Lecture 10

Last time

- Linear programming revisited
- Extreme points
- Intro to Simplex.

Today

- Recap
- Initial point
- Optimality
- Pivoting
- Finishing

Recall our high level description

SIMPLEX (INFORMAL)

- Pick a basis B_0 s.t. $x(B_0)$ is feasible.
- Loop $k \geq 0$:
  - Update $B_{k+1} \leftarrow B_k \cup \{j\} \setminus \{i\}$ s.t.
    1. $x(B_{k+1})$ is feasible.
    2. $c^T x(B_{k+1}) \leq c^T x(B_k)$
  - If $x(B_{k+1})$ is optimal: return $x(B_{k+1})$.

$j \in B_k^c$

$i \in B_k$

How to find this guy?

How to ensure this

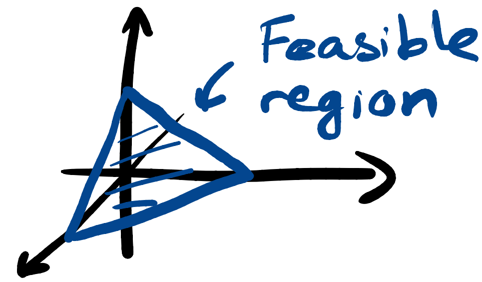
How to check this?

How to guarantee simplex finishes? At what rate of convergence?

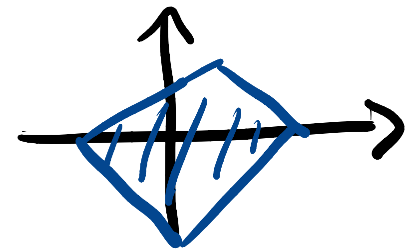
Recall our primal and dual

(P) $p^* = \begin{cases} \inf & \langle c, x \rangle \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$

$A \in \mathbb{R}^{m \times}$



(D) $d^* = \begin{cases} \sup & \langle b, y \rangle \\ \text{s.t.} & A^T y \leq 0. \end{cases}$



Strong duality for LPs

Proposition: There are exactly 4 possibilities for LPs:

1) Both primal and dual are achieved and $p^* = d^*$.
2) The primal is feasible and the dual infeasible $p^* = -\infty = d^*$.

   $P = \{x \mid Ax = b, x \geq 0\} = \emptyset$.
3) The dual is feasible and the dual infeasible.
4) Both primal and dual are infeasible $p^* = \infty$ and $d^* = -\infty$.

Proof: Exercise. $\blacksquare$

How to find an initial feasible point?

To find x_0 we can define an auxiliary problem (Phase I approach):

$$\begin{aligned} \min \ & \sum s_i \\ \text{s.t.} \ & Ax + s = b \\ & x \geq 0,\ s \geq 0. \end{aligned}$$

Note that we can always assume $b \geq 0$ (otherwise we can negate the corresponding constraint in (P)). In this case, we trivially have that

$$x = 0, \quad s = b$$

is a feasible point. So we could use simplex to find an optimal solution. If the solution $\bar{x}, \bar{s}$ such that

(Phase II)

$\bar{s}=0 \Rightarrow$ We run simplex for (P) with $x_0=\bar{x}$.

$\bar{s}>0 \Rightarrow$ Declare infeasibility.

How to check if we reached an optimum?

Recall from last time that each standard form BFS $\bar{x}$ is associated to a basis B

