Relations and transitive closure

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Definition

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A *relation* between sets X, Y is a subset $R \subseteq X \times Y$. A relation on X is a subset $R \subseteq X \times X$.

Properties of relations

A relation R on X is[1]

- **Reflexive** if $\forall a \in X\ (a, a) \in R$
- **Symmetric** if $\forall a, b \in X\ (a, b) \in R \Rightarrow (b, a) \in R$
- **Anti-reflexive** if $\forall a \in X\ (a, a) \notin R$
- **Anti-symmetric** if $\forall a, b \in X\ ((a, b) \in R \wedge (b, a) \in R) \Rightarrow a = b$
- **Total** if $\forall a, b \in X\ (a, b) \in R \vee (b, a) \in R$
- **Transitive** if
  $\forall a, b, c \in X\ ((a, b) \in R \wedge (b, c) \in R) \Rightarrow (a, c) \in R$

[1] $\wedge$ denotes *and*, and $\vee$ denotes *or*.

Examples

Example

The relation $<$ on $\mathbb{N}$ is anti-reflexive, anti-symmetric and transitive.

Example

The relation $\leq$ on $\mathbb{N}$ is reflexive, anti-symmetric, total and transitive.

Exercises

Example

Let $R \subseteq \{0, 1, 2, 3\} \times \{0, 1, 2, 3\}$ be defined as $R = \{(0, 1), (2, 3)\}$. Which properties does R have?

Example

Let $| \subseteq \mathbb{N} \times \mathbb{N}$ be defined as $(n, m) \in |$ iff n divides m. Which properties does $|$ have?

Particular names

Definition

A *linear order* is a reflexive, anti-symmetric, total and transitive relation.

Definition

An *equivalence relation* is a reflexive, symmetric and transitive relation.

Composition of relations

Definition

Given $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$, let $(R \circ Q) \subseteq X \times Z$ be defined as[2]:

$$(R \circ Q) = \{(x, z) \in X \times Z \mid \exists y \in Y \ (x, y) \in R \wedge (y, z) \in Q\}$$

[2] $\exists$ means *there exists*.

Transitive closure

Definition

Let R be a relation on X. We define $R^1 := R$, and $R^{n+1} := R^n \circ R$, and then $R^+ = \bigcup_{n \geq 1} R^n$.

Theorem

The relation R^+ is the smallest transitive relation extending R, and we thus call it the transitive closure of R.

Example

Let $S = \{(n, n+1) \mid n \in \mathbb{N}\}$. Then $S^+ = <$.

Outlook

Next time, we shall use the notion of transitive closure to formally define the derivation process for the language described by a formal grammar.