PART G

Probability, Statistics

Probability theory (Chap. 24) provides models of probability distributions (theoretical models of the observable reality involving chance effects) to be tested by statistical methods, and it will also supply the mathematical foundation of these methods in Chap. 25.

Modern **mathematical statistics** (Chap. 25) has various engineering applications, for instance, in testing materials, control of production processes, quality control of production outputs, performance tests of systems, robotics, and automatization in general, production planning, marketing analysis, and so on.

To this we could add a long list of fields of applications, for instance, in agriculture, biology, computer science, demography, economics, geography, management of natural resources, medicine, meteorology, politics, psychology, sociology, traffic control, urban planning, etc. Although these applications are very heterogeneous, we shall see that most statistical methods are universal in the sense that each of them can be applied in various fields.

Additional Software for Probability and Statistics

See also the list of software at the beginning of Part E on Numerical Analysis.
DATA DESK. Data Description, Inc., Ithaca, NY. Phone 1-800-573-5121 or (607) 257-1000, website at www.datadescription.com.
MINITAB. Minitab, Inc., College Park, PA. Phone 1-800-448-3555 or (814) 238-3280, website at www.minitab.com.
SAS. SAS Institute, Inc., Cary, NC. Phone 1-800-727-0025 or (919) 677-8000, website at www.sas.com.

S-PLUS. Insightful Corporation, Inc., Seattle, WA. Phone 1-800-569-0123 or (206) 283-8802, website at www.insightful.com.

SPSS. SPSS, Inc., Chicago, IL. Phone 1-800-543-2185 or (312) 651-3000, website at www.spss.com.

STATISTICA. StatSoft, Inc., Tulsa, OK. Phone (918) 749-1119, website at www.statsoft.com.

CHAPTER 24

Data Analysis. Probability Theory

We first show how to handle data numerically or in terms of graphs, and how to extract information (average size. spread of data, etc.) from them. If these data are influenced by "chance," by factors whose effect we cannot predict exactly (e.g., weather data, stock prices, lifespans of tires, etc.), we have to rely on **probability theory.** This theory originated in games of chance, such as flipping coins, rolling dice, or playing cards. Nowadays it gives mathematical models of chance processes called *random experiments* or, briefly, **experiments**. In such an experiment we observe a **random variable** X, that is, a function whose values in a **trial** (a performance of an experiment) occur "by chance" (Sec. 24.3) according to a **probability distribution** that gives the individual probabilities with which possible values of X may occur in the long run. (Example: Each of the six faces of a die should occur with the same probability, 1/6.) Or we may simultaneously observe more than one random variable, for instance, height *and* weight of persons or hardness *and* tensile strength of steel. This is discussed in Sec. 24.9, which will also give the basis for the mathematical justification of the statistical methods in Chap. 25.

Prerequisite: Calculus.
References and Answers to Problems: App. 1, Part G, App. 2.

24.1 Data Representation. Average. Spread

Data can be represented numerically or graphically in various ways. For instance, your daily newspaper may contain tables of stock prices and money exchange rates, curves or bar charts illustrating economical or political developments, or pie charts showing how your tax dollar is spent. And there are numerous other representations of data for special purposes.

In this section we discuss the use of standard representations of data in statistics. (For these, software packages, such as DATA DESK and MINITAB, are available, and Maple or Mathematica may also be helpful; see pp. 778 and 991) We explain corresponding concepts and methods in terms of typical examples, beginning with

$$\text{(1)} \qquad 89 \quad 84 \quad 87 \quad 81 \quad 89 \quad 86 \quad 91 \quad 90 \quad 78 \quad 89 \quad 87 \quad 99 \quad 83 \quad 89.$$

These are $n = 14$ measurements of the tensile strength of sheet steel in kg/mm^2, recorded in the order obtained and rounded to integer values. To see what is going on, we **sort** these data, that is, we order them by size,

$$\text{(2)} \qquad 78 \quad 81 \quad 83 \quad 84 \quad 86 \quad 87 \quad 87 \quad 89 \quad 89 \quad 89 \quad 89 \quad 90 \quad 91 \quad 99.$$

Sorting is a standard process on the computer; see Ref. [E25], listed in App. 1.

Graphic Representation of Data

We shall now discuss standard graphic representations used in statistics for obtaining information on properties of data.

Stem-and-Leaf Plot

This is one of the simplest but most useful representations of data. For (1) it is shown in Fig. 506. The numbers in (1) range from 78 to 99; see (2). We divide these numbers into 5 groups, 75–79, 80–84, 85–89, 90–94, 95–99. The integers in the tens position of the groups are 7, 8, 8, 9, 9. These form the *stem* in Fig. 506. The first *leaf* is 8 (representing 78). The second leaf is 134 (representing 81, 83, 84), and so on.

The number of times a value occurs is called its **absolute frequency.** Thus 78 has absolute frequency 1, the value 89 has absolute frequency 4, etc. The column to the extreme left in Fig. 506 shows the **cumulative absolute frequencies,** that is, the sum of the absolute frequencies of the values up to the line of the leaf. Thus, the number 4 in the second line on the left shows that (1) has 4 values up to and including 84. The number 11 in the next line shows that there are 11 values not exceeding 89, etc. Dividing the cumulative absolute frequencies by n (= 14 in Fig. 506) gives the **cumulative relative frequencies.**

Histogram

For large sets of data, histograms are better in displaying the distribution of data than stem-and-leaf plots. The principle is explained in Fig. 507. (An application to a larger data set is shown in Sec. 25.7). The bases of the rectangles in Fig. 507 are the x-intervals (known as **class intervals**) 74.5–79.5, 79.5–84.5, 84.5–89.5, 89.5–94.5, 94.5–99.5, whose midpoints (known as **class marks**) are x = 77, 82, 87. 92, 97, respectively. The height of a rectangle with class mark x is the **relative class frequency** $f_{\mathrm{rel}}(x)$, defined as the number of data values in that class interval, divided by n (= 14 in our case). Hence the areas of the rectangles are proportional to these relative frequencies, so that histograms give a good impression of the distribution of data.

Center and Spread of Data: Median, Quartiles

As a center of the location of data values we can simply take the **median,** the data value that falls in the middle when the values are ordered. In (2) we have 14 values. The seventh of them is 87, the eighth is 89, and we split the difference, obtaining the median 88. (In general, we would get a fraction.)

The spread (variability) of the data values can be measured by the **range** $R = x_{\max} - x_{\min}$, the largest minus the smallest data values, $R = 99 - 78 = 21$ in (2).

Leaf unit = 1.0

1	7	8
4	8	134
11	8	6779999
13	9	01
14	9	9

Fig. 506. Stem-and-leaf plot of the data in (1) and (2)

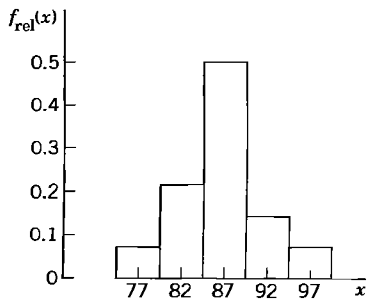


Fig. 507. Histogram of the data in (1) and (2) (grouped as in Fig. 506)

Better information gives the **interquartile range** IQR $= q_U - q_L$. Here the **upper quartile** q_U is the middle value among the data values *above* the median. The **lower quartile** q_L is the middle value among the data values *below* the median. Thus in (2) we have $q_U = 89$ (the fourth value from the end), $q_L = 84$ (the fourth value from the beginning), and IQR $= 89 - 84 = 5$. The median is also called the **middle quartile** and is denoted by q_M. The rule of "splitting the difference" (just applied to the middle quartile) is equally well used for the other quartiles if necessary.

Boxplot

The **boxplot** of (1) in Fig. 508 is obtained from the five numbers $x_{\min}$, q_L, q_M, q_U, $x_{\max}$ just determined. The box extends from q_L to q_U. Hence it has the height IQR. The position of the median in the box shows that the data distribution is not symmetric. The two lines extend from the box to $x_{\min}$ below and to $x_{\max}$ above. Hence they mark the range R.

Boxplots are particularly suitable for making comparisons. For example, Fig. 508 shows boxplots of the data sets (1) and

$$91 \quad 89 \quad 93 \quad 91 \quad 87 \quad 94 \quad 92 \quad 85 \quad 91 \quad 90 \quad 96 \quad 93 \quad 89 \tag{3}$$

(consisting of $n = 13$ values). Ordering gives

$$85 \quad 87 \quad 89 \quad 89 \quad 90 \quad 91 \quad 91 \quad 91 \quad 92 \quad 93 \quad 93 \quad 94 \quad 96 \tag{4}$$

(tensile strength, as before). From the plot we immediately see that the box of (3) is shorter than the box of (1) (indicating the higher quality of the steel sheets!) and that q_M is located in the middle of the box (showing the more symmetric form of the distribution). Finally, $x_{\max}$ is closer to q_U for (3) than it is for (1), a fact that we shall discuss later.

For plotting the box of (3) we took from (4) the values $x_{\min} = 85$, $q_L = 89$, $q_M = 91$, $q_U = 93$, $x_{\max} = 96$.

Outliers

An **outlier** is a value that appears to be uniquely different from the rest of the data set. It might indicate that something went wrong with the data collection process. In connection with quartiles an outlier is conventionally defined as a value more than a distance of 1.5 IQR from either end of the box.

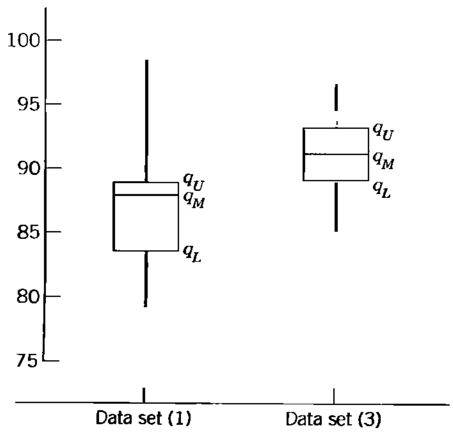


Fig. 508. Boxplots of data sets (1) and (3)

For the data in (1) we have IQR = 5, $q_L = 84$, $q_U = 89$. Hence outliers are smaller than $84 - 7.5$ or larger than $89 + 7.5$, so that 99 is an outlier [see (2)]. The data (3) have no outliers, as you can readily verify.

Mean. Standard Deviation. Variance

Medians and quartiles are easily obtained by ordering and counting, practically without calculation. But they do not give full information on data: you can change data values to some extent without changing the median. Similarly for the quartiles.

The average size of the data values can be measured in a more refined way by the **mean**

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j = \frac{1}{n} (x_1 + x_2 + \cdots + x_n). \tag{5}$$

This is the arithmetic mean of the data values, obtained by taking their sum and dividing by the data *size* n. Thus in (1),

$$\bar{x} = \tfrac{1}{14} (89 + 84 + \cdots + 89) = \tfrac{611}{7} \approx 87.3.$$

Every data value contributes, and changing one of them will change the mean.

Similarly, the spread (variability) of the data values can be measured in a more refined way by the **standard deviation** s or by its square, the **variance**

$$s^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2 = \frac{1}{n-1} \left[(x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2\right]. \tag{6}$$

Thus, to obtain the variance of the data, take the difference $x_j - \bar{x}$ of each data value from the mean, square it, take the sum of these n squares, and divide it by $n - 1$ (not n, as we motivate in Sec. 25.2). To get the standard deviation s, take the square root of s^2.

For example, using $\bar{x} = 611/7$, we get for the data (1) the variance

$$s^2 = \tfrac{1}{13} \left[(89 - \tfrac{611}{7})^2 + (84 - \tfrac{611}{7})^2 + \cdots + (89 - \tfrac{611}{7})^2\right] = \tfrac{176}{7} \approx 25.14.$$

Hence the standard deviation is $s = \sqrt{176/7} \approx 5.014$. Note that the standard deviation has the same dimension as the data values (kg/mm^2, see at the beginning), which is an advantage. On the other hand, the variance is preferable to the standard deviation in developing statistical methods, as we shall see in Chap. 25.

CAUTION! Your CAS (Maple, for instance) may use $1/n$ instead of $1/(n - 1)$ in (6), but the latter is better when n is small (see Sec. 25.2).

1–10 DATA REPRESENTATIONS

Represent the data by a stem-and-leaf plot, a histogram, and a boxplot:

1. 20 21 20 19 20 19 21 19

2. 7 6 4 0 7 1 2 4 6 6

3. 56 58 54 33 41 30 44 37 51 46 56 38 38 49 39

4. 12.1 10 12.4 10.5 9.2 17.2 11.4 11.8 14.7 9.9

5. 70.6 70.9 69.1 71.3 70.5 69.7 71.5 69.8 71.1 68.9 70.3 69.2 71.2 70.4 72.8

6. $-0.52 \quad 0.11 \quad -0.48 \quad 0.94 \quad 0.24 \quad -0.19 \quad -0.55$

7. Reaction time [sec] of an automatic switch
2.3 2.2 2.4 2.5 2.3 2.3 2.4 2.1 2.5 2.4
2.6 2.3 2.5 2.1 2.4 2.2 2.3 2.5 2.4 2.4

8. Carbon content [%] of coal
89 90 89 84 80 88 90 89 88 90 85
87 86 82 85 76 89 87 86 86

9. Weight of filled bottles [g] in an automatic filling process
403 399 398 401 400 401 401

10. Gasoline consumption [gallons per mile] of six cars of the same model
14.0 14.5 13.5 14.0 14.5 14.0

11–16 AVERAGE AND SPREAD

Find the mean and compare it with the median. Find the standard deviation and compare it with the interquartile range.

11. The data in Prob. 1.

12. The data in Prob. 2.

13. The data in Prob. 5.

14. The data in Prob. 6.

15. The data in Prob. 9.

16. 5 22 7 23 6. Why is $|\bar{x} - q_M|$ so large?

17. Construct the simplest possible data with $\bar{x} = 100$ but $q_M = 0$.

18. (Mean) Prove that $\bar{x}$ must always lie between the smallest and the largest data values.

19. (Outlier, reduced data) Calculate s for the data 4 1 3 10 2. Then reduce the data by deleting the outlier and calculate s. Comment.

20. WRITING PROJECT. Average and Spread. Compare Q_M, IQR and $\bar{x}$, s, illustrating the advantages and disadvantages with examples of your own.

24.2 Experiments, Outcomes, Events

We now turn to **probability theory.** This theory has the purpose of providing mathematical models of situations affected or even governed by "chance effects," for instance, in weather forecasting, life insurance, quality of technical products (computers, batteries, steel sheets, etc.), traffic problems, and, of course, games of chance with cards or dice. And the accuracy of these models can be tested by suitable observations or experiments—this is a main purpose of **statistics** to be explained in Chap. 25.

We begin by defining some standard terms. An **experiment** is a process of measurement or observation, in a laboratory, in a factory, on the street, in nature, or wherever; so "experiment" is used in a rather general sense. Our interest is in experiments that involve **randomness,** chance effects, so that we cannot predict a result exactly. A **trial** is a single performance of an experiment. Its result is called an **outcome** or a **sample point.** n trials then give a **sample** of **size** n consisting of n sample points. The **sample space** S of an experiment is the set of all possible outcomes.

EXAMPLES 1–6 Random Experiments. Sample Spaces

(1) Inspecting a lightbulb. $S = \{\text{Defective, Nondefective}\}$.

(2) Rolling a die. $S = \{1, 2, 3, 4, 5, 6\}$.

(3) Measuring tensile strength of wire. S the numbers in some interval.

(4) Measuring copper content of brass. S: 50% to 90%, say.

(5) Counting daily traffic accidents in New York. S the integers in some interval.

(6) Asking for opinion about a new car model. $S = \{\text{Like, Dislike, Undecided}\}$. ■

The subsets of S are called **events** and the outcomes **simple events.**

EXAMPLE 7 Events

In (2), events are $A = \{1, 3, 5\}$ (*"Odd number"*), $B = \{2, 4, 6\}$ (*"Even number"*), $C = \{5, 6\}$, etc. Simple events are $\{1\}, \{2\}, \cdots, \{6\}$.

If in a trial an outcome a happens and $a \in A$ (*a is an element of A*), we say that A happens. For instance, if a die turns up a 3, the event A: *Odd number* happens. Similarly, if C in Example 1 happens (meaning 5 or 6 turns up), then $D = \{4, 5, 6\}$ happens. Also note that S happens in each trial, meaning that *some* event of S always happens. All this is quite natural.

Unions, Intersections, Complements of Events

In connection with basic probability laws we shall need the following concepts and facts about events (subsets) $A, B, C, \cdots$ of a given sample space S.

The **union** $A \cup B$ of A and B consists of all points in A or B or both.

The **intersection** $A \cap B$ of A and B consists of all points that are in both A and B.

If A and B have no points in common, we write

$$A \cap B = \varnothing$$

where $\varnothing$ is the *empty set* (set with no elements) and we call A and B **mutually exclusive** (or **disjoint**) because in a trial the occurrence of A *excludes* that of B (and conversely)—if your die turns up an odd number, it cannot turn up an even number in the same trial. Similarly, a coin cannot turn up *Head* and *Tail* at the same time.

Complement A^c of A. This is the set of all the points of S ***not*** in A. Thus,

$$A \cap A^c = \varnothing, \qquad A \cup A^c = S.$$

In Example 7 we have $A^c = B$, hence $A \cup A^c = \{1, 2, 3, 4, 5, 6\} = S$.

Another notation for the complement of A is $\bar{A}$ (instead of A^c), but we shall not use this because in set theory $\bar{A}$ is used to denote the *closure* of A (not needed in our work).

Unions and intersections of more events are defined similarly. The **union**

$$\bigcup_{j=1}^{m} A_j = A_1 \cup A_2 \cup \cdots \cup A_m$$

of events $A_1, \cdots, A_m$ consists of all points that are in at least one A_j. Similarly for the union $A_1 \cup A_2 \cup \cdots$ of infinitely many subsets $A_1, A_2, \cdots$ of an *infinite* sample space S (that is, S consists of infinitely many points). The **intersection**

$$\bigcap_{j=1}^{m} A_j = A_1 \cap A_2 \cap \cdots \cap A_m$$

of $A_1, \cdots, A_m$ consists of the points of S that are in each of these events. Similarly for the intersection $A_1 \cap A_2 \cap \cdots$ of infinitely many subsets of S.

Working with events can be illustrated and facilitated by **Venn diagrams**[1] for showing unions, intersections, and complements, as in Figs. 509 and 510, which are typical examples that give the idea.

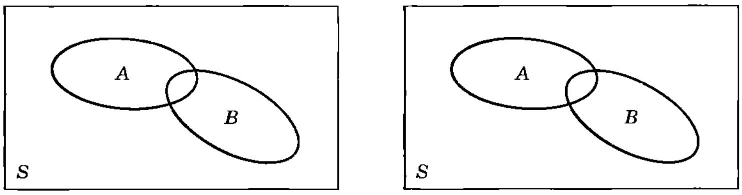
EXAMPLE 8 Unions and Intersections of 3 Events

In rolling a die, consider the events

A: *Number greater than* 3, $\quad$ B: *Number less than* 6, $\quad$ C: *Even number.*

Then $A \cap B = \{4, 5\}$, $B \cap C = \{2, 4\}$, $C \cap A = \{4, 6\}$, $A \cap B \cap C = \{4\}$. Can you sketch a Venn diagram of this? Furthermore, $A \cup B = S$, hence $A \cup B \cup C = S$ (why?). ■

[1]JOHN VENN (1834–1923), English mathematician.



Union $A \cup B$ Intersection $A \cap B$

Fig. 509. Venn diagrams showing two events A and B in a sample space S and their union $A \cup B$ (colored) and intersection $A \cap B$ (colored)

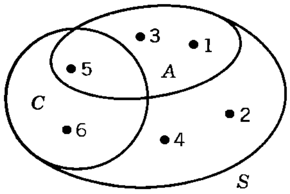


Fig. 510. Venn diagram for the experiment of rolling a die, showing S, $A = \{1, 3, 5\}$, $C = \{5, 6\}$, $A \cup C = \{1, 3, 5, 6\}$, $A \cap C = \{5\}$

1–9 SAMPLE SPACES, EVENTS

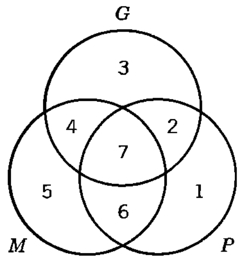
Graph a sample space for the experiment:

1. Tossing 2 coins
2. Drawing 4 screws from a lot of right-handed and left-handed screws
3. Rolling 2 dice
4. Tossing a coin until the first *Head* appears
5. Rolling a die until the first "*Six*" appears
6. Drawing bolts from a lot of 20, containing one defective D, until D is drawn, one at a time and assuming **sampling without replacement,** that is, bolts drawn are not returned to the lot
7. Recording the lifetime of each of 3 lightbulbs
8. Choosing a committee of 3 from a group of 5 people
9. Recording the daily maximum temperature X and the maximum air pressure Y at some point in a city

10. In Prob. 3, circle and mark the events *A: Equal faces, B: Sum exceeds* 9, *C: Sum equals* 7.
11. In rolling 2 dice, are the events *A: Sum divisible by* 3 and *B: Sum divisible by* 5 mutually exclusive?
12. Answer the question in Prob. 11 for rolling 3 dice.
13. In Prob. 5 list the outcomes that make up the event *E:* First "*Six*" in rolling at most 3 times. Describe E^c.
14. List all 8 subsets of the sample space $S = \{a, b, c\}$.

15–20 VENN DIAGRAMS

15. In connection with a trip to Europe by some students, consider the events P that they see Paris, G that they have a good time, and M that they run out of money, and describe in words the events 1, $\cdots$, 7 in the diagram.



Problem 15

16. Using Venn diagrams, graph and check the rules

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

17. **(De Morgan's laws)** Using Venn diagrams, graph and check *De Morgan's laws*

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c.$$

18. Using a Venn diagram, show that $A \subseteq B$ if and only if $A \cap B = A$.

19. Show that, by the definition of complement, for any subset A of a sample space S,

$$(A^c)^c = A, \qquad S^c = \varnothing, \qquad \varnothing^c = S,$$
$$A \cup A^c = S, \qquad A \cap A^c = \varnothing.$$

20. Using a Venn diagram, show that $A \subseteq B$ if and only if $A \cup B = B$.

24.3 Probability

The "probability" of an event A in an experiment is supposed to measure how frequently A is *about* to occur if we make many trials. If we flip a coin, then heads H and tails T will appear *about* equally often—we say that H and T are **"equally likely."** Similarly, for a regularly shaped die of homogeneous material (**"fair die"**) each of the six outcomes $1, \cdots, 6$ will be equally likely. These are examples of experiments in which the sample space S consists of finitely many outcomes (points) that for reasons of some symmetry can be regarded as equally likely. This suggests the following definition.

DEFINITION 1 **First Definition of Probability**

If the sample space S of an experiment consists of finitely many outcomes (points) that are equally likely, then the probability $P(A)$ of an event A is

$$P(A) = \frac{\text{Number of points in } A}{\text{Number of points in } S}. \tag{1}$$

From this definition it follows immediately that, in particular,

$$P(S) = 1. \tag{2}$$

EXAMPLE 1 **Fair Die**

In rolling a fair die once, what is the probability $P(A)$ of A of obtaining a 5 or a 6? The probabilty of B: *"Even number"*?

Solution. The six outcomes are equally likely, so that each has probability 1/6. Thus $P(A) = 2/6 = 1/3$ because $A = \{5, 6\}$ has 2 points, and $P(B) = 3/6 = 1/2$. ■

Definition 1 takes care of many games as well as some practical applications, as we shall see, but certainly not of all experiments, simply because in many problems we do not have finitely many equally likely outcomes. To arrive at a more general definition of probability, we regard ***probability as the counterpart of relative frequency.*** Recall from Sec. 24.1 that the **absolute frequency** $f(A)$ of an event A in n trials is the number of times A occurs, and the **relative frequency** of A in these trials is $f(A)/n$; thus

$$f_{\text{rel}}(A) = \frac{f(A)}{n} = \frac{\text{Number of times } A \text{ occurs}}{\text{Number of trials}}. \tag{3}$$

Now if A did not occur, then $f(A) = 0$. If A always occurred, then $f(A) = n$. These are the extreme cases. Division by n gives

$$(4^*) \qquad 0 \leqq f_{\text{rel}}(A) \leqq 1.$$

In particular, for $A = S$ we have $f(S) = n$ because S always occurs (meaning that some event always occurs; if necessary, see Sec. 24.2, after Example 7). Division by n gives

$$(5^*) \qquad f_{\text{rel}}(S) = 1.$$

Finally, if A and B are mutually exclusive, they cannot occur together. Hence the absolute frequency of their union $A \cup B$ must equal the sum of the absolute frequencies of A and B. Division by n gives the same relation for the relative frequencies,

$$(6^*) \qquad f_{\text{rel}}(A \cup B) = f_{\text{rel}}(A) + f_{\text{rel}}(B) \qquad (A \cap B = \varnothing).$$

We are now ready to extend the definition of probability to experiments in which equally likely outcomes are not available. Of course, the extended definition should include Definition 1. Since probabilities are supposed to be the theoretical counterpart of relative frequencies, we choose the properties in (4*), (5*), (6*) as axioms. (Historically, such a choice is the result of a long process of gaining experience on what might be best and most practical.)

DEFINITION 2 **General Definition of Probability**

Given a sample space S, with each event A of S (subset of S) there is associated a number $P(A)$, called the **probability** of A, such that the following **axioms of probability** are satisfied.

1. For every A in S,

$$(4) \qquad 0 \leqq P(A) \leqq 1.$$

2. The entire sample space S has the probability

$$(5) \qquad P(S) = 1.$$

3. For mutually exclusive events A and B ($A \cap B = \varnothing$; see Sec. 24.2),

$$(6) \qquad P(A \cup B) = P(A) + P(B) \qquad (A \cap B = \varnothing).$$

If S is infinite (has infinitely many points), Axiom 3 has to be replaced by

3′. For mutually exclusive events $A_1, A_2, \cdots$,

$$(6') \qquad P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$

In the infinite case the subsets of S on which $P(A)$ is defined are restricted to form a so-called *σ-algebra*, as explained in Ref. [GR6] (not [G6]!) in App. 1. This is of no practical consequence to us.

Basic Theorems of Probability

We shall see that the axioms of probability will enable us to build up probability theory and its application to statistics. We begin with three basic theorems. The first of them is useful if we can get the probability of the complement A^c more easily than $P(A)$ itself.

THEOREM 1

Complementation Rule

For an event A and its complement A^c in a sample space S,

$$P(A^c) = 1 - P(A). \tag{7}$$

PROOF By the definition of complement (Sec. 24.2), we have $S = A \cup A^c$ and $A \cap A^c = \varnothing$. Hence by Axioms 2 and 3,

$$1 = P(S) = P(A) + P(A^c), \qquad \text{thus} \qquad P(A^c) = 1 - P(A). \qquad \blacksquare$$

EXAMPLE 2 **Coin Tossing**

Five coins are tossed simultaneously. Find the probability of the event *A: At least one head turns up.* Assume that the coins are fair.

Solution. Since each coin can turn up heads or tails, the sample space consists of $2^5 = 32$ outcomes. Since the coins are fair, we may assign the same probability (1/32) to each outcome. Then the event A^c (*No heads turn up*) consists of only 1 outcome. Hence $P(A^c) = 1/32$, and the answer is $P(A) = 1 - P(A^c) = 31/32$. ■

The next theorem is a simple extension of Axiom 3, which you can readily prove by induction.

THEOREM 2

Addition Rule for Mutually Exclusive Events

For mutually exclusive events $A_1, \cdots, A_m$ in a sample space S,

$$P(A_1 \cup A_2 \cup \cdots A_m) = P(A_1) + P(A_2) + \cdots + P(A_m). \tag{8}$$

EXAMPLE 3 **Mutually Exclusive Events**

If the probability that on any workday a garage will get 10–20, 21–30, 31–40, over 40 cars to service is 0.20, 0.35, 0.25, 0.12, respectively, what is the probability that on a given workday the garage gets at least 21 cars to service?

Solution. Since these are mutually exclusive events, Theorem 2 gives the answer $0.35 + 0.25 + 0.12 = 0.72$. Check this by the complementation rule. ■

In many cases, events will not be mutually exclusive. Then we have

THEOREM 3

Addition Rule for Arbitrary Events

For events A and B in a sample space,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{9}$$

PROOF C, D, E in Fig. 511 make up $A \cup B$ and are mutually exclusive (disjoint). Hence by Theorem 2,

$$P(A \cup B) = P(C) + P(D) + P(E).$$

This gives (9) because on the right $P(C) + P(D) = P(A)$ by Axiom 3 and disjointness; and $P(E) = P(B) - P(D) = P(B) - P(A \cap B)$, also by Axiom 3 and disjointness. ■

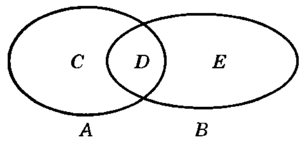


Fig. 511. Proof of Theorem 3

Note that for mutually exclusive events A and B we have $A \cap B = \varnothing$ by definition and, by comparing (9) and (6),

$$P(\varnothing) = 0. \tag{10}$$

(Can you also prove this by (5) and (7)?)

EXAMPLE 4 **Union of Arbitrary Events**

In tossing a fair die, what is the probability of getting an odd number or a number less than 4?

Solution. Let A be the event "*Odd number*" and B the event "*Number less than* 4." Then Theorem 3 gives the answer

$$P(A \cup B) = \tfrac{3}{6} + \tfrac{3}{6} - \tfrac{2}{6} = \tfrac{2}{3}$$

because $A \cap B =$ "*Odd number less than* 4" $= \{1, 3\}$. ■

Conditional Probability. Independent Events

Often it is required to find the probability of an event B under the condition that an event A occurs. This probability is called the **conditional probability** *of B given A* and is denoted by $P(B|A)$. In this case A serves as a new (reduced) sample space, and that probability is the fraction of $P(A)$ which corresponds to $A \cap B$. Thus

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad [P(A) \neq 0]. \tag{11}$$

Similarly, the *conditional probability of A given B* is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad [P(B) \neq 0]. \tag{12}$$

Solving (11) and (12) for $P(A \cap B)$, we obtain

THEOREM 4 **Multiplication Rule**

If A and B are events in a sample space S and $P(A) \neq 0$, $P(B) \neq 0$, then

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B). \tag{13}$$

EXAMPLE 5 Multiplication Rule

In producing screws, let A mean "screw too slim" and B "screw too short." Let $P(A) = 0.1$ and let the conditional probability that a slim screw is also too short be $P(B|A) = 0.2$. What is the probability that a screw that we pick randomly from the lot produced will be both too slim and too short?

Solution. $P(A \cap B) = P(A)P(B|A) = 0.1 \cdot 0.2 = 0.02 = 2\%$, by Theorem 4. ■

Independent Events. If events A and B are such that

$$P(A \cap B) = P(A)P(B), \tag{14}$$

they are called **independent events.** Assuming $P(A) \neq 0$, $P(B) \neq 0$, we see from (11)–(13) that in this case

$$P(A|B) = P(A), \qquad P(B|A) = P(B).$$

This means that the probability of A does not depend on the occurrence or nonoccurrence of B, and conversely. This justifies the term "independent."

Independence of m Events. Similarly, m events $A_1, \cdots, A_m$ are called **independent** if

$$P(A_1 \cap \cdots \cap A_m) = P(A_1) \cdots P(A_m) \tag{15a}$$

as well as for every k different events $A_{j_1}, A_{j_2}, \cdots, A_{j_k}$,

$$P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}) = P(A_{j_1})P(A_{j_2}) \cdots P(A_{j_k}) \tag{15b}$$

where $k = 2, 3, \cdots, m - 1$.

Accordingly, three events A, B, C are independent if and only if

$$\begin{aligned} P(A \cap B) &= P(A)P(B), \\ P(B \cap C) &= P(B)P(C), \\ P(C \cap A) &= P(C)P(A), \\ P(A \cap B \cap C) &= P(A)P(B)P(C). \end{aligned} \tag{16}$$

Sampling. Our next example has to do with randomly drawing objects, *one at a time,* from a given set of objects. This is called **sampling from a population,** and there are two ways of sampling, as follows.

1. In **sampling with replacement,** the object that was drawn at random is placed back to the given set and the set is mixed thoroughly. Then we draw the next object at random.
2. In **sampling without replacement** the object that was drawn is put aside.

EXAMPLE 6 Sampling With and Without Replacement

A box contains 10 screws, three of which are defective. Two screws are drawn at random. Find the probability that none of the two screws is defective.

Solution. We consider the events

A: First drawn screw nondefective.

B: Second drawn screw nondefective.

Clearly, $P(A) = \frac{7}{10}$ because 7 of the 10 screws are nondefective and we sample at random, so that each screw has the same probability ($\frac{1}{10}$) of being picked. If we sample with replacement, the situation before the second drawing is the same as at the beginning, and $P(B) = \frac{7}{10}$. The events are independent, and the answer is

$$P(A \cap B) = P(A)P(B) = 0.7 \cdot 0.7 = 0.49 = 49\%.$$

If we sample without replacement, then $P(A) = \frac{7}{10}$, as before. If A has occurred, then there are 9 screws left in the box, 3 of which are defective. Thus $P(B|A) = \frac{6}{9} = \frac{2}{3}$, and Theorem 4 yields the answer

$$P(A \cap B) = \frac{7}{10} \cdot \frac{2}{3} \approx 47\%.$$

Is it intuitively clear that this value must be smaller than the preceding one? ■

PROBLEM SET 24.3

1. Three screws are drawn at random from a lot of 100 screws, 10 of which are defective. Find the probability that the screws drawn will be nondefective in drawing (a) with replacement, (b) without replacement.
2. In Prob. 1 find the probability of *E: At least* 1 *defective* (i) directly, (ii) by using complements; in both cases (a) and (b).
3. If we inspect paper by drawing 5 sheets without replacement from every batch of 500, what is the probability of getting 5 clean sheets although 2% of the sheets contain spots? First guess.
4. Under what conditions will it make *practically* no difference whether we sample with or without replacement? Give numeric examples.
5. If you need a right-handed screw from a box containing 20 right-handed and 5 left-handed screws, what is the probability that you get at least one right-handed screw in drawing 2 screws with replacement?
6. If in Prob. 5 you draw without replacement, does the probability decrease or increase? First think, then calculate.
7. What gives the greater probability of hitting some target at least once: (a) hitting in a shot with probability 1/2 and firing 1 shot, or (b) hitting in a shot with probability 1/4 and firing 2 shots? First guess. Then calculate.
8. Suppose that we draw cards repeatedly and with replacement from a file of 100 cards, 50 of which refer to male and 50 to female persons. What is the probability of obtaining the second "female" card before the third "male" card?
9. What is the complementary event of the event considered in Prob. 8? Calculate its probability and use it to check your result in Prob. 8.
10. In rolling two fair dice, what is the probability of obtaining a sum greater than 4 but not exceeding 7?
11. In rolling two fair dice, what is the probability of obtaining equal numbers or numbers with an even product?
12. Solve Prob. 11 by considering complements.
13. A motor drives an electric generator. During a 30-day period, the motor needs repair with probability 8% and the generator needs repair with probability 4%. What is the probability that during a given period, the entire apparatus (consisting of a motor and a generator) will need repair?
14. If a circuit contains 3 automatic switches and we want that, with a probability of 95%, during a given time interval they are all working, what probability of failure per time interval can we admit for a single switch?
15. If a certain kind of tire has a life exceeding 25 000 miles with probability 0.95, what is the probability that a set of 4 of these tires on a car will last longer than 25 000 miles?
16. In Prob. 15, what is the probability that at least one of the tires will not last for 25 000 miles?
17. A pressure control apparatus contains 4 valves. The apparatus will not work unless all valves are operative. If the probability of failure of each valve during some interval of time is 0.03, what is the corresponding probability of failure of the apparatus?
18. Show that if B is a subset of A, then $P(B) \leqq P(A)$.
19. Extending Theorem 4, show that $P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$.
20. You may wonder whether in (16) the last relation follows from the others, but the answer is no. To see this, imagine that a chip is drawn from a box containing 4 chips numbered 000, 011, 101, 110, and let A, B, C be the events that the first, second, and third digit, respectively, on the drawn chip is 1. Show that then the first three formulas in (16) hold but the last one does not hold.

