Games and Winning Strategies

1. Consider the following game:

   Starting with a single pile of coins, two players alternate taking either 1 coin or half of the remaining coins, including the leftover coin if there is an odd number of coins remaining. Thus, for example, if there are 25 coins in the pile then a move consists of taking either 1 coin or 13 coins; if 13 coins are taken leaving 12 in the pile, then the next move will consist of taking either 1 coin or 6 coins. The player who takes the last coin wins.

   (a) For each number n from 1 to 10, explain who has the winning strategy in this game starting from a pile of n coins. In the cases in which the first player has the winning strategy, state how many coins the first player should take.

   (b) Argue that the first player has a winning strategy in the game starting with n coins if, and only if, the binary representation of n ends in an even number of 0's. Specifically,

   - if the binary representation of n ends in an odd number of 0's, then every move leaves a number of coins whose binary representation ends in an even number of 0's; and
   - if the binary representation of n ends in an even number of 0's, then either $n=1$ and you can win by taking the single coin, or there is a move which leaves a number of coins whose binary representation ends in an odd number of 0's.

2. In the game of MISÈRE NOUGHTS AND CROSSES, the aim is to *avoid* placing three of your symbols in a row, but rather to force your opponent to place three of their symbols in a row.

   (a) The first player does not have a winning strategy in this game. Explain how the second player can play to avoid losing.

   (Hint: It is a good idea to occupy two adjacent side squares first, and then a square which is aligned with only one of these two side squares. Why is this possible, and why does it work?)

   (b) The second player also does not have a winning strategy in this game. Explain how the first player can play to avoid losing.

   (Hint: Start by placing the first cross in the middle, and then "mirroring" every move of the second player by placing each subsequent cross directly opposite to where the second player places his noughts. Why is this a good idea?)