Week 4

Solving Recurrences

General remarks

- We present a basic tool for analysing algorithms by **Solving Recurrences**.

Reading from CLRS for week 4

- Chapter 4

Analysing divide-and-conquer algorithms

Recall the divide-and-conquer paradigm:

Divide the problem into a number of subproblems, each of them a *smaller instance* of the same problem.

Conquer the subproblems by solving them *recursively*.
Base case: If the subproblems are small enough, just solve them by "brute force".

Combine the subproblem-solutions to give a solution to the original problem.

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We use recurrences to characterise the running time of a divide-and-conquer algorithm. Solving the recurrence gives us the asymptotic running time.

A recurrence is a function defined in terms of

- one or more base cases, and
- itself, with smaller arguments.

Examples for recurrences

- $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + 1 & \text{if } n > 1 \end{cases}$

  Solution: $T(n) = n$.

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  Solution: $T(n) \approx n \lg n + n$.

- $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/3) + T(2n/3) + n & \text{if } n > 1 \end{cases}$

  Solution: $T(n) = \Theta(n \lg n)$.

Main technical issues with recurrences

Floors and ceilings: The recurrence describing worst-case running time of Merge-Sort is really

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n > 1 \end{cases}$$

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Boundary conditions Running time on small inputs is bounded by a constant: $T(n) = \Theta(1)$ for small n. We usually do not mention this constant, as it typically doesn't change the order of growth of $T(n)$. Such constants only play a role if we are interested in exact solutions.

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When we state and solve recurrences, we often omit floors, ceilings, and boundary conditions, as they usually do not matter.

An important fact

For all natural numbers $k \geq 0$ holds:

$$\sum_{i=0}^{k} 2^i = 1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1 = \Theta(2^k).$$

An example for $k = 15$, in hexadecimal notation (16 bits, that is, four hexadecimal places; using the prefix "0x"), with the -1 moved to the other side:

$$(1 + \ldots 2^{15}) + 1 = 0\mathrm{xFFFF} + 1 = 0\mathrm{x}10000 = 2^{16} = 65536.$$

The asymptotic analysis holds in general:

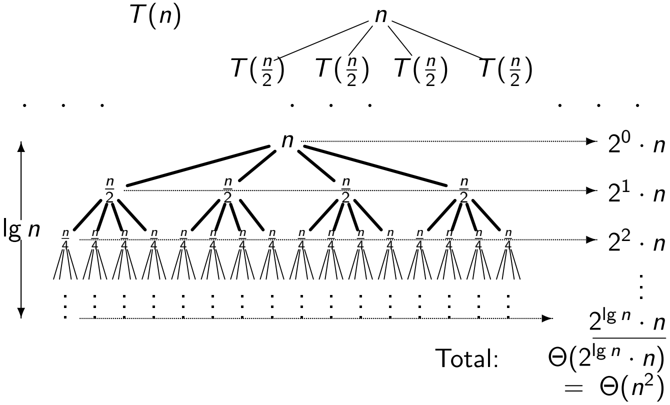
Exponential sums are asymptotically dominated by the last summand.

(This makes them actually useful!)

Recursion trees (quadratic growth)

Unfolding of the recurrence

$$T(n) = n + 4T(n/2):$$



Recursion trees (quasi-linear and linear growth)

What about the MergeSort recurrence

$$T(n) = n + 2T(n/2) \ ?$$

- Again the height of the tree is $\lg n$.
- However now the "workload" of each level is equal to n.
- So all workloads are the same.

So here we get

$$T(n) = \Theta(n \cdot \lg n).$$

And what about the recurrence (as in Min-Max)

$$T(n) = 1 + 2T(n/2) \ ?$$

- Again the height of the tree is $\lg n$.
- The "workload" of the levels are $1, 2, 4, 8, \dots, 2^{\lg n}$.
- Back to the original method, we use the exponential sum.

So here we get

$$T(n) = \Theta(n).$$

Master Theorem (simplified version)

Let $a \geq 1$ and $b > 1$ and $c \geq 0$ be constants.

Let $T(n)$ be defined by the recurrence

$$T(n) = aT(n/b) + \Theta(n^c),$$

where n/b represents either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$.

