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Nonlinear dynamics and chaos in a fractional-order financial system

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Abstract

This study examines the two most attractive characteristics, memory and chaos, in simulations of financial systems. A fractional-order financial system is proposed in this study. It is a generalization of a dynamic financial model recently reported in the literature. The fractional-order financial system displays many interesting dynamic behaviors, such as fixed points, periodic motions, and chaotic motions. It has been found that chaos exists in fractional-order financial systems with orders less than 3. In this study, the lowest order at which this system yielded chaos was 2.35. Period doubling and intermittency routes to chaos in the fractional-order financial system were found.

1. Introduction

In the last few years, econophysics has been raised to an alternative scientific methodology to understand the highly complex dynamics of real financial and economic systems. Researchers are striving to explain the central features of economic data: irregular microeconomic fluctuations, erratic macroeconomic fluctuations (business cycles), irregular growth, structural changes, and overlapping waves of economic development. The economist specifies a model whose endogenous variables, in the absence of external forces, behave in simple ways (attain stationary equilibria, periodic cycles, or steady balanced growth). The model is then augmented with exogenous shock variables whose behavior is assumed to come from forces outside the economic system under consideration but that influence its working. These shocks are often assumed to be random so that endogenous variables display irregular behavior. Typical external influences that are treated as random shocks are weather variables, political events, and other human factors. In contrast to the viewpoint of exogenous shocks mentioned above, chaos supports an endogenous explanation of the complexity observed in economic series. In recent years, the importance of chaos in economics has tremendously increased: "chaos represents a radical change of perspective on business cycles" [1]. Chaos is the inherent randomness in a definite system. The randomness is caused by system internals, not by external disturbances. The main features of deterministic chaos, such as the complex patterns of phase portraits and positive Lyapunov exponents, have been found in many economic

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aggregate data such as the gross national product [2]. Many continuous chaotic models have been proposed to study complex economic dynamics in the literature, e.g., the forced van der Pol model [3–5], the IS-ML model [6–10], and the model proposed in various references [11–13].

Fractional calculus is a 300-year-old mathematical topic. Although it has a long history, it was not used in physics or engineering for many years. However, during the past 10 years, interest has been growing in fractional calculus not only among mathematicians but also among physicists and engineers. Fractional models have been shown by many researchers to adequately describe the operation of a variety of physical and biological processes and systems [14,15]. Examples include viscoelasticity, electroanalytical chemistry, electrical conductance of biological systems, modeling of neurons, diffusion processes, damping laws, and rheology. Recently, research has even applied fractional derivatives to specific fields of research: psychological and life sciences [16]. An extension of fractality concepts in financial mathematics has also been developed [17]. Virtually no area of classical analysis has been left untouched by fractional calculus.

This study examines the two most attractive characteristics, memory and chaos, in simulations of financial systems. One of the major differences between fractional-order and integer-order models is that fractional-order models possess memory; i.e., the fractional-order model depends on the history of the system. The magnitude of the financial variables such as foreign exchange rates, gross domestic product, interest rates, production, and stock market prices can have very long memory, in fact, correlations overlap with the longest time scales in the financial market [18,19]. This means that all fluctuations in financial variables are correlated with all future fluctuations. This was our motivation for describing financial systems using a fractional nonlinear model since it simultaneously possesses memory and chaos. Chaotic attractors have been found in fractional-order systems [20–29] in the past decade. Recently, the present author and Chen [30] investigated the chaotic behavior of the van der Pol equation with physically fractional damping. Even though the fractional concept has been used in financial mathematics as mentioned in the previous paragraph, chaotic behavior has not been studied in fractional financial systems. It is hoped that this paper will trigger more research efforts in this direction.

This paper is organized as follows: in Section 2, we present the integer-order financial model recently reported in the literature, and introduce its fractional version. In Section 3, we present numerical simulation results and discuss the routes to chaos in the fractional-order model. Finally, in Section 4 concluding comments are given.

