Generation of Fractional-Order Chua's Chaotic System and It's Synchronization

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Abstract: In this paper, the chaotic behavior as well as synchronization of fractional-order Chua's system is studied. By using the lyapunov exponent analysis, we find that the the minimal fractional order to generate chaotic behavior is 0.880. Chaos synchronization of this fractional-order system is also investigated based on active control. Finally, the effectiveness of the proposed method is confirmed by numerical simulations.

Key Words: Fractional-order chaotic system; chaos synchronization; active control.

1 Introduction

In recent years, fractional calculus of arbitrary order has been found many applications in the domains of physics, applied mathematics, and engineering. It is shown that the expression of many systems is more accurate when the fractional derivative is employed. For example, electrochemical processes and flexible structures are described by fractional-order models [1], the behavior of some biological systems is explored using fractional calculus [2], the dielectric polarization, and viscoelastic systems are described by fractional-order differential equations [3, 4]. Up to now, many fractional-order systems have been shown behave chaotically, such as the fractional-order Chua's system [5, 6], the fractional-order Duffing system [7], the fractional-order Lu system [8], the fractional-order Chen's system [9]. Pecora and Carroll [10] firstly introduced the concept of synchronization of two chaotic systems with arbitrary initial conditions. Many chaotic synchronization schemes have been used in the last decade such as adaptive control, nonlinear feedback synchronization, and active control [11]. When Ho and Hung [11] presented and applied the concept of active control method on the synchronization of chaotic systems, many recent papers investigated this technique for different systems and in different applications [12,13].

More precisely, in this paper, we numerically investigate the chaotic behaviors of the fractional-order Chua's system.We find that chaos exists in the fractional-order Chua's system with order as low as 0.88. A chaos synchronization approach is also presented for the chaotic fractional-order Chua's systems. Like many other studies of dynamics in fractional-order systems, since no general theory is available in the literature we investigate the dynamics of the fractional-order Chua' system through numerical simulations.

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2 Numerical algorithm for fractional order differential equations

Hear we will introduce the following numerical algorithm on fractional calculus. We applies the Laplace transform to the Riemann-Liouville fractional differential equation and has the form due to the homogeneous conditions:

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{f(t)\}.$$

Then, we can construct a table of the integer order approximate expressions of $\frac{1}{s^q}, 0 < q < 1$. Yet when $q, 0 < q < 1$ is not in the table, one has to enforce the program from stem to stern. However, in this study, we employ the Caputo version and use a predictor-rector algorithm for fractional order differential equations, which is the generalization of Adams-Bashforth-Moulton one. A brief introduction of the algorithm is given as following.

The fractional order differential equation

$$\begin{cases} D^\alpha y(t) = f(t, y(t)), 0 \le t \le T, \\ y^{(k)}(0) = y_0^k, k = 0, 1, \cdots, m-1, \end{cases} \tag{1}$$

is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{2}$$

Let $h = T/N, N \in Z, t_n = nh, n = 0, 1, \cdots, N$. Then (2.2) can be rewritten as

$$y_h(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_n+1, y_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)), \tag{3}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ -2(n-j+1)^{\alpha+1}, 1 \le j \le n, \end{cases}$$

$y_h^p(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j))$, $b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right)$.

The estimation error is $\max|y(t_j) - y_h(t_j)| = o(h^p)$, where $p = \min(2, \alpha+1)$.

3 The description of fractional-order Chua's system

The Chua's circuit, which can be shown in Fig.1, is a electronic circuit that shows nonlinear dynamical phenomena such as chaos and bifurcation.

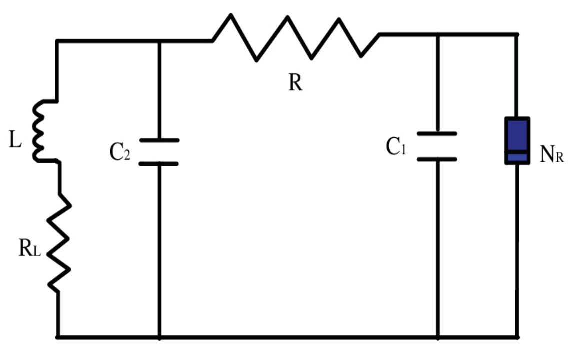

Figure 1: Chua's circuit.

