Mateo Larrea
Signal Processing
HW 1

1) $x(t) = 2\sin\left(w_0 t - \frac{\pi}{4}\right) + \cos(w_0 t)$

a) $x(t) = A\cos(w_0 t + \phi)$, where $\boxed{\phi \in [-\pi, \pi)}$ $\boxed{A > 0}$ $\boxed{\text{in radians}}$

let $w_0 t = \alpha$

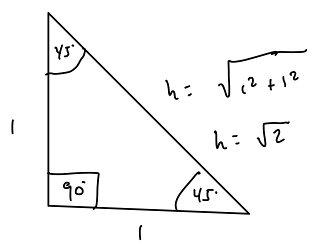
$\hookrightarrow x(t) = 2\sin\left(\alpha - \frac{\pi}{4}\right) + \cos(\alpha)$

- sine subtraction $\rightarrow \sin(u - v) = \sin u \cos v - \cos u \sin v$

$u = \alpha$ & $v = \frac{\pi}{4}$

$$\sin\left(\alpha - \frac{\pi}{4}\right) = \sin\alpha\cos\left(\frac{\pi}{4}\right) - \cos\alpha\sin\left(\frac{\pi}{4}\right)$$

$\downarrow$ 45° $\qquad$ $\downarrow$ 45°



$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$\sin\left(\alpha - \frac{\pi}{4}\right) = \sin\alpha \cdot \frac{1}{\sqrt{2}} - \cos\alpha \cdot \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} \rightarrow \frac{\sqrt{2}}{(\sqrt{2})^2} \rightarrow \frac{\sqrt{2}}{2}$$

$$\sin\left(\alpha - \frac{\pi}{4}\right) = \sin\alpha \cdot \frac{\sqrt{2}}{2} - \cos\alpha \cdot \frac{\sqrt{2}}{2}$$

$$2\sin\left(\alpha - \frac{\pi}{4}\right) = 2\left[\sin\alpha \cdot \frac{\sqrt{2}}{\cancel{2}} - \cos\alpha \cdot \frac{\sqrt{2}}{\cancel{2}}\right]$$

$$2\sin\left(\alpha - \frac{\pi}{4}\right) = \sqrt{2}\sin\alpha - \sqrt{2}\cos\alpha$$

$$x(t) = \left(\sqrt{2}\sin\alpha - \sqrt{2}\cos\alpha\right) + \cos\alpha$$

* cosine terms: $-\sqrt{2}\cos\alpha + \cos\alpha \rightarrow 1 - \sqrt{2}\cos\alpha$

$$x(t) = (1-\sqrt{2})\cos\alpha + \sqrt{2}\sin\alpha$$

we want to write
$x(t) = A\cos(\alpha+\varphi)$

* cosine addition formula: $\cos(\alpha+\varphi) = \cos\alpha\cos\varphi - \sin\alpha\sin\varphi$

-> amplitude

$$A\cos(\alpha+\varphi) = (A\cos\varphi)\cos\alpha + (-A\sin\varphi)\sin\alpha$$

cosine part = $A\cos\varphi = 1-\sqrt{2}$

sine part = $-A\sin\varphi = \sqrt{2} \Rightarrow A\sin\varphi = -\sqrt{2}$

$$\begin{cases} A\cos\varphi = 1-\sqrt{2} \\ A\sin\varphi = -\sqrt{2} \end{cases}$$

$$x(t) = (A\cos\varphi)\cos\alpha + (-A\sin\varphi)\sin\alpha$$

* Solve for Amplitude

↳ $\cos^2\varphi + \sin^2\varphi = 1$

$$(A\cos\varphi)^2 + (A\sin\varphi)^2 = A^2$$

$$(1-\sqrt{2})^2 + (-\sqrt{2})^2 = A^2$$

$$1^2 - 2(1)(\sqrt{2}) + (\sqrt{2})^2$$

$$1 - 2\sqrt{2} + 2 \Rightarrow \boxed{3 - 2\sqrt{2}}$$

$$\boxed{2}$$

$$3 - 2\sqrt{2} + 2 = A^2$$

$$5 - 2\sqrt{2} = A^2 \rightarrow \boxed{A = \sqrt{5 - 2\sqrt{2}}}$$

$$\boxed{A = 1.4736}$$

$$x(t) = \underbrace{\left(\sqrt{5-2\sqrt{2}}\right)}_{A} \cos(\alpha + \varphi), \quad \alpha = \omega_0 t$$