24.4 Permutations and Combinations

Permutations and combinations help in finding probabilities $P(A) = a/k$ by ***systematically counting*** the number a of points of which an event A consists; here, k is the number of points of the sample space S. The practical difficulty is that a may often be surprisingly large, so that actual counting becomes hopeless. For example, if in assembling some instrument you need 10 different screws in a certain order and you want to draw them randomly from a box (which contains nothing else) the probability of obtaining them in the required order is only 1/3 628 800 because there are

$$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 3\,628\,800$$

orders in which they can be drawn. Similarly, in many other situations the numbers of orders, arrangements, etc. are often incredibly large. (If you are unimpressed. take 20 screws—how much bigger will the number be?)

Permutations

A **permutation** of given things (*elements* or *objects*) is an arrangement of these things in a row in some order. For example, for three letters a, b, c there are $3! = 1 \cdot 2 \cdot 3 = 6$ permutations: abc, acb, bac, bca, cab, cba. This illustrates (a) in the following theorem.

THEOREM 1 **Permutations**

(a) ***Different things.*** *The number of permutations of n different things taken all at a time is*

$$n! = 1 \cdot 2 \cdot 3 \cdots n \qquad \text{(read "}n\text{ factorial").} \tag{1}$$

(b) ***Classes of equal things.*** *If n given things can be divided into c classes of alike things differing from class to class, then the number of permutations of these things taken all at a time is*

$$\frac{n!}{n_1!\,n_2! \cdots n_c!} \qquad (n_1 + n_2 + \cdots + n_c = n) \tag{2}$$

where n_j is the number of things in the jth class.

PROOF (a) There are n choices for filling the first place in the row. Then $n - 1$ things are still available for filling the second place, etc.

(b) n_1 alike things in class 1 make $n_1!$ permutations collapse into a single permutation (those in which class 1 things occupy the same n_1 positions), etc., so that (2) follows from (1). ∎

EXAMPLE 1 Illustration of Theorem 1(b)

If a box contains 6 red and 4 blue balls, the probability of drawing first the red and then the blue balls is

$$P = 6!4!/10! = 1/210 \approx 0.5\%.$$ ■

A **permutation of n things taken k at a time** is a permutation containing only k of the n given things. Two such permutations consisting of the same k elements, in a different order, are different, by definition. For example, there are 6 different permutations of the three letters a, b, c, taken two letters at a time, ab, ac, bc, ba, ca, cb.

A **permutation of n things taken k at a time with repetitions** is an arrangement obtained by putting any given thing in the first position, any given thing, including a repetition of the one just used, in the second, and continuing until k positions are filled. For example, there are $3^2 = 9$ different such permutations of a, b, c taken 2 letters at a time, namely, the preceding 6 permutations and aa, bb, cc. You may prove (see Team Project 18):

THEOREM 2

Permutations

The number of different permutations of n different things taken k at a time without repetitions is

(3a) $$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

and with repetitions is

(3b) $$n^k.$$

EXAMPLE 2 Illustration of Theorem 2

In a coded telegram the letters are arranged in groups of five letters, called *words*. From (3b) we see that the number of different such words is

$$26^5 = 11\,881\,376.$$

From (3a) it follows that the number of different such words containing each letter no more than once is

$$26!/(26-5)! = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7\,893\,600.$$ ■

Combinations

In a permutation, the order of the selected things is essential. In contrast, a **combination** of given things means any selection of one or more things *without regard to order.* There are two kinds of combinations, as follows.

The number of **combinations of n different things, taken k at a time, without repetitions** is the number of sets that can be made up from the n given things, each set containing k different things and no two sets containing exactly the same k things.

The number of **combinations of n different things, taken k at a time, with repetitions** is the number of sets that can be made up of k things chosen from the given n things, each being used as often as desired.

For example, there are three combinations of the three letters a, b, c, taken two letters at a time, without repetitions, namely, ab, ac, bc, and six such combinations with repetitions, namely, ab, ac, bc, aa, bb, cc.

THEOREM 3 **Combinations**

The number of different combinations of n different things taken, k at a time, without repetitions, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}. \tag{4a}$$

and the number of those combinations with repetitions is

$$\binom{n+k-1}{k}. \tag{4b}$$

PROOF The statement involving (4a) follows from the first part of Theorem 2 by noting that there are $k!$ *permutations* of k things from the given n things that differ by the order of the elements (see Theorem 1), but there is only a single *combination* of those k things of the type characterized in the first statement of Theorem 3. The last statement of Theorem 3 can be proved by induction (see Team Project 18). ■

EXAMPLE 3 **Illustration of Theorem 3**

The number of samples of five lightbulbs that can be selected from a lot of 500 bulbs is [see (4a)]

$$\binom{500}{5} = \frac{500!}{5!495!} = \frac{500\cdot 499\cdot 498\cdot 497\cdot 496}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 255\,244\,687\,600.$$ ■

Factorial Function

In (1)–(4) the **factorial function** is basic. By definition,

$$0! = 1. \tag{5}$$

Values may be computed recursively from given values by

$$(n+1)! = (n+1)n!. \tag{6}$$

For large n the function is very large (see Table A3 in App. 5). A convenient approximation for large n is the **Stirling formula**[2]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad (e = 2.718\cdots) \tag{7}$$

[2]JAMES STIRLING (1692–1770), Scots mathematician.

where $\sim$ is read **"asymptotically equal"** and means that the ratio of the two sides of (7) approaches 1 as n approaches infinity.

EXAMPLE 4 Stirling Formula

$n!$	By (7)	Exact Value	Relative Error
4!	23.5	24	2.1%
10!	3 598 696	3 628 800	0.8%
20!	$2.422\,79 \cdot 10^{18}$	2 432 902 008 176 640 000	0.4%

■

Binomial Coefficients

The **binomial coefficients** are defined by the formula

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!} \qquad (k \geqq 0, \text{ integer}). \tag{8}$$

The numerator has k factors. Furthermore, we define

$$\binom{a}{0} = 1, \qquad \text{in particular,} \qquad \binom{0}{0} = 1. \tag{9}$$

For integer $a = n$ we obtain from (8)

$$\binom{n}{k} = \binom{n}{n-k} \qquad (n \geqq 0,\ 0 \leqq k \leqq n). \tag{10}$$

Binomial coefficients may be computed recursively, because

$$\binom{a}{k} + \binom{a}{k+1} = \binom{a+1}{k+1} \qquad (k \geqq 0, \text{ integer}). \tag{11}$$

Formula (8) also yields

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k} \qquad (k \geqq 0, \text{ integer}) \ (m > 0). \tag{12}$$

There are numerous further relations; we mention two important ones,

$$\sum_{s=0}^{n-1} \binom{k+s}{k} = \binom{n+k}{k+1} \qquad (k \geqq 0,\ n \geqq 1, \text{ both integer}) \tag{13}$$

and

$$\sum_{k=0}^{r} \binom{p}{k}\binom{q}{r-k} = \binom{p+q}{r} \qquad (r \geqq 0, \text{ integer}). \tag{14}$$

[illegible]

1. List all permutations of four digits 1, 2, 3, 4, taken all at a time.
2. List (a) all permutations, (b) all combinations without repetitions, (c) all combinations with repetitions, of 5 letters a, e, i, o, u taken 2 at a time.
3. In how many ways can we assign 8 workers to 8 jobs (one worker to each job and conversely)?
4. How many samples of 4 objects can be drawn from a lot of 80 objects?
5. In how many different ways can we choose a committee of 3 from 20 persons? First guess.
6. In how many different ways can we select a committee consisting of 3 engineers, 2 biologists, and 2 chemists from 10 engineers, 5 biologists, and 6 chemists? First guess.
7. Of a lot of 10 items, 2 are defective. (a) Find the number of different samples of 4. Find the number of samples of 4 containing (b) no defectives, (c) 1 defective, (d) 2 defectives.
8. If a cage contains 100 mice, two of which are male, what is the probability that the two male mice will be included if 12 mice are randomly selected?
9. An urn contains 2 blue, 3 green, and 4 red balls. We draw 1 ball at random and put it aside. Then we draw the next ball, and so on. Find the probability of drawing at first the 2 blue balls, then the 3 green ones, and finally the red ones.
10. By what factor is the probability in Prob. 9 decreased if the number of balls is doubled (4 blue, etc.)?
11. Determine the number of different bridge hands. (A bridge hand consists of 13 cards selected from a full deck of 52 cards.)
12. In how many different ways can 5 people be seated at a round table?
13. If 3 suspects who committed a burglary and 6 innocent persons are lined up, what is the probability that a witness who is not sure and has to pick three persons will pick the three suspects by chance? That the witness picks 3 innocent persons by chance?
14. **(Birthday problem)** What is the probability that in a group of 20 people (that includes no twins) at least two have the same birthday, if we assume that the probability of having birthday on a given day is 1/365 for every day. First guess.
15. How many different license plates showing 5 symbols, namely, 2 letters followed by 3 digits, could be made?
16. How many automobile registrations may the police have to check in a hit-and-run accident if a witness reports KDP5 and cannot remember the last two digits on the license plate but is certain that all three digits were different?
17. **CAS PROJECT. Stirling formula. (a)** Using (7), compute approximate values of $n!$ for $n = 1, \cdots, 20$.

    (b) Determine the relative error in (a). Find an empirical formula for that relative error.

    (c) An upper bound for that relative error is $e^{1/12n} - 1$. Try to relate your empirical formula to this.

    (d) Search through the literature for further information on Stirling's formula. Write a short report about your findings, arranged in logical order and illustrated with numeric examples.
18. **TEAM PROJECT. Permutations, Combinations.**
    (a) Prove Theorem 2.

    (b) Prove the last statement of Theorem 3.

    (c) Derive (11) from (8).

    (d) By the **binomial theorem,**

    $$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

    so that $a^k b^{n-k}$ has the coefficient $\binom{n}{k}$. Can you conclude this from Theorem 3 or is this a mere coincidence?

    (e) Prove (14) by using the binomial theorem.

    (f) Collect further formulas for binomial coefficients from the literature and illustrate them numerically.

24.5 Random Variables. Probability Distributions

In Sec. 24.1 we considered frequency distributions of data. These distributions show the absolute or relative frequency of the data values. Similarly, a **probability distribution** or, briefly, a **distribution,** shows the probabilities of events in an experiment. The quantity that we observe in an experiment will be denoted by X and called a **random variable** (or

stochastic variable) because the value it will assume in the next trial depends on chance, on **randomness**—if you roll a dice, you get one of the numbers from 1 to 6, but you don't know which one will show up next. Thus X = *Number a die turns up* is a random variable. So is X = *Elasticity of rubber* (elongation at break). ("Stochastic" means related to chance.)

If we ***count*** (cars on a road, defective screws in a production, tosses until a die shows the first Six), we have a **discrete random variable and distribution.** If we ***measure*** (electric voltage, rainfall, hardness of steel), we have a **continuous random variable and distribution.** Precise definitions follow. In both cases the distribution of X is determined by the **distribution function**

$$F(x) = P(X \leqq x); \tag{1}$$

this is the probability that in a trial, X will assume any value not exceeding x.

CAUTION! The terminology is not uniform. $F(x)$ is sometimes also called the **cumulative distribution function.**

For (1) to make sense in both the discrete and the continuous case we formulate conditions as follows.

DEFINITION

Random Variable

A **random variable** X is a function defined on the sample space S of an experiment. Its values are real numbers. For every number a the probability

$$P(X = a)$$

with which X assumes a is defined. Similarly, for any interval I the probability

$$P(X \in I)$$

with which X assumes any value in I is defined.

Although this definition is very general, practically only a very small number of distributions will occur over and over again in applications.

From (1) we obtain the fundamental formula for the probability corresponding to an interval $a < x \leqq b$,

$$P(a < X \leqq b) = F(b) - F(a). \tag{2}$$

This follows because $X \leqq a$ ("*X assumes any value not exceeding a*") and $a < X \leqq b$ ("*X assumes any value in the interval $a < x \leqq b$*") are mutually exclusive events, so that by (1) and Axiom 3 of Definition 2 in Sec. 24.3

$$\begin{aligned} F(b) = P(X \leqq b) &= P(X \leqq a) + P(a < X \leqq b) \\ &= F(a) + P(a < X \leqq b) \end{aligned}$$

and subtraction of $F(a)$ on both sides gives (2).

Discrete Random Variables and Distributions

By definition, a random variable X and its distribution are **discrete** if X assumes only finitely many or at most countably many values $x_1, x_2, x_3, \cdots$, called the **possible values** of X, with positive probabilities $p_1 = P(X = x_1)$, $p_2 = P(X = x_2)$, $p_3 = P(X = x_3), \cdots$, whereas the probability $P(X \in I)$ is zero for any interval I containing no possible value.

Clearly, the discrete distribution of X is also determined by the **probability function** $f(x)$ of X, defined by

$$f(x) = \begin{cases} p_j & \text{if } x = x_j \\ 0 & \text{otherwise} \end{cases} \qquad (j = 1, 2, \cdots), \tag{3}$$

From this we get the values of the **distribution function** $F(x)$ by taking sums,

$$F(x) = \sum_{x_j \leq x} f(x_j) = \sum_{x_j \leq x} p_j \tag{4}$$

where for any given x we sum all the probabilities p_j for which x_j is smaller than or equal to that of x. This is a **step function** with upward jumps of size p_j at the possible values x_j of X and constant in between.

EXAMPLE 1 Probability Function and Distribution Function

Figure 512 shows the probability function $f(x)$ and the distribution function $F(x)$ of the discrete random variable

$$X = \textit{Number a fair die turns up.}$$

X has the possible values $x = 1, 2, 3, 4, 5, 6$ with probability $1/6$ each. At these x the distribution function has upward jumps of magnitude $1/6$. Hence from the graph of $f(x)$ we can construct the graph of $F(x)$, and conversely.

In Figure 512 (and the next one) at each jump the ***fat dot*** indicates the ***function value at the jump!*** ■

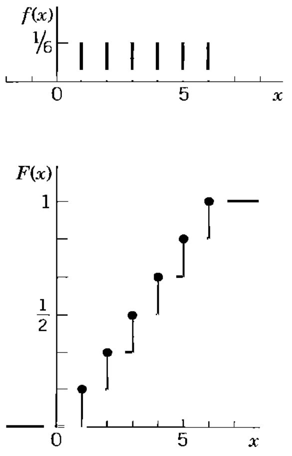


Fig. 512. Probability function $f(x)$ and distribution function $F(x)$ of the random variable X = *Number obtained in tossing a fair die once*

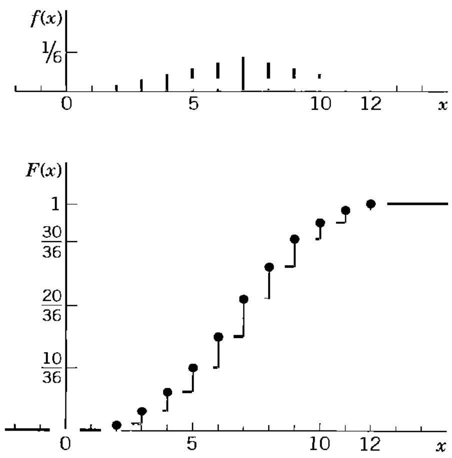


Fig. 513. Probability function $f(x)$ and distribution function $F(x)$ of the random variable X = *Sum of the two numbers obtained in tossing two fair dice once*

EXAMPLE 2 Probability Function and Distribution Function

The random variable X = *Sum of the two numbers two fair dice turn up* is discrete and has the possible values $2\ (= 1 + 1), 3, 4, \cdots, 12\ (= 6 + 6)$. There are $6 \cdot 6 = 36$ equally likely outcomes $(1, 1)\ (1, 2), \cdots, (6, 6)$, where the first number is that shown on the first die and the second number that on the other die. Each such outcome has probability 1/36. Now $X = 2$ occurs in the case of the outcome $(1, 1)$; $X = 3$ in the case of the two outcomes $(1, 2)$ and $(2, 1)$; $X = 4$ in the case of the three outcomes $(1, 3), (2, 2), (3, 1)$; and so on. Hence $f(x) = P(X = x)$ and $F(x) = P(X \leqq x)$ have the values

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
$F(x)$	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36

Figure 513 shows a bar chart of this function and the graph of the distribution function, which is again a step function, with jumps (of different height!) at the possible values of X.

Two useful formulas for discrete distributions are readily obtained as follows. For the probability corresponding to intervals we have from (2) and (4)

$$P(a < X \leqq b) = F(b) - F(a) = \sum_{a < x_j \leqq b} p_j \qquad (X \text{ discrete}). \tag{5}$$

This is the sum of all probabilities p_j for which x_j satisfies $a < x_j \leqq b$. (Be careful about $<$ and $\leqq$!) From this and $P(S) = 1$ (Sec. 24.3) we obtain the following formula.

$$\sum_j p_j = 1 \qquad \text{(sum of all probabilities)}. \tag{6}$$

EXAMPLE 3 Illustration of Formula (5)

In Example 2, compute the probability of a sum of at least 4 and at most 8.

Solution. $P(3 < X \leqq 8) = F(8) - F(3) = \frac{26}{36} - \frac{3}{36} = \frac{23}{36}$. ■

EXAMPLE 4 Waiting Time Problem. Countably Infinite Sample Space

In tossing a fair coin, let X = *Number of trials until the first head appears*. Then, by independence of events (Sec. 24.3),

$$\begin{aligned} P(X = 1) &= P(H) &&= \tfrac{1}{2} && (H = \text{Head}) \\ P(X = 2) &= P(TH) &&= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} && (T = \text{Tail}) \\ P(X = 3) &= P(TTH) &&= \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{8}. && \text{etc.} \end{aligned}$$

and in general $P(X = n) = (\frac{1}{2})^n$, $n = 1, 2, \cdots$. Also, (6) can be confirmed by the sum formula for the geometric series,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = -1 + \frac{1}{1 - \frac{1}{2}}$$

$$= -1 + 2 = 1.$$ ■

Continuous Random Variables and Distributions

Discrete random variables appear in experiments in which we ***count*** (defectives in a production, days of sunshine in Chicago, customers standing in a line, etc.). Continuous random variables appear in experiments in which we ***measure*** (lengths of screws, voltage in a power line, Brinell hardness of steel, etc.). By definition, a random variable X and its distribution are *of continuous type* or, briefly, **continuous,** if its distribution function $F(x)$ [defined in (1)] can be given by an integral

$$F(x) = \int_{-\infty}^{x} f(v)\, dv \tag{7}$$

(we write v because x is needed as the upper limit of the integral) whose integrand $f(x)$, called the **density** of the distribution, is nonnegative, and is continuous, perhaps except for finitely many x-values. Differentiation gives the relation of f to F as

$$f(x) = F'(x) \tag{8}$$

for every x at which $f(x)$ is continuous.

From (2) and (7) we obtain the very important formula for the probability corresponding to an interval:

$$P(a < X \leqq b) = F(b) - F(a) = \int_{a}^{b} f(v)\, dv. \tag{9}$$

This is the analog of (5).

From (7) and $P(S) = 1$ (Sec. 24.3) we also have the analog of (6):

$$\int_{-\infty}^{\infty} f(v)\, dv = 1. \tag{10}$$

Continuous random variables are ***simpler than discrete ones*** with respect to intervals. Indeed, in the continuous case the four probabilities corresponding to $a < X \leqq b$, $a < X < b$, $a \leqq X < b$, and $a \leqq X \leqq b$ with any fixed a and b ($> a$) are all the same. Can you see why? (*Answer.* This probability is the area under the density curve, as in Fig. 514, and does not change by adding or subtracting a single point in the interval of integration.) This is different from the discrete case! (Explain.)

The next example illustrates notations and typical applications of our present formulas.

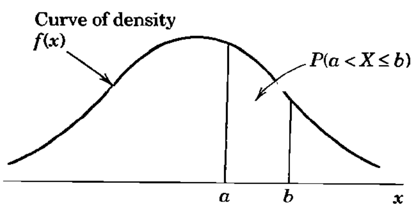


Fig. 514. Example illustrating formula (9)

EXAMPLE 5 Continuous Distribution

Let X have the density function $f(x) = 0.75(1 - x^2)$ if $-1 \leqq x \leqq 1$ and zero otherwise. Find the distribution function. Find the probabilities $P(-\frac{1}{2} \leqq X \leqq \frac{1}{2})$ and $P(\frac{1}{4} \leqq X \leqq 2)$. Find x such that $P(X \leqq x) = 0.95$.

Solution. From (7) we obtain $F(x) = 0$ if $x \leqq -1$,

$$F(x) = 0.75 \int_{-1}^{x} (1 - v^2)\, dv = 0.5 + 0.75x - 0.25x^3 \qquad \text{if } -1 < x \leqq 1,$$

and $F(x) = 1$ if $x > 1$. From this and (9) we get

$$P(-\tfrac{1}{2} \leqq X \leqq \tfrac{1}{2}) = F(\tfrac{1}{2}) - F(-\tfrac{1}{2}) = 0.75 \int_{-1/2}^{1/2} (1 - v^2)\, dv = 68.75\%$$

(because $P(-\frac{1}{2} \leqq X \leqq \frac{1}{2}) = P(-\frac{1}{2} < X \leqq \frac{1}{2})$ for a continuous distribution) and

$$P(\tfrac{1}{4} \leqq X \leqq 2) = F(2) - F(\tfrac{1}{4}) = 0.75 \int_{1/4}^{1} (1 - v^2)\, dv = 31.64\%.$$

(Note that the upper limit of integration is 1, not 2. Why?) Finally,

$$P(X \leqq x) = F(x) = 0.5 + 0.75x - 0.25x^3 = 0.95.$$

Algebraic simplification gives $3x - x^3 = 1.8$. A solution is $x = 0.73$, approximately.

Sketch $f(x)$ and mark $x = -\frac{1}{2}, \frac{1}{2}, \frac{1}{4}$, and 0.73, so that you can see the results (the probabilities) as areas under the curve. Sketch also $F(x)$. ■

Further examples of continuous distributions are included in the next problem set and in later sections.

PROBLEM SET 24.5

1. Graph the probability function $f(x) = kx^2$ ($x = 1, 2, 3, 4, 5$; k suitable) and the distribution function.
2. Graph the density function $f(x) = kx^2$ ($0 \leqq x \leqq 5$; k suitable) and the distribution function.
3. **(Uniform distribution)** Graph f and F when the density is $f(x) = k = const$ if $-4 \leqq x \leqq 4$ and 0 elsewhere.
4. In Prob. 3 find $P(0 \leqq x \leqq 4)$ and c such that $P(-c < X < c) = 95\%$.
5. Graph f and F when $f(-2) = f(2) = 1/8$, $f(-1) = f(1) = 3/8$. Can f have further positive values?
6. Graph the distribution function $F(x) = 1 - e^{-3x}$ if $x > 0$, $F(x) = 0$ if $x \leqq 0$, and the density $f(x)$. Find x such that $F(x) = 0.9$.
7. Let X be the number of years before a particular type of machine will need replacement. Assume that X has the probability function $f(1) = 0.1$, $f(2) = 0.2$, $f(3) = 0.2$, $f(4) = 0.2$, $f(5) = 0.3$. Graph f and F. Find the probability that the machine needs no replacement during the first 3 years.
8. If X has the probability function $f(x) = k/2^x$ ($x = 0, 1, 2, \cdots$), what are k and $P(X \geqq 4)$?
9. Find the probability that none of the three bulbs in a traffic signal must be replaced during the first 1200 hours of operation if the probability that a bulb must be replaced is a random variable X with density $f(x) = 6[0.25 - (x - 1.5)^2]$ when $1 \leqq x \leqq 2$ and $f(x) = 0$ otherwise, where x is time measured in multiples of 1000 hours.
10. Suppose that certain bolts have length $L = 200 + X$ mm, where X is a random variable with density $f(x) = \frac{3}{4}(1 - x^2)$ if $-1 \leqq x \leqq 1$ and 0 otherwise. Determine c so that with a probability of 95% a bolt will have any length between $200 - c$ and $200 + c$. *Hint:* See also Example 5.
11. Let X [millimeters] be the thickness of washers a machine turns out. Assume that X has the density $f(x) = kx$ if $1.9 < x < 2.1$ and 0 otherwise. Find k. What is the probability that a washer will have thickness between 1.95 mm and 2.05 mm?

12. Suppose that in an automatic process of filling oil into cans, the content of a can (in gallons) is $Y = 50 + X$, where X is a random variable with density $f(x) = 1 - |x|$ when $|x| \leqq 1$ and 0 when $|x| > 1$. Graph $f(x)$ and $F(x)$. In a lot of 100 cans, about how many will contain 50 gallons or more? What is the probability that a can will contain less than 49.5 gallons? Less than 49 gallons?

13. Let the random variable X with density $f(x) = ke^{-x}$ if $0 \leqq x \leqq 2$ and 0 otherwise (x = time measured in years) be the time after which certain ball bearings are worn out. Find k and the probability that a bearing will last at least 1 year.

14. Let X be the ratio of sales to profits of some firm. Assume that X has the distribution function $F(x) = 0$ if $x < 2$, $F(x) = (x^2 - 4)/5$ if $2 \leqq x < 3$, $F(x) = 1$ if $x \geqq 3$. Find and graph the density. What is the probability that X is between 2.5 (40% profit) and 5 (20% profit)?

15. Show that $b < c$ implies $P(X \leqq b) \leqq P(X \leqq c)$.

16. If the diameter X of axles has the density $f(x) = k$ if $119.9 \leqq x \leqq 120.1$ and 0 otherwise, how many defectives will a lot of 500 axles approximately contain if defectives are axles slimmer than 119.92 or thicker than 120.08?

17. Let X be a random variable that can assume every real value. What are the complements of the events $X \leqq b$, $X < b$, $X \geqq c$, $X > c$, $b \leqq X \leqq c$, $b < X \leqq c$?

18. A box contains 4 right-handed and 6 left-handed screws. Two screws are drawn at random without replacement. Let X be the number of left-handed screws drawn. Find the probabilities $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, $P(1 < X < 2)$, $P(X \leqq 1)$, $P(X \geqq 1)$, $P(X > 1)$, and $P(0.5 < X < 10)$.

24.6 Mean and Variance of a Distribution

The mean μ and variance σ^2 of a random variable X and of its distribution are the theoretical counterparts of the mean $\bar{x}$ and variance s^2 of a frequency distribution in Sec. 24.1 and serve a similar purpose. Indeed, the mean characterizes the central location and the variance the spread (the variability) of the distribution. The **mean** μ (mu) is defined by

$$(1)\qquad \text{(a)}\quad \mu = \sum_j x_j f(x_j) \qquad \text{(Discrete distribution)}$$

$$\text{(b)}\quad \mu = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \text{(Continuous distribution)}$$

and the **variance** σ^2 (sigma square) by

$$(2)\qquad \text{(a)}\quad \sigma^2 = \sum_j (x_j - \mu)^2 f(x_j) \qquad \text{(Discrete distribution)}$$

$$\text{(b)}\quad \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \qquad \text{(Continuous distribution).}$$

σ (the positive square root of σ^2) is called the **standard deviation** of X and its distribution. f is the probability function or the density, respectively, in (a) and (b).

The mean μ is also denoted by $E(X)$ and is called the **expectation** *of* X because it gives the average value of X to be expected in many trials. Quantities such as μ and σ^2 that measure certain properties of a distribution are called **parameters.** μ and σ^2 are the two most important ones. From (2) we see that

$$(3)\qquad \sigma^2 > 0$$

(except for a discrete "distribution" with only one possible value, so that $\sigma^2 = 0$). We assume that μ and σ^2 exist (are finite), as is the case for practically all distributions that are useful in applications.

EXAMPLE 1 Mean and Variance

The random variable X = *Number of heads in a single toss of a fair coin* has the possible values $X = 0$ and $X = 1$ with probabilities $P(X = 0) = \frac{1}{2}$ and $P(X = 1) = \frac{1}{2}$. From (1a) we thus obtain the mean $\mu = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$, and (2a) yields the variance

$$\sigma^2 = (0 - \tfrac{1}{2})^2 \cdot \tfrac{1}{2} + (1 - \tfrac{1}{2})^2 \cdot \tfrac{1}{2} = \tfrac{1}{4}.$$ ■

EXAMPLE 2 Uniform Distribution. Variance Measures Spread

The distribution with the density

$$f(x) = \frac{1}{b - a} \qquad \text{if} \qquad a < x < b$$

and $f = 0$ otherwise is called the **uniform distribution** on the interval $a < x < b$. From (1b) (or from Theorem 1, below) we find that $\mu = (a + b)/2$, and (2b) yields the variance

$$\sigma^2 = \int_a^b \left(x - \frac{a + b}{2}\right)^2 \frac{1}{b - a}\, dx = \frac{(b - a)^2}{12}.$$

Figure 515 illustrates that the spread is large if and only if σ^2 is large. ■

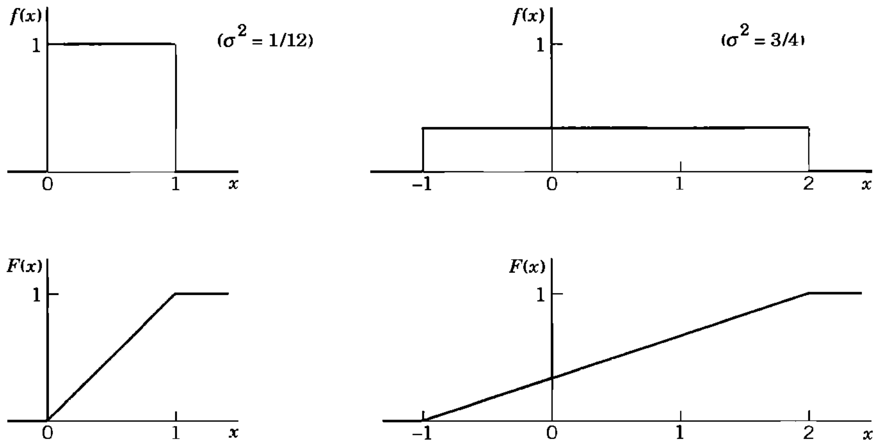


Fig. 515. Uniform distributions having the same mean (0.5) but different variances σ^2

Symmetry. We can obtain the mean μ without calculation if a distribution is symmetric. Indeed, you may prove

THEOREM 1 Mean of a Symmetric Distribution

If a distribution is **symmetric** *with respect to $x = c$, that is, $f(c - x) = f(c + x)$, then $\mu = c$.* (Examples 1 and 2 illustrate this.)

Transformation of Mean and Variance

Given a random variable X with mean μ and variance σ^2, we want to calculate the mean and variance of $X^* = a_1 + a_2 X$, where a_1 and a_2 are given constants. This problem is important in statistics, where it appears often.

THEOREM 2 **Transformation of Mean and Variance**

(a) *If a random variable X has mean μ and variance σ^2, then the random variable*

$$X^* = a_1 + a_2X \qquad (a_2 > 0) \tag{4}$$

has the mean μ^* and variance σ^{*2}, where

$$\mu^* = a_1 + a_2\mu \qquad \text{and} \qquad \sigma^{*2} = a_2{}^2\sigma^2. \tag{5}$$

(b) *In particular, the* **standardized random variable** *Z corresponding to X, given by*

$$Z = \frac{X - \mu}{\sigma} \tag{6}$$

has the mean 0 and the variance 1.

PROOF We prove (5) for a continuous distribution. To a small interval I of length Δx on the x-axis there corresponds the probability $f(x)\Delta x$ [approximately; the area of a rectangle of base Δx and height $f(x)$]. Then the probability $f(x)\Delta x$ must equal that for the corresponding interval on the x^*-axis, that is, $f^*(x^*)\Delta x^*$, where f^* is the density of X^* and Δx^* is the length of the interval on the x^*-axis corresponding to I. Hence for differentials we have $f^*(x^*)\,dx^* = f(x)\,dx$. Also, $x^* = a_1 + a_2x$ by (4), so that (1b) applied to X^* gives

$$\begin{aligned}\mu^* &= \int_{-\infty}^{\infty} x^*f^*(x^*)\,dx^* \\ &= \int_{-\infty}^{\infty} (a_1 + a_2x)f(x)\,dx \\ &= a_1\int_{-\infty}^{\infty} f(x)\,dx + a_2\int_{-\infty}^{\infty} xf(x)\,dx.\end{aligned}$$

On the right the first integral equals 1, by (10) in Sec. 24.5. The second integral is μ. This proves (5) for μ^*. It implies

$$x^* - \mu^* = (a_1 + a_2x) - (a_1 + a_2\mu) = a_2(x - \mu).$$

From this and (2) applied to X^*, again using $f^*(x^*)\,dx^* = f(x)\,dx$, we obtain the second formula in (5),

$$\sigma^{*2} = \int_{-\infty}^{\infty} (x^* - \mu^*)^2f^*(x^*)\,dx^* = a_2{}^2\int_{-\infty}^{\infty} (x - \mu)^2f(x)\,dx = a_2{}^2\sigma^2.$$

For a discrete distribution the proof of (5) is similar.

Choosing $a_1 = -\mu/\sigma$ and $a_2 = 1/\sigma$ we obtain (6) from (4), writing $X^* = Z$. For these a_1, a_2 formula (5) gives $\mu^* = 0$ and $\sigma^{*2} = 1$, as claimed in (b). ■

Expectation, Moments

Recall that (1) defines the expectation (the mean) of X, the value of X to be expected on the average, written $\mu = E(X)$. More generally, if $g(x)$ is nonconstant and continuous for all x, then $g(X)$ is a random variable. Hence its *mathematical expectation* or, briefly, its **expectation** $E(g(X))$ is the value of $g(X)$ to be expected on the average, defined [similarly to (1)] by

$$E(g(X)) = \sum_j g(x_j)f(x_j) \qquad \text{or} \qquad E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx. \tag{7}$$

In the first formula, f is the probability function of the discrete random variable X. In the second formula, f is the density of the continuous random variable X. Important special cases are the ***k*th moment** of X (where $k = 1, 2, \cdots$)

$$E(X^k) = \sum_j x_j^k f(x_j) \qquad \text{or} \qquad \int_{-\infty}^{\infty} x^k f(x)\,dx \tag{8}$$

and the ***k*th central moment** of X ($k = 1, 2, \cdots$)

$$E([X-\mu]^k) = \sum_j (x_j - \mu)^k f(x_j) \qquad \text{or} \qquad \int_{-\infty}^{\infty} (x-\mu)^k f(x)\,dx. \tag{9}$$

This includes the first moment, the **mean** of X

$$\mu = E(X) \qquad [(8) \text{ with } k = 1]. \tag{10}$$

It also includes the second central moment, the **variance** of X

$$\sigma^2 = E([X-\mu]^2) \qquad [(9) \text{ with } k = 2]. \tag{11}$$

For later use you may prove

$$E(1) = 1. \tag{12}$$

PROBLEM SET 24.6

1–6 MEAN, VARIANCE

Find the mean and the variance of the random variable X with probability function or density $f(x)$.