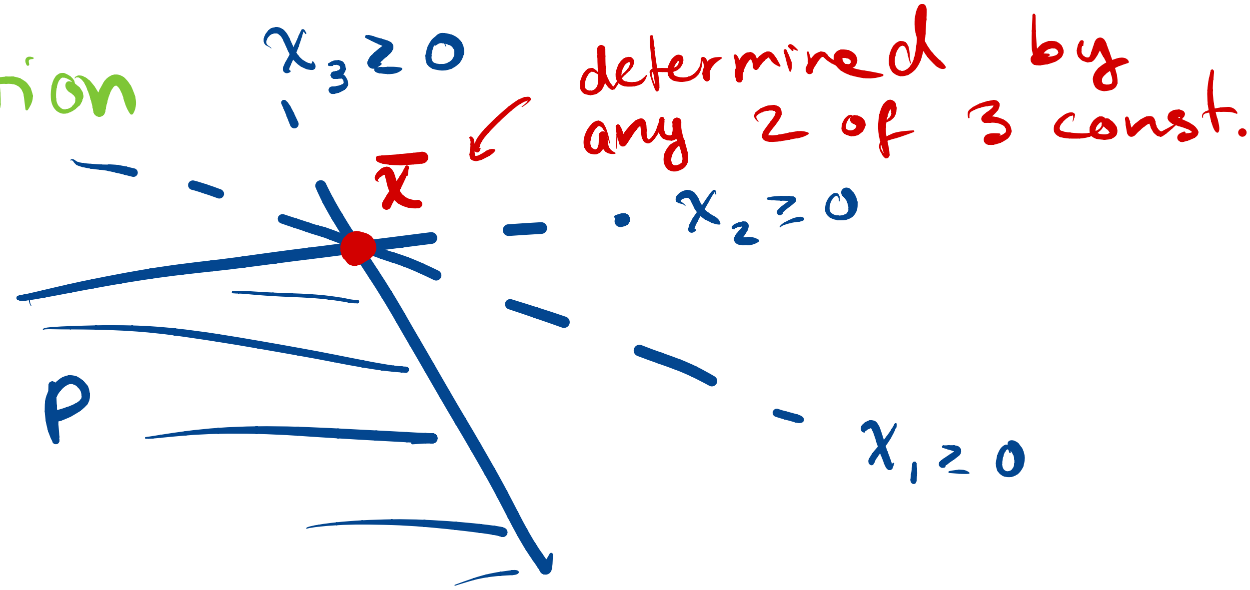
$\bar{x}$ uniquely solves $\begin{cases} A_B x_B = b \\ x_{B^c} = 0. \end{cases}$

This basis doesn't have to be unique! Indeed, we might have two bases B and B' s.t.

$$\begin{cases} A_B \bar{x}_B = b \\ \bar{x}_{B^c} = 0 \end{cases} \quad \text{and} \quad \begin{cases} A_{B'} \bar{x}_{B'} = b \\ \bar{x}_{B'^c} = 0. \end{cases}$$

In such case, all $i \in B^c \cup (B')^c$ have $\bar{x}_i = 0$. Thus if $i \in B \setminus B'$, we have $\bar{x}_i = 0$. (We are getting zeros we didn't enforce).

Intuition



Def: We say that BFS is not degenerate if for an associated basis B, we have $x_B > 0$. ⊣

To check optimality we can use the following dual solution

$$y = A_B^{-T} c_B.$$

Recall y is feasible iff

$$\bar{c} = c - A_B^{-T} c_B \geq 0.$$

Reduced costs

Theorem: Consider a BFS x^* associated with a basis B and reduced costs $\bar{c}$.

1) If $\bar{c} \geq 0 \Rightarrow x^*$ is a minimizer.

2) If x^* is nondegenerate and a minimizer $\Rightarrow \bar{c} \geq 0$.

Proof: 1) If $\bar{c} \geq 0$, then x^* and $y^* = A_B^{-T} c_B$ are feasible solutions and

$$c^T x^* = c_B^T x_B = c_B^T A_B^{-1} b = (A_B^{-T} c_B)^T b = y^{*T} b.$$

Thus, x^* and y^* have to be optimal.

2) Suppose $\exists j$ s.t. $\bar{c}_j < 0$.