2. Dynamic finance model

Refs. [11–13] have recently reported a dynamic model of finance, composed of three first-order differential equations. The model describes the time-variation of three state variables: the interest rate, X, the investment demand, Y, and the price index, Z. The factors that influence changes in X mainly come from two aspects: first, contradictions from the investment market, i.e., the surplus between investment and savings, and second, structural adjustment from good prices. The changing rate of Y is in proportion to the rate of investment, and in proportion to an inversion with the cost of investment and interest rates. Changes in Z, on the one hand, are controlled by a contradiction between supply and demand in commercial markets, and on the other hand, are influenced by inflation rates. By choosing an appropriate coordinate system and setting appropriate dimensions for every state variable, Refs. [11–13] offer the simplified finance model as

$$\begin{aligned} \dot{X} &= Z + (Y - a)X, \\ \dot{Y} &= 1 - bY - X^2, \\ \dot{Z} &= -X - cZ, \end{aligned} \tag{1}$$

where a is the saving amount, b is the cost per investment, and c is the elasticity of demand of commercial markets. It is obvious that all three constants, a, b, and c, are nonnegative.

Here, we consider the generalization of system (1) for the fractional incommensurate-order model which takes the form

$$\begin{aligned} \frac{d^{q_1}X}{dt^{q_1}} &= Z + (Y - a)X, \\ \frac{d^{q_2}Y}{dt^{q_2}} &= 1 - bY - X^2, \\ \frac{d^{q_3}Z}{dt^{q_3}} &= -X - cZ \end{aligned} \tag{2}$$

There are several definitions of fractional derivatives. In this study, we use the Caputo-type fractional derivative defined by [31]:

$$\frac{\mathrm{d}^\alpha y}{\mathrm{d}x^\alpha} = D_*^\alpha y(x) = J^{m-\alpha} y^{(m)}(x), \quad \alpha > 0, \tag{3}$$

where $m = [\alpha]$ is the value of α rounded up to the nearest integer, $y^{(m)}$ is the ordinary mth derivative of y, and

$$J^\beta z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t)\,\mathrm{d}t \tag{4}$$

is the Riemann–Liouville integral operator of order $\beta > 0$, where $\Gamma(\beta)$ is the gamma function.

The Adams–Bashforth–Moulton predictor–corrector scheme is used for numerical solutions of the fractional derivatives in system (2). The details regarding the algorithms of the scheme are available in reference [32].