* Isolate $\cos\alpha$ & $\sin\alpha$

$$\cos\varphi = \frac{1-\sqrt{2}}{A}, \quad \sin\varphi = \frac{-\sqrt{2}}{A}$$

$\cos\varphi \to$ negative, $\sin\varphi \to$ negative

φ is in quadrant 3

$$\Rightarrow \varphi \in \left(-\pi, -\frac{\pi}{2}\right)$$

* find angle size

$y = \sin\varphi$

$x = \cos\varphi$

$$\varphi = \operatorname{atan2}(y, x) =$$

$$\operatorname{atan2}\left(\frac{-\sqrt{2}}{A}, \frac{1-\sqrt{2}}{A}\right) =$$

$$\operatorname{atan2}(-\sqrt{2}, 1-\sqrt{2})$$

$$\boxed{\varphi = -1.8566 \text{ rad}}$$

$$\boxed{x(t) = 1.4736\cos(\omega_0 t - 1.856)}$$

b) The result does not depend on a special property. It's a general property of sine waves that have the same frequency. They can combine even if they have different amplitudes and phases.

if same $\boxed{w_0}$ they can be expressed as a single sinusoid

↳ otherwise = they become a more complex signal (with beats or interference)

c) complex valued signal $\bar{x}(t)$ such that $x(t) = Re\{\hat{x}(t)\}$

real signal

$$x(t) = (1-\sqrt{2})\cos\alpha + \sqrt{2}\sin\alpha, \quad \alpha = w_0 t$$

* find $\bar{x}(t)$ so that $R\{\bar{x}(t)\} = (1-\sqrt{2})\cos\alpha + \sqrt{2}\sin\alpha$

euler → to relate sine/cos to complex exponentials

$$e^{ja} = \cos\alpha + j\sin\alpha$$

↳ we can represent sin/cos with e^{ja}

$$\bar{x}(t) = Ce^{ja}, \text{ where } C \in \mathbb{C} \text{ (complex constant)}$$

$$\bar{x}(t) = (C_r + jC_i)e^{ja}$$

$e^{j\alpha} = \cos\alpha + j\sin\alpha$

$$\bar{x}(t) = (c_r + jc_i)(\cos\alpha + j\sin\alpha)$$

$$(jc_i)(j\sin\alpha) = j^2 c_i \sin\alpha = -c_i \sin\alpha$$

$$\bar{x}(t) = \underbrace{(c_r\cos\alpha - c_i\sin\alpha)}_{\text{real}} + j\underbrace{(c_r\sin\alpha + c_i\cos\alpha)}_{\text{imaginary}}$$

$$\mathcal{R}\{\bar{x}(t)\} = c_r\cos\alpha - c_i\sin\alpha$$

* matching real to $x(t)$

$$c_r\cos\alpha - c_i\sin\alpha = (1-\sqrt{2})\cos\alpha + \sqrt{2}\sin\alpha$$

$$c_r = 1-\sqrt{2}$$

$$-c_i = \sqrt{2}$$

$$c_i = -\sqrt{2}$$

* complex constant

$$C = C_r + jC_i = (1-\sqrt{2}) + j(-\sqrt{2})$$

$$C = (1-\sqrt{2}) - j\sqrt{2}$$

* $\tilde{x}(t)$

$$\hat{x}(t) = C e^{j\alpha} = ((1-\sqrt{2}) - j\sqrt{2}) e^{j\omega_0 t}$$

$$R\{(1-\sqrt{2})(\cos\alpha + j\sin\alpha) - j\sqrt{2}(\cos\alpha + j\sin\alpha)\}$$

$$\boxed{\tilde{x}(t) = ((1-\sqrt{2}) - j\sqrt{2}) e^{j\omega_0 t}, \quad x(t) = R\{\hat{x}(t)\}}$$