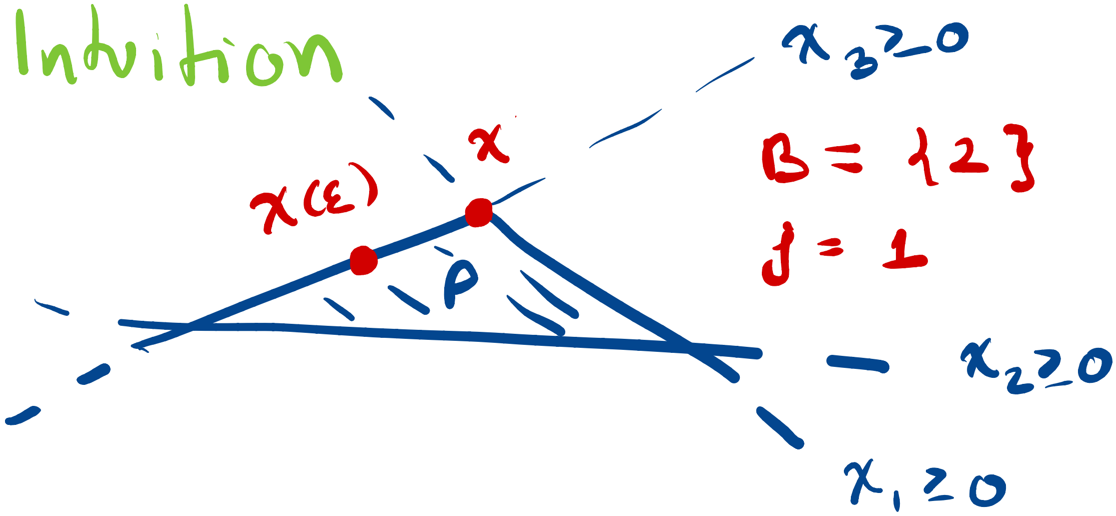
Let's imagine we were to move within the constraint set to make $x_j = \varepsilon > 0$ for $j \in B^c$.

Let $\hat{x}(\varepsilon)$ be the unique solution to

$$\begin{cases} Ax = b \\ x_j = \varepsilon \\ x_{B^c \setminus j} = 0 \end{cases} \Leftrightarrow \begin{cases} A_B x_B = b - A_j x_j \\ x_j = \varepsilon \\ x_{B^c \setminus j} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_B = A_B^{-1}(b - A_j \varepsilon) \\ x_j = \varepsilon \\ x_{B^c \setminus j} = 0 \end{cases}$$

Intuition



Since x^* is not degenerate $x^*_B > 0$.

$\Rightarrow$ For small $\varepsilon > 0$, $x_B(\varepsilon) > 0$, and $x_B(\varepsilon)$ is feasible.

But,

$$c^T x(\varepsilon) = \begin{bmatrix} c_B \\ c_j \\ c_{B^c \setminus j} \end{bmatrix} \begin{bmatrix} A_B^{-1}(b - A_j \varepsilon) \\ \varepsilon \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
&= c_B^T A_B^{-1}(b - A_j \varepsilon) + c_j \varepsilon \\
&= c_B^T x_B^* + (c_j - c_B^T A_B^{-1} A_j)\varepsilon \\
&= c^T x^* + \underbrace{(c_j - A_j^T y)}_{\bar{c}_j < 0}\varepsilon \\
&< c^T x^*,
\end{aligned}
$$

which contradicts the optimality of x^*. $\square$

Thus, we can use $\bar{c} \geq 0$ to check optimality.

Pivoting

Let $\bar{x}$ be a BFS with basis B. and reduced costs $\bar{c}$. Following our nose (using the previous proof) it seems natural to try to move in the direction