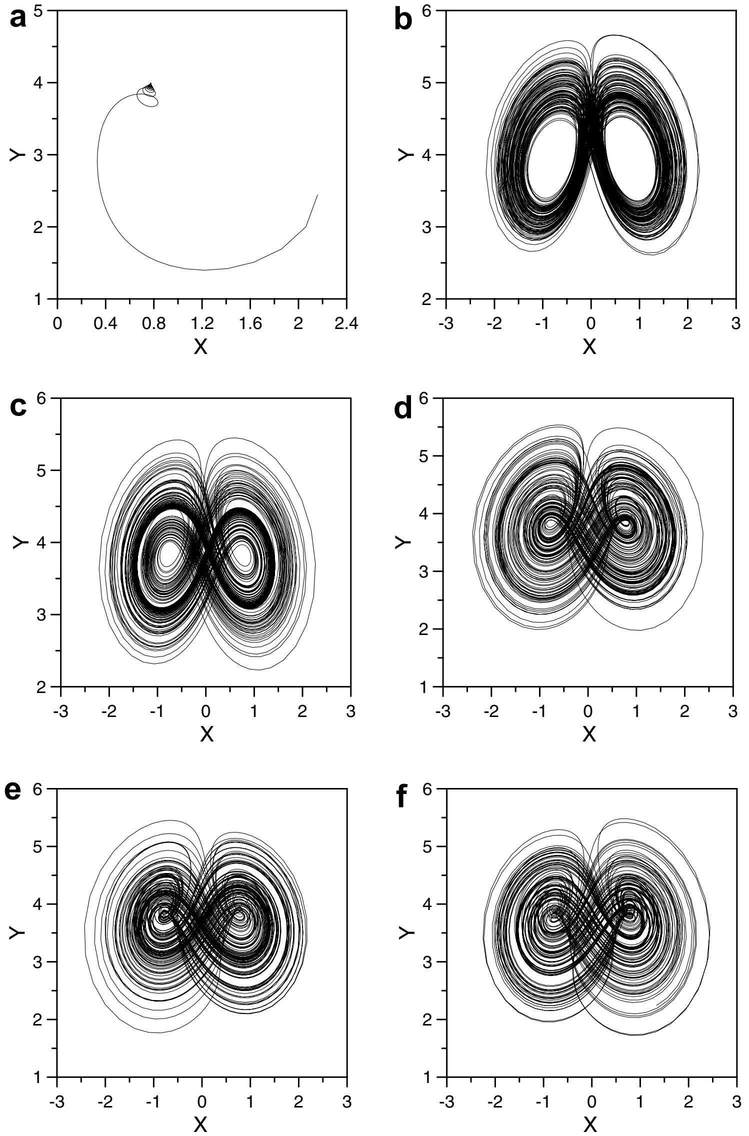

Fig. 1. Phase diagrams for system (2) with fractional-orders at $q_1 = q_2 = q_3 = \alpha$, (a) $\alpha = 0.84$, (b) $\alpha = 0.85$, (c) $\alpha = 0.93$, (d) $\alpha = 0.96$, (e) $\alpha = 0.99$, and (f) $\alpha = 1.0$.

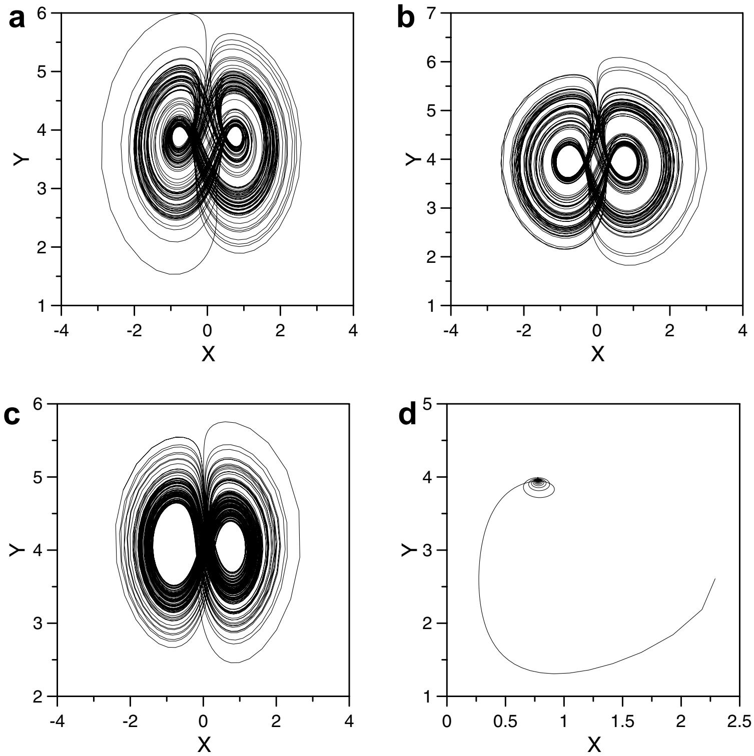

Fig. 2. Phase diagrams for system (2) at $q_2 = q_3 = 1$ and (a) $q_1 = 0.90$, (b) $q_1 = 0.80$, (c) $q_1 = 0.70$, and (d) $q_1 = 0.65$.