1. $f(x) = 2x \quad (0 \leqq x \leqq 1)$
2. $f(0) = 0.512, \quad f(1) = 0.384, \quad f(2) = 0.096, \quad f(3) = 0.008$
3. $X =$ *Number a fair die turns up*
4. $Y = -4X + 5$ with X as in Prob. 1
5. Uniform distribution on [0, 8]
6. $f(x) = 2e^{-2x} \quad (x \geqq 0)$
7. What is the expected daily profit if a store sells X air conditioners per day with probability $f(10) = 0.1$, $f(11) = 0.3$, $f(12) = 0.4$, $f(13) = 0.2$ and the profit per conditioner is \$55?
8. What is the mean life of a light bulb whose life X [hours] has the density $f(x) = 0.001e^{-0.001x}$ $(x \geqq 0)$?
9. If the mileage (in multiples of 1000 mi) after which a tire must be replaced is given by the random variable X with density $f(x) = \theta e^{-\theta x}$ $(x > 0)$, what mileage can you expect to get on one of these tires? Let $\theta = 0.04$ and find the probability that a tire will last at least 40000 mi.

10. What sum can you expect in rolling a fair die 10 times? Do it. Repeat this experiment 20 times and record how the sum varies.

11. A small filling station is supplied with gasoline every Saturday afternoon. Assume that its volume X of sales in ten thousands of gallons has the probability density $f(x) = 6x(1 - x)$ if $0 \leqq x \leqq 1$ and 0 otherwise. Determine the mean, the variance, and the standardized variable.

12. What capacity must the tank in Prob. 11 have in order that the probability that the tank will be emptied in a given week be 5%?

13. Let X [cm] be the diameter of bolts in a production. Assume that X has the density $f(x) = k(x - 0.9)(1.1 - x)$ if $0.9 < x < 1.1$ and 0 otherwise. Determine k, sketch $f(x)$, and find μ and σ^2.

14. Suppose that in Prob. 13, a bolt is regarded as being defective if its diameter deviates from 1.00 cm by more than 0.09 cm. What percentage of defective bolts should we then expect?

15. For what choice of the maximum possible deviation c from 1.00 cm shall we obtain 3% defectives in Probs. 13 and 14?

16. TEAM PROJECT. Means, Variances, Expectations.

(a) Show that $E(X - \mu) = 0$, $\sigma^2 = E(X^2) - \mu^2$.

(b) Prove (10)–(12).

(c) Find all the moments of the uniform distribution on an interval $a \leqq x \leqq b$.

(d) The **skewness** γ of a random variable X is defined by

$$\gamma = \frac{1}{\sigma^3} E([X - \mu]^3). \tag{13}$$

Show that for a symmetric distribution (whose third central moment exists) the skewness is zero.

(e) Find the skewness of the distribution with density $f(x) = xe^{-x}$ when $x > 0$ and $f(x) = 0$ otherwise. Sketch $f(x)$.

(f) Calculate the skewness of a few simple discrete distributions of your own choice.

(g) Find a nonsymmetric discrete distribution with 3 possible values, mean 0, and skewness 0.

24.7 Binomial, Poisson, and Hypergeometric Distributions

These are the three most important ***discrete*** distributions, with numerous applications.

Binomial Distribution

The **binomial distribution** occurs in games of chance (rolling a die, see below, etc.), quality inspection (e.g., counting of the number of defectives), opinion polls (counting number of employees favoring certain schedule changes, etc.), medicine (e.g., recording the number of patients recovered by a new medication), and so on. The conditions of its occurrence are as follows.

We are interested in the number of times an event A occurs in n independent trials. In each trial the event A has the same probability $P(A) = p$. Then in a trial, A will not occur with probability $q = 1 - p$. In n trials the random variable that interests us is

$$X = \textit{Number of times the event A occurs in n trials.}$$

X can assume the values $0, 1, \cdots, n$, and we want to determine the corresponding probabilities. Now $X = x$ means that A occurs in x trials and in $n - x$ trials it does not occur. This may look as follows.

$$\underbrace{A \quad A \cdots A}_{x \text{ times}} \qquad \underbrace{B \quad B \cdots B.}_{n - x \text{ times}} \tag{1}$$

Here $B = A^c$ is the complement of A, meaning that A does not occur (Sec. 24.2). We now use the assumption that the trials are independent, that is, they do not influence each other. Hence (1) has the probability (see Sec. 24.3 on independent events)

$$\underbrace{pp \cdots p}_{x \text{ times}} \cdot \underbrace{qq \cdots q}_{n - x \text{ times}} = p^x q^{n-x}. \tag{1*}$$

Now (1) is just one order of arranging x A's and $n - x$ B's. We now use Theorem 1(b) in Sec. 24.4, which gives the number of permutations of n things (the n outcomes of the n trials) consisting of 2 classes, class 1 containing the $n_1 = x$ A's and class 2 containing the $n - n_1 = n - x$ B's. This number is

$$\frac{n!}{x!(n-x)!} = \binom{n}{x}.$$

Accordingly, (1*) multiplied by this binomial coefficient gives the probability $P(X = x)$ of $X = x$, that is, of obtaining A precisely x times in n trials. Hence X has the probability function

$$f(x) = \binom{n}{x} p^x q^{n-x} \qquad (x = 0, 1, \cdots, n) \tag{2}$$

and $f(x) = 0$ otherwise. The distribution of X with probability function (2) is called the **binomial distribution** or *Bernoulli distribution.* The occurrence of A is called *success* (regardless of what it actually is; it may mean that you miss your plane or lose your watch) and the nonoccurrence of A is called *failure.* Figure 516 shows typical examples. Numeric values can be obtained from Table A5 in App. 5 or from your CAS.

The mean of the binomial distribution is (see Team Project 16)

$$\mu = np \tag{3}$$

and the variance is (see Team Project 16)

$$\sigma^2 = npq. \tag{4}$$

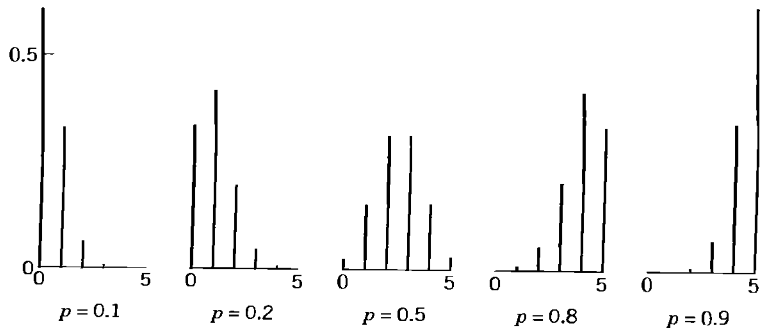


Fig. 516. Probability function (2) of the binomial distribution for $n = 5$ and various values of p

For the ***symmetric case*** of equal chance of success and failure ($p = q = 1/2$) this gives the mean $n/2$, the variance $n/4$, and the probability function

$$f(x) = \binom{n}{x}\left(\frac{1}{2}\right)^n \qquad (x = 0, 1, \cdots, n). \tag{2*}$$

EXAMPLE 1 Binomial Distribution

Compute the probability of obtaining at least two "*Six*" in rolling a fair die 4 times.

Solution. $p = P(A) = P(\text{"}Six\text{"}) = 1/6$, $q = 5/6$, $n = 4$. The event "*At least two 'Six'*" occurs if we obtain 2 or 3 or 4 "Six." Hence the answer is

$$P = f(2) + f(3) + f(4) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 + \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) + \binom{4}{4}\left(\frac{1}{6}\right)^4$$

$$= \frac{1}{6^4}(6 \cdot 25 + 4 \cdot 5 + 1) = \frac{171}{1296} = 13.2\%. \qquad \blacksquare$$

Poisson Distribution

The discrete distribution with infinitely many possible values and probability function

$$f(x) = \frac{\mu^x}{x!} e^{-\mu} \qquad (x = 0, 1, \cdots) \tag{5}$$

is called the **Poisson distribution,** named after S. D. Poisson (Sec. 18.5). Figure 517 shows (5) for some values of μ. It can be proved that this distribution is obtained as a limiting case of the binomial distribution, if we let $p \to 0$ and $n \to \infty$ so that the mean $\mu = np$ approaches a finite value. (For instance, $\mu = np$ may be kept constant.) The Poisson distribution has the mean μ and the variance (see Team Project 16)

$$\sigma^2 = \mu. \tag{6}$$

Figure 517 gives the impression that with increasing mean the spread of the distribution increases, thereby illustrating formula (6), and that the distribution becomes more and more (approximately) symmetric.

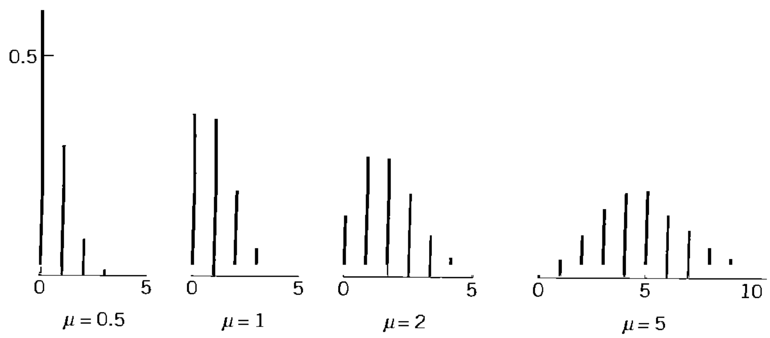


Fig. 517. Probability function (5) of the Poisson distribution for various values of μ

EXAMPLE 2 Poisson Distribution

If the probability of producing a defective screw is $p = 0.01$, what is the probability that a lot of 100 screws will contain more than 2 defectives?

Solution. The complementary event is A^c: *Not more than* 2 *defectives.* For its probability we get from the binomial distribution with mean $\mu = np = 1$ the value [see (2)]

$$P(A^c) = \binom{100}{0} 0.99^{100} + \binom{100}{1} 0.01 \cdot 0.99^{99} + \binom{100}{2} 0.01^2 \cdot 0.99^{98}.$$

Since p is very small, we can approximate this by the much more convenient Poisson distribution with mean $\mu = np = 100 \cdot 0.01 = 1$, obtaining [see (5)]

$$P(A^c) \approx e^{-1}\left(1 + 1 + \frac{1}{2}\right)$$

$$= 91.97\%.$$

Thus $P(A) = 8.03\%$. Show that the binomial distribution gives $P(A) = 7.94\%$, so that the Poisson approximation is quite good. ■

EXAMPLE 3 Parking Problems. Poisson Distribution

If on the average, 2 cars enter a certain parking lot per minute, what is the probability that during any given minute 4 or more cars will enter the lot?

Solution. To understand that the Poisson distribution is a model of the situation, we imagine the minute to be divided into very many short time intervals, let p be the (constant) probability that a car will enter the lot during any such short interval, and assume independence of the events that happen during those intervals. Then we are dealing with a binomial distribution with very large n and very small p, which we can approximate by the Poisson distribution with

$$\mu = np = 2,$$

because 2 cars enter on the average. The complementary event of the event "4 cars or more during a given minute" is "3 *cars or fewer enter the lot*" and has the probability

$$f(0) + f(1) + f(2) + f(3) = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right)$$

$$= 0.857.$$

Answer: 14.3%. (Why did we consider that complement?) ■

Sampling with Replacement

This means that we draw things from a given set one by one, and after each trial we replace the thing drawn (put it back to the given set and mix) before we draw the next thing. This guarantees independence of trials and leads to the **binomial distribution.** Indeed, if a box contains N things, for example, screws, M of which are defective, the probability of drawing a defective screw in a trial is $p = M/N$. Hence the probability of drawing a nondefective screw is $q = 1 - p = 1 - M/N$, and (2) gives the probability of drawing x defectives in n trials in the form

$$f(x) = \binom{n}{x}\left(\frac{M}{N}\right)^x \left(1 - \frac{M}{N}\right)^{n-x} \qquad (x = 0, 1, \cdots, n). \tag{7}$$

Sampling without Replacement. Hypergeometric Distribution

Sampling without replacement means that we return no screw to the box. Then we no longer have independence of trials (why?), and instead of (7) the probability of drawing x defectives in n trials is

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \qquad (x = 0, 1, \cdots, n). \tag{8}$$

The distribution with this probability function is called the **hypergeometric distribution** (because its moment generating function (see Team Project 16) can be expressed by the hypergeometric function defined in Sec. 5.4, a fact that we shall not use).

Derivation of (8). By (4a) in Sec. 24.4 there are

(a) $\binom{N}{n}$ different ways of picking n things from N,

(b) $\binom{M}{x}$ different ways of picking x defectives from M,

(c) $\binom{N-M}{n-x}$ different ways of picking $n - x$ nondefectives from $N - M$,

and each way in (b) combined with each way in (c) gives the total number of mutually exclusive ways of obtaining x defectives in n drawings without replacement. Since (a) is the total number of outcomes and we draw at random, each such way has the probability $1\Big/\binom{N}{n}$. From this, (8) follows. ■

The hypergeometric distribution has the mean (Team Project 16)

$$\mu = n\frac{M}{N} \tag{9}$$

and the variance

$$\sigma^2 = \frac{nM(N-M)(N-n)}{N^2(N-1)}. \tag{10}$$

EXAMPLE 4 Sampling with and without Replacement

We want to draw random samples of two gaskets from a box containing 10 gaskets, three of which are defective. Find the probability function of the random variable X = *Number of defectives in the sample.*

Solution. We have $N = 10$, $M = 3$, $N - M = 7$, $n = 2$. For sampling with replacement, (7) yields

$$f(x) = \binom{2}{x}\left(\frac{3}{10}\right)^x\left(\frac{7}{10}\right)^{2-x}, \qquad f(0) = 0.49, \quad f(1) = 0.42, \quad f(2) = 0.09.$$

For sampling without replacement we have to use (8), finding

$$f(x) = \binom{3}{x}\binom{7}{2-x}\Big/\binom{10}{2}, \qquad f(0) = f(1) = \frac{21}{45} \approx 0.47, \quad f(2) = \frac{3}{45} \approx 0.07.$$ ■

If N, M, and N − M are large compared with n, then it does not matter too much whether we sample with or without replacement, and in this case the hypergeometric distribution may be approximated by the binomial distribution (with $p = M/N$), *which is somewhat simpler.*

Hence in sampling from an indefinitely large population (**"infinite population"**) *we may use the binomial distribution, regardless of whether we sample with or without replacement.*

PROBLEM SET 24.7

1. Four fair coins are tossed simultaneously. Find the probability function of the random variable X = *Number of heads* and compute the probabilities of obtaining no heads, precisely 1 head, at least 1 head, not more than 3 heads.
2. If the probability of hitting a target in a single shot is 10% and 10 shots are fired independently, what is the probability that the target will be hit at least once?
3. In Prob. 2, if the probability of hitting would be 5% and we fired 20 shots, would the probability of hitting at least once be less than, equal to, or greater than in Prob. 2? Guess first, then compute.
4. Suppose that 3% of bolts made by a machine are defective, the defectives occurring at random during production. If the bolts are packaged 50 per box, what is the Poisson approximation of the probability that a given box will contain $x = 0, 1, \cdots, 5$ defectives?
5. Let X be the number of cars per minute passing a certain point of some road between 8 A.M. and 10 A.M. on a Sunday. Assume that X has a Poisson distribution with mean 5. Find the probability of observing 3 or fewer cars during any given minute.
6. Suppose that a telephone switchboard of some company on the average handles 300 calls per hour, and that the board can make at most 10 connections per minute. Using the Poisson distribution, estimate the probability that the board will be overtaxed during a given minute. (Use Table A6 in App. 5 or your CAS.)
7. **(Rutherford–Geiger experiments)** In 1910, E. Rutherford and H. Geiger showed experimentally that the number of alpha particles emitted per second in a radioactive process is a random variable X having a Poisson distribution. If X has mean 0.5, what is the probability of observing two or more particles during any given second?
8. A process of manufacturing screws is checked every hour by inspecting n screws selected at random from that hour's production. If one or more screws are defective, the process is halted and carefully examined. How large should n be if the manufacturer wants the probability to be about 95% that the process will be halted when 10% of the screws being produced are defective? (Assume independence of the quality of any screw of that of the other screws.)
9. Suppose that in the production of 50-Ω resistors, nondefective items are those that have a resistance between 45 Ω and 55 Ω and the probability of a resistor's being defective is 0.2%. The resistors are sold in lots of 100, with the guarantee that all resistors are nondefective. What is the probability that a given lot will violate this guarantee? (Use the Poisson distribution.)
10. Let $p = 1\%$ be the probability that a certain type of lightbulb will fail in a 24-hr test. Find the probability that a sign consisting of 10 such bulbs will burn 24 hours with no bulb failures.
11. Guess how much less the probability in Prob. 10 would be if the sign consisted of 100 bulbs. Then calculate.
12. Suppose that a certain type of magnetic tape contains, on the average, 2 defects per 100 meters. What is the probability that a roll of tape 300 meters long will contain (a) x defects, (b) no defects?
13. Suppose that a test for extrasensory perception consists of naming (in any order) 3 cards randomly drawn from a deck of 13 cards. Find the probability that by chance alone, the person will correctly name (a) no cards, (b) 1 card, (c) 2 cards, (d) 3 cards.
14. A carton contains 20 fuses, 5 of which are defective. Find the probability that, if a sample of 3 fuses is chosen from the carton by random drawing without replacement, x fuses in the sample will be defective.
15. **(Multinomial distribution)** Suppose a trial can result in precisely one of k mutually exclusive events $A_1, \cdots, A_k$ with probabilities $p_1, \cdots, p_k$, respectively, where $p_1 + \cdots + p_k = 1$. Suppose that n independent trials are performed. Show that the probability of getting x_1 A_1's, $\cdots$, x_k A_k's is

$$f(x_1, \cdots, x_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

where $0 \leqq x_j \leqq n$, $j = 1, \cdots, k$, and $x_1 + \cdots + x_k = n$. The distribution having this probability function is called the *multinomial distribution.*

16. TEAM PROJECT. Moment Generating Function. The moment generating function $G(t)$ is defined by

$$G(t) = E(e^{tX_j}) = \sum_j e^{tx_j} f(x_j)$$

or

$$G(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

where X is a discrete or continuous random variable, respectively.

(a) Assuming that termwise differentiation and differentiation under the integral sign are permissible, show that $E(X^k) = G^{(k)}(0)$, where $G^{(k)} = d^kG/dt^k$, in particular, $\mu = G'(0)$.

(b) Show that the binomial distribution has the moment generating function

$$G(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$
$$= (pe^t + q)^n.$$

(c) Using (b), prove (3).

(d) Prove (4).

(e) Show that the Poisson distribution has the moment generating function $G(t) = e^{-\mu} e^{\mu e^t}$ and prove (6).

(f) Prove $x\binom{M}{x} = M\binom{M-1}{x-1}$.

Using this, prove (9).

24.8 Normal Distribution

Turning from discrete to continuous distributions, in this section we discuss the normal distribution. This is the most important continuous distribution because in applications many random variables are **normal random variables** (that is, they have a normal distribution) or they are approximately normal or can be transformed into normal random variables in a relatively simple fashion. Furthermore, the normal distribution is a useful approximation of more complicated distributions, and it also occurs in the proofs of various statistical tests.

The **normal distribution** or *Gauss distribution* is defined as the distribution with the density

$$\text{(1)} \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad (\sigma > 0)$$

where exp is the exponential function with base $e = 2.718 \cdots$. This is simpler than it may at first look. $f(x)$ has these features (see also Fig. 518).

1. μ is the mean and σ the standard deviation.

2. $1/(\sigma\sqrt{2\pi})$ is a constant factor that makes the area under the curve of $f(x)$ from $-\infty$ to ∞ equal to 1, as it must be by (10), Sec. 24.5.

3. The curve of $f(x)$ is symmetric with respect to $x = \mu$ because the exponent is quadratic. Hence for $\mu = 0$ it is symmetric with respect to the y-axis $x = 0$ (Fig. 518, *"bell-shaped curves"*).

4. The exponential function in (1) goes to zero very fast—the faster the smaller the standard deviation σ is, as it should be (Fig. 518).

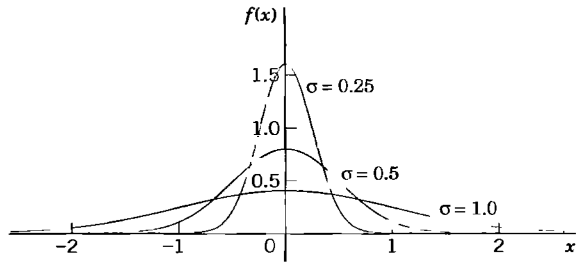


Fig. 518. Density (1) of the normal distribution with $\mu = 0$ for various values of σ

Distribution Function $F(x)$

From (7) in Sec. 24.5 and (1) we see that the normal distribution has the **distribution function**

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2\right] dv. \tag{2}$$

Here we needed x as the upper limit of integration and wrote v (instead of x) in the integrand.

For the corresponding **standardized normal distribution** with mean 0 and standard deviation 1 we denote $F(x)$ by $\Phi(z)$. Then we simply have from (2)

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du. \tag{3}$$

This integral cannot be integrated by one of the methods of calculus. But this is no serious handicap because its values can be obtained from Table A7 in App. 5 or from your CAS. These values are needed in working with the normal distribution. The curve of $\Phi(z)$ is S-shaped. It increases monotone (why?) from 0 to 1 and intersects the vertical axis at 1/2 (why?), as shown in Fig. 519.

Relation Between $F(x)$ and $\Phi(z)$. Although your CAS will give you values of $F(x)$ in (2) with any μ and σ directly, it is important to comprehend that and why any such an $F(x)$ can be expressed in terms of the tabulated standard $\Phi(z)$, as follows.

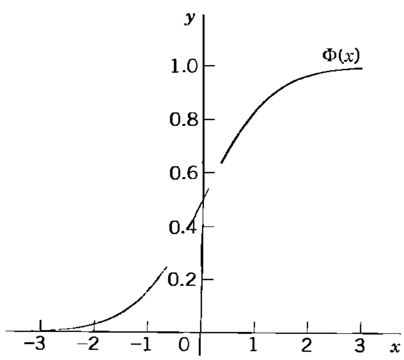


Fig. 519. Distribution function $\Phi(z)$ of the normal distribution with mean 0 and variance 1

THEOREM 1

Use of the Normal Table A7 in App. 5

The distribution function $F(x)$ of the normal distribution with any μ and σ [see (2)] *is related to the standardized distribution function $\Phi(z)$ in* (3) *by the formula*

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right). \tag{4}$$

PROOF Comparing (2) and (3) we see that we should set

$$u = \frac{v-\mu}{\sigma}. \qquad \text{Then } v = x \text{ gives} \qquad u = \frac{x-\mu}{\sigma}$$

as the new upper limit of integration. Also $v - \mu = \sigma u$, thus $dv = \sigma\, du$. Together, since σ drops out.

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-u^2/2}\, \sigma\, du = \Phi\left(\frac{x-\mu}{\sigma}\right) \qquad \blacksquare$$

Probabilities corresponding to intervals will be needed quite frequently in statistics in Chap. 25. These are obtained as follows.

THEOREM 2

Normal Probabilities for Intervals

The probability that a normal random variable X with mean μ and standard deviation σ assume any value in an interval $a < x \leqq b$ is

$$P(a < X \leqq b) = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \tag{5}$$

PROOF Formula (2) in Sec. 24.5 gives the first equality in (5), and (4) in this section gives the second equality. $\blacksquare$

Numeric Values

In practical work with the normal distribution it is good to remember that about 2/3 of all values of X to be observed will lie between $\mu \pm \sigma$, about 95% between $\mu \pm 2\sigma$, and practically all between the **three-sigma limits** $\mu \pm 3\sigma$. More precisely, by Table A7 in App. 5,

$$\begin{aligned}
&\text{(a)} \quad P(\mu - \sigma < X \leqq \mu + \sigma) \approx 68\% \\
(6) \quad &\text{(b)} \quad P(\mu - 2\sigma < X \leqq \mu + 2\sigma) \approx 95.5\% \\
&\text{(c)} \quad P(\mu - 3\sigma < X \leqq \mu + 3\sigma) \approx 99.7\%.
\end{aligned}$$

Formulas (6a) and (6b) are illustrated in Fig. 520.

The formulas in (6) show that a value deviating from μ by more than σ, 2σ, or 3σ will occur in one of about 3, 20, and 300 trials, respectively.

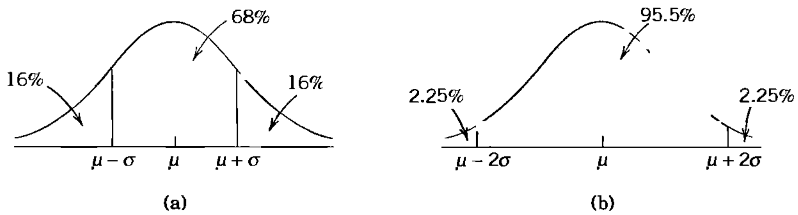


Fig. 520. Illustration of formula (6)

In tests (Chap. 25) we shall ask conversely for the intervals that correspond to certain given probabilities; practically most important are the probabilities of 95%, 99%, and 99.9%. For these, Table A8 in App. 5 gives the answers $\mu \pm 2\sigma$, $\mu \pm 2.5\sigma$, and $\mu \pm 3.3\sigma$, respectively. More precisely,

$$
\text{(7)} \quad \begin{aligned}
&\text{(a)} \quad P(\mu - 1.96\sigma < X \leqq \mu + 1.96\sigma) = 95\% \\
&\text{(b)} \quad P(\mu - 2.58\sigma < X \leqq \mu + 2.58\sigma) = 99\% \\
&\text{(c)} \quad P(\mu - 3.29\sigma < X \leqq \mu + 3.29\sigma) = 99.9\%.
\end{aligned}
$$

Working With the Normal Tables A7 and A8 in App. 5

There are two normal tables in App. 5, Tables A7 and A8. If you want probabilities, use Table A7. If probabilities are given and corresponding intervals or x-values are wanted, use Table A8. The following examples are typical. Do them with care, verifying all values, and don't just regard them as dull exercises for your software. Make sketches of the density to see whether the results look reasonable.

EXAMPLE 1 Reading Entries from Table A7

If X is standardized normal (so that $\mu = 0$, $\sigma = 1$), then

$$P(X \leqq 2.44) = 0.9927 \approx 99\tfrac{1}{4}\%$$

$$P(X \leqq -1.16) = 1 - \Phi(1.16) = 1 - 0.8770 = 0.1230 = 12.3\%$$

$$P(X \geqq 1) = 1 - P(X \leqq 1) = 1 - 0.8413 = 0.1587 \text{ by (7), Sec. 24.3}$$

$$P(1.0 \leqq X \leqq 1.8) = \Phi(1.8) - \Phi(1.0) = 0.9641 - 0.8413 = 0.1228. \qquad \blacksquare$$

EXAMPLE 2 Probabilities for Given Intervals, Table A7

Let X be normal with mean 0.8 and variance 4 (so that $\sigma = 2$). Then by (4) and (5)

$$P(X \leqq 2.44) = F(2.44) = \Phi\left(\frac{2.44 - 0.80}{2}\right) = \Phi(0.82) = 0.7939 \approx 80\%$$

or if you like it better (similarly in the other cases)

$$P(X \leqq 2.44) = P\left(\frac{X - 0.80}{2} \leqq \frac{2.44 - 0.80}{2}\right) = P(Z \leqq 0.82) = 0.7939$$

$$P(X \geqq 1) = 1 - P(X \leqq 1) = 1 - \Phi\left(\frac{1 - 0.8}{2}\right) = 1 - 0.5398 = 0.4602$$

$$P(1.0 \leqq X \leqq 1.8) = \Phi(0.5) - \Phi(0.1) = 0.6915 - 0.5398 = 0.1517. \qquad \blacksquare$$

EXAMPLE 3 Unknown Values c for Given Probabilities, Table A8

Let X be normal with mean 5 and variance 0.04 (hence standard deviation 0.2). Find c or k corresponding to the given probability

$$P(X \leqq c) = 95\%, \qquad \Phi\left(\frac{c-5}{0.2}\right) = 95\%, \qquad \frac{c-5}{0.2} = 1.645, \qquad c = 5.329$$

$$P(5 - k \leqq X \leqq 5 + k) = 90\%, \qquad 5 + k = 5.329 \qquad \text{(as before; why?)}$$

$$P(X \geqq c) = 1\%, \qquad \text{thus } P(X \leqq c) = 99\%, \qquad \frac{c-5}{0.2} = 2.326, \qquad c = 5.465. \qquad \blacksquare$$

EXAMPLE 4 Defectives

In a production of iron rods let the diameter X be normally distributed with mean 2 in. and standard deviation 0.008 in.

(a) What percentage of defectives can we expect if we set the tolerance limits at 2 ± 0.02 in.?

(b) How should we set the tolerance limits to allow for 4% defectives?

Solution. **(a)** $1\frac{1}{4}\%$ because from (5) and Table A7 we obtain for the complementary event the probability

$$\begin{aligned} P(1.98 \leqq X \leqq 2.02) &= \Phi\left(\frac{2.02 - 2.00}{0.008}\right) - \Phi\left(\frac{1.98 - 2.00}{0.008}\right) \\ &= \Phi(2.5) - \Phi(-2.5) \\ &= 0.9938 - (1 - 0.9938) \\ &= 0.9876 \\ &\approx 98\tfrac{3}{4}\%. \end{aligned}$$

(b) 2 ± 0.0164 because for the complementary event we have

$$0.96 = P(2 - c \leqq X \leqq 2 + c)$$

or

$$0.98 = P(X \leqq 2 + c)$$

so that Table A8 gives

$$0.98 = \Phi\left(\frac{2 + c - 2}{0.008}\right),$$

$$\frac{2 + c - 2}{0.008} = 2.054, \qquad c = 0.0164. \qquad \blacksquare$$

Normal Approximation of the Binomial Distribution

The probability function of the binomial distribution is (Sec. 24.7)

$$f(x) = \binom{n}{x} p^x q^{n-x} \qquad (x = 0, 1, \cdots, n). \tag{8}$$

If n is large, the binomial coefficients and powers become very inconvenient. It is of great practical (and theoretical) importance that in this case the normal distribution provides a good approximation of the binomial distribution, according to the following theorem, one of the most important theorems in all probability theory.

THEOREM 3

Limit Theorem of De Moivre and Laplace

For large n,

$$f(x) \sim f^*(x) \qquad (x = 0, 1, \cdots, n). \tag{9}$$

Here f is given by (8). The function

$$f^*(x) = \frac{1}{\sqrt{2\pi}\sqrt{npq}}\, e^{-z^2/2}, \qquad z = \frac{x - np}{\sqrt{npq}} \tag{10}$$

is the density of the normal distribution with mean $\mu = np$ and variance $\sigma^2 = npq$ (the mean and variance of the binomial distribution). The symbol $\sim$ (read **asymptotically equal**) *means that the ratio of both sides approaches 1 as n approaches ∞. Furthermore, for any nonnegative integers a and b ($> a$),*

$$\begin{gathered} P(a \leqq X \leqq b) = \sum_{x=a}^{b} \binom{n}{x} p^x q^{n-x} \sim \Phi(\beta) - \Phi(\alpha), \\ \alpha = \frac{a - np - 0.5}{\sqrt{npq}}, \qquad \beta = \frac{b - np + 0.5}{\sqrt{npq}}. \end{gathered} \tag{11}$$

A proof of this theorem can be found in [G3] listed in App. 1. The proof shows that the term 0.5 in α and β is a correction caused by the change from a discrete to a continuous distribution.

PROBLEM SET 24.8

1–13 NORMAL DISTRIBUTION

1. Let X be normal with mean 80 and variance 9. Find $P(X > 83)$, $P(X < 81)$, $P(X < 80)$, and $P(78 < X < 82)$.

2. Let X be normal with mean 120 and variance 16. Find $P(X \leqq 126)$, $P(X > 116)$, $P(125 < X < 130)$.

3. Let X be normal with mean 14 and variance 4. Determine c such that $P(X \leqq c) = 95\%$, $P(X \leqq c) = 5\%$, $P(X \leqq c) = 99.5\%$.

4. Let X be normal with mean 4.2 and variance 0.04. Find c such that $P(X \leqq c) = 50\%$, $P(X > c) = 10\%$, $P(-c < X - 4.2 \leqq c) = 99\%$.

5. If the lifetime X of a certain kind of automobile battery is normally distributed with a mean of 4 yr and a standard deviation of 1 yr, and the manufacturer wishes to guarantee the battery for 3 yr, what percentage of the batteries will he have to replace under the guarantee?

6. If the standard deviation in Prob. 5 were smaller, would that percentage be smaller or larger?

7. A manufacturer knows from experience that the resistance of resistors he produces is normal with mean $\mu = 150\ \Omega$ and standard deviation $\sigma = 5\ \Omega$. What percentage of the resistors will have resistance between $148\ \Omega$ and $152\ \Omega$? Between $140\ \Omega$ and $160\ \Omega$?

8. The breaking strength X [kg] of a certain type of plastic block is normally distributed with a mean of 1250 kg and a standard deviation of 55 kg. What is the maximum load such that we can expect no more than 5% of the blocks to break?

9. A manufacturer produces airmail envelopes whose weight is normal with mean $\mu = 1.950$ grams and standard deviation $\sigma = 0.025$ grams. The envelopes are sold in lots of 1000. How many envelopes in a lot will be heavier than 2 grams?

10. If the resistance X of certain wires in an electrical network is normal with mean 0.01 Ω and standard deviation 0.001 Ω, how many of 1000 wires will meet the specification that they have resistance between 0.009 and 0.011 Ω?

11. If the mathematics scores of the SAT college entrance exams are normal with mean 480 and standard deviation 100 (these are about the actual values over the past years) and if some college sets 500 as the minimum score for new students, what percent of students will not reach that score?

12. If the monthly machine repair and maintenance cost X in a certain factory is known to be normal with mean \$12000 and standard deviation \$2000, what is the probability that the repair cost for the next month will exceed the budgeted amount of \$15000?

13. If sick-leave time X used by employees of a company in one month is (very roughly) normal with mean 1000 hours and standard deviation 100 hours, how much time t should be budgeted for sick leave during the next month if t is to be exceeded with probability of only 20%?

14. TEAM PROJECT. Normal Distribution. **(a)** Derive the formulas in (6) and (7) from the appropriate normal table.

(b) Show that $\Phi(-z) = 1 - \Phi(z)$. Give an example.

(c) Find the points of inflection of the curve of (1).

(d) Considering $\Phi^2(\infty)$ and introducing polar coordinates in the double integral (a standard trick worth remembering), prove

$$\Phi(\infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\, du = 1. \tag{12}$$

(e) Show that σ in (1) is indeed the standard deviation of the normal distribution. [Use (12).]

(f) **Bernoulli's law of large numbers.** In an experiment let an event A have probability p $(0 < p < 1)$, and let X be the number of times A happens in n independent trials. Show that for any given $\epsilon > 0$,

$$P\left(\left|\frac{X}{n} - p\right| \leqq \epsilon\right) \to 1 \qquad \text{as } n \to \infty.$$

(g) **Transformation.** If X is normal with mean μ and variance σ^2, show that $X^* = c_1X + c_2$ $(c_1 > 0)$ is normal with mean $\mu^* = c_1\mu + c_2$ and variance $\sigma^{*2} = c_1{}^2\sigma^2$.

15. WRITING PROJECT. Use of Tables. Give a systematic discussion of the use of Tables A7 and A8 for obtaining $P(X < b)$, $P(X > a)$, $P(a < X < b)$, $P(X < c) = k$, $P(X > c) = k$, as well as $P(\mu - c < X < \mu + c) = k$; include simple examples. If you have a CAS, describe to what extent it makes the use of those tables superfluous; give examples.

24.9 Distributions of Several Random Variables

Distributions of two or more random variables are of interest for two reasons:

1. They occur in experiments in which we observe several random variables, for example, carbon content X and hardness Y of steel, amount of fertilizer X and yield of corn Y, height X_1, weight X_2, and blood pressure X_3 of persons, and so on.