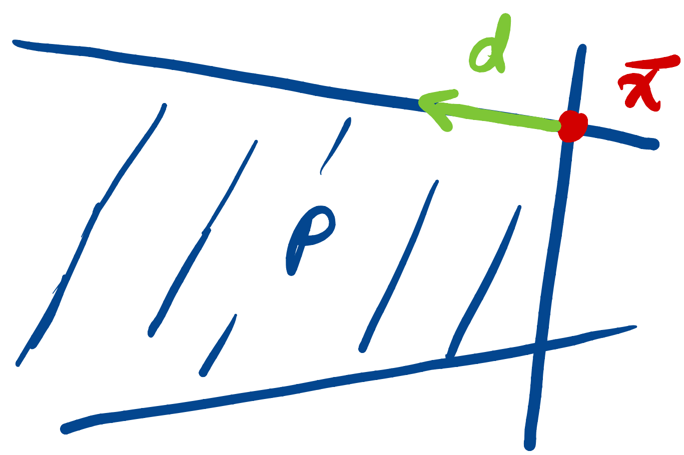
$$x(\varepsilon) = \bar{x} + \varepsilon d \quad \text{with}$$

$$\begin{bmatrix} d_B \\ d_j \\ d_{B^c \setminus j} \end{bmatrix} = \begin{bmatrix} -A_B^{-1} A_j \\ 1 \\ 0 \end{bmatrix}$$ where $\bar{c}_j < 0$.

There are three potential situations:

- **Unbounded case**

If $d \geq 0$, we have a situation like:



By construction $A x(\varepsilon) = b$ and $x(\varepsilon) \geq 0 \ \forall \varepsilon$.

Thus,

$$c^T x(\varepsilon) = c^T \bar{x} + c_j \varepsilon \to -\infty \text{ as } \varepsilon \uparrow \infty.$$

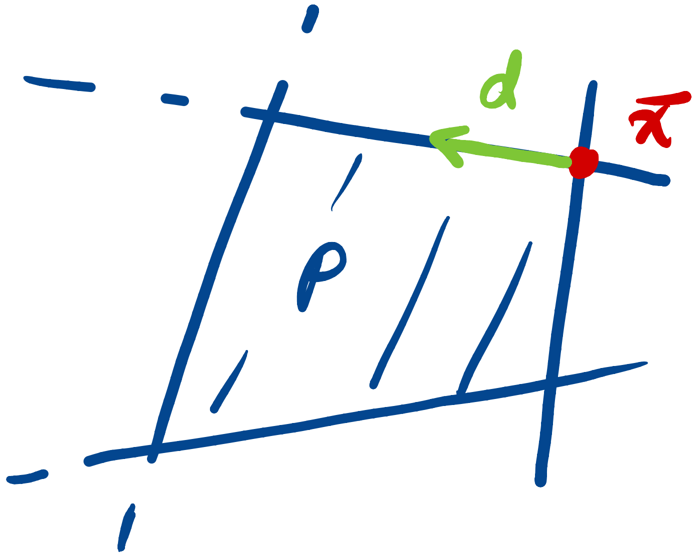
- **Bounded and nondegenerate case**

If $\exists i$ s.t. $d_i < 0$ and $x_B > 0$,

$\Rightarrow$ $x(\varepsilon)$ violates $x(\varepsilon)_i \geq 0$

if, and only if, $\bar{x}_i + \varepsilon d_i < 0$.

$\left(\varepsilon > -\frac{\bar{x}_i}{d_i} \right)$



So we can take

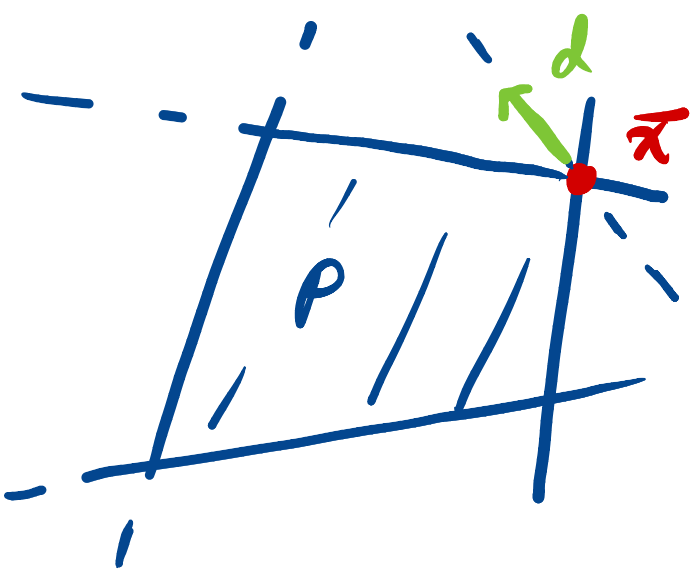
$$\varepsilon^* = \min_{i \in B} \left\{ -\frac{\bar{x}_i}{d_i} \,\middle|\, d_i < 0 \right\}$$

