QUIZ 6
THEORY OF COMPUTATION (DECIDABILITY AND COMPUTABILITY / COMPUTATIONAL COMPLEXITY)

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In the following first 2 questions, a tile is a unit square tile with four edges: North, South, East, West. Each edge is colored by a color between 1 and C. Given a set T of allowed tiles, a tiling is a way to place tiles so that adjacent edges of tiles are colored with the same color. Tiles may be used more than once or none at all, but they are not allowed to be rotated.
Question 1. Prove that the decision problem of finding if a set of tiles can cover completely $\mathbb{Z}^2$ is undecidable. (3 points)
R:
It is enough to show that there exists a sub-set of tile sets for which the Problem is undecidable.
Consider the tile set, including both the starting tiles and the Turing tiles. For each Turing machine there exists a corresponding tile set of this form.
A Turing machine halts if and only if the corresponding tile set doesn't admit a valid tiling of the plane. Since, Any Turing machine either halts or does not halt. Assume the Turing machine halts, then it will have reached the halting state. Recall that there does not exist a merging tile for the halting state. So if the Turing machine reaches the halting state by a given action tile then there is no merging tile that can be placed to the left (or right, respectively) side of it. Since no tile can be placed in this position the tiling cannot be completed and a valid tiling of the plane does not exist. Assume the Turing machine does not halt, then the halting state is never reached. For any other state that is reached there exists a corresponding merging tile that may be placed next to it so that row of the tiling can be completed. Since the halting state isn't reached each row of the tiling can be completed, and hence a valid tiling of the plane exists.

So a valid tiling of the plane by the corresponding tile set exists if and only if the Turing machine doesn't halt.

The problem of the decision is therefore reduced to the Halting Problem and this is undecidable.

Question 2: Prove that deciding whether a set of tiles can form an NxN tiling is NP-complete (3 points).
R:
Let A be an arbitrary set in NP and let U be so. me Turing machine nondeterministically accepting A in time $k.n^k$. Consider the set of tiles which encodes the computations of U

as described above. Let x be an arbitrary input string. We transform this string x to the following instance of BOUNDED TILING:
Let $N = k.|x|^k$; consider the N $\times$ N square with the following colouring on its border; on top the initial configuration q_0x is encoded, extended with $N - |x|$ blank symbols; the left and right border are all white, whereas on the bottom the unique accepting configuration is required. Now a tiling of this square corresponds with an accepting computation, hence x belongs to A if and only if the instance of BOUNDED TILING as described is solvable. Therefore, NP-Complete membership is evident.

In the following question, a vertex cover of an undirected graph $G = (V, E)$ is a subset $S \subset V$ such that for any edge $(u, v) \in E$, either $u \in S$ or $v \in S$ (or both).

Question 3 Prove that given a graph G and an integer k, deciding whether G has a vertex cover of size k is NP-complete. (4 points).

Proof: We can reduce a know problem that is know by being NP-complete to our vertex cover, we can take the independent set which es NP-complete.
So,
If G = (V, E) is a graph, then S is an independent set $\iff$ V - S is a vertex cover.

$\implies$ Suppose S is an independent set, and let $e = (u, v)$ be some edge. Only one of u, v can be in S. Hence, at least one of u, v is in $V - S$. So, $V - S$ is a vertex cover.
$\impliedby$ Suppose $V - S$ is a vertex cover, and let $u, v \in S$. There can't be an edge between u and v. So, S is an independent set. Then,

$$\text{Independent Set} \leq p \text{ Vertex Cover}$$

We must show this, so, we change any instance of Independent Set into an instance of Vertex Cover:

- Given an instance of Independent Set $\langle G, k \rangle$.
- We see in the Vertex Cover if there is a vertex cover $V - S$ of size $\leq |V| - k$

By we state before, S is an independent set iff $V - S$ is a vertex cover. If the Vertex Cover says:

- yes: then S must be an independent set of size $\geq k$.
- no: then there is no vertex cover $V - S$ of size $\leq |V| - k$, hence there is no independent set of size $\geq k$

Actually, we also have:

$$\text{Vertex Cover} \leq_P \text{Independent Set}$$

Thus, to decide if G has an vertex cover of size k, we ask if it has an independent set of size $n - k$. So: VERTEX COVER and INDEPENDENT SET are equivalently, thefore these are NP-complete.