The nonlinear Chua's circuit have five elements: two capacitors C_1 and C_2, an inductor L, a resistor R, and a nonlinear resistor N_R, known as Chua's diode. Based on Kirchhoff's circuit laws, Chua's circuit can be expressed as:

$$\begin{aligned} \frac{\mathrm{d}v_1(t)}{\mathrm{d}t} &= \frac{1}{c_1}[G(v_2(t) - v_1(t)) - f(v_1(t))], \\ \frac{\mathrm{d}v_2(t)}{\mathrm{d}t} &= \frac{1}{c_2}[G(v_1(t) - v_2(t)) + I(t)], \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \frac{1}{L}[-v_2(t) - R_L I(t)], \end{aligned} \tag{4}$$

where $G = \frac{1}{R}$, $v_1(t), v_2(t)$ and $I(t)$ are the voltage across the capacitor C_1, C_2 and the current through the inductor L. $f(v_1(t))$ is the piecewise-linear $v - i$ characteristic of Chua's diode, depicted in Fig.2, which can be written as:

$$f(v_1(t)) = G_b v_1(t) + \frac{1}{2}(G_a - G_b)(|v_1(t) + B_p| - |v_1(t) - B_p|). \tag{5}$$

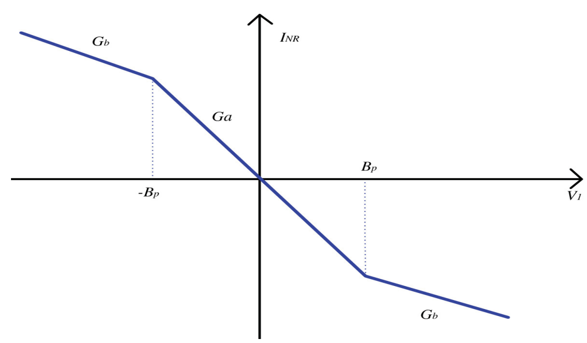

Figure 2: Piecewise-linear characteristic of the nonlinear resistor.

By defining the rescaling: $x_1 = \frac{v_1}{B_p}, x_2 = \frac{v_2}{B_p}, x_3 = \frac{I_L}{B_p G}, m_0 = \frac{G_a}{G}, b = \frac{c_2}{LG^2}, m_1 = \frac{G_b}{G}, m_1 = \frac{G_b}{G}, t = \frac{tG}{c_2}$, we can transform (1) into the following fractional-order form:

$$\begin{aligned} D^q x_1(t) &= a[x_2(t) - x_1(t) - f(x_1)], \\ D^q x_2(t) &= x_1(t) - x_2(t) + x_3(t), \\ D^q x_3(t) &= -bx_2(t) - cx_3(t), \end{aligned} \tag{6}$$

where

$$f(x_1) = m_1 x_1(t) + \frac{1}{2}(m_0 - m_1)(|x_1(t)+1| - |x_1(t)-1|) \tag{7}$$

and $0 < q \le 1, i = 1, 2, 3$ is fractional order.

4 Chaotic dynamics of fractional-order Chua's system

The global dynamics of fractional-order Chua's system may be determined by piecing together the three-dimensional vector fields of regions $D_0 = \{x_1 | x_1 \in [-1, 1]\}$, $D_{-1} = \{x_1 | x_1 < -1\}$ and $D_1 = \{x_1 | x_1 > 1\}$.

In D_0 region, the state equations of fractional-order Chua's oscillator are linear. The Jacobian matrix has the following form:

$$J_{D_0} = \begin{bmatrix} -a(1+m_0) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & -c \end{bmatrix}. \quad (8)$$

The characteristic equation of Chua's system in this region is

$$\begin{aligned} \det|\lambda I - J_{D_0}| = \lambda^3 &+ (a + c + am_0 + 1)\lambda^2 \\ &+ (b + c + ac + am_1 + acm_1)\lambda \quad (9) \\ &+ a(b + bm_0 + cm_0) \end{aligned}$$

In D_{-1} and D_1 regions, the state equations of Chua's oscillator are linear. The Jacobian matrix has the following form:

$$J_D = \begin{bmatrix} -a(1+m_1) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & -c \end{bmatrix}. \quad (10)$$