Then $T(n)$ is bounded asymptotically as follows:

1. If $c < \log_b a$ then $T(n) = \Theta(n^{\log_b a})$.
2. If $c = \log_b a$ then $T(n) = \Theta(n^c \lg n)$.
3. If $c > \log_b a$ then $T(n) = \Theta(n^c)$.

(**General version:** CLRS, Thm 4.1, p94.)

In other words

We start by an equation for $T(n)$ of the form

$$T(n) = b^x \cdot T(\frac{n}{b}) + \Theta(n^c),$$

where the x you have to find: $x = \log_b a$ (so $b^x = b^{\log_b a} = a$).

Then $T(n)$ is bounded asymptotically as follows:

1. If $x > c$ then $T(n) = \Theta(n^x)$.
2. If $x = c$ then $T(n) = \Theta(n^c \lg n)$.
3. If $x < c$ then $T(n) = \Theta(n^c)$.

The meaning of the three parameters in a divide-and-conquer scheme:

- $a = b^x$: the number of subproblems to be solved
- b: how often the subproblems (all of the same size) fit into the full problem
- c: power in the runtime of the local workload, that is, the division- and the combination-computation.

Using the Master Theorem

- The runtime for MIN-MAX satisfies:
  $$T(n) = 2T(n/2) + \Theta(1).$$
  The Master Theorem (case 1) applies:
  $$a = b = 2 \quad \textit{and} \quad c = 0 < 1 = \log_b a,$$
  giving $T(n) = \Theta(n^{\log_b a}) = \Theta(n)$.

- The runtime for MERGE-SORT satisfies:
  $$T(n) = 2T(n/2) + \Theta(n).$$
  The Master Theorem (case 2) applies:
  $$a = b = 2 \quad \textit{and} \quad c = 1 = \log_b a,$$
  giving $T(n) = \Theta(n^c \lg n) = \Theta(n \lg n)$.

What's happening

For the recurrences

$$\begin{aligned} T_1(n) &= 4T(n/2) + n \\ T_2(n) &= 4T(n/2) + n^2 \\ T_3(n) &= 4T(n/2) + n^3 \end{aligned}$$

the Master Theorem with case $i = 1, 2, 3$ applies:

$$a = 4 \text{ and } b = 2 \text{ (so } \log_b a = 2 \text{), and } c = i \ ,$$

giving

$$T_1(n) = \Theta(n^2) \ , \ T_2(n) = \Theta(n^2 \lg n) \ , \text{ and } T_3(n) = \Theta(n^3) \ .$$

Case 1: applies if the workload-cost (n^c) is negligible compared to the number and size of the subproblems.

Case 2: applies if the workload-cost (n^c) is as costly as the subproblems.

Case 3: applies if the workload-cost (n^c) is the dominating factor.

If we have Case 1 or 3, then in general this might indicate, that the divide-and-conquer approach can be replaced by a simpler approach (as we have seen for the min-max algorithm).

Easy decision between the three cases

Consider (again)

$$T(n) = a \cdot T(\frac{n}{b}) + \Theta(n^c).$$

The main question to start with is always:

Which of the three cases applies?

Apparently you needed to compute $x = \log_b a$ for that. But it is actually easier:

1. If $b^c < a$ then Case 1 applies.
2. If $b^c = a$ then Case 2 applies.
3. If $b^c > a$ then Case 3 applies.

(Try to understand why this holds — it's easy.)

Examples

- $T(n) = 5T(n/2) + \Theta(n^2)$
  In Master Theorem: $a = 5,\ b = 2,\ c = 2$.
  As $\log_b a = \log_2 5 > \log_2 4 = 2 = c$, case 1 applies:
  $T(n) = \Theta(n^{\lg 5})$.
- $T(n) = 27T(n/3) + \Theta(n^3)$
  In Master Theorem: $a = 27,\ b = 3,\ c = 3$.
  As $\log_b a = \log_3 27 = 3 = c$, case 2 applies:
  $T(n) = \Theta(n^3 \lg n)$.
- $T(n) = 5T(n/2) + \Theta(n^3)$
  In Master Theorem: $a = 5,\ b = 2,\ c = 3$.
  As $\log_b a = \log_2 5 < \log_2 8 = 3 = c$, case 3 applies:
  $T(n) = \Theta(n^3)$.

