Probability Theory 2, Spring 2024 - Homework 3
Due one hour before lecture on 3/11 (Gradescope)

Your submitted solutions to assignments should be your own work. While discussing homework problems with peers is permitted, the final work and implementation of any discussed ideas must be executed solely by you. Acknowledge any source you consult.

Problem 1 - Things we didn't prove in class

Establish the following facts.

(a) Show that if $X_1 = X_2$ on an event $B \in \mathcal{F}$, then $\mathbb{E}[X_1|\mathcal{F}] = \mathbb{E}[X_2 \mid \mathcal{F}]$ a.s. on the envent B.

(b) Recall the branching process example where ξ_k^n are iid random variable supported on $\mathbb{N}$, and $Z_0 = 1$ and

$$Z_n = \begin{cases} \xi_1^n + \cdots + \xi_{Z_{n-1}}^n & \text{if } Z_{n-1} > 0 \\ 0 & \text{if } Z_{n-1} = 0. \end{cases}$$

Show that $\mathbb{E}[Z_n \mid Z_{n-1}] = \sum_{i=1}^{Z_{n-1}} \xi_i$.

(c) Show that if X_n is a martingale, then $Y_n = |X_n|$ is a submartingale.

Problem 2 - No uniform integrability

Consider the martingale $S_0 = 1$, $S_n = \sum_{k=1}^n \xi_k + S_0$ where ξ_k is iid with $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 1/2$. Define the stopping time $T = \inf\{n \mid S_n = 0\}$.

(a) Show that $\mathbb{E}S_{T\wedge n}\mathbf{1}_{\{T>n\}} = 1$.

(b) Prove that $\mathbb{E}S_{T\wedge n}\mathbf{1}_{\{T>n\}} \leq M\mathbb{P}(T > n) + \mathbb{E}(S_{T\wedge n}\mathbf{1}_{\{X_{T\wedge n}\geq M\}})$.

(c) Leverage (a) and (b) to establish that $S_{T\wedge n}$ is not uniformly integrable.

Problem 3 - Optional stopping

Let $S_n = \xi_1 + \cdots + \xi_n$ be a random walk with $S_0 = 0$, i.e., ξ_k is an iid that are not constant. Suppose $\mathbb{E}\exp(\theta\xi_1) = 1$ for some $\theta < 0$. Prove that $X_n = \exp(\theta S_n)$ is a martingale. Let $T = \inf\{n \mid S_n \in (a, b)\}$, show that $\mathbb{E}X_T = 1$ and $\mathbb{P}(S_T \leq a) \leq \exp(-\theta a)$.

Problem 4 - Almost supermartingale convergence

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^\infty, \mathbb{P})$ be a filtered probability space. Let Z_n, a_n, b_n, and c_n be positive adapted stochastic processes such that

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] \leq (1 - a_n + b_n)Z_n + c_n.$$

Further, assume that almost surely

$$\sum_{n=0}^\infty a_n = \infty, \quad \sum_{n=0}^\infty b_n < \infty, \quad \text{and} \quad \sum_{n=0}^\infty c_n < \infty.$$

In this question, you'll have to prove that $\lim_{n\to\infty} Z_n = 0$ almost surely. To this end, you'll need to show the following.

(a) Show that without loss of generality that we can assume $b_n = 0$. Hint: Multiply the inequality above by $\left(\prod_{k=0}^{n}(1+b_k)\right)^{-1}$.

(b) From now on assume that $b_n = 0$, show that

$$Y_n = Z_n + \sum_{k=0}^{n-1} a_k Z_k - \sum_{k=0}^{n-1} c_k$$

is a supermartingale.

(c) For any $C > 0$ define the stopping time $T_C = \inf\{n \mid \sum_{k=0}^{n} c_k \geq C\}$. Show that $Y_{T_C \wedge n}$ converges almost surely. Use the fact that $\sum_{n=0}^{\infty} c_n < \infty$ to conclude that Y_n converges almost surely

(d) Use the fact that $Z_n - Y_n \geq -Y_n$ to show that Z_n almost surely converges to zero.