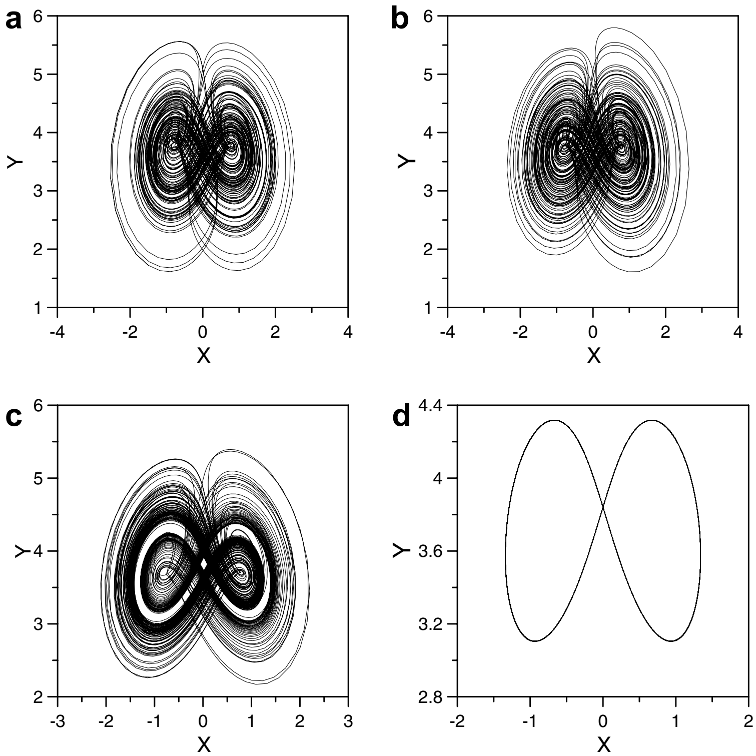

Fig. 3. Phase diagrams of system (2) at $q_1 = q_3 = 1$ and (a) $q_2 = 0.99$, (b) $q_2 = 0.95$, (c) $q_2 = 0.90$, and (d) $q_2 = 0.89$.

