}_{\{|X_{T\wedge n}|>M\}}\right] = \underbrace{\mathbb{E}\left[|X_T| \mathbb{1}_{\{|X_T|>M;\, T\le n\}}\right]}_{T_1} + \underbrace{\mathbb{E}\left[X_n \mathbb{1}_{\{|X_n|>M;\, n\le T\}}\right]}_{T_2}$$

Since $\mathbb{E}|X_T| < \infty$, then we can pick M large so that $T_1 \le \varepsilon$, similarly since X_n is UI we can pick M large so $T_2 \le \varepsilon$. □

Theorem If X_n is a UI (sub) martingale, and T is a stopping time s.t. $T<\infty$ a.s. Then

$$\mathbb{E}X_0 \overset{(\le)}{=} \mathbb{E}X_T \overset{(\le)}{=} \mathbb{E}X_\infty.$$

Proof: We know that $X_n \to X_\infty$ a.s and in $\mathcal{L}^1$ so $|\mathbb{E}X_n - \mathbb{E}X_\infty| \le \mathbb{E}|X_n - X_\infty| \to 0$ and $\mathbb{E}X_0 = \mathbb{E}X_n \ \forall n \Rightarrow \mathbb{E}X_\infty = \mathbb{E}X_0$.

Further, by Theorem (☺) $X_{T\wedge n}$ is UI

so $X_{T\wedge n} \to X_T$ a.s. and in $\mathcal{L}^1$, by the same reasoning $\mathbb{E}X_T = \mathbb{E}X_\infty$.
A similar argument follows for sub and super martingales.

$\square$

Backwards martingale

Def: Let $\dots \subseteq \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1}$ be filtration indexed by the negative integers. A **backward submartingale** is a process X_n adapted to $\mathcal{F}_n$ such that

(⊠) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n \quad \forall n \leq -1.$

Because $\mathcal{F}_n$ decreases as $n \downarrow -\infty$, the theory turns out to be much simpler.

Theorem (⇐) $X_{-\infty} = \lim_{n \to -\infty} X_n$ exists a.s. and in $\mathcal{L}^1$.

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To prove this result we need an auxiliary proposition.

Proposition (♡) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space and $X \in \mathcal{L}^1$. Then, the collection $\{\mathbb{E}[X \mid \mathcal{F}_\alpha] \mid \mathcal{F}_\alpha \text{ is a sub-}\sigma\text{-algebra of } \mathcal{F}\}$

is UI.

$\dashv$

Proof of Theorem $(\leftarrow)$:

Let $U_n[a,b]$ be the number of upcrossings from a to b in $X_{-n}, \dots, X_0$. using the same reasoning as in Lecture 8, $(b-a)\,\mathbb{E} U_n \leq \mathbb{E}(X_0 - a)^+$. Thus, $\mathbb{E} U_n$ is uniformly bounded and by BCT $\mathbb{E} U_\infty < \infty$. The same proof strategy as in Doob's convergence thm yields $X_n \to X_{-\infty}$ a.s. Further, $(\bowtie)$ implies $X_n = \mathbb{E}[X_0 \mid \mathcal{F}_n]$, and so $\{X_n\}_{n \in \mathbb{N}}$ is UI. By the Theorem $(\heartsuit)$ in Lecture 11 $X_n \to X_{-\infty}$ in $\mathcal{L}^1$.

$\square$

Proof of Proposition $(\heartsuit)$: By DCT if a sequence of sets is such that $\mathbb{P}(A_n) \to 0$ $\Rightarrow$ $\mathbb{E}[|X| \mathbb{1}_{A_n}] = 0$. Thus for every $\varepsilon > 0$ $\exists \delta > 0$ $\forall A$ st if $\mathbb{P}(A) < \delta \Rightarrow \mathbb{E}[|X| \mathbb{1}_A] \leq \varepsilon$. If this wasn't the case we could find a sequence A_n with $\mathbb{P}(A_n) \leq \frac{1}{n}$ and $\mathbb{E}[|X| \mathbb{1}_{A_n}] > \varepsilon$ ↯.

Pick M so that $\mathbb{E}|X|/M \leq \delta$. Jensen's give

$$\mathbb{E}\left[|\mathbb{E}[X|\mathcal{F}_\alpha]| \, \mathbb{1}_{\{|\mathbb{E}[X|\mathcal{F}_\alpha]|>M\}} \right]$$

$$\leq \mathbb{E}\left[\mathbb{E}[|X||\mathcal{F}_\alpha] \, \mathbb{1}_{\{|\mathbb{E}[X|\mathcal{F}_\alpha]|>M\}} \right]$$

(★) $$\leq \mathbb{E}\left[\mathbb{E}[|X||\mathcal{F}_\alpha] \, \mathbb{1}_{\{\mathbb{E}[|X||\mathcal{F}_\alpha]>M\}} \right]$$

$$= \mathbb{E}\left[|X| \, \mathbb{1}_{\{\mathbb{E}[|X||\mathcal{F}_\alpha]>M\}} \right]$$

Follows by def of $\mathbb{E}[|X||\mathcal{F}_\alpha]$ since this set $\in \mathcal{F}_\alpha$.

Using Markov's

$$P(\mathbb{E}[|X||\mathcal{F}_\alpha] > M) \leq \frac{\mathbb{E}[\mathbb{E}[|X||\mathcal{F}_\alpha]]}{M}$$

$$\leq \frac{\mathbb{E}|X|}{M} \leq \delta.$$

Thus, we conclude from (★) that

$$\mathbb{E}\left[|\mathbb{E}[X|\mathcal{F}_\alpha]| \, \mathbb{1}_{\{|\mathbb{E}[X|\mathcal{F}_\alpha]|>M\}} \right] < \varepsilon.$$

Since ε was arbitrary $\Rightarrow$ the collection is UI.

$\square$

Let's conclude by identifying the limit

Theorem If $X_{-\infty} = \lim_{n\to-\infty} X_n$ and $\mathcal{F}_{-\infty} = \bigcap_{n=1}^{-\infty} \mathcal{F}_n$, then $X_{-\infty} = \mathbb{E}[X_0 | \mathcal{F}_{-\infty}]$.

Proof: By construction $X_{-\infty} \in m\mathcal{F}_{-\infty}$, and $X_n = \mathbb{E}[X_0 | \mathcal{F}_n]$, so if $A \in \mathcal{F}_{-\infty} \subseteq \mathcal{F}_n$ then

$$\int_A X_n \, d\mathbb{P} = \int_A X_0 \, d\mathbb{P}.$$

Moreover, since $X_n \to X_{-\infty}$ in $\mathcal{L}^1$, then

$$\begin{aligned} |\mathbb{E} X_n \mathbb{1}_A - \mathbb{E} X_{-\infty} \mathbb{1}_A| &\leq \mathbb{E}|X_n \mathbb{1}_A - X_{-\infty}\mathbb{1}_A| \\ &\leq \mathbb{E}|X_n - X_{-\infty}| \to 0. \end{aligned}$$

So

$$\int_A X_{-\infty} \, d\mathbb{P} = \int_A X_0 \, d\mathbb{P}.$$

$\square$

Theorem. Le $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$ as $n \to -\infty$ ($\mathcal{F}_{-\infty} = \bigcap_{n=1}^{-\infty} \mathcal{F}_n$). Then, if Y in $\mathcal{L}^1$ then

$$\mathbb{E}[Y | \mathcal{F}_n] \to \mathbb{E}[Y | \mathcal{F}_{-\infty}] \text{ a.s. and in } \mathcal{L}^1.$$

Proof: Exercise