2) $x(t) = A\cos(2\pi f_0 t + \phi) =$

$A\cos(2\pi f_0(t - t_1))$

$T_0 = 12$ seconds

$$x(t) = A\cos(2\pi f_0 - 2\pi f_0 t_1)$$

* match inside of the cosine

$$\cancel{2\pi f_0 t} + \varphi = \cancel{2\pi f_0 t} - 2\pi f_0 t_1$$

$$\boxed{\varphi = -2\pi f_0 t_1}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{12} \text{ Hz}$$

$$\varphi = -2\pi\left(\frac{1}{12}\right)t_1 = -\frac{\pi}{6}t_1$$

for every time shift the corresponding phase shift is: $\varphi = -\frac{\pi}{6} t_1$

case $t_1 = -3$ sec

$$\varphi = -\frac{\pi}{6}(-3) = \frac{\cancel{3}\pi}{\cancel{6}_{2}} \rightarrow \boxed{\frac{\pi}{2}}$$

positive since signal moved left (advancing)

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case $t_1 = 0$

$$\varphi = -\frac{\pi}{6}(0) = \boxed{0}$$

Case $t_1 = 3 \sec$

$$\varphi = -\frac{\pi}{6}(3) = \frac{-3\pi}{6} = \boxed{-\frac{\pi}{2}}$$

(phase decreases)

Case $t_1 = 7 \sec$

$$\varphi = -\frac{\pi}{6}(7) = \boxed{-\frac{7\pi}{6}}$$

to make inside $[-\pi, \pi]$

$$-\frac{7\pi}{6} + \overset{\text{full cycle}}{2\pi} = -\frac{7\pi}{6} + \frac{12\pi}{6} =$$

$$\boxed{\varphi = \frac{5\pi}{6}}$$

3.

a) sine with period $T_0 = 1/10\,s$

freq f_0? (Hz)

freq ω_0? (radians/second)

$$\boxed{f_0 = \frac{1}{T_0}} \leftarrow \text{formula}$$

$$T_0 = \frac{1}{10} \text{ seconds}$$

$$\left(\frac{\frac{1}{1}}{\frac{1}{10}}\right) \rightarrow \boxed{10}$$

$$\boxed{f_0 = 10 \text{ Hz}}$$

$$\boxed{\omega_0 = 2\pi(10) = 20\pi \text{ rad/seconds}}$$

$\hookrightarrow$ 62.83 rad/s

b)

$$x(t) = A\sin[\omega(t-T)]$$

$$x(t) = A\sin[(\omega t - \omega T)]$$

standard sine $\rightarrow$ $x(t) = A\sin(\omega t + \phi)$

$$\omega t + \phi = \omega t - \omega T$$

$$\boxed{\phi = -\omega T}$$

c) $\boxed{\phi = -\omega T}$

$$\boxed{\begin{aligned} T &= 0.25 \text{ seconds} \\ f &= 10 \text{ Hz} \\ \omega &= 20 \text{ rad/s} \end{aligned}}$$