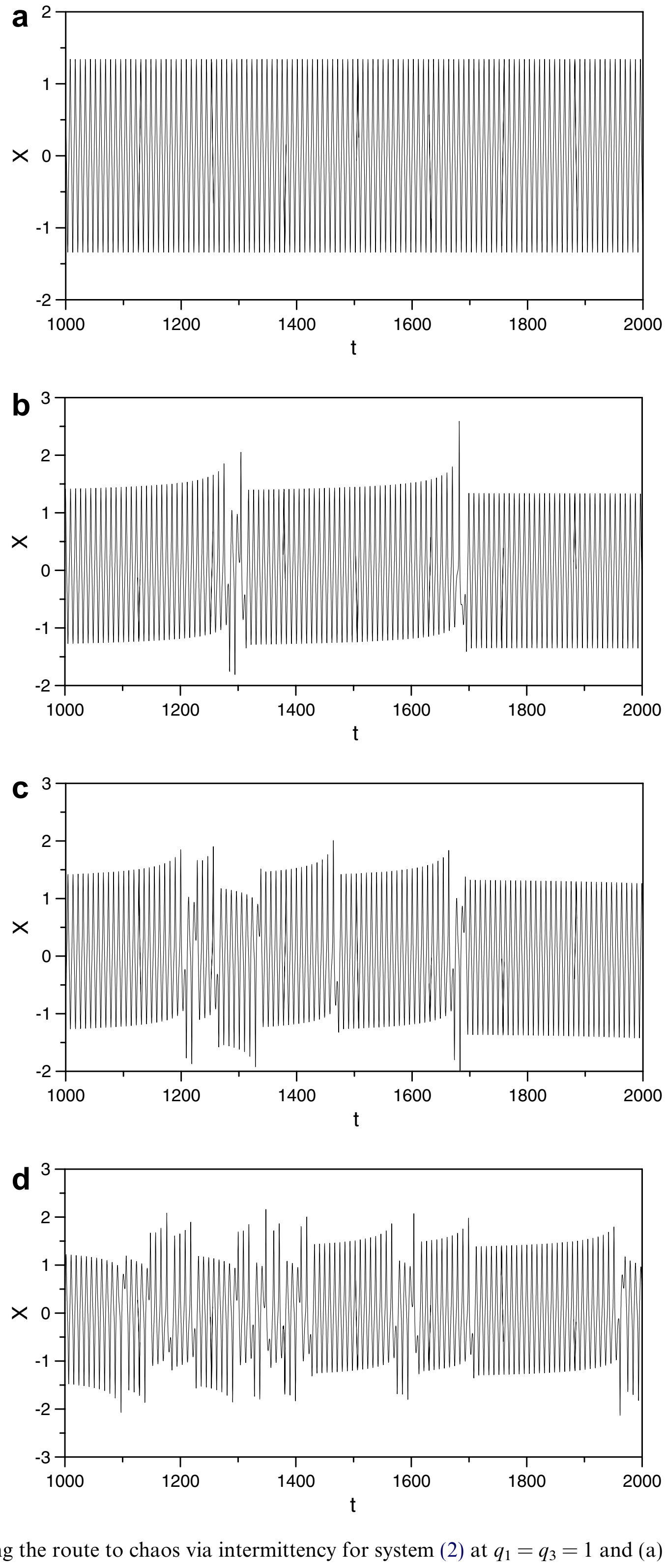

Fig. 4. Time histories showing the route to chaos via intermittency for system (2) at $q_1 = q_3 = 1$ and (a) $q_2 = 0.890$, (b) $q_2 = 0.891$, (c) $q_2 = 0.893$, and (d) $q_2 = 0.895$.

3. Simulation results

Numerical simulations were conducted on both the integer- and fractional-order models. The parameters were chosen to be $a = 3.0$, $b = 0.1$, and $c = 1.0$, with an initial state $(X_0, Y_0, Z_0) = (2.0, 3.0, 2.0)$. The time histories, phase diagrams, and the largest Lyapunov exponents (LEs) were used to identify the dynamics of the system. The largest LEs were calculated using the scheme proposed by Wolf et al. [33]. Following are the results of our investigations of various cases studied.

3.1. Commensurate order $q_1 = q_2 = q_3 = \alpha$

This system was calculated numerically against $\alpha \in [0.84, 1]$, while the incremental value of α was 0.01. System (2) corresponds to the integer-order system with $\alpha = 1.0$, where the chaotic behavior is confirmed by the largest LE, $\lambda_{\max} = 0.229$. It was found that when $0.85 \leqslant \alpha \leqslant 1$, system (2) behaves chaotically. When $\alpha = 0.84$, the chaotic motion disappears, and the system stabilizes to a fixed point. The phase plot of $X - Y$ is depicted in Fig. 1a. It is obvious that the trajectory for $\alpha = 0.84$ is attracted to a fixed point. The chaotic attractors projected onto the two-dimensional phase space, X–Y, when $\alpha = 0.85$, 0.93, 0.96, 0.99, and 1.0, are shown in Fig. 1b–f, respectively. The simulation results demonstrate that chaos indeed exists in the fractional-order financial system with orders less than 3. The lowest order for which this case generated chaotic motion was 2.55, where $\alpha = 0.85$ and the largest LE is $\lambda_{\max} = 0.130$.

3.2. $q_2 = q_3 = 1$, and let q_1 reduce to values less than 1

The system was calculated numerically against $q_1 \in [0.4, 1]$, while the incremental value of q_1 is 0.01. It was found that chaotic motions exist in the range $q_1 \in [0.69, 1]$. Fig. 2a–d shows the phase diagram at $q_1 = 0.9$, 0.8, 0.7, and 0.67, respectively. It is seen that system (2) degenerates to a fixed point at $q_1 = 0.67$. The lowest value for which q_1 in this case generates chaotic motion was 0.69, where the largest LE is $\lambda_{\max} = 0.150$. Results indicate that system (2)

