Lecture 11

Monday Feb/26/24

Last time

- Branching Process continued.
- Convergence in $\mathcal{L}^p$.

Today

- Uniform integrability
- Convergence in $\mathcal{L}^1$.

Uniform integrability

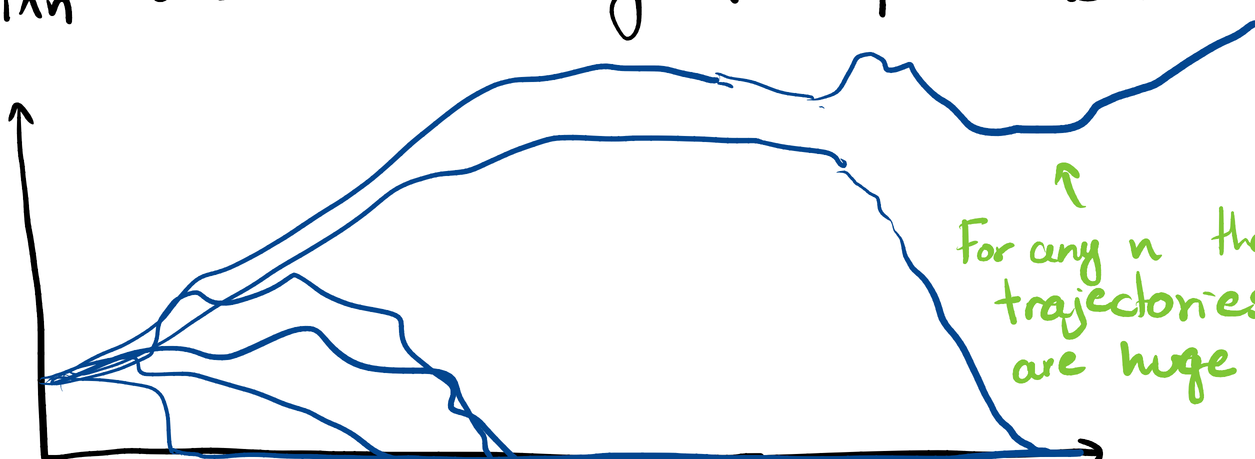
Why is it that we can fail to converge in $\mathcal{L}^1$?

Recall our example

$$S_0 = 1, \quad S_n = \sum_{k=1}^{n} \xi_k \quad \text{and} \quad T = \inf\{k \mid S_k = 0\}.$$

Then $S_{T\wedge n} \xrightarrow{a.s.} S_T$ (Convergence theorem.

BUT $\mathbb{E} S_{T\wedge n} = \mathbb{E} S_0 = 1 \neq 0 = \mathbb{E} S_T$. Thus $S_{T\wedge n}$ does not converge to S_T in $\mathcal{L}^1$.



For any n there are trajectories that are huge!

Although they always come back

The issue here is heavy tails!

Def: A collection of random variables $\{X_\alpha\}_{\alpha\in\Delta}$ is *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\Delta} \mathbb{E}[|X_\alpha| \mathbb{1}_{\{|X_\alpha|\geq M\}}] = 0.$$

$\dashv$

Remarks:

- Δ does not have to be countable.
- This definition is general (no martingales).
- Intuitively we are asking the tail averages to decrease at similar rates.
- There are large collections of r.v. that are UI, e.g., let Y be integrable and let $\{X_\alpha\}_{\alpha\in\Delta}$ be the collection of r.v. dominated by Y ($|X_\alpha| < Y$), then $\{X_\alpha\}_{\alpha\in\Delta}$ is UI.

We care about UI because it characterizes L^1 convergence.

No martingales!

Theorem (♡) Suppose $\mathbb{E}|X_n| < \infty$ $\forall n$ and $X_n \to X$ in probability (i.e., $\lim \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$). Then, the following are equivalent:

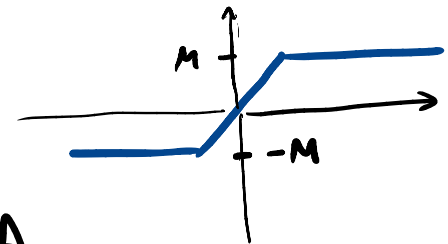
a) $\{X_n : n \geq 0\}$ is UI.

b) $X_n \to X$ in L^1.

c) $\mathbb{E}|X_n| \to \mathbb{E}|X| < \infty$.

Proof: a) $\Rightarrow$ b) For any M, define the function

$$\varphi_M(x) = \begin{cases} M & \text{if } x \geq M, \\ x & \text{if } |x| \leq M, \\ -M & \text{if } x \leq -M. \end{cases}$$



We now apply triangle inequality

$$|X_n - X| \leq |X_n - \varphi_M(X_n)| + |\varphi_M(X_n) - \varphi_M(X)| + |\varphi_M(X) - \varphi(X)|.$$

Notice that $|y - \varphi_M(y)| = (|y| - M)^+ \leq |y| \mathbb{1}_{\{\ \geq M\}}$.

Thus,

$$\mathbb{E}|X_n - X| \leq \underbrace{\mathbb{E}|X_n| \mathbb{1}_{\{|X_n| \geq M\}}}_{T_1} + \underbrace{\mathbb{E}|\varphi_M(X_n) - \varphi_M(X)|}_{T_2} + \underbrace{\mathbb{E}|X| \mathbb{1}_{\{|X| \geq M\}}}_{T_3}$$

Let's show that we can make each term small

T_1) By UI we can pick M large enough to ensure that $T_1 \leq \varepsilon$.