$\phi = -20\pi(0.25)$

$\boxed{\phi = -5\pi \text{ radians}}$

* adjust to standard range $[-\pi, \pi]$

$-5\pi + 2\pi = -3\pi$

$-3\pi + 2\pi = -\pi$

at $t = 0$

$$x(0) = A\sin[-\omega T] = A\sin(-5\pi)$$

$$\phi(t=0) = -\omega T$$

$$= -(20\pi)(0.25) = -5\pi \text{ radians}$$

$$= -\pi \text{ radians } (\text{mod } 2\pi)$$

4) AM

$$x(t) = [12 + 7\sin(\pi t - \pi/3)]\cos(13\pi t)$$

as

$$x(t) = A_1\cos(\omega_1 t + \varphi_1) + A_2\cos(\omega_2 t + \varphi_2) + A_3\cos(\omega_3 t + \varphi_3),$$ with $\omega_1 < \omega_2 < \omega_2$

$$x(t) = 12\cos(13\pi t) + 7\sin(\pi t - \pi/3)\cos(13\pi t)$$

* euler → complex sines

$$\boxed{\sin a = \frac{e^{ja} - e^{-ja}}{2j}, \quad \cos b = \frac{e^{jb} + e^{-jb}}{2}}$$

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$a = \pi t - \frac{\pi}{3}$ $\qquad b = 13\pi t$

$a + b = \left(\pi t - \frac{\pi}{3}\right) + 13\pi t = \boxed{14\pi t - \frac{\pi}{3}}$

$a - b = \left(\pi t - \frac{\pi}{3}\right) - 13\pi t = \boxed{-12\pi t - \frac{\pi}{3}}$

$\rightarrow \sin\left(\pi t - \frac{\pi}{3}\right)\cos\left(13\pi t\right) = \frac{1}{2}\left[\sin\left(14\pi t - \frac{\pi}{3}\right)\right.$

$\left. + \sin\left(-12\pi t - \frac{\pi}{3}\right)\right]$

$\rightarrow \sin\left(-12\pi t - \frac{\pi}{3}\right) = -\sin\left(12\pi t + \frac{\pi}{3}\right)$

$\rightarrow x(t) = 12\cos\left(13\pi t\right) + \frac{7}{2}\sin\left(14\pi t - \frac{\pi}{3}\right) - \frac{7}{2}$
$\sin\left(12\pi t + \frac{\pi}{3}\right)$

$\omega_1 = 12\pi, \ \omega_2 = 13\pi, \ \omega_3 = 14\pi$

* convert to cosine with phase

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

$$\sin\left(14\pi t - \frac{\pi}{3}\right) = \cos\left(\frac{14\pi t}{1} - \frac{\pi}{3} - \frac{\pi}{2}\right)$$

$$\boxed{\frac{-2\pi}{6} - \frac{3\pi}{6} = -\frac{5\pi}{6}}$$

$$\boxed{\sin\left(14\pi t - \frac{\pi}{3}\right) = \cos\left(14\pi t - \frac{5\pi}{6}\right)}$$

$$\sin\left(12\pi t + \frac{\pi}{3}\right) = \cos\left(12\pi t + \frac{\pi}{3} - \frac{\pi}{2}\right)$$

$$\boxed{\sin\left(12\pi t + \frac{\pi}{3}\right) = \cos\left(12\pi t - \frac{\pi}{6}\right)}$$

$$x(t) = 12\cos(13\pi t) + \frac{7}{2}\cos\left(14\pi t - \frac{5\pi}{6}\right) - \frac{7}{2}\cos\left(12\pi t - \frac{\pi}{6}\right)$$

$$-\frac{7}{2}\cos\left(12\pi t - \frac{\pi}{6}\right) = \frac{7}{2}\cos\left(12\pi t - \frac{\pi}{6} + \pi\right)$$

$$\hookrightarrow \frac{7}{2}\cos\left(12\pi t - \frac{5\pi}{6}\right) \checkmark$$

$$\hookrightarrow \boxed{\frac{7}{2}\cos\left(12\pi t + \frac{5\pi}{6}\right) + 12\cos(13\pi t) + \frac{7}{2}\cos\left(14\pi t - \frac{5\pi}{6}\right)}$$

low sideband (lowest freq)