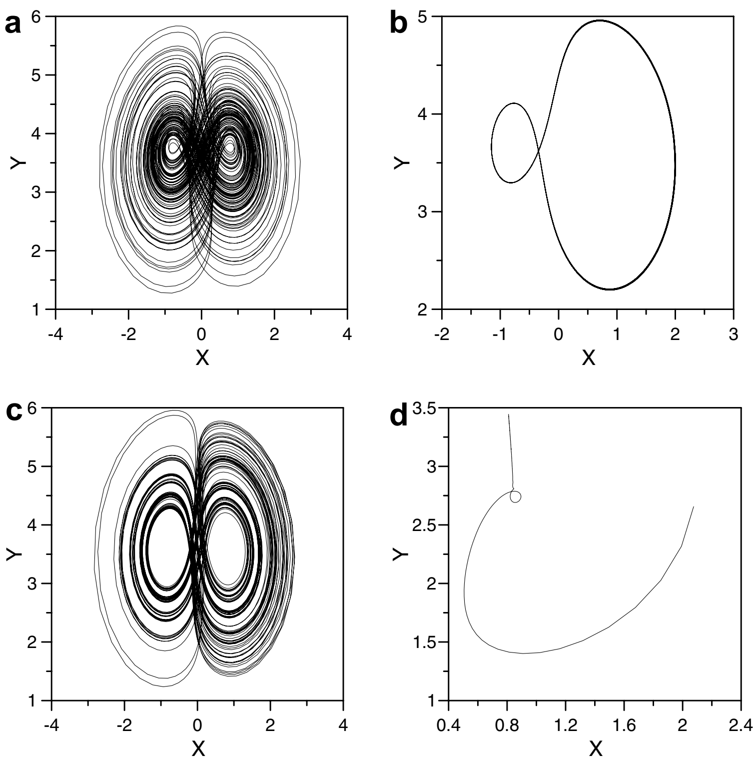

Fig. 5. Phase diagrams for system (2) at $q_1 = q_2 = 1$ and (a) $q_3 = 0.90$, (b) $q_3 = 0.80$, (c) $q_3 = 0.50$, and (d) $q_3 = 0.20$.