More examples

1. $T(n) = 3T(n/3) + 1$:

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4. $T(n) = 4T(n/3) + 1 :$

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7. $T(n) = 2T(n/3) + n^{\log_3 2}:$

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Final examples

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Thus Case I: $T(n) = \Theta(n^{\log_3 10})$ (note $\log_3(10) > 2$).

Final examples

$$T(n) = 10\,T(n/3) + \boxed{n^2}$$

$$3^2 < 10$$

Thus Case I: $T(n) = \Theta(n^{\log_3 10})$ (note $\log_3(10) > 2$).

The other cases:

- $T(n) = 9\,T(n/3) + n^2$: Case II, thus $T(n) = \Theta(n^2 \cdot \lg n)$.
- $T(n) = 8\,T(n/3) + n^2$: Case III, thus $T(n) = n^2$.

Growth rates as response rates

The fundamental setting:

- input size $n > 0$
- some abstract "runtime" $f(n) > 0$.

Understanding the "growth rate" of $f(n)$ means:

understanding how a change of n effects $f(n)$ —
increasing n in a certain way, how much is $f(n)$ increased?

Thus we get the "dictionary":

slow growth a big change of n causes only a small change of $f(n)$

large growth a small change of n causes a large change of $f(n)$

intermediate growth the change of $f(n)$ is kind of proportional to the change of n.

Ways of change

The two most basic forms of quantitative change are:

additive: the quantity (n or $f(n)$) is changed by adding a constant:

$$n \rightsquigarrow n + 1$$

$$f(n) \rightsquigarrow f(n) + c$$

multiplicative: the quantity (n or $f(n)$) is changed by multiplying a constant:

$$n \rightsquigarrow 2n$$

$$f(n) \rightsquigarrow c \cdot f(n)$$

For us "additive" means "small", and "multiplicative" means "big".

Linear growth

Considering all growth rates, **linear growth** is the middle (while for pure algorithms it is the bottom):

- a small change of n yields a small change of $f(n)$:

$$n \rightsquigarrow n+1 \Longrightarrow f(n) \rightsquigarrow c + f(n)$$

- a big change of n yields the *same* big change of $f(n)$:

$$n \rightsquigarrow 2 \cdot n \Longrightarrow f(n) \rightsquigarrow 2 \cdot f(n).$$

For the proof let $f(n) = \alpha \cdot n$:

$$\begin{aligned} f(n+1) &= \alpha \cdot (n+1) = \alpha \cdot n + \alpha = f(n) + \alpha \\ f(2n) &= \alpha \cdot (2n) = 2 \cdot (\alpha \cdot n) = 2f(n). \end{aligned}$$

Polynomial growth

Polynomial growth means (roughly) a function $f(n) = n^{\alpha}$ for some (constant) $\alpha > 0$:

- sublinear $\alpha < 1$
- superlinear $\alpha > 1$.

Characteristic here is that a big change of n yields a proportional change of $f(n)$:

$$n \rightsquigarrow 2n \Longrightarrow f(n) \rightsquigarrow 2^{\alpha} \cdot f(n).$$

The proof is simple:

$$f(2n) = (2n)^{\alpha} = 2^{\alpha} \cdot n^{\alpha} = 2^{\alpha} \cdot f(n).$$

(Note that 2^{α} is constant.)

Logarithmic growth

Logarithmic growth means (roughly) a function $f(n) = \log_b(n)$ for some (constant) $b > 1$:

Characteristic here is that a big change of n yields a small change of $f(n)$:

$$n \rightsquigarrow 2n \Longrightarrow f(n) \rightsquigarrow \log_b(2) + f(n).$$

The proof is again simple:

$$f(2n) = \log_b(2n) = \log_b(2) + \log_b(n) = \log_b(2) + f(n).$$

(Note that $\log_b(2)$ is constant.)

Exponential growth

Exponential growth means (roughly) a function $f(n) = b^n$ for some (constant) $b > 1$:

Characteristic here is that a small change of n yields a big change of $f(n)$:

$$n \rightsquigarrow n + 1 \Longrightarrow f(n) \rightsquigarrow b \cdot f(n).$$

The proof is, as always, simple:

$$f(n + 1) = b^{n+1} = b^n \cdot b^1 = b \cdot f(n).$$

(Note that b is constant.)