$\boxed{w_1 = 12\pi \quad A_1 = \frac{7}{2} \quad \varphi = +\frac{5\pi}{6}}$

$\boxed{w_2 = 13\pi t \quad A_2 = 12 \quad \varphi = 0}$ ← carrier

$\boxed{w_3 = 14\pi t \quad A_3 = \frac{7}{2} \quad \varphi_3 = -\frac{5\pi}{6}}$ highest freq

$$\boxed{x(t) = \frac{7}{2}\cos\left(12\pi + \frac{5\pi}{6}\right) + 12\cos\left(13\pi\right) + \frac{7}{2}\cos\left(14\pi t - \frac{5\pi}{6}\right)}$$

* using euler

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right), \quad \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\hookrightarrow$$

$$x(t) = 12\cos(13\pi t) + 7\sin\left(\pi t - \frac{\pi}{3}\right)\cos(13\pi t)$$

$$\hookrightarrow 6\left(e^{j13\pi t} + e^{-j13\pi t}\right) + \frac{7}{4j}\left(e^{j(\pi t - \pi/3)} - e^{-j(\pi t - \pi/3)}\right)\left(e^{j13\pi t} + e^{-j13\pi t}\right)$$

* the carrier:

$$12\cos(13\pi t) = 6e^{j13\pi t} + 6e^{-j13\pi t}$$

$$\boxed{\omega = 13\pi}$$

* $7\sin(\cdot)\cos(\cdot)$

$$7\sin(\cdot)\cos(\cdot) = \frac{7}{4j}\left(e^{j(\pi t - \pi/3)} - e^{-j(\pi t - \pi/3)}\right)\left(e^{j13\pi t} + e^{-j13\pi t}\right)$$

$$\boxed{e^{j(\pi t - \pi/3)} \cdot e^{j13\pi t} = e^{j(14\pi t - \pi/3)} = 14\pi}$$

$$\boxed{e^{j(\pi t - \pi/3)} \cdot e^{-j13\pi t} = e^{j(-12\pi t - \pi/3)} = -12\pi}$$

$$\boxed{-e^{-j(\pi t - \pi/3)} \cdot e^{j13\pi t} = -e^{j(12\pi t + \pi/3)} = 12\pi}$$

$$\boxed{-e^{-j(\pi t - \pi/3)} \cdot e^{-j13\pi t} = -e^{j(-14\pi t + \pi/3)} = -14\pi}$$

$$C e^{j\Omega t} + \bar{C} e^{-j\Omega t} = 2|C| \cos(\Omega t + \arg C)$$

* coefficient of $e^{+j14\Omega t}$:

$$C_{14} = \frac{7}{4j} e^{-j\pi/3} = -\frac{7j}{4} e^{-j\pi/3}$$

* coefficient of $e^{-j14\Omega t}$

$$\bar{C}_{14} = \frac{7}{4j}\left(-e^{+j\pi/3}\right) = \frac{7j}{4} e^{+j\pi/3}$$

* magnitude & angle

$$|C_{14}| = \frac{7}{4}, \quad \arg(C_{14}) = \arg(-j) + (-\pi/3)$$

$$\hookrightarrow -\frac{\pi}{2} - \frac{\pi}{3} = \boxed{-\frac{5\pi}{6}}$$

$\hookrightarrow$ into cosine

$$C_{14} e^{j14\Omega t} + \bar{C}_{14} e^{-j14\Omega t} = 2|C_{14}| \cos(14\Omega t + \arg C_{14}) = \frac{7}{2} \cos\left(14\Omega t - \frac{5\pi}{6}\right)$$

↳ high band:

$$\boxed{\omega_3 = 14\pi, \quad A_3 = \frac{7}{2}, \quad \varphi_3 = -\frac{5\pi}{6}}$$

* low sideband: $\Omega = 12\pi$

* coefficient of $e^{-j12\pi t}$ comes from $-e^{j(12\pi t + \pi/3)}$

$$C_{12} = \frac{7}{4j} \cdot \left(-e^{j\pi/3}\right) = \frac{7j}{4} e^{j\pi/3}$$