2. They will be needed in the mathematical justification of the methods of statistics in Chap. 25.

In this section we consider two random variables X and Y or, as we also say, a **two-dimensional random variable** (X, Y). For (X, Y) the outcome of a trial is a pair of numbers $X = x$, $Y = y$, briefly $(X, Y) = (x, y)$, which we can plot as a point in the XY-plane.

The **two-dimensional probability distribution** of the random variable (X, Y) is given by the **distribution function**

$$F(x, y) = P(X \leqq x,\ Y \leqq y). \tag{1}$$

This is the probability that in a trial, X will assume any value not greater than x and in the same trial, Y will assume any value not greater than y. This corresponds to the blue region in Fig. 521, which extends to $-\infty$ to the left and below. $F(x, y)$ determines the

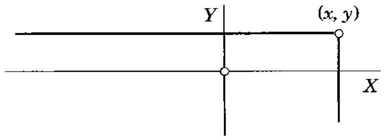


Fig. 521. Formula (1)

probability distribution uniquely, because in analogy to formula (2) in Sec. 24.5, that is, $P(a < X \leqq b) = F(b) - F(a)$, we now have for a rectangle (see Prob. 14)

$$(2) \quad P(a_1 < X \leqq b_1, \quad a_2 < Y \leqq b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2).$$

As before, in the two-dimensional case we shall also have discrete and continuous random variables and distributions.

Discrete Two-Dimensional Distributions

In analogy to the case of a single random variable (Sec. 24.5), we call (X, Y) and its distribution **discrete** if (X, Y) can assume only finitely many or at most countably infinitely many pairs of values (x_1, y_1), (x_2, y_2), $\cdots$ with positive probabilities, whereas the probability for any domain containing none of those values of (X, Y) is zero.

Let (x_i, y_j) be any of those pairs and let $P(X = x_i, Y = y_j) = p_{ij}$ (where we admit that p_{ij} may be 0 for certain pairs of subscripts i, j). Then we define the **probability function** $f(x, y)$ of (X, Y) by

$$(3) \qquad f(x, y) = p_{ij} \quad \text{if} \quad x = x_i, \ y = y_j \qquad \text{and} \qquad f(x, y) = 0 \quad \text{otherwise;}$$

here, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots$ independently. In analogy to (4), Sec. 24.5, we now have for the distribution function the formula

$$(4) \qquad F(x, y) = \sum_{x_i \leqq x} \sum_{y_j \leqq y} f(x_i, y_j).$$

Instead of (6) in Sec. 24.5 we now have the condition

$$(5) \qquad \sum_i \sum_j f(x_i, y_j) = 1.$$

EXAMPLE 1 Two-Dimensional Discrete Distribution

If we simultaneously toss a dime and a nickel and consider

$$X = \textit{Number of heads the dime turns up,}$$
$$Y = \textit{Number of heads the nickel turns up,}$$

then X and Y can have the values 0 or 1, and the probability function is

$$f(0, 0) = f(1, 0) = f(0, 1) = f(1, 1) = \tfrac{1}{4}, \quad f(x, y) = 0 \text{ otherwise.}$$ ■

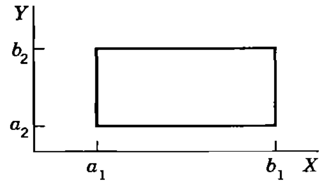


Fig. 522. Notion of a two-dimensional distribution

Continuous Two-Dimensional Distributions

In analogy to the case of a single random variable (Sec. 24.5) we call (X, Y) and its distribution **continuous** if the corresponding distribution function $F(x, y)$ can be given by a double integral

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x^*, y^*)\, dx^*\, dy^* \tag{6}$$

whose integrand f, called the **density** of (X, Y), is nonnegative everywhere, and is continuous, possibly except on finitely many curves.

From (6) we obtain the probability that (X, Y) assume any value in a rectangle (Fig. 522) given by the formula

$$P(a_1 < X \leqq b_1,\ \ a_2 < Y \leqq b_2) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y)\, dx\, dy. \tag{7}$$

EXAMPLE 2 Two-Dimensional Uniform Distribution in a Rectangle

Let R be the rectangle $\alpha_1 < x \leqq \beta_1$, $\alpha_2 < y \leqq \beta_2$. The density (see Fig. 523)

$$f(x, y) = 1/k \quad \text{if } (x, y) \text{ is in } R, \qquad f(x, y) = 0 \text{ otherwise} \tag{8}$$

defines the so-called **uniform distribution** *in the rectangle R*; here $k = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)$ is the area of R. The distribution function is shown in Fig. 524. ■

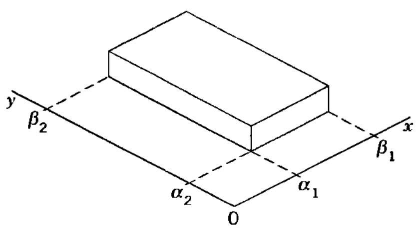


Fig. 523. Density function (8) of the uniform distribution

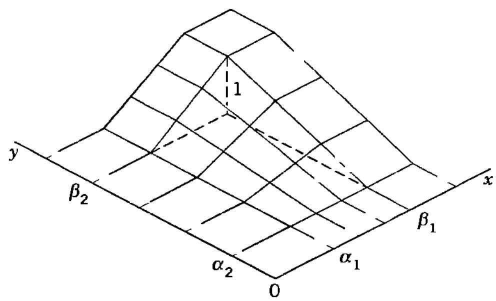


Fig. 524. Distribution function of the uniform distribution defined by (8)

Marginal Distributions of a Discrete Distribution

This is a rather natural idea, without counterpart for a single random variable. It amounts to being interested only in one of the two variables in (X, Y), say, X, and asking for its distribution, called the **marginal distribution** of X in (X, Y). So we ask for the probability $P(X = x, Y \text{ arbitrary})$. Since (X, Y) is discrete, so is X. We get its probability function, call it $f_1(x)$, from the probability function $f(x, y)$ of (X, Y) by summing over y:

$$f_1(x) = P(X = x, Y \text{ arbitrary}) = \sum_y f(x, y) \tag{9}$$

where we sum all the values of $f(x, y)$ that are not 0 for that x.

From (9) we see that the distribution function of the marginal distribution of X is

$$F_1(x) = P(X \leqq x, Y \text{ arbitrary}) = \sum_{x^* \leqq x} f_1(x^*). \tag{10}$$

Similarly, the probability function

$$f_2(y) = P(X \text{ arbitrary}, Y = y) = \sum_x f(x, y) \tag{11}$$

determines the **marginal distribution** of Y in (X, Y). Here we sum all the values of $f(x, y)$ that are not zero for the corresponding y. The distribution function of this marginal distribution is

$$F_2(y) = P(X \text{ arbitrary}, Y \leqq y) = \sum_{y^* \leqq y} f_2(y^*). \tag{12}$$

EXAMPLE 3 Marginal Distributions of a Discrete Two-Dimensional Random Variable

In drawing 3 cards with replacement from a bridge deck let us consider

$$(X, Y), \qquad X = \textit{Number of queens}, \qquad Y = \textit{Number of kings or aces}.$$

The deck has 52 cards. These include 4 queens, 4 kings, and 4 aces. Hence in a single trial a queen has probability $4/52 = 1/13$ and a king or ace $8/52 = 2/13$. This gives the probability function of (X, Y),

$$f(x, y) = \frac{3!}{x!\, y!\, (3 - x - y)!} \left(\frac{1}{13}\right)^x \left(\frac{2}{13}\right)^y \left(\frac{10}{13}\right)^{3-x-y} \qquad (x + y \leqq 3)$$

and $f(x, y) = 0$ otherwise. Table 24.1 shows in the center the values of $f(x, y)$ and on the right and lower margins the values of the probability functions $f_1(x)$ and $f_2(y)$ of the marginal distributions of X and Y, respectively. ■

Table 24.1 Values of the Probability Functions $f(x, y)$, $f_1(x)$, $f_2(y)$ in Drawing Three Cards with Replacement from a Bridge Deck, where X is the Number of Queens Drawn and Y is the Number of Kings or Aces Drawn

x \ y	0	1	2	3	$f_1(x)$
0	$\frac{1000}{2197}$	$\frac{600}{2197}$	$\frac{120}{2197}$	$\frac{8}{2197}$	$\frac{1728}{2197}$
1	$\frac{300}{2197}$	$\frac{120}{2197}$	$\frac{12}{2197}$	0	$\frac{432}{2197}$
2	$\frac{30}{2197}$	$\frac{6}{2197}$	0	0	$\frac{36}{2197}$
3	$\frac{1}{2197}$	0	0	0	$\frac{1}{2197}$
$f_2(y)$	$\frac{1331}{2197}$	$\frac{726}{2197}$	$\frac{132}{2197}$	$\frac{8}{2197}$	

Marginal Distributions of a Continuous Distribution

This is conceptually the same as for discrete distributions, with probability functions and sums replaced by densities and integrals. For a continuous random variable (X, Y) with density $f(x, y)$ we now have the **marginal distribution** of X in (X, Y), defined by the distribution function

$$F_1(x) = P(X \leqq x, -\infty < Y < \infty) = \int_{-\infty}^{x} f_1(x^*)\, dx^* \tag{13}$$

with the density f_1 of X obtained from $f(x, y)$ by integration over y,

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy. \tag{14}$$

Interchanging the roles of X and Y, we obtain the **marginal distribution** of Y in (X, Y) with the distribution function

$$F_2(y) = P(-\infty < X < \infty, Y \leqq y) = \int_{-\infty}^{y} f_2(y^*)\, dy^* \tag{15}$$

and density

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx. \tag{16}$$

Independence of Random Variables

X and Y in a (discrete or continuous) random variable (X, Y) are said to be **independent** if

$$F(x, y) = F_1(x)F_2(y) \tag{17}$$

holds for all (x, y). Otherwise these random variables are said to be **dependent.** These definitions are suggested by the corresponding definitions for events in Sec. 24.3.

Necessary and sufficient for independence is

$$f(x, y) = f_1(x)f_2(y) \tag{18}$$

for all x and y. Here the f's are the above probability functions if (X, Y) is discrete or those densities if (X, Y) is continuous. (See Prob. 20.)

EXAMPLE 4 Independence and Dependence

In tossing a dime and a nickel, X = *Number of heads on the dime*, Y = *Number of heads on the nickel* may assume the values 0 or 1 and are independent. The random variables in Table 24.1 are dependent. ■

Extension of Independence to n-Dimensional Random Variables. This will be needed throughout Chap. 25. The distribution of such a random variable $\mathbf{X} = (X_1, \cdots, X_n)$ is determined by a **distribution function** of the form

$$F(x_1, \cdots, x_n) = P(X_1 \leqq x_1, \cdots, X_n \leqq x_n).$$

The random variables $X_1, \cdots, X_n$ are said to be **independent** if

$$F(x_1, \cdots, x_n) = F_1(x_1)F_2(x_2) \cdots F_n(x_n) \tag{19}$$

for all $(x_1, \cdots, x_n)$. Here $F_j(x_j)$ is the distribution function of the marginal distribution of X_j in **X**, that is,

$$F_j(x_j) = P(X_j \leqq x_j, X_k \text{ arbitrary}, k = 1, \cdots, n, k \neq j).$$

Otherwise these random variables are said to be **dependent.**

Functions of Random Variables

When $n = 2$, we write $X_1 = X, X_2 = Y, x_1 = x, x_2 = y$. Taking a nonconstant continuous function $g(x, y)$ defined for all x, y, we obtain a random variable $Z = g(X, Y)$. For example, if we roll two dice and X and Y are the numbers the dice turn up in a trial, then $Z = X + Y$ is the sum of those two numbers (see Fig. 513 in Sec. 24.5).

In the case of a ***discrete*** random variable (X, Y) we may obtain the probability function $f(z)$ of $Z = g(X, Y)$ by summing all $f(x, y)$ for which $g(x, y)$ equals the value of z considered; thus

$$f(z) = P(Z = z) = \sum_{g(x,y)=z}\sum f(x, y). \tag{20}$$

Hence the distribution function of Z is

$$F(z) = P(Z \leqq z) = \sum_{g(x,y)\leqq z}\sum f(x, y) \tag{21}$$

where we sum all values of $f(x, y)$ for which $g(x, y) \leqq z$.

In the case of a ***continuous*** random variable (X, Y) we similarly have

$$F(z) = P(Z \leqq z) = \iint_{g(x,y)\leqq z} f(x, y)\, dx\, dy \tag{22}$$

where for each z we integrate the density $f(x, y)$ of (X, Y) over the region $g(x, y) \leqq z$ in the xy-plane, the boundary curve of this region being $g(x, y) = z$.

Addition of Means

The number

$$E(g(X, Y)) = \begin{cases} \displaystyle\sum_x \sum_y g(x, y)f(x, y) & [(X, Y) \text{ discrete}] \\ \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)\, dx\, dy & [(X, Y) \text{ continuous}] \end{cases} \tag{23}$$

is called the *mathematical expectation* or, briefly, the **expectation of** $g(X, Y)$. Here it is assumed that the double series converges absolutely and the integral of $|g(x, y)|f(x, y)$ over the xy-plane exists (is finite). Since summation and integration are linear processes, we have from (23)

$$E(ag(X, Y) + bh(X, Y)) = aE(g(X, Y)) + bE(h(X, Y)). \tag{24}$$

An important special case is

$$E(X + Y) = E(X) + E(Y),$$

and by induction we have the following result.

THEOREM 1 **Addition of Means**

The mean (expectation) of a sum of random variables equals the sum of the means (expectations), that is,

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n). \tag{25}$$

Furthermore, we readily obtain

THEROEM 2 **Multiplication of Means**

The mean (expectation) of the product of ***independent*** *random variables equals the product of the means (expectations), that is,*

$$E(X_1X_2 \cdots X_n) = E(X_1)E(X_2) \cdots E(X_n). \tag{26}$$

PROOF If X and Y are independent random variables (both discrete or both continuous), then $E(XY) = E(X)E(Y)$. In fact, in the discrete case we have

$$E(XY) = \sum_x \sum_y xyf(x, y) = \sum_x xf_1(x) \sum_y yf_2(y) = E(X)E(Y),$$

and in the continuous case the proof of the relation is similar. Extension to n independent random variables gives (26), and Theorem 2 is proved. ∎

Addition of Variances

This is another matter of practical importance that we shall need. As before, let $Z = X + Y$ and denote the mean and variance of Z by μ and σ^2. Then we first have (see Team Project 16(a) in Problem Set 24.6)

$$\sigma^2 = E([Z - \mu]^2) = E(Z^2) - [E(Z)]^2.$$

From (24) we see that the first term on the right equals

$$E(Z^2) = E(X^2 + 2XY + Y^2) = E(X^2) + 2E(XY) + E(Y^2).$$

For the second term on the right we obtain from Theorem 1

$$[E(Z)]^2 = [E(X) + E(Y)]^2 = [E(X)]^2 + 2E(X)E(Y) + [E(Y)]^2.$$

By substituting these expressions into the formula for σ^2 we have

$$\begin{aligned}\sigma^2 = E(X^2) - [E(X)]^2 + E(Y^2) - [E(Y)]^2 \\ + 2[E(XY) - E(X)E(Y)].\end{aligned}$$

From Team Project 16, Sec. 24.6, we see that the expression in the first line on the right is the sum of the variances of X and Y, which we denote by σ_1^2 and σ_2^2, respectively. The quantity in the second line (except for the factor 2) is

$$\sigma_{XY} = E(XY) - E(X)E(Y) \tag{27}$$

and is called the **covariance** of X and Y. Consequently, our result is

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\sigma_{XY}. \tag{28}$$

If X and Y are independent, then

$$E(XY) = E(X)E(Y);$$

hence $\sigma_{XY} = 0$, and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2. \tag{29}$$

Extension to more than two variables gives the basic

THEOREM 3

Addition of Variances

*The variance of the sum of **independent** random variables equals the sum of the variances of these variables.*

CAUTION! In the numerous applications of Theorems 1 and 3 we must always remember that Theorem 3 holds only for ***independent*** variables.

This is the end of Chap. 24 on probability theory. Most of the concepts, methods, and special distributions discussed in this chapter will play a fundamental role in the next chapter, which deals with methods of **statistical inference,** that is, conclusions from samples to populations, whose unknown properties we want to know and try to discover by looking at suitable properties of samples that we have obtained.

PROBLEM SET

1. Let $f(x, y) = k$ when $8 \leqq x \leqq 12$ and $0 \leqq y \leqq 2$ and zero elsewhere. Find k. Find $P(X \leqq 11, 1 \leqq Y \leqq 1.5)$ and $P(9 \leqq X \leqq 13, Y \leqq 1)$.

2. Find $P(X > 2, Y > 2)$ and $P(X \leqq 1, Y \leqq 1)$ if (X, Y) has the density $f(x, y) = 1/8$ if $x \geqq 0, y \geqq 0, x + y \leqq 4$.

3. Let $f(x, y) = k$ if $x > 0, y > 0, x + y < 3$ and 0 otherwise. Find k. Sketch $f(x, y)$. Find $P(X + Y \leqq 1)$, $P(Y > X)$.

4. Find the density of the marginal distribution of X in Prob 2.

5. Find the density of the marginal distribution of Y in Fig. 523.

6. If certain sheets of wrapping paper have a mean weight of 10 g each, with a standard deviation of 0.05 g, what are the mean weight and standard deviation of a pack of 10 000 sheets?

7. What are the mean thickness and the standard deviation of transformer cores each consisting of 50 layers of sheet metal and 49 insulating paper layers if the metal sheets have mean thickness 0.5 mm each with a standard deviation of 0.05 mm and the paper layers have mean 0.05 mm each with a standard deviation of 0.02 mm?

8. If the weight of certain (empty) containers has mean 2 lb and standard deviation 0.1 lb, and if the filling of the containers has mean weight 75 lb and standard deviation 0.8 lb, what are the mean weight and standard deviation of filled containers?

9. A 5-gear assembly is put together with spacers between the gears. The mean thickness of the gears is 5.020 cm with a standard deviation of 0.003 cm. The mean thickness of the spacers is 0.040 cm with a standard deviation of 0.002 cm. Find the mean and standard deviation of the assembled units consisting of 5 randomly selected gears and 4 randomly selected spacers.

10. Give an example of two different discrete distributions that have the same marginal distributions.

11. Show that the random variables with the densities

$$f(x, y) = x + y$$

and

$$g(x, y) = (x + \tfrac{1}{2})(y + \tfrac{1}{2})$$

if $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ and $f(x, y) = 0$ and $g(x, y) = 0$ elsewhere, have the same marginal distribution.

12. Let X [cm] and Y [cm] be the diameter of a pin and hole, respectively. Suppose that (X, Y) has the density

$$f(x, y) = 2500 \qquad \text{if}$$

$$0.99 < x < 1.01, \ 1.00 < y < 1.02$$

and 0 otherwise. (a) Find the marginal distributions. (b) What is the probability that a pin chosen at random will fit a hole whose diameter is 1.00?

13. An electronic device consists of two components. Let X and Y [months] be the length of time until failure of the first and second component, respectively. Assume that (X, Y) has the probability density

$$f(x, y) = 0.01e^{-0.1(x+y)} \qquad \text{if } x > 0 \text{ and } y > 0$$

and 0 otherwise. (a) Are X and Y dependent or independent? (b) Find the densities of the marginal distributions. (c) What is the probability that the first component has a lifetime of 10 months or longer?

14. Prove (2).

15. Find $P(X > Y)$ when (X, Y) has the density

$$f(x, y) = 0.25e^{-0.5(x+y)} \text{ if } x \geqq 0, y \geqq 0$$

and 0 otherwise.

16. Let (X, Y) have the density

$$f(x, y) = k \text{ if } x^2 + y^2 < 1$$

and 0 otherwise. Determine k. Find the densities of the marginal distributions. Find the probability

$$P(X^2 + Y^2 < 1/4).$$

17. Let (X, Y) have the probability function

$$f(0, 0) = f(1, 1) = 1/8,$$
$$f(0, 1) = f(1, 0) = 3/8.$$

Are X and Y independent?

18. Using Theorem 1, obtain the formula for the mean of the hypergeometric distribution. Can you use Theorem 3 to obtain the variance of that distribution?

19. Using Theorems 1 and 3, obtain the formulas for the mean and the variance of the binomial distribution.

20. Prove the statement involving (18).

CHAPTER 24 REVIEW QUESTIONS AND PROBLEMS

1. Why did we begin the chapter with a section on handling data?
2. What are stem-and-leaf plots? Boxplots? Histograms? Compare their advantages.
3. What quantities measure the average size of data? The spread?
4. Why did we consider probability theory? What is its role in statistics?
5. What do we mean by an experiment? By a random variable related with it? What are outcomes? Events?
6. Give examples of experiments in which you have equally likely cases and others in which you don't.
7. State the definition of probability from memory.
8. What is the difference between the concepts of a permutation and a combination?
9. State the main theorems on probability. Illustrate them by simple examples.
10. What is the distribution of a random variable? The distribution function? The probability function? The density?
11. State the definitions of mean and variance of a random variable from memory.
12. If $P(A) = P(B)$ and $A \subseteq B$, can $A \neq B$?
13. If $E \neq S$ (= the sample space), can $P(E) = 1$?
14. What distributions correspond to sampling with replacement and without replacement?
15. When will an experiment involve a binomial distribution? A hypergeometric distribution?
16. When will the Poisson distribution be a good approximation of the binomial distribution?
17. What do you know about the approximation of the binomial distribution by the normal distribution?
18. Explain the use of the tables of the normal distribution. If you have a CAS, how would you proceed without the tables?
19. Can the probability function of a discrete random variable have infinitely many positive values?
20. State the most important facts about distributions of two random variables and their marginal distributions.
21. Make a stem-and-leaf plot, histogram, and boxplot of the data 22.5. 23.2, 22.1, 23.6, 23.3, 23.4, 24.0, 20.6, 23.3.
22. Do the same task as in Prob. 21, for the data 210, 213, 209, 218, 210, 215, 204, 211, 216, 213.
23. Find the mean, standard deviation, and variance in Prob. 21.
24. Find the mean, standard deviation, and variance in Prob. 22.
25. What are the outcomes of the sample space of *X: Tossing a coin until the first Head appears*?
26. What are the outcomes in the sample space of the experiment of simultaneously tossing three coins?
27. A box contains 50 screws, five of which are defective. Find the probability function of the random variable X = *Number of defective screws in drawing two screws without replacement* and compute its values.
28. Find the values of the distribution function in Prob. 27.
29. Using a Venn diagram, show that $A \subseteq B$ if and only if $A \cup B = B$.
30. Using a Venn diagram, show that $A \subseteq B$ if and only if $A \cap B = A$.
31. If X has the density $f(x) = 0.5x$ $(0 \leqq x \leqq 2)$ and 0 otherwise, what are the mean and the variance of $X^* = -2X + 5$?
32. If 6 different inks are available, in how many ways can we select two colors for a printing job? Four colors?
33. Compute 5! by the Stirling formula and find the absolute and relative errors.
34. Two screws are randomly drawn without replacement from a box containing 7 right-handed and 3 left-handed screws. Let X be the number of left-handed screws drawn. Find $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, $P(1 < X < 2)$, $P(0 < X < 5)$.
35. Find the mean and the variance of the distribution having the density $f(x) = \frac{1}{2}e^{-|x|}$.
36. Find the skewness of the distribution with density $f(x) = 2(1 - x)$ if $0 < x < 1$, $f(x) = 0$ otherwise.
37. Sketch the probability function $f(x) = x^2/30$ $(x = 1, 2, 3, 4)$ and the distribution function. Find μ.
38. Sketch $F(x) = 0$ if $x \leqq 0$, $F(x) = 0.2x$ if $0 < x \leqq 5$, $F(x) = 1$ if $x > 5$, and its density $f(x)$.
39. If the life of tires is normal with mean 25 000 km and variance 25 000 000 km^2, what is the probability that a given one of those tires will last at least 30 000 km? At least 35 000 km?
40. If the weight of bags of cement is normal with mean 50 kg and standard deviation 1 kg, what is the probability that 100 bags will be heavier than 5030 kg?

Data Analysis. Probability Theory

A *random experiment,* briefly called **experiment,** is a process in which the result (**"outcome"**) depends on "chance" (effects of factors unknown to us). Examples are games of chance with dice or cards, measuring the hardness of steel, observing weather conditions, or recording the number of accidents in a city. (Thus the word "experiment" is used here in a much wider sense than in common language.) The outcomes are regarded as points (elements) of a set S, called the **sample space,** whose subsets are called **events.** For events E we define a **probability** $P(E)$ by the axioms (Sec. 24.3)

$$\begin{aligned} &0 \leqq P(E) \leqq 1 \\ (1) \qquad &P(S) = 1 \\ &P(E_1 \cup E_2 \cup \cdots) = P(E_1) + P(E_2) + \cdots \qquad (E_j \cap E_k = \varnothing). \end{aligned}$$

These axioms are motivated by properties of frequency distributions of data (Sec. 24.1).

The complement E^c of E has the probability

$$P(E^c) = 1 - P(E). \tag{2}$$

The **conditional probability** of an event B under the condition that an event A happens is (Sec. 24.3)

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad [P(A) > 0]. \tag{3}$$

Two events A and B are called **independent** if the probability of their simultaneous appearance in a trial equals the product of their probabilities, that is, if

$$P(A \cap B) = P(A)P(B). \tag{4}$$

With an experiment we associate a **random variable** X. This is a function defined on S whose values are real numbers; furthermore, X is such that the probability $P(X = a)$ with which X assumes any value a, and the probability $P(a < X \leqq b)$ with which X assumes any value in an interval $a < X \leqq b$ are defined (Sec. 24.5). The **probability distribution** of X is determined by the distribution function

$$F(x) = P(X \leqq x). \tag{5}$$

In applications there are two important kinds of random variables: those of the **discrete** type, which appear if we count (defective items, customers in a bank, etc.) and those of the **continuous** type, which appear if we measure (length, speed, temperature, weight, etc.).

A discrete random variable has a **probability function**

$$f(x) = P(X = x). \tag{6}$$

Its **mean** μ and **variance** σ^2 are (Sec. 24.6)

$$\mu = \sum_j x_j f(x_j) \qquad \text{and} \qquad \sigma^2 = \sum_j (x_j - \mu)^2 f(x_j) \tag{7}$$

where the x_j are the values for which X has a positive probability. Important discrete random variables and distributions are the **binomial, Poisson,** and **hypergeometric distributions** discussed in Sec. 24.7.

A continuous random variable has a **density**

$$f(x) = F'(x) \qquad \text{[see (5)].} \tag{8}$$

Its mean and variance are (Sec. 24.6)

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \text{and} \qquad \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,dx. \tag{9}$$

Very important is the **normal distribution** (Sec. 24.8), whose density is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{10}$$

and whose distribution function is (Sec. 24.8; Tables A7, A8 in App. 5)

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right). \tag{11}$$

A **two-dimensional random variable** (X, Y) occurs if we simultaneously observe two quantities (for example, height X and weight Y of adults). Its distribution function is (Sec. 24.9)

$$F(x, y) = P(X \leqq x,\ Y \leqq y). \tag{12}$$

X and Y have the distribution functions (Sec. 24.9)

$$F_1(x) = P(X \leqq x,\ Y \text{ arbitrary}) \qquad \text{and} \qquad F_2(y) = P(x \text{ arbitrary},\ Y \leqq y) \tag{13}$$

respectively; their distributions are called **marginal distributions.** If both X and Y are discrete, then (X, Y) has a probability function

$$f(x, y) = P(X = x,\ Y = y).$$

If both X and Y are continuous, then (X, Y) has a density $f(x, y)$.

CHAPTER 25

Mathematical Statistics

In probability theory we set up mathematical models of processes that are affected by "chance". In mathematical statistics or, briefly, **statistics,** we check these models against the observable reality. This is called **statistical inference.** It is done by **sampling,** that is, by drawing random samples, briefly called **samples.** These are sets of values from a much larger set of values that could be studied, called the **population.** An example is 10 diameters of screws drawn from a large lot of screws. Sampling is done in order to see whether a model of the population is accurate enough for practical purposes. If this is the case, the model can be used for predictions, decisions, and actions, for instance, in planning productions, buying equipment, investing in business projects, and so on.

Most important methods of statistical inference are **estimation of parameters** (Secs. 25.2), determination of **confidence intervals** (Sec. 25.3), and **hypothesis testing** (Secs. 25.4, 25.7, 25.8), with application to *quality control* (Sec. 25.5) and *acceptance sampling* (Sec. 25.6).

In the last section (25.9) we give an introduction to **regression** and **correlation analysis,** which concern experiments involving two variables.

Prerequisite: Chap. 24.
Sections that may be omitted in a shorter course: 25.5, 25.6, 25.8.
References, Answers to Problems, and Statistical Tables: App. 1 Part G, App. 2, App. 5.

25.1 Introduction. Random Sampling

Mathematical statistics consists of methods for designing and evaluating random experiments to obtain information about practical problems, such as exploring the relation between iron content and density of iron ore, the quality of raw material or manufactured products, the efficiency of air-conditioning systems, the performance of certain cars, the effect of advertising, the reactions of consumers to a new product, etc.

Random variables occur more frequently in engineering (and elsewhere) than one would think. For example, properties of mass-produced articles (screws, lightbulbs, etc.) always show **random variation,** due to small (uncontrollable!) differences in raw material or manufacturing processes. Thus the diameter of screws is a random variable X and we have *nondefective screws,* with diameter between given tolerance limits, and *defective screws,* with diameter outside those limits. We can ask for the distribution of X, for the percentage of defective screws to be expected, and for necessary improvements of the production process.

Samples are selected from populations—20 screws from a lot of 1000, 100 of 5000 voters, 8 beavers in a wildlife conservation project—because inspecting the entire population would be too expensive, time-consuming, impossible or even senseless (think of destructive testing of lightbulbs or dynamite). To obtain meaningful conclusions, samples must be **random selections.** Each of the 1000 screws must have the same chance of being sampled (of being drawn when we sample), at least approximately. Only then will the sample mean $\bar{x} = (x_1 + \cdots + x_{20})/20$ (Sec. 24.1) of a sample of size $n = 20$ (or any other n) be a good approximation of the population mean μ (Sec. 24.6); and the accuracy of the approximation will generally improve with increasing n, as we shall see. Similarly for other parameters (standard deviation, variance, etc.).

Independent sample values will be obtained in experiments with an infinite sample space S (Sec. 24.2), certainly for the normal distribution. This is also true in sampling with replacement. It is approximately true in drawing *small* samples from a large finite population (for instance, 5 or 10 of 1000 items). However, if we sample without replacement from a small population, the effect of dependence of sample values may be considerable.

Random numbers help in obtaining samples that are in fact random selections. This is sometimes not easy to accomplish because there are many subtle factors that can bias sampling (by personal interviews, by poorly working machines, by the choice of nontypical observation conditions, etc.). Random numbers can be obtained from a **random number generator** in Maple, Mathematica, or other systems listed on p. 991. (The numbers are not truly random, as they would be produced in flipping coins or rolling dice, but are calculated by a tricky formula that produces numbers that do have practically all the essential features of true randomness.)

EXAMPLE 1 Random Numbers from a Random Number Generator

To select a sample of size $n = 10$ from 80 given ball bearings, we number the bearings from 1 to 80. We then let the generator randomly produce 10 of the integers from 1 to 80 and include the bearings with the numbers obtained in our sample, for example.

44 55 53 03 52 61 67 78 39 54

or whatever.

Random numbers are also contained in (older) statistical tables. ■

Representing and processing data were considered in Sec. 24.1 in connection with frequency distributions. These are the empirical counterparts of probability distributions and helped motivating axioms and properties in probability theory. The new aspect in this chapter is **randomness:** the data are samples selected **randomly** from a population. Accordingly, we can immediately make the connection to Sec. 24.1, using stem-and-leaf plots, box plots, and histograms for representing samples graphically.

Also, we now call the mean $\bar{x}$ in (5), Sec. 24.1, the **sample mean**

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}(x_1 + x_2 + \cdots + x_n). \tag{1}$$

We call n the **sample size,** the variance s^2 in (6), Sec. 24.1, the **sample variance**

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2 = \frac{1}{n-1}\left[(x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2\right], \tag{2}$$

and its positive square root s the **sample standard deviation.** $\bar{x}$, s^2, and s are called **parameters** *of a sample;* they will be needed throughout this chapter.

25.2 Point Estimation of Parameters

Beginning in this section, we shall discuss the most basic practical tasks in statistics and corresponding statistical methods to accomplish them. The first of them is point estimation of **parameters,** that is, of quantities appearing in distributions, such as p in the binomial distribution and μ and σ in the normal distribution.

A **point estimate** of a parameter is a number (point on the real line), which is computed from a given sample and serves as an approximation of the unknown exact value of the parameter of the population. An **interval estimate** is an interval ("*confidence interval*") obtained from a sample; such estimates will be considered in the next section. Estimation of parameters is of great practical importance in many applications.

As an approximation of the mean μ of a population we may take the mean $\bar{x}$ of a corresponding sample. This gives the estimate $\hat{\mu} = \bar{x}$ for μ, that is,

$$\hat{\mu} = \bar{x} = \frac{1}{n}(x_1 + \cdots + x_n) \tag{1}$$

where n is the sample size. Similarly, an estimate $\hat{\sigma}^2$ for the variance of a population is the variance s^2 of a corresponding sample, that is,

$$\hat{\sigma}^2 = s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2. \tag{2}$$

Clearly, (1) and (2) are estimates of parameters for distributions in which μ or σ^2 appear explicity as parameters. such as the normal and Poisson distributions. For the binomial distribution, $p = \mu/n$ [see (3) in Sec. 24.7]. From (1) we thus obtain for p the estimate

$$\hat{p} = \frac{\bar{x}}{n}. \tag{3}$$

We mention that (1) is a special case of the so-called **method of moments.** In this method the parameters to be estimated are expressed in terms of the moments of the distribution (see Sec. 24.6). In the resulting formulas those moments of the distribution are replaced by the corresponding moments of the sample. This gives the estimates. Here the **kth moment of a sample** $x_1, \cdots, x_n$ is

$$m_k = \frac{1}{n}\sum_{j=1}^{n} x_j^{\,k}.$$

Maximum Likelihood Method

Another method for obtaining estimates is the so-called **maximum likelihood method** of R. A. Fisher [*Messenger Math.* **41** (1912), 155–160]. To explain it, we consider a discrete (or continuous) random variable X whose probability function (or density) $f(x)$ depends on a single parameter θ. We take a corresponding sample of n ***independent*** values $x_1, \cdots, x_n$. Then in the discrete case the probability that a sample of size n consists precisely of those n values is

$$l = f(x_1)f(x_2) \cdots f(x_n). \tag{4}$$

In the continuous case the probability that the sample consists of values in the small intervals $x_j \leqq x \leqq x_j + \Delta x$ $(j = 1, 2, \cdots, n)$ is

$$f(x_1)\Delta x\, f(x_2)\Delta x \cdots f(x_n)\Delta x = l(\Delta x)^n. \tag{5}$$

Since $f(x_j)$ depends on θ, the function l in (5) given by (4) depends on $x_1, \cdots, x_n$ and θ. We imagine $x_1, \cdots, x_n$ to be given and fixed. Then l is a function of θ, which is called the **likelihood function.** The basic idea of the maximum likelihood method is quite simple, as follows. We choose *that* approximation for the unknown value of θ for which l is as large as possible. If l is a differentiable function of θ, a necessary condition for l to have a maximum in an interval (not at the boundary) is

$$\frac{\partial l}{\partial \theta} = 0. \tag{6}$$

(We write a *partial* derivative, because l depends also on $x_1, \cdots, x_n$.) A solution of (6) depending on $x_1, \cdots, x_n$ is called a **maximum likelihood estimate** for θ. We may replace (6) by

$$\frac{\partial \ln l}{\partial \theta} = 0, \tag{7}$$

because $f(x_j) > 0$, a maximum of l is in general positive, and $\ln l$ is a monotone increasing function of l. This often simplifies calculations.

