Defining Turing machines

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The idea

Imagine a person, equipped with pen and an unlimited amount of paper. What kind of calculations could you have this person perform for you, if you need to fully specify what they ought to do?

Similarities and differences to pushdown automata

- We have a double-linked list as data structure instead of a stack (we call this list *tape*).
- The input is initially written on the tape (rather than being fed from the outside symbol by symbol).
- A Turing machine needs to be explicit about when it is done.

Formal definition

Definition

A Turing machine with tape alphabet $\Sigma' \supseteq \Sigma \cup \{\bot\}$[1] is specified by a set of states Q including three special states $q_0, q_{\text{yes}}, q_{\text{no}}$ and a transition function $\delta : Q \times \Sigma' \to Q \times \Sigma' \times \{\mathrm{L}, \mathrm{R}, \mathrm{S}\}$.

The idea behind the transition function is that we look at our current state and the symbol at the active position in the list. We then move to a new state, and change the current list position to a new symbol. We also go **l**eft or **r**ight on the list, or **s**tay where we are.

[1] This means that the tape alphabet Σ' contains every symbol from our input alphabet Σ, as well as a special bottom symbol $\bot$, and maybe additional stuff.

Configurations

- A configuration of a Turing machine is a triple $(q, k, p) \in Q \times \mathbb{Z} \times (\Sigma')^{\mathbb{Z}}$.
- If the input is $w \in \Sigma^*$, then the initial configuration is $(q_0, 0, p_w)$ where $p_w(i) = w(i)$ for $0 \leq i < |w|$, and $p_w(i) = \perp$ otherwise (i.e. w is written on the tape, we look at its first letter, and elsewhere we have the special $\perp$ symbol)
- If our current configuration is (q, k, p) we look at $\delta(q, p(k)) = (q', x, d)$ to determine what is next. The next configuration is (q', k', p') where $p'(k) = x$ and $p'(i) = p(i)$ for $i \neq k$, and $k' = k + 1$ if $d = \mathrm{R}$, $k' = k - 1$ if $d = \mathrm{L}$ and $k' = k$ if $d = \mathrm{S}$.
- If the current state is either q_{yes} or q_{no}, the computation ends and outputs yes or no respectively.