

## Lecture 9

### Last time

- ▷ Doob's optimal -stopping Theorem.
- ▷ Doob's convergence Theorem.

Mon Feb 19 / 2024

### Today

- ▷ A word of warning
- ▷ Symmetric random walk
- ▷ Branching process.

### A word of Warning

The last result from the previous lecture gives:

**Corollary (♡)** If  $X_n \geq 0$  is a supermartingale, then  $X_\infty = \lim_{n \rightarrow \infty} X_n < \infty$  exists almost surely.  $\rightarrow$

**Warning** We cannot conclude that  $X_n$  converges to  $X_\infty$  in  $L^1$ .  $\leftarrow$  Will come back to  $L^1$  next class.

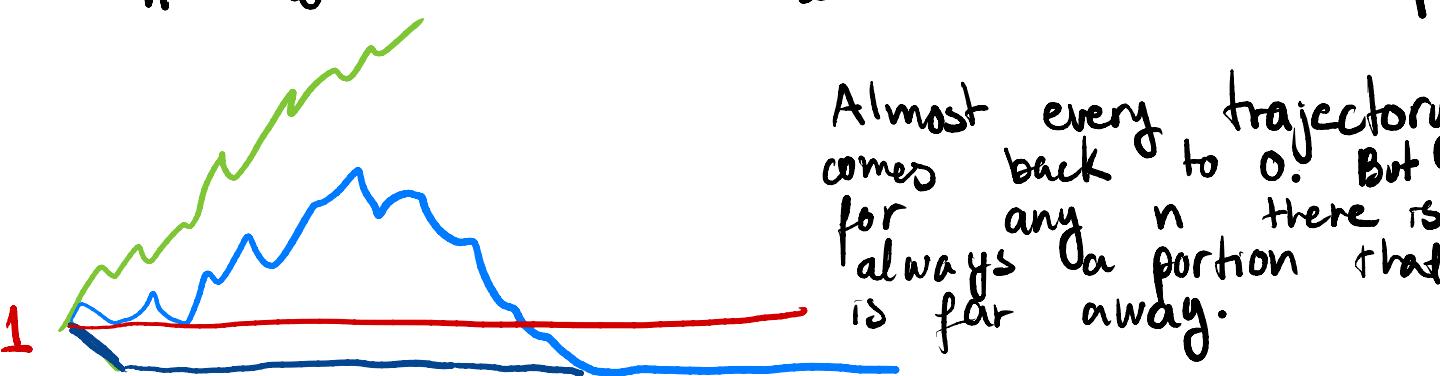
**Example:** Consider a simple random walk  $K$   $S_n = \sum_{k=1}^n \xi_k$  with  $S_0 = 1$ , and  $P(\xi_k = 1) = P(\xi_k = -1) = \frac{1}{2}$ . Define

$$T = \inf \{ n \mid S_n = 0 \} \text{ and } X_n = S_{T \wedge n}.$$

Since  $X_n$  is a martingale  $E X_n = E X_0 = 1$ .

On the other hand,  $X_n \xrightarrow{\text{a.s.}} X_\infty$ .  
 Notice that  $X_\infty(w) = 0 \quad \forall w$  since  
 if  $X_n(w) = K > 0 \Rightarrow X_{n+1}(w) = K \pm 1$ , and so  
 $\lim X_n \neq K$ .

This example illustrates the warning as  
 $\mathbb{E}|X_n - X_\infty| \geq |\mathbb{E}X_n - \mathbb{E}X_\infty| = 1.$



## Applications to Random Walks

Consider  $S_0 = 0$ ,  $S_n = \sum_{k=1}^n \xi_k$  with iid r.v.  $\xi_k$   
 s.t.  $\mathbb{E}\xi_k = \mu$  and  $\mathbb{E}\xi_k^2 = \sigma^2$ .

Claim The following are martingales:

a)  $M_n = S_n - \mu n$ .

b)  $M_n = S_n^2 - \sigma^2 n$ .

c)  $M_n = \exp(\theta S_n) / \phi(\theta)^n$  with  $\phi(\theta) = \mathbb{E} \exp(\theta \xi_1) < \infty$ .

Proof: a)  $\mathbb{E}[S_n - \mu n | \mathcal{F}_{n-1}] = \mathbb{E}[S_{n-1} + \xi_n - \mu n | \mathcal{F}_{n-1}] = S_{n-1} - \mu(n-1)$

b)  $\mathbb{E}[S_n^2 - \sigma^2 n | \mathcal{F}_{n-1}] = \mathbb{E}[(S_{n-1} + \xi_n)^2 - \sigma^2 n | \mathcal{F}_{n-1}]$

$= \mathbb{E}[S_{n-1}^2 + 2\xi_n S_{n-1} + \xi_n^2 - \sigma^2 n | \mathcal{F}_{n-1}]$

0 because  $\mathbb{E}\xi_n = 0$  and  $\xi_n \perp \mathcal{F}_{n-1}$ .

Useful to analyze  
MGF (we will not cover it)

$$= S_{n-1}^2 + \sigma^2 - \sigma^2 n \\ = S_{n-1}^2 - \sigma^2(n-1).$$

c) Follows from the fact that  $X_k = \exp(\theta \xi_k) / \phi(s)$   
are i.i.d. positive random variables with mean 1, and  
 $M_n = \prod_{k=1}^n X_k$ . (Recall the example from Lecture 7)  $\square$

We will leverage these to obtain explicit formulas for probabilities and expectations of stopped random walks.