The characteristic equation of Chua's system in this region is

$$\begin{aligned} \det|\lambda I - J_D| = \lambda^3 &+ (a + c + am_1 + 1)\lambda^2 \\ &+ (b + c + ac + am_1 + acm_1)\lambda \quad (11) \\ &+ a(b + bm_1 + cm_1). \end{aligned}$$

The dynamic behavior of Chua's circuit is determined by the six eigenvalues. And the can be obtained by solving the characteristic equations (4.2) and (4.4). Let $a = 10.725, b = 10.593, c = 0.268, m_0 = -1.1726, m_1 = -0.7872$, then the solutions of (4.2) and (4.4) satisfy

$$\min\{|\arg(\lambda_i)|\} = 1.3816, \quad (12)$$

where $\arg(\cdot)$ denotes angle function.
Then we know when $0.88 < q \le 1$, the system (3.3) exhibits chaotic behavior, which can be seen in Fig.3 and Fig.4.

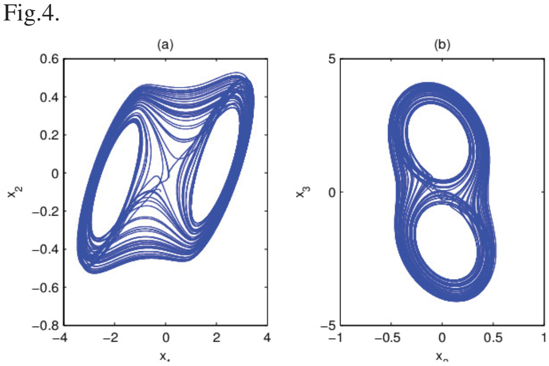

Fig.3. Chaotic attractor of the fractional-order Chua's system with fractional order $q = 0.95$: (a) $x_1 - x_2$ plane, (b)$x_2 - x_3$ plane.

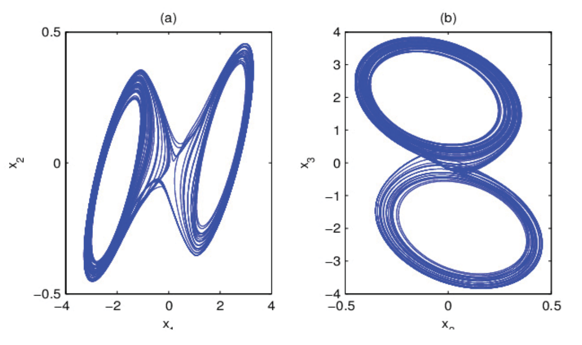

Fig.4. Chaotic attractor of the fractional-order Chua's system with fractional order $q = 0.90$: (a) $x_1 - x_2$ plane, (b)$x_2 - x_3$ plane.

5 Synchronization of fractional-order Chua's system

In this section, we briefly discuss the synchronization between two fractional-order Chua's systems. Now, we write the master system as (3.3) and the slave system as the following form:

$$\begin{aligned} D^q\hat{x}_1(t) &= a[\hat{x}_2(t) - \hat{x}_1(t) - f(\hat{x}_1)] + k_1(x_1 - \hat{x}_1), \\ D^q\hat{x}_2(t) &= \hat{x}_1(t) - \hat{x}_2(t) + \hat{x}_3(t) + k_2(x_2 - \hat{x}_2), \\ D^q\hat{x}_3(t) &= -b\hat{x}_2(t) - c\hat{x}_3(t) + k_3(x_3 - \hat{x}_3). \end{aligned} \quad (13)$$

The constant parameters $k_i, i = 1, 2, 3$ are coupling strength. Let us define the synchronization errors as:

$$e_i = x_i - \hat{x}_i, i = 1, 2, 3. \quad (14)$$

Then we have the dynamic equations of the synchronization errors:

$$\begin{aligned} D^q e_1(t) &= -(a + k_1)e_1 - ae_2 + a(f(\hat{x}_1) - f(x_1)), \\ D^q\hat{x}_2(t) &= e_1 - (1 + k_2)e_2 + e_3, \\ D^q\hat{x}_3(t) &= -be_2 - (c + k_3)e_3. \end{aligned} \quad (15)$$

If we can chose $k_i, i = 1, 2, 3$ appropriately, system (5.3) will be asymptotic stable. Let $k_i = 0.68, i = 1, 2, 3$, fractional order $q = 0.95$, and the initial conditions $x(0) = [-0.2, -0.1, 0.1]^T$, $\hat{x}(0) = [2, 3, -2]^T$. Then simulation results are shown in Fig. 5- Fig. 8. From the results we can conclude that good synchronization performance has been achieved and the synchronization errors converge fast.

