Counting with infinities

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Pre-preview

Claim: There are more formal languages than there are grammars.

Injections, surjections and bijections

Definition

We call a function $f : X \to Y$

- **an injection** if $\forall x_1, x_2 \in X \; f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- **plain reading** every element in Y is the image of at most element from X
- **a surjection** if $\forall y \in Y \exists x \in X \; f(x) = y$
- **plain reading** every element in Y is the image of some element from X
- **a bijection** if it is both an injection and a surjection

Connection to cardinality

Let $\mathbf{n} = \{0, 1, \ldots, n-1\}$. Realize that

- There is an injection from $\mathbf{n}$ to X if and only if X has at least n elements.
- There is a surjection from $\mathbf{n}$ to X if and only if X has at most n elements.
- There is a bijection from $\mathbf{n}$ to X if and only if X has exactly n elements.

Some facts

1. There is an injection from X to Y if and only if there is a surjection from Y to X.
2. If there is no injection from X to Y, then there is an injection from Y to X.
3. If there is an injection from X to Y and an injection from Y to X, then there is a bijection from X to Y.

Cardinality

Definition

We write $|X| \leq |Y|$ if there is a surjection from Y to X, and $|X| = |Y|$ if there is a bijection from X to Y.

- For any two sets, $|X| \leq |Y|$ or $|Y| \leq |X|$.
- If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.
- If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$,

We usually write $|X| = n$ rather than $|X| = |\mathbf{n}|$.

Infinite cardinalities

Proposition

$|\{2n \mid n \in \mathbb{N}\}| = |\mathbb{N}|$ *(plain reading: there are as many even natural numbers as there are natural numbers)*

Theorem

$|\mathbb{Q}| = |\mathbb{N}|$ *(plain reading: there are as many rational numbers as there are natural numbers)*

So its all just infinity?

Theorem

Let X be a set. There is no surjection $\phi : X \to \mathcal{P}(X)$. (Recall $\mathcal{P}(X)$ is the set of all subsets of X).

Proof.

- Let $\phi : X \to \mathcal{P}(X)$ be a function.
- Consider $D_\phi := \{x \in X \mid x \notin \phi(x)\}$.
- Assume that there is some $x_0 \in X$ with $\phi(x_0) = D_\phi$ (if ϕ were surjection, there would need to be one).
- Does $x_0 \in D_\phi = \phi(x_0)$ hold?
- Contradiction!

$\square$

Preview

- What are the possible cardinalities of formal languages?
- How many grammars are there?
- How many formal languages are there, i.e. what is $|\mathcal{P}(\Sigma^*)|$?