Several Parameters. If the distribution of X involves r parameters $\theta_1, \cdots, \theta_r$, then instead of (6) we have the r conditions $\partial l/\partial\theta_1 = 0, \cdots, \partial l/\partial\theta_r = 0$, and instead of (7) we have

$$\frac{\partial \ln l}{\partial \theta_1} = 0, \qquad \cdots, \qquad \frac{\partial \ln l}{\partial \theta_r} = 0. \tag{8}$$

EXAMPLE 1 Normal Distribution

Find maximum likelihood estimates for $\theta_1 = \mu$ and $\theta_2 = \sigma$ in the case of the normal distribution.

Solution. From (1), Sec. 24.8, and (4) we obtain the likelihood function

$$l = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sigma}\right)^n e^{-h} \qquad \text{where} \qquad h = \frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j - \mu)^2.$$

Taking logarithms, we have

$$\ln l = -n \ln \sqrt{2\pi} - n \ln \sigma - h.$$

The first equation in (8) is $\partial(\ln l)/\partial\mu = 0$, written out

$$\frac{\partial \ln l}{\partial \mu} = -\frac{\partial h}{\partial \mu} = \frac{1}{\sigma^2} \sum_{j=1}^{n} (x_j - \mu) = 0, \qquad \text{hence} \qquad \sum_{j=1}^{n} x_j - n\mu = 0.$$

The solution is the desired estimate $\hat{\mu}$ for μ; we find

$$\hat{\mu} = \frac{1}{n} \sum_{j=1}^{n} x_j = \bar{x}.$$

The second equation in (8) is $\partial(\ln l)/\partial\sigma = 0$, written out

$$\frac{\partial \ln l}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\partial h}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{j=1}^{n} (x_j - \mu)^2 = 0.$$

Replacing μ by $\hat{\mu}$ and solving for σ^2, we obtain the estimate

$$\tilde{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} (x_j - \bar{x})^2$$

which we shall use in Sec. 25.7. Note that this differs from (2). We cannot discuss criteria for the goodness of estimates but want to mention that for small n, formula (2) is preferable. ■

PROBLEM SET 25.6

1. Find the maximum likelihood estimate for the parameter μ of a normal distribution with known variance $\sigma^2 = \sigma_0^2$.
2. Apply the maximum likelihood method to the normal distribution with $\mu = 0$.
3. **(Binomial distribution)** Derive a maximum likelihood estimate for p.
4. Extend Prob. 3 as follows. Suppose that m times n trials were made and in the first n trials A happened k_1 times, in the second n trials A happened k_2 times, $\cdots$, in the mth n trials A happened k_m times. Find a maximum likelihood estimate of p based on this information.
5. Suppose that in Prob. 4 we made 4 times 5 trials and A happened 2, 1, 4, 4 times, respectively. Estimate p.
6. Consider X = *Number of independent trials until an event A occurs.* Show that X has the probability function $f(x) = pq^{x-1}$, $x = 1, 2, \cdots$, where p is the probability of A in a single trial and $q = 1 - p$. Find the maximum likelihood estimate of p corresponding to a sample $x_1, \cdots, x_n$ of observed values of X.
7. In Prob. 6 find the maximum likelihood estimate of p corresponding to a single observation x of X.
8. In rolling a die, suppose that we get the first Six in the 7th trial and in doing it again we get it in the 6th trial. Estimate the probability p of getting a Six in rolling that die once.
9. **(Poisson distribution)** Apply the maximum likelihood method to the Poisson distribution.
10. **(Uniform distribution)** Show that in the case of the parameters a and b of the uniform distribution (see Sec. 24.6), the maximum likelihood estimate cannot be obtained by equating the first derivative to zero. How can we obtain maximum likelihood estimates in this case?
11. Find the maximum likelihood estimate of θ in the density $f(x) = \theta e^{-\theta x}$ if $x \geqq 0$ and $f(x) = 0$ if $x < 0$.
12. In Prob. 11, find the mean μ, substitute it in $f(x)$, find the maximum likelihood estimate of μ, and show that it is identical with the estimate for μ which can be obtained from that for θ in Prob. 11.
13. Compute $\hat{\theta}$ in Prob. 11 from the sample 1.8, 0.4, 0.8, 0.6, 1.4. Graph the sample distribution function $\hat{F}(x)$ and the distribution function $F(x)$ of the random variable, with $\theta = \hat{\theta}$, on the same axes. Do they agree reasonably well? (We consider goodness of fit systematically in Sec. 25.7.)

14. Do the same task as in Prob. 13 if the given sample is 0.5. 0.7. 0.1. 1.1. 0.1.

15. CAS EXPERIMENT. Maximum Likelihood Estimates. (MLEs). Find experimentally how much MLEs can differ depending on the sample size. *Hint.* Generate many samples of the same size n, e.g., of the standardized normal distribution, and record $\bar{x}$ and s^2. Then increase n.

25.3 Confidence Intervals

Confidence intervals[1] for an unknown parameter θ of some distribution (e.g., $\theta = \mu$) are intervals $\theta_1 \leqq \theta \leqq \theta_2$ that contain θ, not with certainty but with a high probability γ, which we can choose (95% and 99% are popular). Such an interval is calculated from a sample. $\gamma = 95\%$ means probability $1 - \gamma = 5\% = 1/20$ of being wrong—one of about 20 such intervals will not contain θ. Instead of writing $\theta_1 \leqq \theta \leqq \theta_2$, we denote this more distinctly by writing

$$\text{CONF}_\gamma \{\theta_1 \leqq \theta \leqq \theta_2\}. \tag{1}$$

Such a special symbol, CONF, seems worthwhile in order to avoid the misunderstanding that θ *must* lie between θ_1 and θ_2.

γ is called the **confidence level,** and θ_1 and θ_2 are called the **lower** and **upper confidence limits.** They depend on γ. The larger we choose γ, the smaller is the error probability $1 - \gamma$, but the longer is the confidence interval. If $\gamma \rightarrow 1$, then its length goes to infinity. The choice of γ depends on the kind of application. In taking no umbrella, a 5% chance of getting wet is not tragic. In a medical decision of life or death, a 5% chance of being wrong may be too large and a 1% chance of being wrong ($\gamma = 99\%$) may be more desirable.

Confidence intervals are more valuable than point estimates (Sec. 25.2). Indeed, we can take the midpoint of (1) as an approximation of θ and half the length of (1) as an "error bound" (not in the strict sense of numerics, but except for an error whose probability we know).

θ_1 and θ_2 in (1) are calculated from a sample $x_1, \cdots, x_n$. These are n observations of a random variable X. Now comes a **standard trick.** *We regard* $x_1, \cdots, x_n$ *as* ***single*** *observations of n random variables* $X_1, \cdots, X_n$ *(with the same distribution, namely, that of* X*).* Then $\theta_1 = \theta_1(x_1, \cdots, x_n)$ and $\theta_2 = \theta_2(x_1, \cdots, x_n)$ in (1) are observed values of two random variables $\Theta_1 = \Theta_1(X_1, \cdots, X_n)$ and $\Theta_2 = \Theta_2(X_1, \cdots, X_n)$. The condition (1) involving γ can now be written

$$P(\Theta_1 \leqq \theta \leqq \Theta_2) = \gamma. \tag{2}$$

Let us see what all this means in concrete practical cases.

In each case in this section we shall first state the steps of obtaining a confidence interval in the form of a table, then consider a typical example, and finally justify those steps theoretically.

[1]JERZY NEYMAN (1894–1981), American statistician, developed the theory of confidence intervals (*Annals of Mathematical Statistics* **6** (1935), 111–116).

Confidence Interval for μ of the Normal Distribution with Known σ^2

Table 25.1 Determination of a Confidence Interval for the Mean μ of a Normal Distribution with Known Variance σ^2

Step 1. Choose a confidence level γ (95%, 99%, or the like).

Step 2. Determine the corresponding c:

γ	0.90	0.95	0.99	0.999
c	1.645	1.960	2.576	3.291

Step 3. Compute the mean $\bar{x}$ of the sample $x_1, \cdots, x_n$.

Step 4. Compute $k = c\sigma/\sqrt{n}$. The confidence interval for μ is

$$\text{CONF}_{\gamma}\,\{\bar{x} - k \leqq \mu \leqq \bar{x} + k\}. \tag{3}$$

EXAMPLE 1 Confidence Interval for μ of the Normal Distribution with Known σ^2

Deterimine a 95% confidence interval for the mean of a normal distribution with variance $\sigma^2 = 9$, using a sample of $n = 100$ values with mean $\bar{x} = 5$.

Solution. ***Step 1.*** $\gamma = 0.95$ is required. ***Step 2.*** The corresponding c equals 1.960; see Table 25.1. ***Step 3.*** $\bar{x} = 5$ is given. ***Step 4.*** We need $k = 1.960 \cdot 3/\sqrt{100} = 0.588$. Hence $\bar{x} - k = 4.412$, $\bar{x} + k = 5.588$ and the confidence interval is $\text{CONF}_{0.95}\,\{4.412 \leqq \mu \leqq 5.588\}$.

This is sometimes written $\mu = 5 \pm 0.588$, but we shall not use this notation, which can be misleading.

With your CAS you can determine this interval more directly. Similarly for the other examples in this section. ■

Theory for Table 25.1. The method in Table 25.1 follows from the basic

THEOREM 1 Sum of Independent Normal Random Variables

*Let $X_1, \cdots, X_n$ be **independent** normal random variables each of which has mean μ and variance σ^2. Then the following holds.*

(a) *The sum $X_1 + \cdots + X_n$ is normal with mean $n\mu$ and variance $n\sigma^2$.*

(b) The following random variable $\bar{X}$ *is normal with mean μ and variance σ^2/n.*

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n) \tag{4}$$

(c) *The following random variable Z is normal with mean* 0 *and variance* 1.

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \tag{5}$$

PROOF The statements about the mean and variance in (a) follow from Theorems 1 and 3 in Sec. 24.9. From this and Theorem 2 in Sec. 24.6 we see that $\bar{X}$ has the mean $(1/n)n\mu = \mu$ and the variance $(1/n)^2 n\sigma^2 = \sigma^2/n$. This implies that Z has the mean 0 and variance 1, by Theorem 2(b) in Sec. 24.6. The normality of $X_1 + \cdots + X_n$ is proved in Ref. [G3] listed in App. 1. This implies the normality of (4) and (5). ■

Derivation of (3) in Table 25.1. Sampling from a normal distribution gives independent sample values (see Sec. 25.1), so that Theorem 1 applies. Hence we can choose γ and then determine c such that

$$P(-c \leqq Z \leqq c) = P\left(-c \leqq \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leqq c\right) = \Phi(c) - \Phi(-c) = \gamma. \tag{6}$$

For the value $\gamma = 0.95$ we obtain $z(D) = 1.960$ from Table A8 in App. 5, as used in Example 1. For $\gamma = 0.9, 0.99, 0.999$ we get the other values of c listed in Table 25.1. Finally, all we have to do is to convert the inequality in (6) into one for μ and insert observed values obtained from the sample. We multiply $-c \leqq Z \leqq c$ by -1 and then by $\sigma/\sqrt{n}$, writing $c\sigma/\sqrt{n} = k$ (as in Table 25.1),

$$P(-c \leqq Z \leqq c) = P(c \geqq -Z \geqq -c) = P\left(c \geqq \frac{\mu - \overline{X}}{\sigma/\sqrt{n}} \geqq -c\right)$$

$$= P(k \geqq \mu - \overline{X} \geqq -k) = \gamma.$$

Adding $\overline{X}$ gives $P(\overline{X} + k \geqq \mu \geqq \overline{X} - k) = \gamma$ or

$$P(\overline{X} - k \leqq \mu \leqq \overline{X} + k) = \gamma. \tag{7}$$

Inserting the observed value $\bar{x}$ of $\overline{X}$ gives (3). Here we have regarded $x_1, \cdots, x_n$ as single observations of $X_1, \cdots, X_n$ (the standard trick!), so that $x_1 + \cdots + x_n$ is an observed value of $X_1 + \cdots + X_n$ and $\bar{x}$ is an observed value of $\overline{X}$. Note further that (7) is of the form (2) with $\Theta_1 = \overline{X} - k$ and $\Theta_2 = \overline{X} + k$. ■

EXAMPLE 2 Sample Size Needed for a Confidence Interval of Prescribed Length

How large must n be in Example 1 if we want to obtain a 95% confidence interval of length $L = 0.4$?

Solution. The interval (3) has the length $L = 2k = 2c\sigma/\sqrt{n}$. Solving for n, we obtain

$$n = (2c\sigma/L)^2.$$

In the present case the answer is $n = (2 \cdot 1.960 \cdot 3/0.4)^2 \approx 870$.

Figure 525 shows how L decreases as n increases and that for $\gamma = 99\%$ the confidence interval is substantially longer than for $\gamma = 95\%$ (and the same sample size n). ■

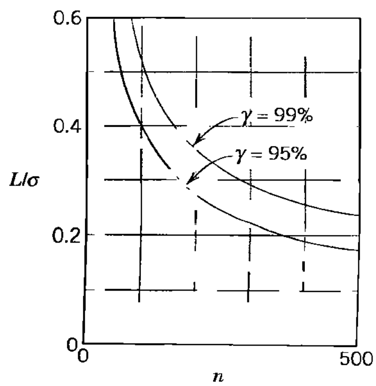


Fig. 525. Length of the confidence interval (3) (measured in multiples of σ) as a function of the sample size n for $\gamma = 95\%$ and $\gamma = 99\%$

Confidence Interval for μ of the Normal Distribution With Unknown σ^2

In practice σ^2 is frequently unknown. Then the method in Table 25.1 does not help and the whole theory changes, although the steps of determining a confidence interval for μ remain quite similar. They are shown in Table 25.2. We see that k differs from that in Table 25.1, namely, the sample standard deviation s has taken the place of the unknown standard deviation σ of the population. And c now depends on the sample size n and must be determined from Table A9 in App. 5 or from your CAS. That table lists values z for given values of the distribution function (Fig. 526)

$$F(z) = K_m \int_{-\infty}^{z} \left(1 + \frac{u^2}{m}\right)^{-(m+1)/2} du \tag{8}$$

of the ***t*-distribution**. Here, m ($= 1, 2, \cdots$) is a parameter, called the **number of degrees of freedom** of the distribution (***abbreviated* d.f.**). In the present case, $m = n - 1$; see Table 25.2. The constant K_m is such that $F(\infty) = 1$. By integration it turns out that $K_m = \Gamma(\frac{1}{2}m + \frac{1}{2})/[\sqrt{m\pi}\,\Gamma(\frac{1}{2}m)]$, where Γ is the gamma function (see (24) in App. A3.1).

Table 25.2 Determination of a Confidence Interval for the Mean μ of a Normal Distribution with Unknown Variance σ^2

Step 1. Choose a confidence level γ (95%, 99%, or the like).

Step 2. Determine the solution c of the equation

$$F(c) = \tfrac{1}{2}(1 + \gamma) \tag{9}$$

from the table of the t-distribution with $n - 1$ degrees of freedom (Table A9 in App. 5; or use a CAS; n = sample size).

Step 3. Compute the mean $\bar{x}$ and the variance s^2 of the sample $x_1, \cdots, x_n$.

Step 4. Compute $k = cs/\sqrt{n}$. The confidence interval is

$$\text{CONF}_\gamma\ \{\bar{x} - k \leqq \mu \leqq \bar{x} + k\}. \tag{10}$$

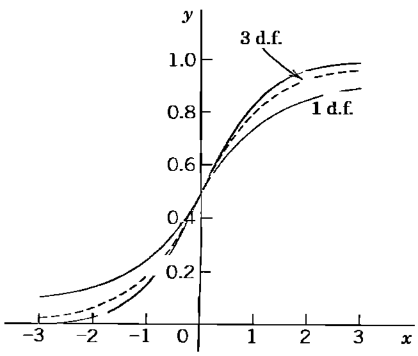


Fig. 526. Distribution functions of the t-distribution with 1 and 3 d.f. and of the standardized normal distribution (steepest curve)

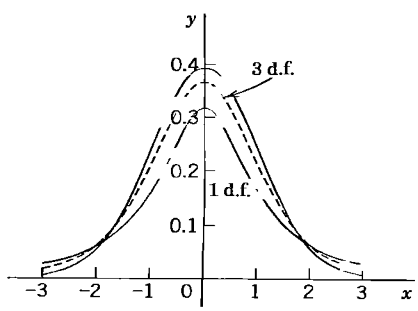


Fig. 527. Densities of the t-distribution with 1 and 3 d.f. and of the standardized normal distribution

Figure 527 compares the curve of the density of the t-distribution with that of the normal distribution. The latter is steeper. This illustrates that Table 25.1 (which uses more information, namely, the known value of σ^2) yields shorter confidence intervals than Table 25.2. This is confirmed in Fig. 528, which also gives an idea of the gain by increasing the sample size.

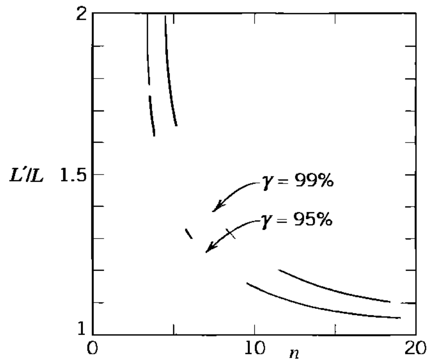


Fig. 528. Ratio of the lengths L' and L of the confidence intervals (10) and (3) with $\gamma = 95\%$ and $\gamma = 99\%$ as a function of the sample size n for equal s and σ

EXAMPLE 3 Confidence Interval for μ of the Normal Distribution with Unknown σ^2

Five independent measurements of the point of inflammation (flash point) of Diesel oil (D-2) gave the values (in °F) 144 147 146 142 144. Assuming normality, determine a 99% confidence interval for the mean.

Solution. *Step 1.* $\gamma = 0.99$ is required.

Step 2. $F(c) = \frac{1}{2}(1 + \gamma) = 0.995$, and Table A9 in App. 5 with $n - 1 = 4$ d.f. gives $c = 4.60$.

Step 3. $\bar{x} = 144.6$, $s^2 = 3.8$.

Step 4. $k = \sqrt{3.8} \cdot 4.60/\sqrt{5} = 4.01$. The confidence interval is $\text{CONF}_{0.99}\{140.5 \leqq \mu \leqq 148.7\}$.

If the variance σ^2 were known and equal to the sample variance s^2, thus $\sigma^2 = 3.8$, then Table 25.1 would give $k = c\sigma/\sqrt{n} = 2.576\sqrt{3.8}/\sqrt{5} = 2.25$ and $\text{CONF}_{0.99}\{142.35 \leqq \mu \leqq 146.85\}$. We see that the present interval is almost twice as long as that obtained from Table 25.1 (with $\sigma^2 = 3.8$). Hence for small samples the difference is considerable! See also Fig. 528. ■

Theory for Table 25.2. For deriving (10) in Table 25.2 we need from Ref. [G3]

THEOREM 2 Student's t-Distribution

Let $X_1, \cdots, X_n$ be independent normal random variables with the same mean μ and the same variance σ^2. Then the random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \tag{11}$$

has a t-distribution [see (8)] *with $n - 1$ degrees of freedom* (d.f.): *here $\bar{X}$ is given by* (4) *and*

$$S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2. \tag{12}$$

Derivation of (10). This is similar to the derivation of (3). We choose a number γ between 0 and 1 and determine a number c from Table A9 in App. 5 with $n - 1$ d.f. (or from a CAS) such that

$$P(-c \leqq T \leqq c) = F(c) - F(-c) = \gamma. \tag{13}$$

Since the t-distribution is symmetric, we have

$$F(-c) = 1 - F(c),$$

and (13) assumes the form (9). Substituting (11) into (13) and transforming the result as before, we obtain

$$P(\overline{X} - K \leqq \mu \leqq \overline{X} + K) = \gamma \tag{14}$$

where

$$K = cS/\sqrt{n}.$$

By inserting the observed values $\bar{x}$ of $\overline{X}$ and s^2 of S^2 into (14) we finally obtain (10). ■

Confidence Interval for the Variance σ^2 of the Normal Distribution

Table 25.3 shows the steps, which are similar to those in Tables 25.1 and 25.2.

Table 25.3 Determination of a Confidence Interval for the Variance σ^2 of a Normal Distribution, Whose Mean Need Not Be Known

Step 1. Choose a confidence level γ (95%, 99%, or the like).
Step 2. Determine solutions c_1 and c_2 of the equations
(15) $F(c_1) = \frac{1}{2}(1 - \gamma), \qquad F(c_2) = \frac{1}{2}(1 + \gamma)$
from the table of the chi-square distribution with $n - 1$ degrees of freedom (Table A10 in App. 5; or use a CAS; n = sample size).
Step 3. Compute $(n - 1)s^2$, where s^2 is the variance of the sample $x_1, \cdots, x_n$.
Step 4. Compute $k_1 = (n - 1)s^2/c_1$ and $k_2 = (n - 1)s^2/c_2$. The confidence interval is
(16) $\mathrm{CONF}_{\gamma}\ \{k_2 \leqq \sigma^2 \leqq k_1\}.$

EXAMPLE 4 Confidence Interval for the Variance of the Normal Distribution

Determine a 95% confidence interval (16) for the variance, using Table 25.3 and a sample (tensile strength of sheet steel in kg/mm^2, rounded to integer values)

$$89 \quad 84 \quad 87 \quad 81 \quad 89 \quad 86 \quad 91 \quad 90 \quad 78 \quad 89 \quad 87 \quad 99 \quad 83 \quad 89.$$

Solution. *Step 1.* $\gamma = 0.95$ is required.

Step 2. For $n - 1 = 13$ we find

$$c_1 = 5.01 \quad \text{and} \quad c_2 = 24.74.$$

Step 3. $13s^2 = 326.9$.

Step 4. $13s^2/c_1 = 65.25$, $13s^2/c_2 = 13.21$.

The confidence interval is

$$\text{CONF}_{0.95}\,\{13.21 \leqq \sigma^2 \leqq 65.25\}.$$

This is rather large, and for obtaining a more precise result, one would need a much larger sample. ■

Theory for Table 25.3. In Table 25.1 we used the normal distribution, in Table 25.2 the t-distribution, and now we shall use the **χ^2-distribution** (*chi-square distribution*), whose distribution function is $F(z) = 0$ if $z < 0$ and

$$F(z) = C_m \int_0^z e^{-u/2} u^{(m-2)/2}\, du \qquad \text{if } z \geqq 0 \qquad \text{(Fig. 529).}$$

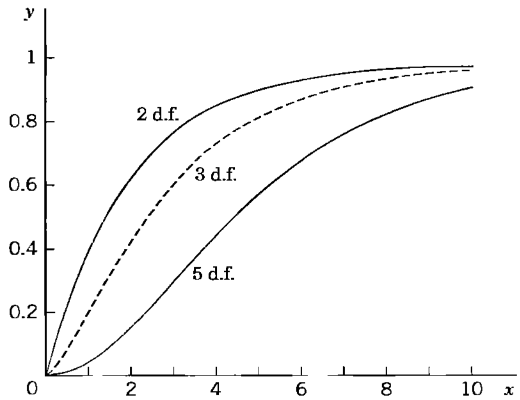


Fig. 529. Distribution function of the chi-square distribution with 2, 3, 5 d.f.

The parameter m ($= 1, 2, \cdots$) is called the **number of degrees of freedom** (d.f.), and

$$C_m = 1/[2^{m/2}\Gamma(\tfrac{1}{2}m)].$$

Note that the distribution is not symmetric (see also Fig. 530).

For deriving (16) in Table 25.3 we need the following theorem.

THEOREM 3

Chi-Square Distribution

Under the assumptions in Theorem 2 the random variable

$$Y = (n-1)\frac{S^2}{\sigma^2} \tag{17}$$

with S^2 given by (12) *has a chi-square distribution with $n - 1$ degrees of freedom*

Proof in Ref. [G3], listed in App. 1.

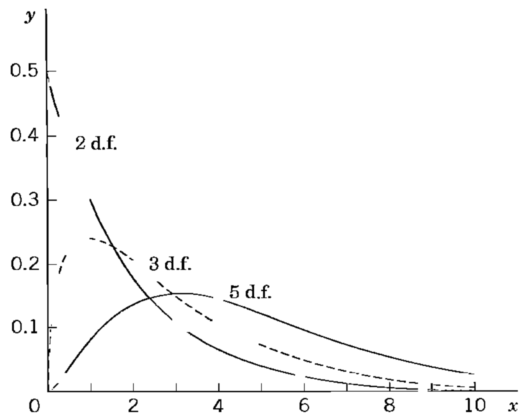


Fig. 530. Density of the chi-square distribution with 2, 3, 5 d.f.

Derivation of (16). This is similar to the derivation of (3) and (10). We choose a number γ between 0 and 1 and determine c_1 and c_2 from Table A10, App. 5, such that [see (15)]

$$P(Y \leqq c_1) = F(c_1) = \tfrac{1}{2}(1 - \gamma), \qquad P(Y \leqq c_2) = F(c_2) = \tfrac{1}{2}(1 + \gamma).$$

Subtraction yields

$$P(c_1 \leqq Y \leqq c_2) = P(Y \leqq c_2) - P(Y \leqq c_1) = F(c_2) - F(c_1) = \gamma.$$

Transforming $c_1 \leqq Y \leqq c_2$ with Y given by (17) into an inequality for σ^2, we obtain

$$\frac{n-1}{c_2} S^2 \leqq \sigma^2 \leqq \frac{n-1}{c_1} S^2.$$

By inserting the observed value s^2 of S^2 we obtain (16). ■

Confidence Intervals for Parameters of Other Distributions

The methods in Tables 25.1–25.3 for confidence intervals for μ and σ^2 are designed for the normal distribution. We now show that they can also be applied to other distributions if we use large samples.

We know that if $X_1, \cdots, X_n$ are independent random variables with the same mean μ and the same variance σ^2, then their sum $Y_n = X_1 + \cdots + X_n$ has the following properties.

(A) Y_n has the mean $n\mu$ and the variance $n\sigma^2$ (by Theorems 1 and 3 in Sec. 24.9).

(B) If those variables are normal, then Y_n is normal (by Theorem 1).

If those random variables are not normal, then **(B)** is not applicable. However, for large n the random variable Y_n is still *approximately* normal. This follows from the central limit theorem, which is one of the most fundamental results in probability theory.

THEOREM 4 **Central Limit Theorem**

Let $X_1, \cdots, X_n, \cdots$ be independent random variables that have the same distribution function and therefore the same mean μ and the same variance σ^2. Let $Y_n = X_1 + \cdots + X_n$. Then the random variable

$$Z_n = \frac{Y_n - n\mu}{\sigma\sqrt{n}} \tag{18}$$

is **asymptotically normal** *with mean* 0 *and variance* 1; *that is, the distribution function $F_n(x)$ of Z_n satisfies*

$$\lim_{n\to\infty} F_n(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

A proof can be found in Ref. [G3] listed in App. 1.

Hence when applying Tables 25.1–25.3 to a nonnormal distribution, we must use ***sufficiently large samples.*** As a rule of thumb, if the sample indicates that the skewness of the distribution (the asymmetry; see Team Project 16(d), Problem Set 24.6) is small, use at least $n = 20$ for the mean and at least $n = 50$ for the variance.

1–7 MEAN (VARIANCE KNOWN)

1. Find a 95% confidence interval for the mean μ of a normal population with standard deviation 4.00 from the sample 30, 42, 40, 34, 48, 50.
2. Does the interval in Prob. 1 get longer or shorter if we take $\gamma = 0.99$ instead of 0.95? By what factor?
3. By what factor does the length of the interval in Prob. 1 change if we double the sample size?
4. Find a 90% confidence interval for the mean μ of a normal population with variance 0.25, using a sample of 100 values with mean 212.3.
5. What sample size would be needed for obtaining a 95% confidence interval (3) of length 2σ? Of length σ?
6. **(Use of Fig. 525)** Find a 95% confidence interval for a sample of 200 values with mean 120 from a normal distribution with variance 4, using Fig. 525.
7. What sample size is needed to obtain a 99% confidence interval of length 2.0 for the mean of a normal population with variance 25? Use Fig. 525. Check by calculation.

8–12 MEAN (VARIANCE UNKNOWN)

Find a 99% confidence interval for the mean of a normal population from the sample:

8. 425, 420, 425, 435
9. Length of 20 bolts with sample mean 20.2 cm and sample variance 0.04 cm^2
10. Knoop hardness of diamond 9500, 9800, 9750, 9200, 9400, 9550
11. Copper content (%) of brass 66. 66. 65. 64. 66. 67, 64. 65, 63, 64
12. Melting point (°C) of aluminum 660, 667, 654, 663, 662
13. Find a 95% confidence interval for the percentage of cars on a certain highway that have poorly adjusted brakes. using a random sample of 500 cars stopped at a roadblock on that highway. 87 of which had poorly adjusted brakes.
14. Find a 99% confidence interval for p in the binomial distribution from a classical result by K. Pearson, who in 24000 trials of tossing a coin obtained 12012 Heads. Do you think that the coin was fair?

15–20 **VARIANCE**

Find a 95% confidence interval for the variance of a normal population from the sample:

15. A sample of 30 values with variance 0.0007

16. The sample in Prob. 9

17. The sample in Prob. 11

18. Carbon monoxide emission (grams per mile) of a certain type of passenger car (cruising at 55 mph): 17.3, 17.8, 18.0, 17.7, 18.2, 17.4. 17.6. 18.1

19. Mean energy (keV) of delayed neutron group (Group 3, half-life 6.2 sec.) for uranium U^{235} fission: 435, 451, 430, 444, 438

20. Ultimate tensile strength (k psi) of alloy steel (Maraging H) at room temperature: 251, 255, 258, 253, 253, 252, 250, 252, 255, 256

21. If X is normal with mean 27 and variance 16, what distributions do $-X$, $3X$, and $5X - 2$ have?

22. If X_1 and X_2 are independent normal random variables with mean 23 and 4 and variance 3 and 1, respectively, what distribution does $4X_1 - X_2$ have? *Hint.* Use Team Project 14(g) in Sec. 24.8.

23. A machine fills boxes weighing Y lb with X lb of salt, where X and Y are normal with mean 100 lb and 5 lb and standard deviation 1 lb and 0.5 lb, respectively. What percent of filled boxes weighing between 104 lb and 106 lb are to be expected?

24. If the weight X of bags of cement is normally distributed with a mean of 40 kg and a standard deviation of 2 kg, how many bags can a delivery truck carry so that the probability of the total load exceeding 2000 kg will be 5%?

25. **CAS EXPERIMENT. Confidence Intervals.** Obtain 100 samples of size 10 of the standardized normal distribution. Calculate from them and graph the corresponding 95% confidence intervals for the mean and count how many of them do not contain 0. Does the result support the theory? Repeat the whole experiment. compare and comment.

25.4 Testing of Hypotheses. Decisions

The ideas of confidence intervals and of tests[2] are the two most important ideas in modern statistics. In a statistical **test** we make inference from sample to population through testing a **hypothesis,** resulting from experience or observations, from a theory or a quality requirement, and so on. In many cases the result of a test is used as a basis for a **decision,** for instance, to buy (or not to buy) a certain model of car, depending on a test of the fuel efficiency (miles/gal) (and other tests, of course), to apply some medication, depending on a test of its effect; to proceed with a marketing strategy, depending on a test of consumer reactions, etc.

Let us explain such a test in terms of a typical example and introduce the corresponding standard notions of statistical testing.

EXAMPLE 1 **Test of a Hypothesis. Alternative. Significance Level α**

We want to buy 100 coils of a certain kind of wire, provided we can verify the manufacturer's claim that the wire has a breaking limit $\mu = \mu_0 = 200$ lb (or more). This is a test of the **hypothesis** (also called *null hypothesis*) $\mu = \mu_0 = 200$. We shall not buy the wire if the (statistical) test shows that actually $\mu = \mu_1 < \mu_0$, the wire is weaker, the claim does not hold. μ_1 is called the **alternative** (or *alternative hypothesis*) of the test. We shall **accept** the hypothesis if the test suggests that it is true, except for a small error probability α, called the **significance level** of the test. Otherwise we **reject** the hypothesis. Hence α is the probability of rejecting a hypothesis although it is true. The choice of α is up to us. 5% and 1% are popular values.

For the test we need a sample. We randomly select 25 coils of the wire, cut a piece from each coil, and determine the breaking limit experimentally. Suppose that this sample of $n = 25$ values of the breaking limit has the mean $\bar{x} = 197$ lb (somewhat less than the claim!) and the standard deviation $s = 6$ lb.

[2]Beginning around 1930, a systematic theory of tests was developed by NEYMAN (see Sec. 25.3) and EGON SHARPE PEARSON (1895–1980), English statistician, the son of Karl Pearson (see the footnote on p. 1066).

At this point we could only speculate whether this difference $197 - 200 = -3$ is due to randomness, is a chance effect, or whether it is **significant,** due to the actually inferior quality of the wire To continue beyond speculation requires probability theory, as follows.

We assume that the breaking limit is normally distributed. (This assumption could be tested by the method in Sec. 25.7. Or we could remember the central limit theorem (Sec. 25.3) and take a still larger sample.) Then

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

in (11), Sec. 25.3, with $\mu = \mu_0$ has a ***t*-distribution** with $n - 1$ degrees of freedom ($n - 1 = 24$ for our sample). Also $\bar{x} = 197$ and $s = 6$ are observed values of $\bar{X}$ and S to be used later. We can now choose a significance level, say, $\alpha = 5\%$. From Table A9 in App. 5 or from a CAS we then obtain a critical value c such that $P(T \leqq c) = \alpha = 5\%$. For $P(T \leqq \tilde{c}) = 1 - \alpha = 95\%$ the table gives $\tilde{c} = 1.71$, so that $c = -\tilde{c} = -1.71$ because of the symmetry of the distribution (Fig. 531).

We now reason as follows—this is the ***crucial idea*** of the test. If the hypothesis is true, we have a chance of only α (= 5%) that we observe a value t of T (calculated from a sample) that will fall between $-\infty$ and -1.71. Hence if we nevertheless do observe such a t, we assert that the hypothesis cannot be true and we reject it. Then we accept the alternative. If, however, $t \geqq c$, we accept the hypothesis.

A simple calculation finally gives $t = (197 - 200)/(6/\sqrt{25}) = -2.5$ as an observed value of T. Since $-2.5 < -1.71$, we reject the hypothesis (the manufacturer's claim) and accept the alternative $\mu = \mu_1 < 200$, the wire seems to be weaker than claimed. ■

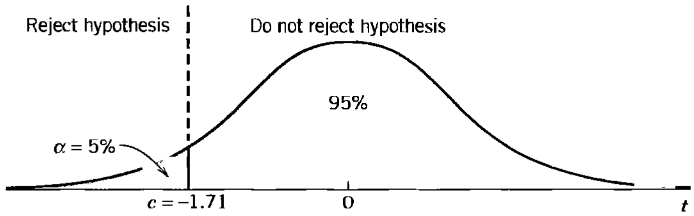


Fig. 531. t-distribution in Example 1

This example illustrates the ***steps of a test:***

1. Formulate the **hypothesis** $\theta = \theta_0$ to be tested. ($\theta_0 = \mu_0$ in the example.)

2. Formulate an **alternative** $\theta = \theta_1$. ($\theta_1 = \mu_1$ in the example.)

3. Choose a **significance level** α (5%, 1%, 0.1%).

4. Use a random variable $\hat{\Theta} = g(X_1, \cdots, X_n)$ whose distribution depends on the hypothesis and on the alternative, and this distribution is known in both cases. Determine a critical value c from the distribution of $\hat{\Theta}$, assuming the hypothesis to be true. (In the example, $\hat{\Theta} = T$, and c is, obtained from $P(T \leqq c) = \alpha$.)