Theorem (Wald's equation) Consider a random walk

$S_n = \sum_{k=1}^n \xi_k$  with  $P(\xi_k = 1) = 1 - P(\xi_k = -1)$ . Let  $N$  be a stopping time with  $E N < \infty$ . Then,

$$E S_N = \mu E N, \quad \text{where } \mu = E \xi_k.$$

Proof: Optional Stopping Thm applied to  $S_n - \mu n$ .  $\square$

Theorem: Consider the symmetric random walk

$S_n = \sum_{k=1}^n \xi_k$  with  $\{\xi_k\}$  i.i.d. s.t.  $P(\xi_k = 1) = P(\xi_k = -1) = \frac{1}{2}$ .

Let  $N = \inf \{n \mid S_n \notin (a, b)\}$ . Then,

a)  $P(S_N = a) = \frac{b}{b-a}$  and  $P(S_N = b) = \frac{a}{b-a}$ .

b)  $E[N] = -ab$ .

Proof: a) We want to apply Wald's and so we need

$\mathbb{E} N < \infty$ . Notice that if we have a (b-a) consecutive losses +1, then we stop.

Then,  $N$  is dominated by  $(b-a)X$

where  $X$  is a geometric random

variable with success probability  $p = 2^{-(b-a)}$ . Thus

$$\mathbb{E} N < (b-a) \mathbb{E} X = (b-a) 2^{(b-a)} < \infty.$$

Thus, Wald's equation tell us

$$0 = \mathbb{E} S_N = a P(S_N = a) + b (1 - P(S_N = a)),$$

rearranging

$$P(S_N = a) = \frac{b}{b-a}.$$

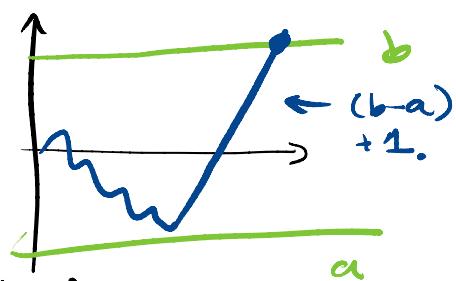
Analogously, we derive  $P(S_N = b) = \frac{a}{b-a}$ .

b) Using that  $S_{N \wedge n}^2 - N \wedge n$  is a martingale, we obtain using the Optional Stopping Theorem and the fact that

$$\begin{aligned} |S_{N \wedge n}^2 - S_{N \wedge (n-1)}^2| &\leq |(S_{N \wedge n-1} + g_n)^2 - S_{N \wedge n-1}^2| \\ &= |2S_{N \wedge n-1}g_n + 1| \\ &\leq 2 \max\{b, -a\} + 1, \end{aligned}$$

we conclude that

$$\mathbb{E} N = \mathbb{E} S_N^2 = a^2 \frac{b}{b-a} - b^2 \frac{a}{b-a} = -ab \left( \frac{b}{b-a} - \frac{a}{b-a} \right) = -ab. \quad \square$$



## Branching Processes

Suppose we want to study the dynamics of a population throughout generations. A simple model considers iid nonnegative r.v.  $\xi_k^n$  = "number of children of  $k^{\text{th}}$  individual at generation  $n$ ".

Define  $Z_0 = 1$  and

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \dots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0. \end{cases}$$

This is known as the Galton - Watson process, and  $p_l = P(\xi_k^n = l)$  is called offspring distribution.

Lemma: Let  $\mathcal{F}_n = \sigma(\xi_k^m : 1 \leq k, 1 \leq m \leq n)$  and  $\mu = \mathbb{E}\xi_1^m \in (0, \infty)$ . Then  $Z_n/\mu^n$  is a martingale.

Proof: By construction  $Z_n \in \mathcal{F}_n$  and  $\mathbb{E}|Z_n| < \infty$ . Then,

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = \mathbb{E}[\xi_1^{n+1} + \dots + \xi_{Z_n}^{n+1} | \mathcal{F}_n]$$

$$= Z_n \mathbb{E}\xi_1^{n+1}$$

Proved in  $\xrightarrow{\text{HW 3}}$  independence.

$$= z_n \mu.$$

Thus  $E[z_n/\mu^n | \mathcal{F}_{n-1}] = z_{n-1}/\mu^{n-1}$ . □

Thus, by corollary (3) we have that  $z_n/\mu^n \rightarrow$  a limit a.s.

Theorem If  $\mu < 1$ , then  $z_n = 0$  for all large  $n$ , and so  $z_n/\mu^n \rightarrow 0$  a.s.

Proof:  $E[z_n/\mu^n] = E[z_0] = 1 \Rightarrow E z_n = \mu^n$ .

Now, by Markov's inequality

$$P(z_n > 0) = P(z_n \geq 1) \leq E z_n = \mu^n \rightarrow 0.$$

The result follows by Borell-Cantelli. □

Theorem If  $\mu = 1$  and  $P(\xi_k^m = 1) < 1$ , then  $z_n = 0$  for all large  $n$  a.s.

Proof: When  $\mu = 1$ ,  $z_n$  is a nonnegative martingale, thus  $z_n \rightarrow z_\infty$  a.s. Since  $z_n$  is integer-valued  $z_n = z_\infty$  for large enough  $n$ . If  $P(\xi_k^m = 1) < 1$ , then

$$P(z_m = k \text{ for all } n \geq N)$$

$$= P(z_n = k, z_{n+1} = k, \dots)$$

$$= P(z_n = k) \prod_{l=n}^{\infty} P(z_{l+1} = k | z_l = k) \quad (\infty)$$

Note that  $P(Z_{\ell+1} = k \mid Z_K = k) = P(\xi_1^{\ell+1} = 1)^{k-1} P(\xi_k^{\ell+1} \neq 1) < 1$ . Thus the infinite product in  $(\infty) \rightarrow 0$ .  $\square$