T_2) By the Continuous Mapping Thm $\varphi_M(X_n) \to \varphi_M(X)$ in probability. Since $Y_n = |\varphi_M(X_n) - \varphi_M(X)|$ converges in prob to 0 $\Rightarrow$ By BCT for convergence in prob (Exercise 2.3.5 Durrett) $\mathbb{E} Y_n \to 0$.

T_3) By Fatou's Lemma for convergence in probability (Exercise 2.3.4 Durret)

$$\begin{aligned}\mathbb{E}|X| &= \mathbb{E}|\liminf X| \\ &\le \liminf \mathbb{E}|X_n| \\ &\le \sup \mathbb{E}|X_n| < \infty,\end{aligned}$$

Thus $\lim_{M\to\infty} \mathbb{E}|X| \mathbb{1}_{\{|X|\ge M\}} = 0$, and for large M, $T_3 \le \varepsilon$.

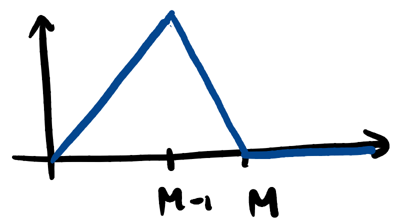
Thus we derive $\lim_{n\to\infty} \mathbb{E}|X_n - X| \le 2\varepsilon \quad \forall \varepsilon$, which shows the implication after $\varepsilon \downarrow 0$.

b) $\Rightarrow$ c) By Jensen's

$$\big|\mathbb{E}|X_n| - \mathbb{E}|X|\big| \le \mathbb{E}\big||X_n| - |X|\big| \le \mathbb{E}|X_n - X| \to 0.$$

c) $\Rightarrow$ a) Define the function

$$\psi_M(x) = \begin{cases} x & \text{if } x \in [0, M-1] \\ \text{linear} & \text{if } x \in [M-1, M] \\ 0 & \text{otherwise} \end{cases}$$



We can bound

$$\begin{aligned}\mathbb{E}(|X_n| \mathbb{1}_{\{|X_n| > M\}}) &\le \mathbb{E}|X_n| - \mathbb{E}\psi_M(|X_n|) \\ &\le \underbrace{\mathbb{E}|X| - \mathbb{E}\psi_M(|X|)}_{T_1} + \underbrace{\big(\mathbb{E}|X_n| - \mathbb{E}\psi_M(|X_n|) - \mathbb{E}|X| - \mathbb{E}\psi_M(|X_n|)\big)}_{T_2}\end{aligned}$$

T_1) By DCT, for large M we have $T_1 \le \varepsilon$.

T_2) By the same reasoning as in a) $\Rightarrow$ b)
$\mathbb{E}\,\Psi_M(|X_n|) \to \mathbb{E}\,\Psi(|X|)$, so $T_2 \leq \varepsilon$ for large $n \geq n_0$.

Then for all $n \geq n_0$ we have
$\mathbb{E}(|X_n| \mathbb{1}_{\{|X_n| > M\}}) \leq \varepsilon$. For $1 \leq n \leq n_0$ we can take M larger to ensure that the same inequality holds $\Rightarrow \{X_n\}$ is UI. $\square$

$\mathcal{L}^1$ convergence

Theorem Let X_n be a supermartingale. Then the following are equivalent:

a) X_n is UI.

b) X_n converges a.s. and in $\mathcal{L}^1$.

c) X_n converges in $\mathcal{L}^1$.

Proof:

a) $\Rightarrow$ b) Pick M so that $\sup \mathbb{E}|X_n| \mathbb{1}_{\{|X_n| \geq M\}} \leq 1$. Then

$$\sup \mathbb{E}|X_n| = \sup \mathbb{E}|X_n| \mathbb{1}_{\{|X_n| \leq M\}} + \mathbb{E}|X_n| \mathbb{1}_{\{|X_n| \geq M\}}$$
$$\leq M + \sup |X_n| \mathbb{1}_{\{|X_n| \geq M\}}$$
$$\leq M + 1 < \infty.$$

Then by Doob's Convergence Thm, $X_n \to X$

a.s. Recall that a.s. convergence implies convergence in prob. Then by Theorem (♡) $X_n \to X$ in $\mathcal{L}^1$.

b) $\Rightarrow$ c) Trivial.

c) $\Rightarrow$ a) Recall that $\mathcal{L}^1$ convergence implies convergence in probability. Then by Theorem (♡) X_n is UI.

$\square$

Q: How can we check uniform integrability?

Theorem: Let φ be any nonnegative function with $x/\varphi(x) \to 0$ as $x \to \infty$. Suppose $\mathbb{E}\,\varphi(|X_\alpha|) \leq C$ for all $\alpha \in \Delta$. Then, $\{X_\alpha\}_{\alpha \in \Delta}$ is UI.

Proof: Let $\varepsilon_M := \sup\{x/\varphi(x) \mid x \geq M\}$. For $\alpha \in \Delta$ we have

$$\mathbb{E}(|X_\alpha| \mathbb{1}_{\{|X_\alpha| \geq M\}}) \leq \varepsilon_M \, \mathbb{E}(\varphi(|X_\alpha|) \mathbb{1}_{\{|X_\alpha| \geq M\}})$$
$$\leq \varepsilon_M C.$$

By assumption $\varepsilon_M \to 0$ as $M \to \infty$ so the result follows.

$\square$

Useful functions are $\varphi(x) = x^p$ (which yields the same condition as in the $\mathcal{L}^p$ convergence Thm from last class) and $\varphi(x) = x \log^+ x$ (which will appear in the homework).