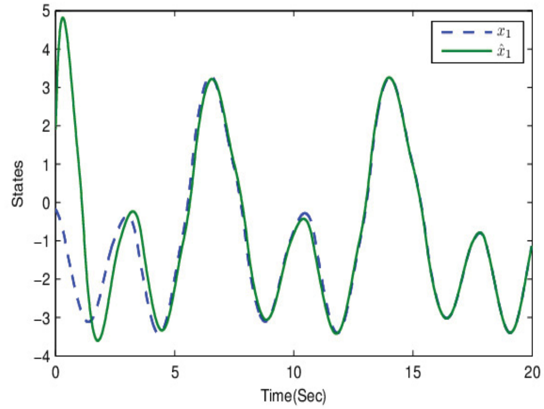

Fig.5. Time response of x_1 and $\hat{x}_1$.

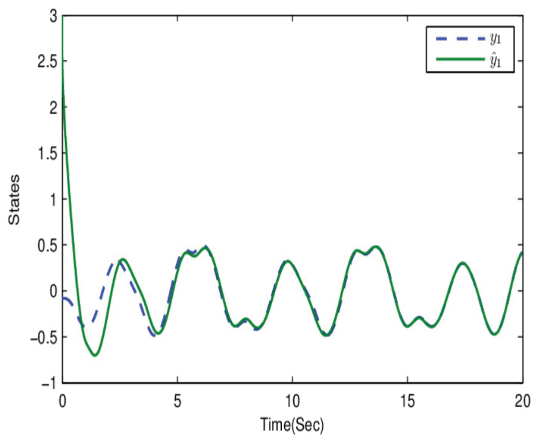

Fig.5. Time response of x_2 and $\hat{x}_2$.

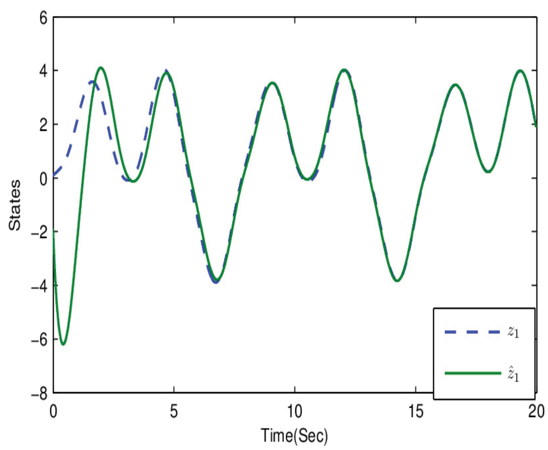

Fig.5. Time response of x_3 and $\hat{x}_3$.

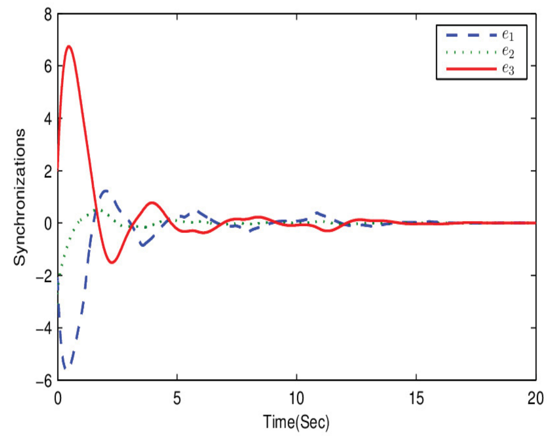

Fig.5. Time response of synchronization errors.

6 Conclusions

In this paper, we study the chaotic behaviors of the fractional-order Chua's system. We find that chaos exists in the system with fractional order $q > 0.88$. A simple, but effective, master-salve synchronization method is also introduced to achieve synchronization between two identical fractional-order Chua's systems with arbitrary initial conditions.

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