5. Use a sample $x_1, \cdots, x_n$ to determine an observed value $\hat{\theta} = g(x_1, \cdots, x_n)$ of $\hat{\Theta}$. (t in the example.)

6. Accept or reject the hypothesis, depending on the size of $\hat{\theta}$ relative to c. ($t < c$ in the example, rejection of the hypothesis.)

Two important facts require further discussion and careful attention. The first is the choice of an alternative. In the example, $\mu_1 < \mu_0$, but other applications may require $\mu_1 > \mu_0$ or $\mu_1 \neq \mu_0$. The second fact has to do with errors. We know that α (the significance level of the test) is the probability of *rejecting* a *true* hypothesis. And we shall discuss the probability β of *accepting* a *false* hypothesis.

One-Sided and Two-Sided Alternatives (Fig. 532)

Let θ be an unknown parameter in a distribution, and suppose that we want to test the hypothesis $\theta = \theta_0$. Then there are three main kinds of alternatives, namely,

$$\theta > \theta_0 \tag{1}$$

$$\theta < \theta_0 \tag{2}$$

$$\theta \neq \theta_0. \tag{3}$$

(1) and (2) are **one-sided alternatives,** and (3) is a **two-sided alternative.**

We call **rejection region** (or **critical region**) the region such that we reject the hypothesis if the observed value in the test falls in this region. In ① the critical c lies to the right of θ_0 because so does the alternative. Hence the rejection region extends to the right. This is called a **right-sided test.** In ② the critical c lies to the left of θ_0 (as in Example 1), the rejection region extends to the left, and we have a **left-sided test** (Fig. 532, middle part). These are **one-sided tests.** In ③ we have two rejection regions. This is called a **two-sided test** (Fig. 532, lower part).

All three kinds of alternatives occur in practical problems. For example, (1) may arise if θ_0 is the maximum tolerable inaccuracy of a voltmeter or some other instrument. Alternative (2) may occur in testing strength of material, as in Example 1. Finally, θ_0 in (3) may be the diameter of axle-shafts, and shafts that are too thin or too thick are equally undesirable, so that we have to watch for deviations in both directions.

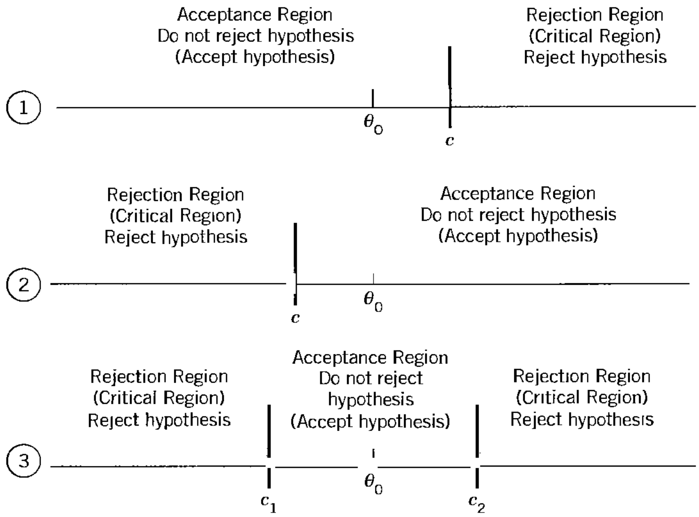


Fig. 532. Test in the case of alternative (1) (upper part of the figure), alternative (2) (middle part), and alternative (3)

Errors in Tests

Tests always involve **risks of making false decisions:**

(I) Rejecting a true hypothesis **(Type I error).**
α = Probability of making a Type I error.

(II) Accepting a false hypothesis **(Type II error).**
β = Probability of making a Type II error.

Clearly, we cannot avoid these errors because no absolutely certain conclusions about populations can be drawn from samples. But we show that there are ways and means of choosing suitable levels of risks, that is, of values α and β. The choice of α depends on the nature of the problem (e.g., a small risk $\alpha = 1\%$ is used if it is a matter of life or death).

Let us discuss this systematically for a test of a hypothesis $\theta = \theta_0$ against an alternative that is a single number θ_1, for simplicity. We let $\theta_1 > \theta_0$, so that we have a right-sided test. For a left-sided or a two-sided test the discussion is quite similar.

We choose a critical $c > \theta_0$ (as in the upper part of Fig. 532, by methods discussed below). From a given sample $x_1, \cdots, x_n$ we then compute a value

$$\hat{\theta} = g(x_1, \cdots, x_n)$$

with a suitable g (whose choice will be a main point of our further discussion; for instance, take $g = (x_1 + \cdots + x_n)/n$ in the case in which θ is the mean). If $\hat{\theta} > c$, we reject the hypothesis. If $\hat{\theta} \leqq c$, we accept it. Here, the value $\hat{\theta}$ can be regarded as an observed value of the random variable

$$\hat{\Theta} = g(X_1, \cdots, X_n) \tag{4}$$

because x_j may be regarded as an observed value of X_j, $j = 1, \cdots, n$. In this test there are two possibilities of making an error, as follows.

Type I Error (see Table 25.4). The hypothesis is true but is rejected (hence the alternative is accepted) because Θ assumes a value $\hat{\theta} > c$. Obviously, the probability of making such an error equals

$$P(\hat{\Theta} > c)_{\theta=\theta_0} = \alpha. \tag{5}$$

α is called the **significance level** of the test, as mentioned before.

Type II Error (see Table 25.4). The hypothesis is false but is accepted because $\hat{\Theta}$ assumes a value $\hat{\theta} \leqq c$. The probability of making such an error is denoted by β; thus

$$P(\hat{\Theta} \leqq c)_{\theta=\theta_1} = \beta. \tag{6}$$

$\eta = 1 - \beta$ is called the **power** of the test. Obviously, the power η is the probability of avoiding a Type II error.

Table 25.4 Type I and Type II Errors in Testing a Hypothesis $\theta = \theta_0$ Against an Alternative $\theta = \theta_1$

		Unknown Truth	
		$\theta = \theta_0$	$\theta = \theta_1$
Accepted	$\theta = \theta_0$	True decision $P = 1 - \alpha$	Type II error $P = \beta$
	$\theta = \theta_1$	Type I error $P = \alpha$	True decision $P = 1 - \beta$

Formulas (5) and (6) show that both α and β depend on c, and we would like to choose c so that these probabilities of making errors are as small as possible. But the important Figure 533 shows that these are conflicting requirements because to let α decrease we must shift c to the right, but then β increases. In practice we first choose α (5%, sometimes 1%), then determine c, and finally compute β. If β is large so that the power $\eta = 1 - \beta$ is small, we should repeat the test, choosing a larger sample, for reasons that will appear shortly.

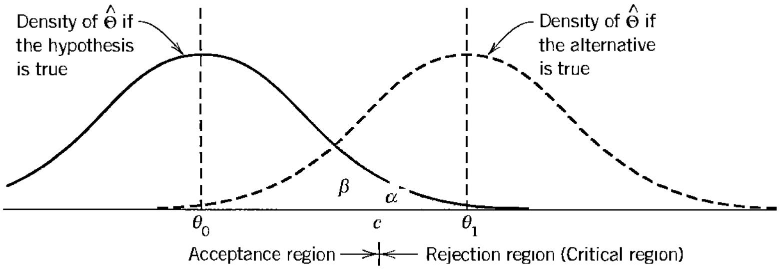


Fig. 533. Illustration of Type I and II errors in testing a hypothesis $\theta = \theta_0$ against an alternative $\theta = \theta_1$ ($> \theta_0$, right-sided test)

If the alternative is not a single number but is of the form (1)–(3), then β becomes a function of θ. This function $\beta(\theta)$ is called the **operating characteristic** (OC) of the test and its curve the **OC curve.** Clearly, in this case $\eta = 1 - \beta$ also depends on θ. This function $\eta(\theta)$ is called the **power function** of the test. (Examples will follow.)

Of course, from a test that leads to the acceptance of a certain hypothesis θ_0, it does ***not*** follow that this is the only possible hypothesis or the best possible hypothesis. Hence the terms **"not reject"** or **"fail to reject"** are perhaps better than the term **"accept."**

Test for μ of the Normal Distribution with Known σ^2

The following example explains the three kinds of hypotheses.

EXAMPLE 2 Test for the Mean of the Normal Distribution with Known Variance

Let X be a normal random variable with variance $\sigma^2 = 9$. Using a sample of size $n = 10$ with mean $\bar{x}$, test the hypothesis $\mu = \mu_0 = 24$ against the three kinds of alternatives, namely,

$$\text{(a)}\quad \mu > \mu_0 \qquad \text{(b)}\quad \mu < \mu_0 \qquad \text{(c)}\quad \mu \neq \mu_0.$$

Solution. We choose the significance level $\alpha = 0.05$. An estimate of the mean will be obtained from

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n).$$

If the hypothesis is true, $\bar{X}$ is normal with mean $\mu = 24$ and variance $\sigma^2/n = 0.9$, see Theorem 1, Sec. 25.3. Hence we may obtain the critical value c from Table A8 in App. 5.

Case (a). ***Right-Sided Test.*** We determine c from $P(\bar{X} > c)_{\mu=24} = \alpha = 0.05$, that is,

$$P(\bar{X} \leqq c)_{\mu=24} = \Phi\left(\frac{c - 24}{\sqrt{0.9}}\right) = 1 - \alpha = 0.95.$$

Table A8 in App. 5 gives $(c - 24)/\sqrt{0.9} = 1.645$, and $c = 25.56$, which is greater than μ_0, as in the upper part of Fig. 532. If $\bar{x} \leqq 25.56$, the hypothesis is accepted. If $\bar{x} > 25.56$, it is rejected. The power function of the test is (Fig. 534)

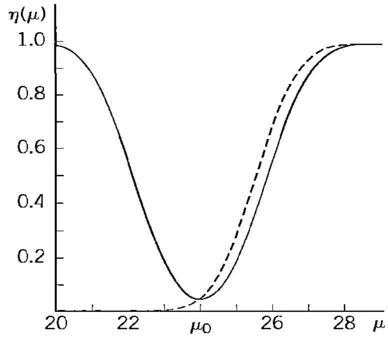


Fig. 534. Power function $\eta(\mu)$ in Example 2, case (a) (dashed) and case (c)

$$\eta(\mu) = P(\bar{X} > 25.56)_\mu = 1 - P(\bar{X} \leqq 25.56)_\mu$$

(7)
$$= 1 - \Phi\left(\frac{25.56 - \mu}{\sqrt{0.9}}\right) = 1 - \Phi(26.94 - 1.05\mu)$$

Case (b). ***Left-Sided Test.*** The critical value c is obtained from the equation

$$P(\bar{X} \leqq c)_{\mu=24} = \Phi\left(\frac{c - 24}{\sqrt{0.9}}\right) = \alpha = 0.05.$$

Table A8 in App. 5 yields $c = 24 - 1.56 = 22.44$. If $\bar{x} \geqq 22.44$, we accept the hypothesis. If $\bar{x} < 22.44$, we reject it. The power function of the test is

(8)
$$\eta(\mu) = P(\bar{X} \leqq 22.44)_\mu = \Phi\left(\frac{22.44 - \mu}{\sqrt{0.9}}\right) = \Phi(23.65 - 1.05\mu).$$

Case (c). ***Two-Sided Test.*** Since the normal distribution is symmetric, we choose c_1 and c_2 equidistant from $\mu = 24$, say, $c_1 = 24 - k$ and $c_2 = 24 + k$, and determine k from

$$P(24 - k \leqq \bar{X} \leqq 24 + k)_{\mu=24} = \Phi\left(\frac{k}{\sqrt{0.9}}\right) - \Phi\left(-\frac{k}{\sqrt{0.9}}\right) = 1 - \alpha = 0.95.$$

Table A8 in App. 5 gives $k/\sqrt{0.9} = 1.960$, hence $k = 1.86$. This gives the values $c_1 = 24 - 1.86 = 22.14$ and $c_2 = 24 + 1.86 = 25.86$. If $\bar{x}$ is not smaller than c_1 and not greater than c_2, we accept the hypothesis. Otherwise we reject it. The power function of the test is (Fig. 534)

$$\eta(\mu) = P(\bar{X} < 22.14)_\mu + P(\bar{X} > 25.86)_\mu = P(\bar{X} < 22.14)_\mu + 1 - P(\bar{X} \leqq 25.86)_\mu$$

(9)
$$= 1 + \Phi\left(\frac{22.14 - \mu}{\sqrt{0.9}}\right) - \Phi\left(\frac{25.86 - \mu}{\sqrt{0.9}}\right)$$

$$= 1 + \Phi(23.34 - 1.05\mu) - \Phi(27.26 - 1.05\mu).$$

Consequently, the operating characteristic $\beta(\mu) = 1 - \eta(\mu)$ (see before) is (Fig. 535)

$$\beta(\mu) = \Phi(27.26 - 1.05\mu) - \Phi(23.34 - 1.05\mu).$$

If we take a larger sample, say, of size $n = 100$ (instead of 10), then $\sigma^2/n = 0.09$ (instead of 0.9) and the critical values are $c_1 = 23.41$ and $c_2 = 24.59$, as can be readily verified. Then the operating characteristic of the test is

$$\beta(\mu) = \Phi\left(\frac{24.59 - \mu}{\sqrt{0.09}}\right) - \Phi\left(\frac{23.41 - \mu}{\sqrt{0.09}}\right)$$

$$= \Phi(81.97 - 3.33\mu) - \Phi(78.03 - 3.33\mu).$$

Figure 535 shows that the corresponding OC curve is steeper than that for $n = 10$. This means that the increase of n has led to an improvement of the test. In any practical case, n is chosen as small as possible but so large that the test brings out deviations between μ and μ_0 that are of practical interest. For instance, if deviations of ± 2 units are of interest, we see from Fig. 535 that $n = 10$ is much too small because when $\mu = 24 - 2 = 22$ or $\mu = 24 + 2 = 26$ β is almost 50%. On the other hand, we see that $n = 100$ is sufficient for that purpose. ■

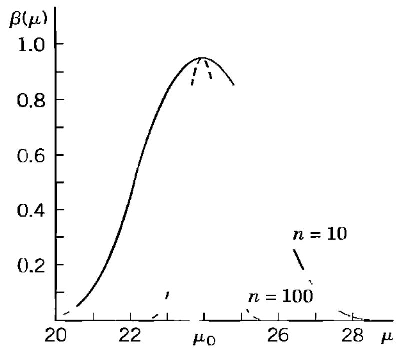


Fig. 535. Curves of the operating characteristic (OC curves) in Example 2, case (c), for two different sample sizes n

Test for μ When σ^2 is Unknown, and for σ^2

EXAMPLE 3 Test for the Mean of the Normal Distribution with Unknown Variance

The tensile strength of a sample of $n = 16$ manila ropes (diameter 3 in.) was measured. The sample mean was $\bar{x} = 4482$ kg, and the sample standard deviation was $s = 115$ kg (N. C. Wiley, 41st Annual Meeting of the American Society for Testing Materials). Assuming that the tensile strength is a normal random variable, test the hypothesis $\mu_0 = 4500$ kg against the alternative $\mu_1 = 4400$ kg. Here μ_0 may be a value given by the manufacturer, while μ_1 may result from previous experience.

Solution. We choose the significance level $\alpha = 5\%$. If the hypothesis is true, it follows from Theorem 2 in Sec. 25.3, that the random variable

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{\bar{X} - 4500}{S/4}$$

has a t-distribution with $n - 1 = 15$ d.f. The test is left-sided. The critical value c is obtained from $P(T < c)_{\mu_0} = \alpha = 0.05$. Table A9 in App. 5 gives $c = -1.75$. As an observed value of T we obtain from the sample $t = (4482 - 4500)/(115/4) = -0.626$. We see that $t > c$ and accept the hypothesis. For obtaining numeric values of the power of the test, we would need tables called noncentral Student t-tables; we shall not discuss this question here. ■

EXAMPLE 4 Test for the Variance of the Normal Distribution

Using a sample of size $n = 15$ and sample variance $s^2 = 13$ from a normal population, test the hypothesis $\sigma^2 = \sigma_0{}^2 = 10$ against the alternative $\sigma^2 = \sigma_1{}^2 = 20$.

Solution. We choose the significance level $\alpha = 5\%$. If the hypothesis is true, then

$$Y = (n - 1)\frac{S^2}{\sigma_0{}^2} = 14\,\frac{S^2}{10} = 1.4S^2$$

has a chi-square distribution with $n - 1 = 14$ d.f. by Theorem 3, Sec. 25.3. From

$$P(Y > c) = \alpha = 0.05, \qquad \text{that is,} \qquad P(Y \leqq c) = 0.95,$$

and Table A10 in App. 5 with 14 degrees of freedom we obtain $c = 23.68$. This is the critical value of Y. Hence to $S^2 = \sigma_0^2 Y/(n-1) = 0.714Y$ there corresponds the critical value $c^* = 0.714 \cdot 23.68 = 16.91$. Since $s^2 < c^*$, we accept the hypothesis.

If the alternative is true, the random variable $Y_1 = 14S^2/\sigma_1^2 = 0.7S^2$ has a chi-square distribution with 14 d.f. Hence our test has the power

$$\eta = P(S^2 > c^*)_{\sigma^2=20} = P(Y_1 > 0.7c^*)_{\sigma^2=20} = 1 - P(Y_1 \leqq 11.84)_{\sigma^2=20}.$$

From a more extensive table of the chi-square distribution (e.g. in Ref. [G3] or [G8]) or from your CAS, you see that $\eta \approx 62\%$. Hence the Type II risk is very large, namely, 38%. To make this risk smaller, we would have to increase the sample size. ■

Comparison of Means and Variances

EXAMPLE 5 Comparison of the Means of Two Normal Distributions

Using a sample $x_1, \cdots, x_{n_1}$ from a normal distribution with unknown mean μ_x and a sample $y_1, \cdots, y_{n_2}$ from another normal distribution with unknown mean μ_y, we want to test the hypothesis that the means are equal, $\mu_x = \mu_y$, against an alternative, say, $\mu_x > \mu_y$. The variances need not be known but are assumed to be equal.[3]

Two cases of comparing means are of practical importance:

Case A. *The samples have the **same size**. Furthermore, each value of the first sample corresponds to precisely one value of the other*, because corresponding values result from the same person or thing **(paired comparison)**—for example, two measurements of the same thing by two different methods or two measurements from the two eyes of the same person. More generally, they may result from pairs of *similar* individuals or things, for example, identical twins, pairs of used front tires from the same car, etc. Then we should form the differences of corresponding values and test the hypothesis that the population corresponding to the differences has mean 0, using the method in Example 3. If we have a choice, this method is better than the following.

Case B. *The two samples are independent and not necessarily of the same size.* Then we may proceed as follows. Suppose that the alternative is $\mu_x > \mu_y$. We choose a significance level α. Then we compute the sample means $\bar{x}$ and $\bar{y}$ as well as $(n_1 - 1)s_x^2$ and $(n_2 - 1)s_y^2$, where s_x^2 and s_y^2 are the sample variances. Using Table A9 in App. 5 with $n_1 + n_2 - 2$ degrees of freedom, we now determine c from

$$P(T \leqq c) = 1 - \alpha. \tag{10}$$

We finally compute

$$t_0 = \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}} \, \frac{\bar{x} - \bar{y}}{\sqrt{(n_1 - 1)s_x^2 + (n_2 - 1)s_y^2}} . \tag{11}$$

It can be shown that this is an observed value of a random variable that has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom, provided the hypothesis is true. If $t_0 \leqq c$, the hypothesis is accepted. If $t_0 > c$, it is rejected.

If the alternative is $\mu_x \neq \mu_y$, then (10) must be replaced by

$$P(T \leqq c_1) = 0.5\alpha, \qquad P(T \leqq c_2) = 1 - 0.5\alpha. \tag{10*}$$

Note that for samples of equal size $n_1 = n_2 = n$, formula (11) reduces to

$$t_0 = \sqrt{n} \, \frac{\bar{x} - \bar{y}}{\sqrt{s_x^2 + s_y^2}} \tag{12}$$

[3]This assumption of equality of variances can be tested, as shown in the next example. If the test shows that they differ significantly, choose two samples of the same size $n_1 = n_2 = n$ (not too small, > 30, say), use the test in Example 2 together with the fact that (12) is an observed value of an approximately standardized normal random variable.

To illustrate the computations, let us consider the two samples $(x_1, \cdots, x_{n_1})$ and $(y_1, \cdots, y_{n_2})$ given by

105 108 86 103 103 107 124 105

and

89 92 84 97 103 107 111 97

showing the relative output of tin plate workers under two different working conditions [J. J. B. Worth. *Journal of Industrial Engineering* **9**, 249–253). Assuming that the corresponding populations are normal and have the same variance, let us test the hypothesis $\mu_x = \mu_y$ against the alternative $\mu_x \neq \mu_y$. (Equality of variances will be tested in the next example.)

Solution. We find

$$\bar{x} = 105.125, \qquad \bar{y} = 97.500, \qquad s_x^2 = 106.125, \qquad s_y^2 = 84.000.$$

We choose the significance level $\alpha = 5\%$. From (10*) with $0.5\alpha = 2.5\%$, $1 - 0.5\alpha = 97.5\%$ and Table A9 in App. 5 with 14 degrees of freedom we obtain $c_1 = -2.14$ and $c_2 = 2.14$. Formula (12) with $n = 8$ gives the value

$$t_0 = \sqrt{8} \cdot 7.625/\sqrt{190.125} = 1.56.$$

Since $c_1 \leqq t_0 \leqq c_2$, we ***accept the hypothesis*** $\mu_x = \mu_y$ that under both conditions the mean output is the same.

Case A applies to the example because the two first sample values correspond to a certain type of work, the next two were obtained in another kind of work, etc. So we may use the differences

16 16 2 6 0 0 13 8

of corresponding sample values and the method in Example 3 to test the hypothesis $\mu = 0$, where μ is the mean of the population corresponding to the differences. As a logical alternative we take $\mu \neq 0$. The sample mean is $\bar{d} = 7.625$, and the sample variance is $s^2 = 45.696$. Hence

$$t = \sqrt{8}\,(7.625 - 0)/\sqrt{45.696} = 3.19.$$

From $P(T \leqq c_1) = 2.5\%$, $P(T \leqq c_2) = 97.5\%$ and Table A9 in App. 5 with $n - 1 = 7$ degrees of freedom we obtain $c_1 = -2.36$, $c_2 = 2.36$ and ***reject the hypothesis*** because $t = 3.19$ does not lie between c_1 and c_2. Hence our present test, in which we used more information (but the same samples), shows that the difference in output is significant. ■

EXAMPLE 6 Comparison of the Variance of Two Normal Distributions

Using the two samples in the last example, test the hypothesis $\sigma_x^2 = \sigma_y^2$; assume that the corresponding populations are normal and the nature of the experiment suggests the alternative $\sigma_x^2 > \sigma_y^2$.

Solution. We find $s_x^2 = 106.125$, $s_y^2 = 84.000$. We choose the significance level $\alpha = 5\%$. Using $P(V \leqq c) = 1 - \alpha = 95\%$ and Table A11 in App. 5, with $(n_1 - 1, n_2 - 1) = (7, 7)$ degrees of freedom, we determine $c = 3.79$. We finally compute $v_0 = s_x^2/s_y^2 = 1.26$. Since $v_0 \leqq c$, we accept the hypothesis. If $v_0 > c$, we would reject it.

This test is justified by the fact that v_0 is an observed value of a random variable that has a so-called ***F*-distribution** with $(n_1 - 1, n_2 - 1)$ degrees of freedom, provided the hypothesis is true. (Proof in Ref. [G3] listed in App. 1.) The F-distribution with (m, n) degrees of freedom was introduced by R. A. Fisher[4] and has the distribution function $F(z) = 0$ if $z < 0$ and

$$F(z) = K_{mn} \int_0^z t^{(m-2)/2}(mt + n)^{-(m+n)/2}\, dt \qquad (z \geqq 0). \tag{13}$$

where $K_{mn} = m^{m/2} n^{n/2} \Gamma(\frac{1}{2}m + \frac{1}{2}n)/\Gamma(\frac{1}{2}m)\Gamma(\frac{1}{2}n)$. (For Γ see App. A3.1.) ■

[4]After the pioneering work of the English statistician and biologist, KARL PEARSON (1857–1936), the founder of the English school of statistics, and WILLIAM SEALY GOSSET (1876–1937), who discovered the t-distribution (and published under the name 'Student'), the English statistician Sir RONALD AYLMER FISHER (1890–1962), professor of eugenics in London (1933–1943) and professor of genetics in Cambridge, England (1943–1957) and Adelaide, Australia (1957–1962), had great influence on the further development of modern statistics.

This long section contained the basic ideas and concepts of testing, along with typical applications and you may perhaps want to review it quickly before going on, because the next sections concern an adaption of these ideas to tasks of great practical importance and resulting tests in connection with quality control, acceptance (or rejection) of goods produced, and so on.

PROBLEM SET 25.4

1. Test $\mu = 0$ against $\mu > 0$, assuming normality and using the sample 1, -1, 1, 3, -8, 6, 0 (deviations of the azimuth [multiples of 0.01 radian] in some revolution of a satellite). Choose $\alpha = 5\%$.

2. In one of his classical experiments Buffon obtained 2048 heads in tossing a coin 4040 times. Was the coin fair?

3. Do the same test as in Prob. 2, using a result by K. Pearson, who obtained 6 019 heads in 12 000 trials.

4. Assuming normality and known variance $\sigma^2 = 4$, test the hypothesis $\mu = 30.0$ against the alternative (a) $\mu = 28.5$, (b) $\mu = 30.7$, using a sample of size 10 with mean $\bar{x} = 28.5$ and choosing $\alpha = 5\%$.

5. How does the result in Prob. 4(a) change if we use a smaller sample, say, of size 4, the other data ($\bar{x} = 28.5$, $\alpha = 5\%$, etc.) remaining as before?

6. Detemine the power of the test in Prob. 4(a).

7. What is the rejection region in Prob. 4 in the case of a two-sided test with $\alpha = 5\%$?

8. Using the sample 0.80, 0.81, 0.81, 0.82, 0.81, 0.82, 0.80, 0.82, 0.81, 0.81 (length of nails in inches), test the hypothesis $\mu = 0.80$ in. (the length indicated on the box) against the alternative $\mu \neq 0.80$ in. (Assume normality, choose $\alpha = 5\%$.)

9. A firm sells oil in cans containing 1000 g oil per can and is interested to know whether the mean weight differs significantly from 1000 g at the 5% level, in which case the filling machine has to be adjusted. Set up a hypothesis and an alternative and perform the test, assuming normality and using a sample of 20 fillings with mean 996 g and standard deviation 5 g.

10. If a sample of 50 tires of a certain kind has a mean life of 32 000 mi and a standard deviation of 4000 mi, can the manufacturer claim that the true mean life of such tires is greater than 30 000 mi? Set up and test a corresponding hypothesis at a 5% level, assuming normality.

11. If simultaneous measurements of electric voltage by two different types of voltmeter yield the differences (in volts) 0.8, 0.2, -0.3, 0.1, 0.0, 0.5, 0.7, 0.2, can we assert at the 5% level that there is no significant difference in the calibration of the two types of instruments? (Assume normality.)

12. If a standard medication cures about 70% of patients with a certain disease and a new medication cured 148 of the first 200 patients on whom it was tried, can we conclude that the new medication is better? (Choose $\alpha = 5\%$.)

13. Suppose that in the past the standard deviation of weights of certain 25.0-oz packages filled by a machine was 0.4 oz. Test the hypothesis H_0: $\sigma = 0.4$ against the alternative H_1: $\sigma > 0.4$ (an undesirable increase), using a sample of 10 packages with standard deviation 0.5 oz and assuming normality. (Choose $\alpha = 5\%$.)

14. Suppose that in operating battery-powered electrical equipment, it is less expensive to replace all batteries at fixed intervals than to replace each battery individually when it breaks down, provided the standard deviation of the lifetime is less than a certain limit, say, less than 5 hours. Set up and apply a suitable test, using a sample of 28 values of lifetimes with standard deviation $s = 3.5$ hours and assuming normality; choose $\alpha = 5\%$.

15. Brand A gasoline was used in 9 automobiles of the same model under identical conditions. The corresponding sample of 9 values (miles per gallon) had mean 20.2 and standard deviation 0.5. Under the same conditions, high-power brand B gasoline gave a sample of 10 values with mean 21.8 and standard deviation 0.6. Is the mileage of B significantly better than that of A? (Test at the 5% level; assume normality.)

16. The two samples 70, 80, 30, 70, 60, 80 and 140, 120, 130, 120, 120, 130, 120 are values of the differences of temperatures (°C) of iron at two stages of casting, taken from two different crucibles. Is the variance of the first population larger than that of the second? (Assume normality. Choose $\alpha = 5\%$.)

17. Using samples of sizes 10 and 16 with variances $s_x^2 = 50$ and $s_y^2 = 30$ and assuming normality of the corresponding populations, test the hypothesis H_0: $\sigma_x^2 = \sigma_y^2$ against the alternative $\sigma_x^2 > \sigma_y^2$. Choose $\alpha = 5\%$.

18. Assuming normality and equal variance and using independent samples with $n_1 = 9$, $\bar{x} = 12$, $s_x = 2$, $n_2 = 9$, $\bar{y} = 15$, $s_y = 2$, test H_0: $\mu_x = \mu_y$ against $\mu_x \neq \mu_y$; choose $\alpha = 5\%$.

19. Show that for a normal distribution the two types of errors in a test of a hypothesis H_0: $\mu = \mu_0$ against an alternative H_1: $\mu = \mu_1$ can be made as small as one pleases (not zero) by taking the sample sufficiently large.

20. CAS EXPERIMENT. Tests of Means and Variances. **(a)** Obtain 100 samples of size 10 each from the normal distribution with mean 100 and variance 25. For each sample test the hypothesis $\mu_0 = 100$ against the alternative $\mu_1 > 100$ at the level of $\alpha = 10\%$ Record the number of rejections of the hypothesis. Do the whole experiment once more and compare.

(b) Set up a similar experiment for the variance of a normal distribution and perform it 100 times.

25.5 Quality Control

The ideas on testing can be adapted and extended in various ways to serve basic practical needs in engineering and other fields. We show this in the remaining sections for some of the most important tasks solvable by statistical methods. As a first such area of problems, we discuss industrial quality control, a highly successful method used in various industries.

No production process is so perfect that all the products are completely alike. There is always a small variation that is caused by a great number of small. uncontrollable factors and must therefore be regarded as a chance variation. It is important to make sure that the products have required values (for example, length, strength, or whatever property may be essential in a particular case). For this purpose one makes a test of the hypothesis that the products have the required property, say, $\mu = \mu_0$, where μ_0 is a required value. If this is done after an entire lot has been produced (for example, a lot of 100 000 screws), the test will tell us how good or how bad the products are, but it it obviously too late to alter undesirable results. It is much better to test during the production run. This is done at regular intervals of time (for example, every hour or half-hour) and is called **quality control.** Each time a sample of the same size is taken, in practice 3 to 10 times. If the hypothesis is rejected. we stop the production and look for the cause of the trouble.

If we stop the production process even though it is progressing properly, we make a Type I error. If we do not stop the process even though something is not in order, we make a Type II error (see Sec. 25.4). The result of each test is marked in graphical form on what is called a **control chart.** This was proposed by W. A. Shewhart in 1924 and makes quality control particularly effective.

Control Chart for the Mean

An illustration and example of a control chart is given in the upper part of Fig. 536. This control chart for the mean shows the **lower control limit** LCL, the **center control line** CL, and the **upper control limit** UCL. The two **control limits** correspond to the critical values c_1 and c_2 in case (c) of Example 2 in Sec. 25.4. As soon as a sample mean falls outside the range between the control limits, we reject the hypothesis and assert that the production process is "out of control"; that is, we assert that there has been a shift in process level. Action is called for whenever a point exceeds the limits.

If we choose control limits that are too loose, we shall not detect process shifts. On the other hand, if we choose control limits that are too tight, we shall be unable to run the process because of frequent searches for nonexistent trouble. The usual significance level is $\alpha = 1\%$. From Theorem 1 in Sec. 25.3 and Table A8 in App. 5 we see that in the case of the normal distribution the corresponding control limits for the mean are

$$\text{(1)} \qquad \text{LCL} = \mu_0 - 2.58 \frac{\sigma}{\sqrt{n}}, \qquad \text{UCL} = \mu_0 + 2.58 \frac{\sigma}{\sqrt{n}}.$$

Here σ is assumed to be known. If σ is unknown, we may compute the standard deviations of the first 20 or 30 samples and take their arithmetic mean as an approximation of σ. The broken line connecting the means in Fig. 536 is merely to display the results.

Additional, more subtle controls are often used in industry. For instance, one observes the motions of the sample means above and below the centerline, which should happen frequently. Accordingly, long runs (conventionally of length 7 or more) of means all above (or all below) the centerline could indicate trouble.

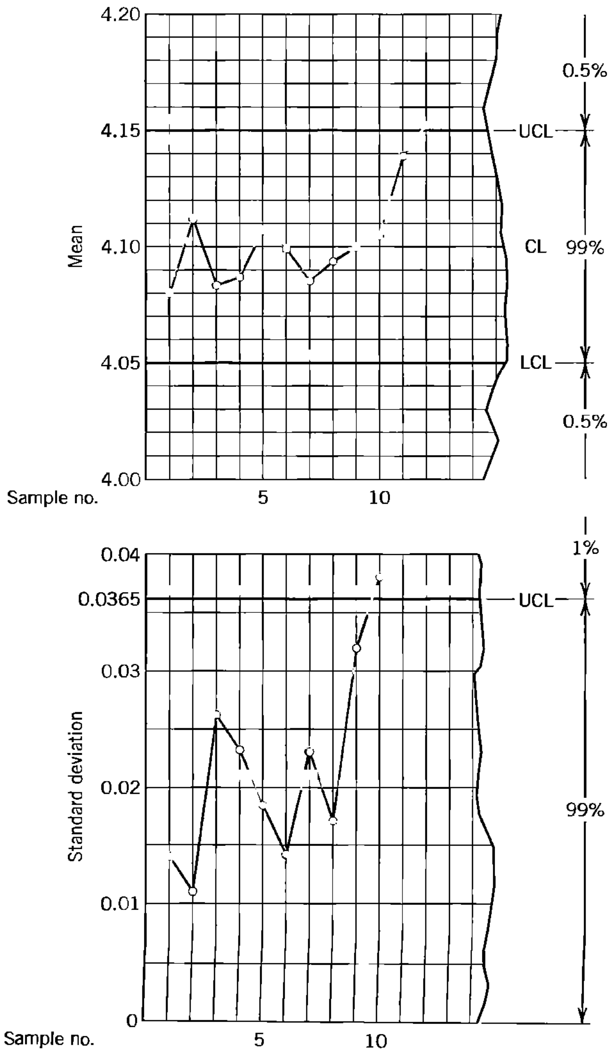


Fig. 536. Control charts for the mean (upper part of figure) and the standard deviation in the case of the samples on p. 1070

Table 25.5 Twelve Samples of Five Values Each (Diameter of Small Cylinders, Measured in Millimeters)

Sample Number	Sample Values					$\bar{x}$	s	R
1	4.06	4.08	4.08	4.08	4.10	4.080	0.014	0.04
2	4.10	4.10	4.12	4.12	4.12	4.112	0.011	0.02
3	4.06	4.06	4.08	4.10	4.12	4.084	0.026	0.06
4	4.06	4.08	4.08	4.10	4.12	4.088	0.023	0.06
5	4.08	4.10	4.12	4.12	4.12	4.108	0.018	0.04
6	4.08	4.10	4.10	4.10	4.12	4.100	0.014	0.04
7	4.06	4.08	4.08	4.10	4.12	4.088	0.023	0.06
8	4.08	4.08	4.10	4.10	4.12	4.096	0.017	0.04
9	4.06	4.08	4.10	4.12	4.14	4.100	0.032	0.08
10	4.06	4.08	4.10	4.12	4.16	4.104	0.038	0.10
11	4.12	4.14	4.14	4.14	4.16	4.140	0.014	0.04
12	4.14	4.14	4.16	4.16	4.16	4.152	0.011	0.02

Control Chart for the Variance

In addition to the mean, one often controls the variance, the standard deviation, or the range. To set up a control chart for the variance in the case of a normal distribution, we may employ the method in Example 4 of Sec. 25.4 for determining control limits. It is customary to use only one control limit, namely, an upper control limit. Now from Example 4 of Sec. 25.4 we have $S^2 = \sigma_0{}^2 Y/(n-1)$, where because of our normality assumption the random variable Y has a chi-square distribution with $n - 1$ degrees of freedom. Hence the desired control limit is

$$\text{UCL} = \frac{\sigma^2 c}{n-1} \tag{2}$$

where c is obtained from the equation

$$P(Y > c) = \alpha, \qquad \text{that is,} \qquad P(Y \leqq c) = 1 - \alpha$$

and the table of the chi-square distribution (Table A10 in App. 5) with $n - 1$ degrees of freedom (or from your CAS); here α (5% or 1%, say) is the probability that in a properly running process an observed value s^2 of S^2 is greater than the upper control limit.

