Register and counter machines

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March 23, 2021

The topic of today

- All models of computation we considered so far were acting on finite strings.
- We can still use them for computations on numbers, graphs, etc, via encodings.
- But today, we will consider models of computation that natively act on natural numbers.

Register machines

Idea: Register machines have access to infinitely many registers, which each contain a natural number. They can do arithmetic operations on these. Powerful model of computation.

Register machines – definition

1. A configuration of a register machine is specifying for each register R_i its value, a natural number.
2. Initially, R_1, R_2, ... contain the input, every other register is empty.
3. The computation is given by a program (executed line by line) consisting of the following:
4. Addition $R_i := R_j + R_k$, increment $R_i := R_i + 1$, decrement $R_i := \max\{R_i - 1, 0\}$, assignment $R_i := n$
5. Direct and indirect addressing: $R_i := R_{R_0}$
6. And conditional branching "If $(R_i = R_j)$ then GOTO line ℓ"
7. If the program reaches its end, the output is the content of R_0.

Example – Computing multiplication

1. If $(R_2 = R_4)$ GOTO 5
2. $R_0 = R_0 + R_1$
3. $R_2 := \max\{R_2 - 1, 0\}$
4. If $(R_4 = R4)$ GOTO 1
5. END

Very robust notion

The definition of register machines is very robust. We could add way more functions, remove direct or indirect addressing, allowed only for comparison to 0, etc. The resulting model retains the same computational power.

Theorem

A function $f : \mathbb{N}^k \to \mathbb{N}$ is computable if and only if it is computable by a register machine.

Counter machines

Let's now make it very simplistic – how much do we **need** to get all computable functions $f : \mathbb{N} \to \mathbb{N}$?

3CM

- We use 3 counters, the first starting with the input, the other two at 0.
- Our commands are $R_i := R_i + 1$ and "If ($R_i = 0$) GOTO line ℓ else $R_i := R_i - 1$."
- That's it.
- Side note: Counter and register means more or less the same, word choice depends on the commands we use.

It's powerful enough

Theorem

A function $f : \mathbb{N} \to \mathbb{N}$ is computable if and only if it is computable by a 3CM.

Sketch.

- If we have 2 stacks, we can simulate a tape.
- if we have an auxiliary counter, a counter can simulate a stack (least significant bit is at the top)

□