· coefficient of $e^{-j12\pi t}$ comes from $e^{j(-12\pi t - \pi/3)}$

$$\bar{C}_{12} = \frac{7}{4j} e^{-j\pi/3} = -\frac{7j}{4} e^{-j\pi/3}$$

· magnitude & angle

$$|C_{12}| = \frac{7}{4}, \quad \arg(C_{12}) = \arg(j) + \frac{\pi}{3} = \frac{\pi}{2} + \frac{\pi}{3} = \boxed{\frac{5\pi}{6}}$$

$\hookrightarrow$ into cosine

$$C_{12} e^{j12\pi t} + C_{12} e^{-j12\pi t} = 2|C_{12}| \cos$$

$$(12\pi t + \arg(C_{12})) = \frac{7}{2} \cos\left(12\pi t + \frac{5\pi}{6}\right)$$

low sideband

$$\boxed{\omega_1 = 12\pi, \; A_1 = \frac{7}{2}, \; \varphi_1 = \frac{5\pi}{6}}$$

* carrier: $\Omega = 13\pi$

$$6e^{j13\pi t} + 6e^{-j13\pi t} = 12\cos(13\pi t)$$

$\hookrightarrow$ $\boxed{\omega_2 = 13\pi, \; A_2 = 12, \; \varphi_2 = 0}$

$$x(t) = \frac{7}{2}\cos\left(12\pi t + \frac{5\pi}{6}\right) + 12\cos(13\pi t) + \frac{7}{2}\cos\left(14\pi t - \frac{5\pi}{6}\right),$$

$$(\omega_1, A_1, \varphi_1) = \left(12\pi, \frac{7}{2}, +\frac{5\pi}{6}\right),$$

$$(\omega_2, A_2, \varphi_2) = (13\pi, 12, 0),$$

$$(\omega_3, A_3, \varphi_3) = \left(14\pi, \frac{7}{2}, -\frac{5\pi}{6}\right)$$

5) e^x, $\cos(x)$, $\sin(x)$ for x near 0

written as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

using terms up to x^{10}

$\hookrightarrow$ $e^{jx} = \cos(x) + j\sin(x)$

$x \to jx$

$$e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!}$$

$n = 0$ to 10

$$e^{jx} = \frac{(jx)^0}{0!} + \frac{(jx)^1}{1!} + \frac{(jx)^2}{2!} \cdots \frac{(jx)^{10}}{10!}$$

$(jx)^n = j^n x^n \to$ cycle of j

$n=0 : (jx)^0 = 1$ ✓

$n=1 = (jx)^1 = jx$ ✓

$n=2 = (jx)^2 : \cancel{jx} \; -x^2$

$n=3 : (jx)^3 = j^2 x^2 = -jx^3$

$n=4 : (jx)^4 = x^4$

$n=5 : jx^5$

$n=6 = -x^6$

$n=7 = -jx^7$

$n=8 : j^8 x^8 = x^8$

$n=9 : jx^9$

$n=10 = -x^{10}$

$$e^{jx} \approx \frac{1}{0!} + \frac{jx}{1!} + \frac{-x^2}{2!} + \frac{-jx^3}{3!} + \frac{x^4}{4!} + \frac{jx^5}{5!} + \frac{-x^6}{6!} + \frac{-jx^7}{7!} + \frac{x^8}{8!} + \frac{jx^9}{9!} + \frac{-x^{10}}{10!}$$

* real vs. imaginary parts

real:

$$\boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}}$$

imaginary:

$$j\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right)$$

through x^{10}:

$$\boxed{\begin{aligned} e^{jx} = &\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}\right) + \\ &j\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right)\end{aligned}}$$

* Cosine & sine patterns

$$= \cos x + j\sin x$$

(through degree x^{10})

$$\boxed{e^{jx} = \cos x + j\sin x}$$