If we wanted a control chart for the variance with both an upper control limit UCL and a lower control limit LCL, these limits would be

$$\text{LCL} = \frac{\sigma^2 c_1}{n-1} \qquad \text{and} \qquad \text{UCL} = \frac{\sigma^2 c_2}{n-1}. \tag{3}$$

where c_1 and c_2 are obtained from Table A10 with $n - 1$ d.f. and the equations

$$P(Y \leqq c_1) = \frac{\alpha}{2} \qquad \text{and} \qquad P(Y \leqq c_2) = 1 - \frac{\alpha}{2}. \tag{4}$$

Control Chart for the Standard Deviation

To set up a control chart for the standard deviation, we need an upper control limit

$$\text{(5)} \qquad \text{UCL} = \frac{\sigma\sqrt{c}}{\sqrt{n-1}}$$

obtained from (2). For example, in Table 25.5 we have $n = 5$. Assuming that the corresponding population is normal with standard deviation $\sigma = 0.02$ and choosing $\alpha = 1\ \%$, we obtain from the equation

$$P(Y \leqq c) = 1 - \alpha = 99\%$$

and Table A10 in App. 5 with 4 degrees of freedom the critical value $c = 13.28$ and from (5) the corresponding value

$$\text{UCL} = \frac{0.02\sqrt{13.28}}{\sqrt{4}} = 0.0365,$$

which is shown in the lower part of Fig. 536.

A control chart for the standard deviation with both an upper and a lower control limit is obtained from (3).

Control Chart for the Range

Instead of the variance or standard deviation, one often controls the **range** R (= largest sample value minus smallest sample value). It can be shown that in the case of the normal distribution, the standard deviation σ is proportional to the expectation of the random variable R^* for which R is an observed value, say, $\sigma = \lambda_n E(R^*)$, where the factor of proportionality λ_n depends on the sample size n and has the values

n	2	3	4	5	6	7	8	9	10
$\lambda_n = \sigma/E(R^*)$	0.89	0.59	0.49	0.43	0.40	0.37	0.35	0.34	0.32

n	12	14	16	18	20	30	40	50
$\lambda_n = \sigma/E(R^*)$	0.31	0.29	0.28	0.28	0.27	0.25	0.23	0.22

Since R depends on two sample values only, it gives less information about a sample than s does. Clearly, the larger the sample size n is, the more information we lose in using R instead of s. A practical rule is to use s when n is larger than 10.

[illegible]

1. Suppose a machine for filling cans with lubricating oil is set so that it will generate fillings which form a normal population with mean 1 gal and standard deviation 0.03 gal. Set up a control chart of the type shown in Fig. 536 for controlling the mean (that is, find LCL and UCL), assuming that the sample size is 6.
2. **(Three-sigma control chart)** Show that in Prob. 1, the requirement of the significance level $\alpha = 0.3\%$ leads to $\text{LCL} = \mu - 3\sigma/\sqrt{n}$ and $\text{UCL} = \mu + 3\sigma/\sqrt{n}$, and find the corresponding numeric values.
3. What sample size should we choose in Prob. 1 if we want LCL and UCL somewhat closer together, say, UCL − LCL = 0.05, without changing the significance level?

4. How does the meaning of the control limits (1) change if we apply a control chart with these limits in the case of a population that is not normal?

5. How should we change the sample size in controlling the mean of a normal population if we want the difference

$$\text{UCL} - \text{LCL}$$

to decrease to half its original value?

6. What LCL and UCL should we use instead of (1) if instead of $\bar{x}$ we use the sum $x_1 + \cdots + x_n$ of the sample values? Determine these limits in the case of Fig. 536.

7. Ten samples of size 2 were taken from a production lot of bolts. The values (length in mm) are as shown. Assuming that the population is normal with mean 27.5 and variance 0.024 and using (1), set up a control chart for the mean and graph the sample means on the chart.

Sample No.	1	2	3	4	5	6	7	8	9	10
Length	27.4	27.4	27.5	27.3	27.9	27.6	27.6	27.8	27.5	27.3
	27.6	27.4	27.7	27.4	27.5	27.5	27.4	27.3	27.4	27.7

8. Graph the means of the following 10 samples (thickness of washers, coded values) on a control chart for means, assuming that the population is normal with mean 5 and standard deviation 1.55.

Time	8:00	8:30	9:00	9:30	10:00	10:30	11:00	11:30	12:00	12:30
Sample Values	3	3	5	7	7	4	5	6	5	5
	4	6	2	5	3	4	6	4	5	2
	8	6	5	4	6	3	4	6	6	5
	4	8	6	4	5	6	6	4	4	3

9. Graph the ranges of the samples in Prob. 8 on a control chart for ranges.

10. What effect on UCL − LCL does it have if we double the sample size? If we switch from $\alpha = 1\%$ to $\alpha = 5\%$?

11. Since the presence of a point outside control limits for the mean indicates trouble ("the process is out of control"), how often would we be making the mistake of looking for nonexistent trouble if we used (a) 1-sigma limits, (b) 2-sigma limits? (Assume normality.)

12. Graph $\lambda_n = \sigma/E(R^*)$ as a function of n. Why is λ_n a monotone decreasing function of n?

13. **(Number of defectives)** Find formulas for the UCL, CL, and LCL (corresponding to 3σ-limits) in the case of a control chart for the number of defectives, assuming that in a state of statistical control the fraction of defectives is p.

14. How would progressive tool wear in an automatic lathe operation be indicated by a control chart of the mean? Answer the same question for a sudden change in the position of the tool in that operation.

15. **(Number of defects per unit)** A so-called *c-chart* or *defects-per-unit chart* is used for the control of the number X of defects per unit (for instance, the number of defects per 10 meters of paper, the number of missing rivets in an airplane wing, etc.) (a) Set up formulas for CL and LCL, UCL corresponding to

$$\mu \pm 3\sigma,$$

assuming that X has a Poisson distribution. (b) Compute CL, LCL, and UCL in a control process of the number of imperfections in sheet glass; assume that this number is 2.5 per sheet on the average when the process is under control.

16. **(Attribute control charts).** Twenty samples of size 100 were taken from a production of containers. The numbers of defectives (leaking containers) in those samples (in the order observed) were

3 7 6 1 4 5 4 9 7 0 5 6 13 4

9 0 2 1 12 8.

From previous experience it was known that the average fraction defective is $p = 5\%$ provided that the process of production is running properly. Using the binomial distribution, set up a *fraction defective chart* (also called a ***p*-chart**), that is, choose the LCL = 0 and determine the UCL for the fraction defective (in percent) by the use of 3-sigma limits, where σ^2 is the variance of the random variable

$$\bar{X} = \textit{Fraction defective in a sample of size } 100.$$

Is the process under control?

17. **CAS PROJECT. Control Charts. (a)** Obtain 100 samples of 4 values each from the normal distribution with mean 8.0 and variance 0.16 and their means, variances, and ranges.

(b) Use these samples for making up a control chart for the mean.

(c) Use them on a control chart for the standard deviation.

(d) Make up a control chart for the range.

(e) Describe quantitative properties of the samples that you can see from those charts (e.g., whether the corresponding process is under control, whether the quantities observed vary randomly, etc.).

25.6 Acceptance Sampling

Acceptance sampling is usually done when products leave the factory (or in some cases even within the factory). The standard situation in acceptance sampling is that a **producer** supplies to a **consumer** (a buyer or wholesaler) a lot of N items (a carton of screws, for instance). The decision to **accept** or **reject** the lot is made by determining the number x of **defectives** (= defective items) in a sample of size n from the lot. The lot is accepted if $x \leqq c$, where c is called the **acceptance number,** giving the allowable number of defectives. If $x > c$, the consumer rejects the lot. Clearly, producer and consumer must agree on a certain **sampling plan** giving n and c.

From the hypergeometric distribution we see that the event A: *"Accept the lot"* has probability (see Sec. 24.7)

$$P(A) = P(X \leqq c) = \sum_{x=0}^{c} \binom{M}{x}\binom{N-M}{n-x} \bigg/ \binom{N}{n} \tag{1}$$

where M is the number of defectives in a lot of N items. In terms of the **fraction defective** $\theta = M/N$ we can write (1) as

$$P(A;\ \theta) = \sum_{x=0}^{c} \binom{N\theta}{x}\binom{N-N\theta}{n-x} \bigg/ \binom{N}{n}. \tag{2}$$

$P(A;\ \theta)$ can assume $n + 1$ values corresponding to $\theta = 0, 1/N, 2/N, \cdots, N/N$; here, n and c are fixed. A monotone smooth curve through these points is called the **operating characteristic curve (OC curve)** of the sampling plan considered.

EXAMPLE 1 Sampling Plan

Suppose that certain tool bits are packaged 20 to a box, and the following sampling plan is used. A sample of two tool bits is drawn, and the corresponding box is accepted if and only if both bits in the sample are good. In this case, $N = 20$, $n = 2$, $c = 0$, and (2) takes the form (a factor 2 drops out)

$$P(A;\ \theta) = \binom{20\theta}{0}\binom{20-20\theta}{2} \bigg/ \binom{20}{2}$$

$$= \frac{(20-20\theta)(19-20\theta)}{380}.$$

The values of $P(A, \theta)$ for $\theta = 0, 1/20, 2/20, \cdots, 20/20$ and the resulting OC curve are shown in Fig. 537 on p. 1074. (Verify!) ■

In most practical cases θ will be small (less than 10%). Then if we take small samples compared to N, we can approximate (2) by the Poisson distribution (Sec. 24.7); thus

$$P(A;\ \theta) \sim e^{-\mu} \sum_{x=0}^{c} \frac{\mu^x}{x!} \qquad (\mu = n\theta). \tag{3}$$

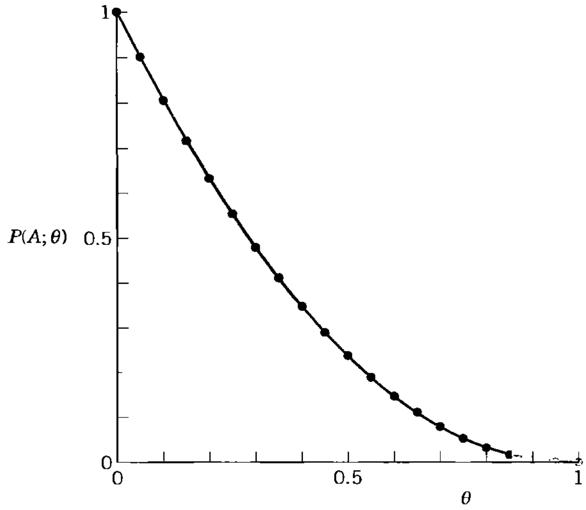


Fig. 537. OC curve of the sampling plan with $n = 2$ and $c = 0$ for lots of size $N = 20$

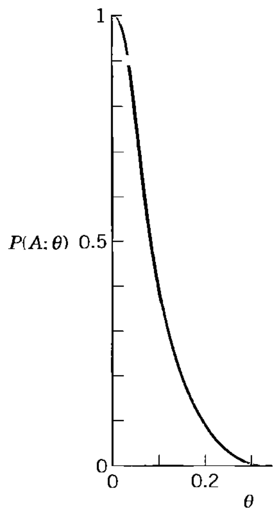


Fig. 538. OC curve in Example 2

EXAMPLE 2 Sampling Plan. Poisson Distribution

Suppose that for large lots the following sampling plan is used. A sample of size $n = 20$ is taken. If it contains not more than one defective, the lot is accepted. If the sample contains two or more defectives, the lot is rejected. In this plan, we obtain from (3)

$$P(A;\ \theta) \sim e^{-20\theta}(1 + 20\theta).$$

The corresponding OC curve is shown in Fig. 538. ■

Errors in Acceptance Sampling

We show how acceptance sampling fits into general test theory (Sec. 25.4) and what this means from a practical point of view. The producer wants the probability α of rejecting an **acceptable lot** (a lot for which θ does not exceed a certain number θ_0 on which the two parties agree) to be small. θ_0 is called the **acceptable quality level** (AQL). Similarly,

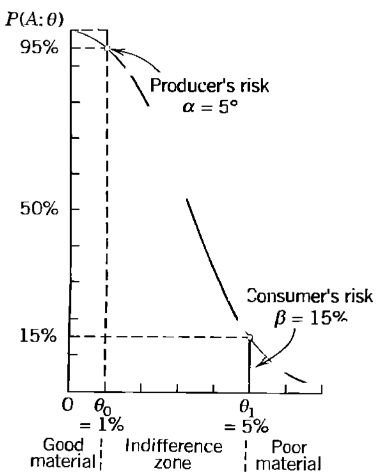


Fig. 539. OC curve, producer's and consumer's risks

the consumer (the buyer) wants the probability β of accepting an **unacceptable lot** (a lot for which θ is greater than or equal to some θ_1) to be small. θ_1 is called the **lot tolerance percent defective** (LTPD) or the **rejectable quality level** (RQL). α is called **producer's risk.** It corresponds to a Type I error in Sec. 25.4. β is called **consumer's risk** and corresponds to a Type II error. Figure 539 shows an example. We see that the points $(\theta_0, 1 - \alpha)$ and (θ_1, β) lie on the OC curve. It can be shown that for large lots we can choose θ_0, θ_1 $(> \theta_0)$, α, β and then determine n and c such that the OC curve runs very close to those prescribed points. Table 25.6 shows the analogy between acceptance sampling and hypothesis testing in Sec. 25.4.

Table 25.6 Acceptance Sampling and Hypothesis Testing

Acceptance Sampling	Hypothesis Testing
Acceptable quality level (AQL) $\theta = \theta_0$	Hypothesis $\theta = \theta_0$
Lot tolerance percent defectives (LTPD) $\theta = \theta_1$	Alternative $\theta = \theta_1$
Allowable number of defectives c	Critical value c
Producer's risk α of rejecting a lot with $\theta \leqq \theta_0$	Probability α of making a Type I error (significance level)
Consumer's risk β of accepting a lot with $\theta \geqq \theta_1$	Probability β of making a Type II error

Rectification

Rectification of a *rejected* lot means that the lot is inspected item by item and all defectives are removed and replaced by nondefective items. (This may be too expensive if the lot is cheap; in this case the lot may be sold at a cut-rate price or scrapped.) If a production turns out $100\theta\%$ defectives, then in K lots of size N each, $KN\theta$ of the KN items are defectives. Now $KP(A;\ \theta)$ of these lots are accepted. These contain $KPN\theta$ defectives, whereas the rejected and rectified lots contain no defectives, because of the rectification. Hence after the rectification the fraction defective in all K lots equals $KPN\theta/KN$. This is called the **average outgoing quality** (AOQ); thus

(4) $$\mathrm{AOQ}(\theta) = \theta P(A;\ \theta).$$

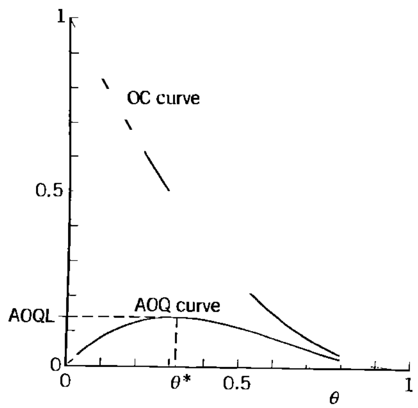


Fig. 540. OC curve and AOQ curve for the sampling plan in Fig. 537

Figure 540 on p. 1075 shows an example. Since AOQ(0) = 0 and $P(A;\ 1) = 0$, the AOQ curve has a maximum at some $\theta = \theta^*$, giving the **average outgoing quality limit** (AOQL). This is the worst average quality that may be expected to be accepted under rectification.

1. Lots of knives are inspected by a sampling plan that uses a sample of size 20 and the acceptance number $c = 1$. What are probabilities of accepting a lot with 1%, 2%, 10% defectives (dull blades)? Use Table A6 in App. 5. Graph the OC curve.
2. What happens in Prob. 1 if the sample size is increased to 50? First guess. Then calculate. Graph the OC curve and compare.
3. How will the probabilities in Prob. 1 with $n = 20$ change (up or down) if we decrease c to zero? First guess.
4. What are the producer's and consumer's risks in Prob. 1 if the AQL is 1.5% and the RQL is 7.5%?
5. Large lots of batteries are inspected according to the following plan. $n = 30$ batteries are randomly drawn from a lot and tested. If this sample contains at most $c = 1$ defective battery, the lot is accepted. Otherwise it is rejected. Graph the OC curve of the plan, using the Poisson approximation.
6. Graph the AOQ curve in Prob. 5. Determine the AOQL, assuming that rectification is applied.
7. Do the work required in Prob. 5 if $n = 50$ and $c = 0$.
8. Find the binomial approximation of the hypergeometric distribution in Example 1 and compare the approximate and the accurate values.
9. In Example 1, what are the producer's and consumer's risks if the AQL is 0.1 and the RQL is 0.6?
10. Calculate $P(A;\ \theta)$ in Example 1 if the sample size is increased from $n = 2$ to $n = 3$, the other data remaining as before. Compute $P(A;\ 0.10)$ and $P(A;\ 0.20)$ and compare with Example 1.
11. Samples of 5 screws are drawn from a lot with fraction defective θ. The lot is accepted if the sample contains (a) no defective screws, (b) at most 1 defective screw. Using the binomial distribution, find, graph, and compare the OC curves.
12. Find the risks in the single sampling plan with $n = 5$ and $c = 0$, assuming that the AQL is $\theta_0 = 1\%$ and the RQL is $\theta_1 = 15\%$.
13. Why is it impossible for an OC curve to have a vertical portion separating good from poor quality?
14. If in a single sampling plan for large lots of spark plugs, the sample size is 100 and we want the AQL to be 5% and the producer's risk 2%, what acceptance number c should we choose? (Use the normal approximation.)
15. What is the consumer's risk in Prob. 14 if we want the RQL to be 12%?
16. Graph and compare sampling plans with $c = 1$ and increasing values of n, say, $n = 2, 3, 4$. (Use the binomial distribution.)
17. Samples of 3 fuses are drawn from lots and a lot is accepted if in the corresponding sample we find no more than 1 defective fuse. Criticize this sampling plan. In particular, find the probability of accepting a lot that is 50% defective. (Use the binomial distribution.)
18. Graph the OC curve and the AOQ curve for the single sampling plan for large lots with $n = 5$ and $c = 0$, and find the AOQL.

25.7 Goodness of Fit. χ^2-Test

To test for **goodness of fit** means that we wish to test that a certain function $F(x)$ is the distribution function of a distribution from which we have a sample $x_1, \cdots, x_n$. Then we test whether the **sample distribution function** $\tilde{F}(x)$ defined by

$$\tilde{F}(x) = \textit{Sum of the relative frequencies of all sample values } x_j \textit{ not exceeding } x$$

fits $F(x)$ "sufficiently well." If this is so, we shall accept the hypothesis that $F(x)$ is the distribution function of the population; if not, we shall reject the hypothesis.

This test is of considerable practical importance, and it differs in character from the tests for parameters (μ, σ^2, etc.) considered so far.

To test in that fashion, we have to know how much $\widetilde{F}(x)$ can differ from $F(x)$ if the hypothesis is true. Hence we must first introduce a quantity that measures the deviation of $\widetilde{F}(x)$ from $F(x)$, and we must know the probability distribution of this quantity under the assumption that the hypothesis is true. Then we proceed as follows. We determine a number c such that if the hypothesis is true, a deviation greater than c has a small preassigned probability. If, nevertheless, a deviation greater than c occurs, we have reason to doubt that the hypothesis is true and we reject it. On the other hand, if the deviation does not exceed c, so that $\widetilde{F}(x)$ approximates $F(x)$ sufficiently well, we accept the hypothesis. Of course, if we accept the hypothesis, this means that we have insufficient evidence to reject it, and this does not exclude the possibility that there are other functions that would not be rejected in the test. In this respect the situation is quite similar to that in Sec. 25.4.

Table 25.7 shows a test of that type, which was introduced by R. A. Fisher. This test is justified by the fact that if the hypothesis is true, then χ_0^2 is an observed value of a random variable whose distribution function approaches that of the chi-square distribution with $K - 1$ degrees of freedom (or $K - r - 1$ degrees of freedom if r parameters are estimated) as n approaches infinity. The requirement that at least five sample values lie in each interval in Table 25.7 results from the fact that for finite n that random variable has only *approximately* a chi-square distribution. A proof can be found in Ref. [G3] listed in App. 1. If the sample is so small that the requirement cannot be satisfied, one may continue with the test, but then use the result with caution.

Table 25.7 Chi-square Test for the Hypothesis That $F(x)$ is the Distribution Function of a Population from Which a Sample $x_1, \cdots, x_n$ is Taken

Step 1. Subdivide the x-axis into K intervals $I_1, I_2, \cdots, I_K$ such that each interval contains at least 5 values of the given sample $x_1, \cdots, x_n$. Determine the number b_j of sample values in the interval I_j, where $j = 1, \cdots, K$. If a sample value lies at a common boundary point of two intervals, add 0.5 to each of the two corresponding b_j.

Step 2. Using $F(x)$, compute the probability p_j that the random variable X under consideration assumes any value in the interval I_j, where $j = 1, \cdots, K$. Compute

$$e_j = np_j.$$

(This is the number of sample values theoretically expected in I_j if the hypothesis is true.)

Step 3. Compute the deviation

$$\chi_0^2 = \sum_{j=1}^{K} \frac{(b_j - e_j)^2}{e_j}. \tag{1}$$

Step 4. Choose a significance level (5%, 1%, or the like).

Step 5. Determine the solution c of the equation

$$P(\chi^2 \leqq c) = 1 - \alpha$$

from the table of the chi-sqare distribution with $K - 1$ degrees of freedom (Table A10 in App. 5). If r parameters of $F(x)$ are unknown and their maximum likelihood estimates (Sec. 25.2) are used, then use $K - r - 1$ degrees of freedom (instead of $K - 1$). If $\chi_0^2 \leqq c$, accept the hypothesis. If $\chi_0^2 > c$, reject the hypothesis.

Table 25.8 Sample of 100 Values of the Splitting Tensile Strength (lb/in.^{2}) of Concrete Cylinders

320	380	340	410	380	340	360	350	320	370
350	340	350	360	370	350	380	370	300	420
370	390	390	440	330	390	330	360	400	370
320	350	360	340	340	350	350	390	380	340
400	360	350	390	400	350	360	340	370	420
420	400	350	370	330	320	390	380	400	370
390	330	360	380	350	330	360	300	360	360
360	390	350	370	370	350	390	370	370	340
370	400	360	350	380	380	360	340	330	370
340	360	390	400	370	410	360	400	340	360

D. L. IVEY. Splitting tensile tests on structural lightweight aggregate concrete. Texas Transportation Institute, College Station, Texas.

EXAMPLE 1 Test of Normality

Test whether the population from which the sample in Table 25.8 was taken is normal.

Solution. Table 25.8 shows the values (column by column) in the order obtained in the experiment. Table 25.9 gives the frequency distribution and Fig. 541 the histogram. It is hard to guess the outcome of the test—does the histogram resemble a normal density curve sufficiently well or not?

The maximum likelihood estimates for μ and σ^2 are $\hat{\mu} = \bar{x} = 364.7$ and $\tilde{\sigma}^2 = 712.9$. The computation in Table 25.10 yields $\chi_0{}^2 = 2.942$. It is very interesting that the interval $375 \cdots 385$ contributes over 50% of $\chi_0{}^2$. From the histogram we see that the corresponding frequency looks much too small. The second largest contribution comes from $395 \cdots 405$, and the histogram shows that the frequency seems somewhat too large, which is perhaps not obvious from inspection.

Table 25.9 Frequency Table of the Sample in Table 25.8

1 Tensile Strength x [lb/in.^{2}]	2 Absolute Frequency	3 Relative Frequency $\tilde{f}(x)$	4 Cumulative Absolute Frequency	5 Cumulative Relative Frequency $\tilde{F}(x)$
300	2	0.02	2	0.02
310	0	0.00	2	0.02
320	4	0.04	6	0.06
330	6	0.06	12	0.12
340	11	0.11	23	0.23
350	14	0.14	37	0.37
360	16	0.16	53	0.53
370	15	0.15	68	0.68
380	8	0.08	76	0.76
390	10	0.10	86	0.86
400	8	0.08	94	0.94
410	2	0.02	96	0.96
420	3	0.03	99	0.99
430	0	0.00	99	0.99
440	1	0.01	100	1.00

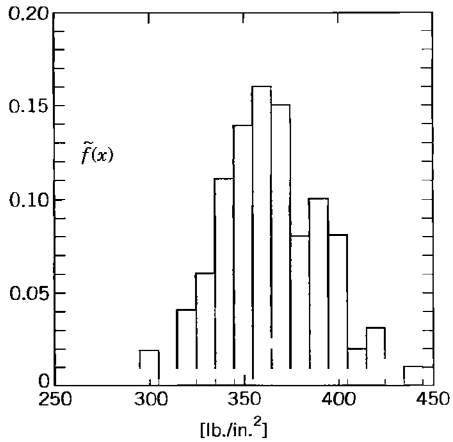


Fig. 541. Frequency histogram of the sample in Table 25.8

We choose $\alpha = 5\%$. Since $K = 10$ and we estimated $r = 2$ parameters we have to use Table A10 in App. 5 with $K - r - 1 = 7$ degrees of freedom. We find $c = 14.07$ as the solution of $P(\chi^2 \leqq c) = 95\%$. Since $\chi_0^2 < c$, we accept the hypothesis that the population is normal. ■

Table 25.10 Computations in Example 1

x_j	$\dfrac{x_j - 364.7}{26.7}$	$\Phi\left(\dfrac{x_j - 364.7}{26.7}\right)$	e_j	b_j	Term in (1)
$-\infty \cdots 325$	$-\infty \cdots -1.49$	$0.0000 \cdots 0.0681$	6.81	6	0.096
$325 \cdots 335$	$-1.49 \cdots -1.11$	$0.0681 \cdots 0.1335$	6.54	6	0.045
$335 \cdots 345$	$-1.11 \cdots -0.74$	$0.1335 \cdots 0.2296$	9.61	11	0.201
$345 \cdots 355$	$-0.74 \cdots -0.36$	$0.2296 \cdots 0.3594$	12.98	14	0.080
$355 \cdots 365$	$-0.36 \cdots 0.01$	$0.3594 \cdots 0.4960$	13.66	16	0.401
$365 \cdots 375$	$0.01 \cdots 0.39$	$0.4960 \cdots 0.6517$	15.57	15	0.021
$375 \cdots 385$	$0.39 \cdots 0.76$	$0.6517 \cdots 0.7764$	12.47	8	1.602
$385 \cdots 395$	$0.76 \cdots 1.13$	$0.7764 \cdots 0.8708$	9.44	10	0.033
$395 \cdots 405$	$1.13 \cdots 1.51$	$0.8708 \cdots 0.9345$	6.37	8	0.417
$405 \cdots \infty$	$1.51 \cdots \infty$	$0.9345 \cdots 1.0000$	6.55	6	0.046

$\chi_0^2 = 2.942$

PROBLEM SET 25.7

1. If 100 flips of a coin result in 30 heads and 70 tails, can we assert on the 5% level that the coin is fair?
2. If in 10 flips of a coin we get the same ratio as in Prob. 1 (3 heads and 7 tails), is the conclusion the same as in Prob. 1? First conjecture, then compute.
3. What would be the smallest number of heads in Prob. 1 under which the hypothesis "Fair coin" is still accepted (with $\alpha = 5\%$)?
4. If in rolling a die 180 times we get 39, 22, 41, 26, 20, 32, can we claim on the 5% level that the die is fair?
5. Solve Prob. 4 if the sample is 25, 31, 33, 27, 29, 35.
6. A manufacturer claims that in a process of producing kitchen knives, only 2.5% of the knives are dull. Test the claim against the alternative that more than 2.5% of the knives are dull, using a sample of 400 knives containing 17 dull ones. (Use $\alpha = 5\%$.)
7. Between 1 P.M. and 2 P.M. on five consecutive days (Monday through Friday) a certain service station has 92, 60, 66, 62, and 90 customers, respectively. Test the hypothesis that the expected number of customers during that hour is the same on those days. (Use $\alpha = 5\%$.)

8. Test for normality at the 1% level using a sample of $n = 79$ (rounded) values x (tensile strength [kg/mm^2] of steel sheets of 0.3 mm thickness). $a = a(x)$ = absolute frequency. (Take the first two values together, also the last three, to get $K = 5$.)

x	57	58	59	60	61	62	63	64
a	4	10	17	27	8	9	3	1

9. In a sample of 100 patients having a certain disease 45 are men and 55 women. Does this support the claim that the disease is equally common among men and women? Choose $\alpha = 5\%$.

10. In Prob. 9 find the smallest number (>50) of women that leads to the rejection of the hypothesis on the levels 5%, 1%, 0.5%.

11. Verify the calculations in Example 1 of the text.

12. Does the random variable X = *Number of accidents per week in a certain foundry* have a Poisson distribution if within 50 weeks, 33 were accident-free, 1 accident occurred in 11 of the 50 weeks, 2 in 6 of the weeks and more than 2 accidents in no week? (Choose $\alpha = 5\%$.)

13. Using the given sample, test that the corresponding population has a Poisson distribution. x is the number of alpha particles per 7.5-sec intervals observed by E. Rutherford and H. Geiger in one of their classical experiments in 1910, and $a(x)$ is the absolute frequency (= number of time periods during which exactly x particles were observed). (Use $\alpha = 5\%$.)

x	0	1	2	3	4	5	6
a	57	203	383	525	532	408	273

x	7	8	9	10	11	12	$\geqq 13$
a	139	45	27	10	4	2	0

14. Can we assert that the traffic on the three lanes of an expressway (in one direction) is about the same on each lane if a count gives 910, 850, 720 cars on the right, middle, and left lanes, respectively, during a particular time interval? (Use $\alpha = 5\%$.)

15. If it is known that 25% of certain steel rods produced by a standard process will break when subjected to a load of 5000 lb, can we claim that a new process yields the same breakage rate if we find that in a sample of 80 rods produced by the new process, 27 rods broke when subjected to that load? (Use $\alpha = 5\%$.)

16. Three samples of 200 rivets each were taken from a large production of each of three machines. The numbers of defective rivets in the samples were 7, 8, and 12. Is this difference significant? (Use $\alpha = 5\%$.)

17. In a table of properly rounded function values, even and odd last decimals should appear about equally often. Test this for the 90 values of $J_1(x)$ in Table A1 in App. 5.

18. Are the 5 tellers in a certain bank equally time-efficient if during the same time interval on a certain day they serve 120, 95, 110, 108, 102 customers? (Use $\alpha = 5\%$.)

19. CAS EXPERIMENT. Random Number Generator. Check your generator experimentally by imitating results of n trials of rolling a fair die, with a convenient n (e.g., 60 or 300 or the like). Do this many times and see whether you can notice any "nonrandomness" features, for example, too few Sixes, too many even numbers, etc., or whether your generator seems to work properly. Design and perform other kinds of checks.

20. TEAM PROJECT. Difficulty with Random Selection. 77 students were asked to choose 3 of the integers 11, 12, 13, $\cdots$, 30 completely arbitrarily. The amazing result was as follows.

Number	11	12	13	14	15	16	17	18	19	20
Frequ.	11	10	20	8	13	9	21	9	16	8

Number	21	22	23	24	25	26	27	28	29	30
Frequ.	12	8	15	10	10	9	12	8	13	9

If the selection were completely random, the following hypotheses should be true.

(a) The 20 numbers are equally likely.

(b) The 10 even numbers together are as likely as the 10 odd numbers together.

(c) The 6 prime numbers together have probability 0.3 and the 14 other numbers together have probability 0.7. Test these hypotheses, using $\alpha = 5\%$. Design further experiments that illustrate the difficulties of random selection.

25.8 Nonparametric Tests

Nonparametric tests, also called **distribution-free tests,** are valid for any distribution. Hence they are used in cases when the kind of distribution is unknown, or is known but such that no tests specifically designed for it are available. In this section we shall explain the basic idea of these tests, which are based on **"order statistics"** and are rather simple.

If there is a choice, then tests designed for a specific distribution generally give better results than do nonparametric tests. For instance, this applies to the tests in Sec. 25.4 for the normal distribution.

We shall discuss two tests in terms of typical examples. In deriving the distributions used in the test, it is essential that the distributions from which we sample are continuous. (Nonparametric tests can also be derived for discrete distributions, but this is slightly more complicated.)

EXAMPLE 1 Sign Test for the Median

A **median** of the population is a solution $x = \tilde{\mu}$ of the equation $F(x) = 0.5$, where F is the distribution function of the population.

Suppose that eight radio operators were tested, first in rooms without air-conditioning and then in air-conditioned rooms over the same period of time, and the difference of errors (unconditioned minus conditioned) were

$$9 \quad 4 \quad 0 \quad 6 \quad 4 \quad 0 \quad 7 \quad 11.$$

Test the hypothesis $\tilde{\mu} = 0$ (that is, air-conditioning has no effect) against the alternative $\tilde{\mu} > 0$ (that is, inferior performance in unconditioned rooms).

Solution. We choose the significance level $\alpha = 5\%$. If the hypothesis is true, the probability p of a positive difference is the same as that of a negative difference. Hence in this case, $p = 0.5$, and the random variable

$$X = \textit{Number of positive values among } n \textit{ values}$$

has a binomial distribution with $p = 0.5$. Our sample has eight values. We omit the values 0, which do not contribute to the decision. Then six values are left, all of which are positive. Since

$$\begin{aligned} P(X = 6) &= \binom{6}{6} (0.5)^6 (0.5)^0 \\ &= 0.0156 \\ &= 1.56\% \end{aligned}$$

we do have observed an event whose probability is very small if the hypothesis is true; in fact $1.56\% < \alpha = 5\%$. Hence we assert that the alternative $\tilde{\mu} > 0$ is true. That is, the number of errors made in unconditioned rooms is significantly higher, so that installation of air conditioning should be considered. ■

EXAMPLE 2 Test for Arbitrary Trend

A certain machine is used for cutting lengths of wire. Five successive pieces had the lengths

$$29 \quad 31 \quad 28 \quad 30 \quad 32.$$

Using this sample, test the hypothesis that there is **no trend**, that is, the machine does not have the tendency to produce longer and longer pieces or shorter and shorter pieces. Assume that the type of machine suggests the alternative that there is *positive trend*, that is, there is the tendency of successive pieces to get longer.