and

$$i^* \in \arg\min_{i \in B} \left\{ -\frac{\bar{x}_i}{d_i} \,\middle|\, d_i < 0 \right\}.$$

▷ Bounded and degenerate

If $\exists i$ s.t. $d_i < 0$ and $\bar{x}$ is degenerate.



Then, we can have that $\varepsilon^* = 0$. In which case we should take a different d.

Lemma: Pick $j \in B^c$ if $d \ngtr 0$,

then $x(\varepsilon^*)$ is a BFS with associated basis $B' = B \cup \{j\} \setminus \{i^*\}$.

Proof: $x(\varepsilon^*)$ solves

$$\begin{cases} Ax = b \\ x_{\bar{B}'} = 0 \end{cases} \Leftrightarrow \begin{cases} A_{B'} x_{B'} = b \\ x_{\bar{B}'} = 0. \end{cases}$$

We need to show that $A_{B'}$ is inverible. Note that

$$A_{B'} = \begin{bmatrix} | & & | & & | \\ A_{B_1} & \cdots & A_j & \cdots & A_{B_m} \\ | & & | & & | \end{bmatrix}$$

replaced A_{i^*} (assume it was in the kth column)

$$= A_B + (A_j - A_{i^*}) e_k^T$$

Sherman-Morrison states this is invertible if A_B is invertible ✓ and $1 + e_k^T A_B^{-1}(A_j - A_{i^*}) \neq 0$.

To check the last condition note

$$1 + e_k^T A_B^{-1}(A_j - A_{i^*}) = 1 + e_k^T(-d_B - e_k)$$

$= -d_{i^*} > 0.$

This completes the proof. $\square$

Finishing

With what we learned we can write a complete simplex method

Simplex method

- Start with a BFS $x(B_0)$ associated to B_0 (Use Phase 1)
- Loop for $k = 0, 1, \ldots$
  - Compute dual solution
    $$y(B_k) = A_{B_k}^{-T} c_{B_k}.$$
  - If $y(B_k)$ is feasible ($\bar{c} \geq 0$) return $x(B_k)$ and $y(B_k)$.
  - Else
    - Pick any j with $\bar{c}_j < 0$ and compute
      $$d = \begin{pmatrix} -A_{B_k}^{-1} A_j \\ 1 \\ 0 \end{pmatrix}.$$

- If $d \geq 0$
  Return "unbounded LP."

- Else set
  $B_{k+1} = B_k \cup \{j\} \setminus \{i\}$
  with $i \in \operatorname{argmin}\{-\frac{x(B_k)_i}{d_i} \mid d_i < 0\}$.

Q: How do we know that the loop stops?

In order to guarantee that we need to use Bland's rule:

- Pick $j \in B^c$ with $\bar{c}_j < 0$ to be the smallest such index.

- Pick i the smallest index in $\operatorname{argmin}\{-\frac{x(B_k)_i}{d_i} \mid d_i < 0\}$.

Using this rule, simplex always finishes. We will not prove that in this class.

Q. How fast is simplex?

- In the worst case it can take exponential time in the dimension (Klee and Minty '72).

- For random problems it takes $O(n+m)$ iterations on average (Borgwardt '87, Smale '83)

- For randomly perturbed problems, it finishes after poly(n, m) many iterations.
  (Smoothed Analysis, Spielman & Tang '04)