Solution. We count the number of **transpositions** in the sample, that is, the number of times a larger value precedes a smaller value:

29 precedes 28	(1 transposition),
31 precedes 28 and 30	(2 transpositions).

The remaining three sample values follow in ascending order. Hence in the sample there are $1 + 2 = 3$ transpositions. We now consider the random variable

$$T = \textit{Number of transpositions.}$$

If the hypothesis is true (no trend), then each of the $5! = 120$ permutations of five elements 1 2 3 4 5 has the same probability (1/120). We arrange these permutations according to their number of transpositions:

$T = 0$	$T = 1$	$T = 2$	$T = 3$
1 2 3 4 5	1 2 3 5 4	1 2 4 5 3	1 2 5 4 3
	1 2 4 3 5	1 2 5 3 4	1 3 4 5 2
	1 3 2 4 5	1 3 2 5 4	1 3 5 2 4
	2 1 3 4 5	1 3 4 2 5	1 4 2 5 3
		1 4 2 3 5	1 4 3 2 5
		2 1 3 5 4	1 5 2 3 4
		2 1 4 3 5	2 1 4 5 3
		2 3 1 4 5	2 1 5 3 4 etc.
		3 1 2 4 5	2 3 1 5 4
			2 3 4 1 5
			2 4 1 3 5
			3 1 2 5 4
			3 1 4 2 5
			3 2 1 4 5
			4 1 2 3 5

From this we obtain

$$P(T \leqq 3) = \tfrac{1}{120} + \tfrac{4}{120} + \tfrac{9}{120} + \tfrac{15}{120} = \tfrac{29}{120} = 24\%.$$

We accept the hypothesis because we have observed an event that has a relatively large probability (certainly much more than 5%) if the hypothesis is true.

Values of the distribution function of T in the case of no trend are shown in Table A12, App. 5. For instance, if $n = 3$, then $F(0) = 0.167$, $F(1) = 0.500$, $F(2) = 1 - 0.167$. If $n = 4$, then $F(0) = 0.042$, $F(1) = 0.167$, $F(2) = 0.375$, $F(3) = 1 - 0.375$, $F(4) = 1 - 0.167$, and so on.

Our method and those values refer to *continuous* distributions. Theoretically, we may then expect that all the values of a sample are different. Practically, some sample values may still be equal, because of rounding: If m values are equal, add $m(m-1)/4$ (= mean value of the transpositions in the case of the permutations of m elements), that is, $\frac{1}{2}$ for each pair of equal values, $\frac{3}{2}$ for each triple, etc. ■

[illegible]

1. What would change in Example 1, had we observed only 5 positive values? Only 4?

2. Does a process of producing plastic pipes of length $\mu = 2$ meters need adjustment if in a sample, 4 pipes have the exact length and 15 are shorter and 3 longer than 2 meters? (Use the normal approximation of the binomial distribution.)

3. Do the computations in Prob. 2 without the use of the DeMoivre–Laplace limit theorem (in Sec. 24.8).

4. Test whether a thermostatic switch is properly set to 20°C against the alternative that its setting is too low. Use a sample of 9 values, 8 of which are less than 20°C and 1 is greater than 20°C.

5. Are air filters of type A better than type B filters if in 10 trials, A gave cleaner air than B in 7 cases, B gave cleaner air than A in 1 case, whereas in 2 of the trials the results for A and B were practically the same?

6. In a clinical experiment, each of 10 patients were given two different sedatives A and B. The following table shows the effect (increase of sleeping time, measured in hours). Using the sign test, find out whether the difference is significant.

A	1.9	0.8	1.1	0.1	−0.1	4.4	5.5	1.6	4.6	3.4
B	0.7	−1.6	−0.2	−1.2	−0.1	3.4	3.7	0.8	0.0	2.0
Difference	1.2	2.4	1.3	1.3	0.0	1.0	1.8	0.8	4.6	1.4

7. Assuming that the populations corresponding to the samples in Prob. 6 are normal, apply a suitable test for the normal distribution.

8. Thirty new employees were grouped into 15 pairs of similar intelligence and experience and were then instructed in data processing by an old method (A) applied to one (randomly selected) person of each pair, and by a new presumably better method (B) applied to

the other person of each pair. Test for equality of methods against the alternative that (B) is better than (A), using the following scores obtained after the end of the training period.

A	60	70	80	85	75	40	70	45	95	80	90	60	80	75	65
B	65	85	85	80	95	65	100	60	90	85	100	75	90	60	80

9. Assuming normality, solve Prob. 8 by a suitable test from Sec. 25.4.

10. Set up a sign test for the lower quartile q_{25} (defined by the condition $F(q_{25}) = 0.25$).

11. How would you proceed in the sign test if the hypothesis is $\tilde{\mu} = \tilde{\mu}_0$ (any number) instead of $\tilde{\mu} = 0$?

12. Check the table in Example 2 of the text.

13. Apply the test in Example 2 to the following data (x = disulfide content of a certain type of wool, measured in percent of the content in unreduced fibers; y = saturation water content of the wool, measured in percent). Test for no trend against negative trend.

x	10	15	30	40	50	55	80	100
y	50	46	43	42	36	39	37	33

14. Test the hypothesis that for a certain type of voltmeter, readings are independent of temperature T [°C] against the alternative that they tend to increase with T. Use a sample of values obtained by applying a constant voltage:

Temperature T [°C]	10	20	30	40	50
Reading V [volts]	99.5	101.1	100.4	100.8	101.6

15. In a swine-feeding experiment, the following gains in weight [kg] of 10 animals (ordered according to increasing amounts of food given per day) were recorded:

20 17 19 18 23 16 25 28 24 22.

Test for no trend against positive trend.

16. Apply the test explained in Example 2 to the following data (x = diastolic blood pressure [mm Hg], y = weight of heart [in grams] of 10 patients who died of cerebral hemorrhage).

x	121	120	95	123	140	112	92	100	102	91
y	521	465	352	455	490	388	301	395	375	418

17. Does an increase in temperature cause an increase of the yield of a chemical reaction from which the following sample was taken?

Temperature [°C]	10	20	30	40	60	80
Yield [kg/min]	0.6	1.1	0.9	1.6	1.2	2.0

18. Does the amount of fertilizer increase the yield of wheat X [kg/plot]? Use a sample of values ordered according to increasing amounts of fertilizer:

41.4 43.3 39.6 43.0 44.1 45.6 44.5 46.7.

25.9 Regression. Fitting Straight Lines. Correlation

So far we were concerned with random experiments in which we observed a single quantity (random variable) and got samples whose values were single numbers. In this section we discuss experiments in which we observe or measure two quantities simultaneously, so that we get samples of *pairs* of values $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Most applications involve one of two kinds of experiments, as follows.

1. In **regression analysis** one of the two variables, call it x, can be regarded as an ordinary variable because we can measure it without substantial error or we can even give it values we want. x is called the **independent variable,** or sometimes the **controlled variable** because we can control it (set it at values we choose). The other variable, Y, is a random variable, and we are interested in the dependence of Y on x. Typical examples are the dependence of the blood pressure Y on the age x of a person or, as we shall now say, the regression of Y on x, the regression of the gain of weight Y of certain animals on the daily ration of food x, the regression of the heat conductivity Y of cork on the specific weight x of the cork, etc.

2. In **correlation analysis** both quantities are random variables and we are interested in relations between them. Examples are the relation (one says "correlation") between wear X and wear Y of the front tires of cars, between grades X and Y of students in mathematics and in physics, respectively, between the hardness X of steel plates in the center and the hardness Y near the edges of the plates, etc.

Regression Analysis

In regression analysis the dependence of Y on x is a dependence of the mean μ of Y on x, so that $\mu = \mu(x)$ is a function in the ordinary sense. The curve of $\mu(x)$ is called the **regression curve** of Y on x.

In this section we discuss the simplest case, namely, that of a straight **regression line**

$$\mu(x) = \kappa_0 + \kappa_1 x. \tag{1}$$

Then we may want to graph the sample values as n points in the xY-plane, fit a straight line through them, and use it for estimating $\mu(x)$ at values of x that interest us, so that we know what values of Y we can expect for those x. Fitting that line by eye would not be good because it would be subjective; that is, different persons' results would come out differently, particularly if the points are scattered. So we need a mathematical method that gives a unique result depending only on the n points. A widely used procedure is the method of least squares by Gauss and Legendre. For our task we may formulate it as follows.

Least Squares Principle

The straight line should be fitted through the given points so that the sum of the squares of the distances of those points from the straight line is minimum, where the distance is measured in the vertical direction (the y-direction). (Formulas below.)

To get uniqueness of the straight line, we need some extra condition. To see this, take the sample $(0, 1)$, $(0, -1)$. Then all the lines $y = k_1 x$ with any k_1 satisfy the principle. (Can you see it?) The following assumption will imply uniqueness, as we shall find out.

General Assumption (A1)

The x-values $x_1, \cdots, x_n$ in our sample $(x_1, y_1), \cdots, (x_n, y_n)$ are not all equal.

From a given sample $(x_1, y_1), \cdots, (x_n, y_n)$ we shall now determine a straight line by least squares. We write the line as

$$y = k_0 + k_1 x \tag{2}$$

and call it the **sample regression line** because it will be the counterpart of the population regression line (1).

Now a sample point (x_j, y_j) has the vertical distance (distance measured in the y-direction) from (2) given by

$$|y_j - (k_0 + k_1 x_j)| \qquad \text{(see Fig. 542).}$$

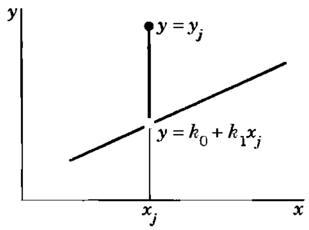


Fig. 542. Vertical distance of a point (x_j, y_j) from a straight line $y = k_0 + k_1 x$

Hence the sum of the squares of these distances is

$$q = \sum_{j=1}^{n} (y_j - k_0 - k_1 x_j)^2. \tag{3}$$

In the method of least squares we now have to determine k_0 and k_1 such that q is minimum. From calculus we know that a necessary condition for this is

$$\frac{\partial q}{\partial k_0} = 0 \quad \text{and} \quad \frac{\partial q}{\partial k_1} = 0. \tag{4}$$

We shall see that from this condition we obtain for the sample regression line the formula

$$y - \bar{y} = k_1(x - \bar{x}). \tag{5}$$

Here $\bar{x}$ and $\bar{y}$ are the means of the x- and the y-values in our sample, that is,

$$\text{(6)} \qquad \begin{aligned} &\text{(a)} \quad \bar{x} = \frac{1}{n}(x_1 + \cdots + x_n) \\ &\text{(b)} \quad \bar{y} = \frac{1}{n}(y_1 + \cdots + y_n). \end{aligned}$$

The slope k_1 in (5) is called the **regression coefficient** of the sample and is given by

$$k_1 = \frac{s_{xy}}{s_x^{\,2}}. \tag{7}$$

Here the **"sample covariance"** s_{xy} is

$$s_{xy} = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})(y_j - \bar{y}) = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j y_j - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} y_j\right)\right] \tag{8}$$

and $s_x^{\,2}$ is given by

$$s_x^{\,2} = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j^{\,2} - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)^2\right]. \tag{9a}$$

From (5) we see that the sample regression line passes through the point $(\bar{x}, \bar{y})$, by which it is determined, together with the regression coefficient (7). We may call $s_x{}^2$ the *variance* of the x-values, but we should keep in mind that x is an ordinary variable, not a random variable.

We shall soon also need

$$\text{(9b)} \qquad s_y{}^2 = \frac{1}{n-1} \sum_{j=1}^{n} (y_j - \bar{y})^2 = \frac{1}{n-1} \left[\sum_{j=1}^{n} y_j{}^2 - \frac{1}{n} \left(\sum_{j=1}^{n} y_j \right)^2 \right].$$

Derivation of (5) and (7). Differentiating (3) and using (4), we first obtain

$$\frac{\partial q}{\partial k_0} = -2 \sum (y_j - k_0 - k_1 x_j) = 0,$$

$$\frac{\partial q}{\partial k_1} = -2 \sum x_j (y_j - k_0 - k_1 x_j) = 0$$

where we sum over j from 1 to n. We now divide by 2, write each of the two sums as three sums, and take the sums containing y_j and $x_j y_j$ over to the right. Then we get the **"normal equations"**

$$\text{(10)} \qquad \begin{aligned} k_0 n + k_1 \sum x_j \quad &= \sum y_j \\ k_0 \sum x_j + k_1 \sum x_j^2 &= \sum x_j y_j. \end{aligned}$$

This is a linear system of two equations in the two unknowns k_0 and k_1. Its coefficient determinant is [see (9)]

$$\begin{vmatrix} n & \sum x_j \\ \sum x_j & \sum x_j^2 \end{vmatrix} = n \sum x_j^2 - \left(\sum x_j \right)^2 = n(n-1)s_x{}^2 = n \sum (x_j - \bar{x})^2$$

and is not zero because of Assumption (A1). Hence the system has a unique solution. Dividing the first equation of (10) by n and using (6), we get $k_0 = \bar{y} - k_1 \bar{x}$. Together with $y = k_0 + k_1 x$ in (2) this gives (5). To get (7), we solve the system (10) by Cramer's rule (Sec. 7.6) or elimination, finding

$$\text{(11)} \qquad k_1 = \frac{n \sum x_j y_j - \sum x_i \sum y_j}{n(n-1)s_x{}^2}.$$

This gives (7)–(9) and completes the derivation. [The equality of the two expressions in (8) and in (9) may be shown by the student; see Prob. 14]. ■

EXAMPLE 1 Regression Line

The decrease of volume y [%] of leather for certain fixed values of high pressure x [atmospheres] was measured. The results are shown in the first two columns of Table 25.11. Find the regression line of y on x.

Solution. We see that $n = 4$ and obtain the values $\bar{x} = 28\,000/4 = 7000$, $\bar{y} = 19.0/4 = 4.75$, and from (9) and (8)

Table 25.11 Regression of the Decrease of Volume y [%] of Leather on the Pressure x [Atmospheres]

Given Values		Auxiliary Values	
x_j	y_j	x_j^2	$x_j y_j$
4 000	2.3	16 000 000	9 200
6 000	4.1	36 000 000	24 600
8 000	5.7	64 000 000	45 600
10 000	6.9	100 000 000	69 000
28 000	19.0	216 000 000	148 400

$$s_x^2 = \frac{1}{3}\left(216\,000\,000 - \frac{28\,000^2}{4}\right) = \frac{20\,000\,000}{3}$$

$$s_{xy} = \frac{1}{3}\left(148\,400 - \frac{28\,000 \cdot 19}{4}\right) = \frac{15\,400}{3}.$$

Hence $k_1 = 15\,400/20\,000\,000 = 0.000\,77$ from (7), and the regression line is

$$y - 4.75 = 0.000\,77(x - 7000) \qquad \text{or} \qquad y = 0.000\,77x - 0.64.$$

Note that $y(0) = -0.64$, which is physically meaningless, but typically indicates that a linear relation is merely an approximation valid on some restricted interval. ■

Confidence Intervals in Regression Analysis

If we want to get confidence intervals, we have to make assumptions about the distribution of Y (which we have not made so far; least squares is a "geometric principle," nowhere involving probabilities!). We assume normality and independence in sampling:

Assumption (A2)

For each fixed x the random variable Y is normal with mean (1), *that is,*

$$\mu(x) = \kappa_0 + \kappa_1 x \tag{12}$$

and variance σ^2 independent of x.

Assumption (A3)

The n performances of the experiment by which we obtain a sample

$$(x_1, y_1), \qquad (x_2, y_2), \qquad \cdots, \qquad (x_n, y_n)$$

are independent.

κ_1 in (12) is called the **regression coefficient** of the population because it can be shown that under Assumptions (A1)–(A3) the maximum likelihood estimate of κ_1 is the sample regression coefficient k_1 given by (11).

Under Assumptions (A1)–(A3) we may now obtain a confidence interval for κ_1, as shown in Table 25.12.

Table 25.12 Determination of a Confidence Interval for κ_1 in (1) under Assumptions (A1)–(A3)

Step 1. Choose a confidence level γ(95%, 99%, or the like).

Step 2. Determine the solution c of the equation

$$F(c) = \tfrac{1}{2}(1 + \gamma) \tag{13}$$

from the table of the t-distribution with $n - 2$ degrees of freedom (Table A9 in App. 5; n = sample size).

Step 3. Using a sample $(x_1, y_1), \cdots, (x_n, y_n)$, compute $(n-1)s_x^{\,2}$ from (9a), $(n-1)s_{xy}$ from (8), k_1 from (7),

$$(n-1)s_y^{\,2} = \sum_{j=1}^{n} y_j^{\,2} - \frac{1}{n}\left(\sum_{j=1}^{n} y_j\right)^2 \tag{14}$$

[as in (9b)], and

$$q_0 = (n-1)(s_y^{\,2} - k_1^{\,2} s_x^{\,2}). \tag{15}$$

Step 4. Compute

$$K = c\sqrt{\frac{q_0}{(n-2)(n-1)s_x^{\,2}}}\,.$$

The confidence interval is

$$\mathrm{CONF}_\gamma\ \{k_1 - K \leqq \kappa_1 \leqq k_1 + K\}. \tag{16}$$

EXAMPLE 2 Confidence Interval for the Regression Coefficient

Using the sample in Table 25.11, determine a confidence interval for κ_1 by the method in Table 25.12.

Solution. *Step 1.* We choose $\gamma = 0.95$.

Step 2. Equation (13) takes the form $F(c) = 0.975$, and Table A9 in App. 5 with $n - 2 = 2$ degrees of freedom gives $c = 4.30$.

Step 3. From Example 1 we have $3s_x^{\,2} = 20\,000\,000$ and $k_1 = 0.00077$. From Table 25.11 we compute

$$\begin{aligned} 3s_y^{\,2} &= 102.2 - \frac{19^2}{4} \\ &= 11.95, \\ q_0 &= 11.95 - 20\,000\,000 \cdot 0.00077^2 \\ &= 0.092. \end{aligned}$$

Step 4. We thus obtain

$$\begin{aligned} K &= 4.30\sqrt{0.092/(2 \cdot 20\,000\,000)} \\ &= 0.000\,206 \end{aligned}$$

and

$$\mathrm{CONF}_{0.95}\ \{0.00056 \leqq \kappa_1 \leqq 0.00098\}.$$

■

Correlation Analysis

We shall now give an introduction to the basic facts in correlation analysis; for proofs see Ref. [G2] or [G8] in App. 1.

Correlation analysis is concerned with the relation between X and Y in a two-dimensional random variable (X, Y) (Sec. 24.9). A sample consists of n ordered pairs of values $(x_1, y_1), \cdots, (x_n, y_n)$, as before. The interrelation between the x and y values in the sample is measured by the sample covariance s_{xy} in (8) or by the sample **correlation coefficient**

$$r = \frac{s_{xy}}{s_x s_y} \tag{17}$$

with s_x and s_y given in (9). Here r has the advantage that it does not change under a multiplication of the x and y values by a factor (in going from feet to inches, etc.).

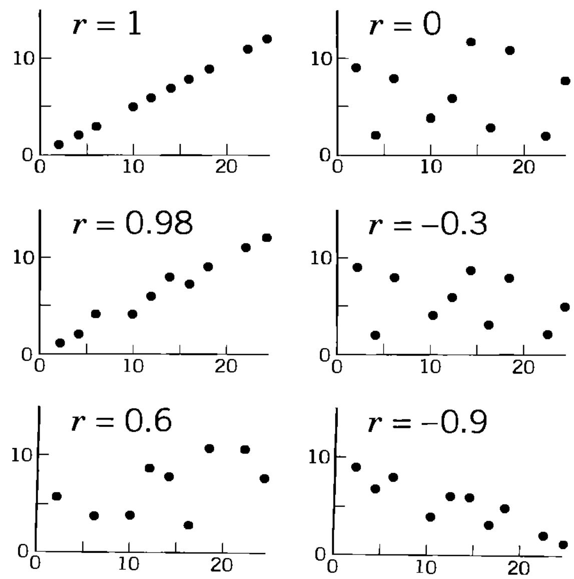
THEOREM 1 **Sample Correlation Coefficient**

The sample correlation coefficient r satisfies $-1 \leqq r \leqq 1$. In particular, $r = \pm 1$ if and only if the sample values lie on a straight line. (See Fig. 543.)

The theoretical counterpart of r is the **correlation coefficient** ρ of X and Y,

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{18}$$

where $\mu_X = E(X)$, $\mu_Y = E(Y)$, $\sigma_X{}^2 = E([X - \mu_X]^2)$, $\sigma_Y{}^2 = E([Y - \mu_Y]^2)$ (the means and variances of the marginal distributions of X and Y; see Sec. 24.9), and σ_{XY} is the



.g. 543. Samples with various values of the correlation coefficient r

covariance of X and Y given by (see Sec. 24.9)

(19) $$\sigma_{XY} = E([X - \mu_X][Y - \mu_Y]) = E(XY) - E(X)E(Y).$$

The analog of Theorem 1 is

THEOREM 2 **Correlation Coefficient**

The correlation coefficient ρ satisfies $-1 \leqq \rho \leqq 1$. In particular, $\rho = \pm 1$ if and only if X and Y are **linearly related,** *that is, $Y = \gamma X + \delta$, $X = \gamma^* Y + \delta^*$.*

X and Y are called **uncorrelated** if $\rho = 0$.

THEOREM 3 **Independence. Normal Distribution**

(a) *Independent X and Y* (see Sec. 24.9) *are uncorrelated.*

(b) *If (X, Y) is normal* (see below), *then uncorrelated X and Y are independent.*

Here the two-dimensional normal distribution can be introduced by taking two independent standardized normal random variables X^*, Y^*, whose joint distribution thus has the density

(20) $$f^*(x^*, y^*) = \frac{1}{2\pi} e^{-(x^{*2}+y^{*2})/2}$$

(representing a surface of revolution over the x^*y^*-plane with a bell-shaped curve as cross section) and setting

$$X = \mu_X + \sigma_X X^*$$
$$Y = \mu_Y + \rho\sigma_Y X^* + \sqrt{1 - \rho^2}\, \sigma_Y Y^*.$$

This gives the general **two-dimensional normal distribution** with the density

(21a) $$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1 - \rho^2}} e^{-h(x,y)/2}$$

where

(21b) $$h(x, y) = \frac{1}{1 - \rho^2}\left[\left(\frac{x - \mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x - \mu_X}{\sigma_X}\right)\left(\frac{y - \mu_Y}{\sigma_Y}\right) + \left(\frac{y - \mu_Y}{\sigma_Y}\right)^2\right].$$

In Theorem 3(b), normality is important, as we can see from the following example.

EXAMPLE 3 **Uncorrelated but Dependent Random Variables**

If X assumes -1, 0, 1 with probability 1/3 and $Y = X^2$, then $E(X) = 0$ and in (3)

$$\sigma_{XY} = E(XY) = E(X^3) = (-1)^3 \cdot \frac{1}{3} + 0^3 \cdot \frac{1}{3} + 1^3 \cdot \frac{1}{3} = 0,$$

so that $\rho = 0$ and X and Y are uncorrelated. But they are certainly not independent since they are even functionally related. ■

Test for the Correlation Coefficient ρ

Table 25.13 shows a test for ρ in the case of the two-dimensional normal distribution. t is an observed value of a random variable that has a t-distribution with $n - 2$ degrees of freedom. This was shown by R. A. Fisher (*Biometrika* **10** (1915), 507–521).

Table 25.13 Test of the Hypothesis $\rho = 0$ Against the Alternative $\rho > 0$ in the Case of the Two-Dimensional Normal Distribution

Step 1. Choose a significance level α (5%, 1%, or the like).

Step 2. Determine the solution c of the equation

$$P(T \leqq c) = 1 - \alpha$$

from the t-distribution (Table A9 in App. 5) with $n - 2$ degrees of freedom.

Step 3. Compute r from (17), using a sample $(x_1, y_1), \cdots, (x_n, y_n)$.

Step 4. Compute

$$t = r\left(\sqrt{\frac{n-2}{1-r^2}}\right).$$

If $t \leqq c$, accept the hypothesis. If $t > c$, reject the hypothesis.

EXAMPLE 4 Test for the Correlation Coefficient ρ

Test the hypothesis $\rho = 0$ (independence of X and Y, because of Theorem 3) against the alternative $\rho > 0$, using the data in the lower left corner of Fig. 543, where $r = 0.6$ (manual soldering errors on 10 two-sided circuit boards done by 10 workers; x = front, y = back of the boards).

Solution. We choose $\alpha = 5\%$; thus $1 - \alpha = 95\%$. Since $n = 10$, $n - 2 = 8$, the table gives $c = 1.86$. Also, $t = 0.6\sqrt{8/0.64} = 2.12 > c$. We reject the hypothesis and assert that there is a **positive correlation.** A worker making few (many) errors on the front side also tends to make few (many) errors on the reverse side of the board. ■

1–10 SAMPLE REGRESSION LINE

Find and sketch or graph the sample regression line of y and x and the given data as points on the same axes.

1. $(-1, 1), (0, 1.7), (1, 3)$
2. $(3, 3.5), (5, 2), (7, 4.5), (9, 3)$
3. $(2, 12), (5, 24), (9, 33), (14, 50)$
4. $(11, 22), (15, 18), (17, 16), (20, 9), (22, 10)$
5.

Speed x [mph] of a car	30	40	50	60
Stopping distance y [ft]	150	195	240	295

Also find the stopping distance at 35 mph.

6. x = Deformation of a certain steel [mm], y = Brinell hardness [kg/mm^2]

x	6	9	11	13	22	26	28	33	35
y	68	67	65	53	44	40	37	34	32

7. x = Revolutions per minute, y = Power of a Diesel engine [hp]

x	400	500	600	700	750
y	580	1030	1420	1880	2100

8.

Humidity of air x [%]	10	20	30	40
Expansion of gelatin y [%]	0.8	1.6	2.3	2.8

9.

Voltage x [V]	40	40	80	80	110	110
Current y [A]	5.1	4.8	10.0	10.3	13.0	12.7

Also find the resistance R [Ω] by **Ohms' law** (Sec. 2.9).

10.

Force x [lb]	2	4	6	8
Extension y [in] of a spring	4.1	7.8	12.3	15.8

Also find the spring modulus by **Hooke's law** (Sec. 2.4).

11–13 **CONFIDENCE INTERVALS**

Find a 95% confidence interval for the regression coefficient κ_1, assuming that (A2) and (A3) hold and using the sample:

11. In Prob. 6

12. In Prob. 7

13. In Prob. 8

14. Derive the second expression for $s_x{}^2$ in (9a) from the first one.

15. CAS EXPERIMENT. Moving Data. Take a sample, for instance, that in Prob. 6, and investigate and graph the effect of changing y-values (a) for small x, (b) for large x, (c) in the middle of the sample.

[illegible] STIONS AND PROBLEMS

1. What is a sample? Why do we take samples?

2. What is the role of probability theory in statistics?

3. Will you get better results by taking larger samples? Explain.

4. Do several samples from a certain population have the same mean? The same variance?

5. What is a parameter? How can we estimate it? Give an example.

6. What is a statistical test? What errors occur in testing?

7. How do we test in quality control?

8. What is the χ^2-test? Give a simple example from memory.

9. What are nonparametric tests? When would you apply them?

10. In what tests did we use the t-distribution? The χ^2-distribution?

11. What are one-sided and two-sided tests? Give typical examples.

12. List some areas of application of statistical tests.

13. What do we mean by "goodness of fit"?

14. Acceptance sampling uses principles of testing. Explain.

15. What is the power of a test? What can you do if the power is low?

16. Explain the idea of a maximum likelihood estimate from memory.

17. How does the length of a confidence interval depend on the sample size? On the confidence level?

18. Couldn't we make the error in interval estimation zero simply by choosing the confidence level 1?

19. What is the least squares principle? Give applications.

20. What is the difference between regression and correlation analysis?

21. Find the maximum likelihood estimates of mean and variance of a normal distribution using the sample 5, 4, 6, 5, 3, 5, 7, 4, 6, 5, 8, 6.

22. Determine a 95% confidence interval for the mean μ of a normal population with variance $\sigma^2 = 16$, using a sample of size 400 with mean 53.

23. What will happen to the length of the interval in Prob. 22 if we reduce the sample size to 100?

24. Determine a 99% confidence interval for the mean of a normal population with standard deviation 2.2, using the sample 28, 24, 31, 27, 22.

25. What confidence interval do we obtain in Prob. 24 if we assume the variance to be unknown?

26. Assuming normality, find a 95% confidence interval for the variance from the sample 145.3, 145.1, 145.4, 146.2.

27–29 Find a 95% confidence interval for the mean μ, assuming normality and using the sample:

27. Nitrogen content [%] of steel 0.74. 0.75. 0.73, 0.75, 0.74. 0.72

28. Diameters of 10 gaskets with mean 4.37 cm and standard deviation 0.157 cm

29. Density [g/cm^3] of coke 1.40, 1.45, 1.39, 1.44, 1.38

30. What sample size should we use in Prob. 28 if we want to obtain a confidence interval of length 0.1, assuming that the standard deviation of the samples is (about) the same?

31–32 Find a 99% confidence interval for the variance σ^2, assuning normality and using the sample:

31. Rockwell hardness of tool bits 64.9, 64.1, 63.8, 64.0

32. A sample of size $n = 128$ with variance $s^2 = 1.921$

33. Using a sample of 10 values with mean 14.5 from a normal population with variance $\sigma^2 = 0.25$, test the hypothesis $\mu_0 = 15.0$ against the alternative $\mu_1 = 14.4$ on the 5% level.

34. In Prob. 33, change the alternative to $\mu \neq 15.0$ and test as before.

35. Find the power in Prob. 33.

36. Using a sample of 15 values with mean 36.2 and variance 0.9, test the hypothesis $\mu_0 = 35.0$ against the alternative $\mu_1 = 37.0$, assuming normality and taking $\alpha = 1\%$.

37. Using a sample of 20 values with variance 8.25 from a normal population, test the hyothesis $\sigma_0^2 = 5.0$ against the alternative $\sigma_1^2 = 8.1$, choosing $\alpha = 5\%$.

38. A firm sells paint in cans containing 1 kg of paint per can and is interested to know whether the mean weight differs significantly from 1 kg, in which case the filling machine must be adjusted. Set up a hypothesis and an alternative and perform the test, assuming normality and using a sample of 20 fillings having a mean of 991 g and a standard deviation of 8 g. (Choose $\alpha = 5\%$.)

39. Using samples of sizes 10 and 5 with variances $s_x^2 = 50$ and $s_y^2 = 20$ and assuming normality of the corresponding populations, test the hypothesis H_0: $\sigma_x^2 = \sigma_y^2$ against the alternative $\sigma_x^2 > \sigma_y^2$. Choose $\alpha = 5\%$.

40. Assume the thickness X of washers to be normal with mean 2.75 mm and variance 0.00024 mm^2. Set up a control chart for μ, choosing $\alpha = 1\%$, and graph the means of the five samples (2.74, 2.76), (2.74, 2.74), (2.79, 2.81), (2.78, 2.76), (2.71, 2.75) on the chart.

41. What effect on UCL − LCL in a control chart for the mean does it have if we double the sample size? If we switch from $\alpha = 1\%$ to $\alpha = 5\%$?

42. The following samples of screws (length in inches) were taken from an ongoing production. Assuming that the population is normal with mean 3.500 and variance 0.0004, set up a control chart for the mean, choosing $\alpha = 1\%$, and graph the sample means on the chart.

Sample No.	1	2	3	4	5	6	7	8
Length	3.49	3.48	3.52	3.50	3.51	3.49	3.52	3.53
	3.50	3.47	3.49	3.51	3.48	3.50	3.50	3.49

43. A purchaser checks gaskets by a single sampling plan that uses a sample size of 40 and an acceptance number of 1. Use Table A6 in App. 5 to compute the probability of acceptance of lots containing the following percentages of defective gaskets $\frac{1}{4}\%$, $\frac{1}{2}\%$, 1%, 2%, 5%, 10%. Graph the OC curve. (Use the Poisson approximation.)

44. Does an automatic cutter have the tendency of cutting longer and longer pieces of wire if the lengths of subsequent pieces [in.] were 10.1, 9.8, 9.9, 10.2, 10.6, 10.5?

45. Find the least squares regression line to the data (−2, 1), (0, 1), (2, 3), (4, 4), (6, 5).

SUMMARY OF CHAPTER 25

Mathematical Statistics

We recall from Chap. 24 that with an experiment in which we observe some quantity (number of defectives, height of persons, etc.) there is associated a random variable X whose probability distribution is given by a distribution function

$$F(x) = P(X \leqq x) \qquad (1) \qquad \text{(Sec. 24.5)}$$

which for each x gives the probability that X assumes any value not exceeding x.

In statistics we take random samples $x_1, \cdots, x_n$ of size n by performing that experiment n times (Sec. 25.1) and draw conclusions from properties of samples about properties of the distribution of the corresponding X. We do this by calculating *point estimates* or *confidence intervals* or by performing a *test* for **parameters** (μ and σ^2 in the normal distribution, p in the binomial distribution, etc.) or by a test for distribution functions.

A **point estimate** (Sec. 25.2) is an approximate value for a parameter in the distribution of X obtained from a sample. Notably, the **sample mean** (Sec. 25.1)

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}(x_1 + \cdots + x_n) \tag{2}$$

is an estimate of the mean μ of X, and the **sample variance** (Sec. 25.1)

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2 = \frac{1}{n-1}[(x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2] \tag{3}$$

is an estimate of the variance σ^2 of X. Point estimation can be done by the basic **maximum likelihood method** (Sec. 25.2).

Confidence intervals (Sec. 25.3) are intervals $\theta_1 \leqq \theta \leqq \theta_2$ with endpoints calculated from a sample such that with a high probability γ we obtain an interval that contains the unknown true value of the parameter θ in the distribution of X. Here, γ is chosen at the beginning, usually 95% or 99%. We denote such an interval by $\text{CONF}_\gamma\ \{\theta_1 \leqq \theta \leqq \theta_2\}$.

In a **test** for a parameter we test a *hypothesis* $\theta = \theta_0$ against an *alternative* $\theta = \theta_1$ and then, on the basis of a sample, accept the hypothesis, or we reject it in favor of the alternative (Sec. 25.4). Like any conclusion about X from samples, this may involve errors leading to a false decision. There is a small probability α (which we can choose, 5% or 1%, for instance) that we reject a true hypothesis, and there is a probability β (which we can compute and decrease by taking larger samples) that we accept a false hypothesis. α is called the **significance level** and $1 - \beta$ the **power** of the test. Among many other engineering applications, testing is used in **quality control** (Sec. 25.5) and **acceptance sampling** (Sec. 25.6).

If not merely a parameter but the kind of distribution of X is unknown, we can use the **chi-square test** (Sec. 25.7) for testing the hypothesis that some function $F(x)$ is the unknown distribution function of X. This is done by determining the discrepancy between $F(x)$ and the distribution function $\widetilde{F}(x)$ of a given sample.

"Distribution-free" or **nonparametric tests** are tests that apply to any distribution, since they are based on combinatorial ideas. These tests are usually very simple. Two of them are discussed in Sec. 25.8.

The last section deals with samples of **pairs of values,** which arise in an experiment when we simultaneously observe two quantities. In **regression analysis,** one of the quantities, x, is an ordinary variable and the other, Y, is a random variable whose mean μ depends on x, say, $\mu(x) = \kappa_0 + \kappa_1 x$. In **correlation analysis** the relation between X and Y in a two-dimensional random variable (X, Y) is investigated, notably in terms of the **correlation coefficient** ρ.