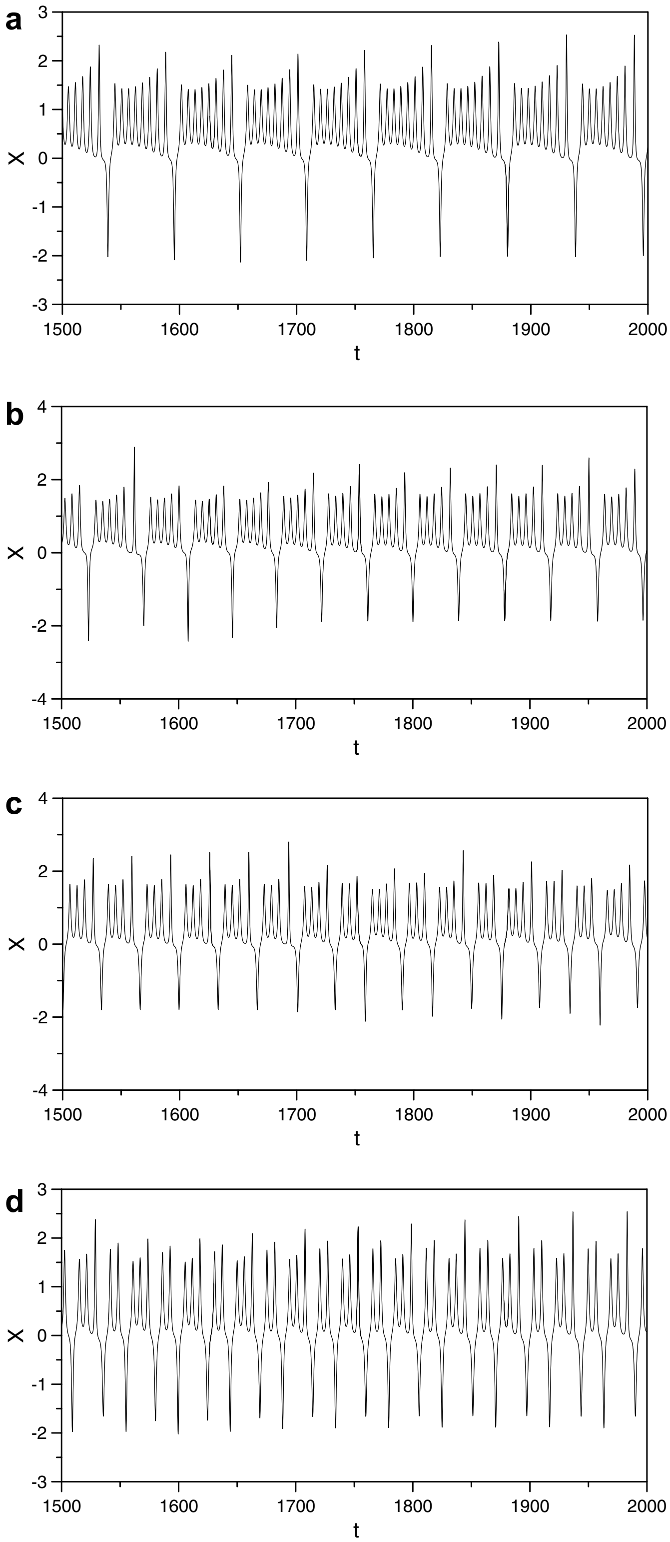

Fig. 6. Time histories showing the route to chaos via intermittency for system (2) at $q_1 = q_2 = 1$ and (a) $q_3 = 0.28$, (b) $q_3 = 0.30$, (c) $q_3 = 0.32$, and (d) $q_3 = 0.35$.

eventually converges to a fixed point in the range $q_1 \in [0.40, 0.68]$ when $q_2 = q_3 = 1$. Hence, the lowest order in which this case generates chaotic motion is 2.69.

3.3. $q_1 = q_3 = 1$, and let q_2 reduce to values less than 1

The system was calculated numerically against $q_2 \in [0.7, 1]$, while the incremental value of q_2 was 0.01. Fig. 3a–d shows the phase diagram in the X–Y-plane at $q_2 = 0.99$, 0.95, 0.90, and 0.89, respectively. Results show that chaotic motions exist in the range $q_2 \in [0.9, 1]$. At $q_2 = 0.89$, system (2) exhibits periodic motion. The lowest order at which this case generates chaotic motion is 2.90. To identify the route to chaos, the cases of q_2 between 0.89 and 0.90 were evaluated with an interval of 0.001 in q_2. Fig. 4a–d shows the time history of X at $q_2 =$ 0.890, 0.891, 0.893, and 0.895, respectively. When q_2 slightly exceeds the intermittency threshold ($q_2 = 0.890$), intermittent dynamic behavior was observed, with the intermittency growing as q_2 increases. The periodic behavior just before the onset of intermittency is shown in Fig. 4a. System (2) has a limited cycle at $q_2 = 0.890$. As q_2 exceeds the intermittency threshold, intermittency sets in as shown in Fig. 4b. The time history consists of oscillations that appear regular but are interrupted from time to time by abnormal fluctuations or bursts. The frequency of bursts increases with q_2 as shown in Fig. 4c. As q_2 was further increased, the periodic behavior almost completely disappeared, and the motion became chaotic as shown for $q_2 = 0.895$, where the largest LE is $\lambda_{\max} = 0.104$.

3.4. $q_1 = q_2 = 1$, and let q_3 reduce to values less than 1

The system was calculated numerically against $q_3 \in [0.1, 1]$, while the incremental value of q_3 was 0.01. In this case, the dynamic behaviors of system (2) were more complex than that of the previous cases. Fig. 5a–d shows the phase diagram in the X–Y-plane at $q_3 = 0.90$, 0.80, 0.50, and 0.20, respectively. It can be seen that the system is chaotic at $q_3 = 0.9$. As q_3 decreases to 0.8, the system becomes periodic. As q_3 is further decreased to 0.5, chaotic motion again

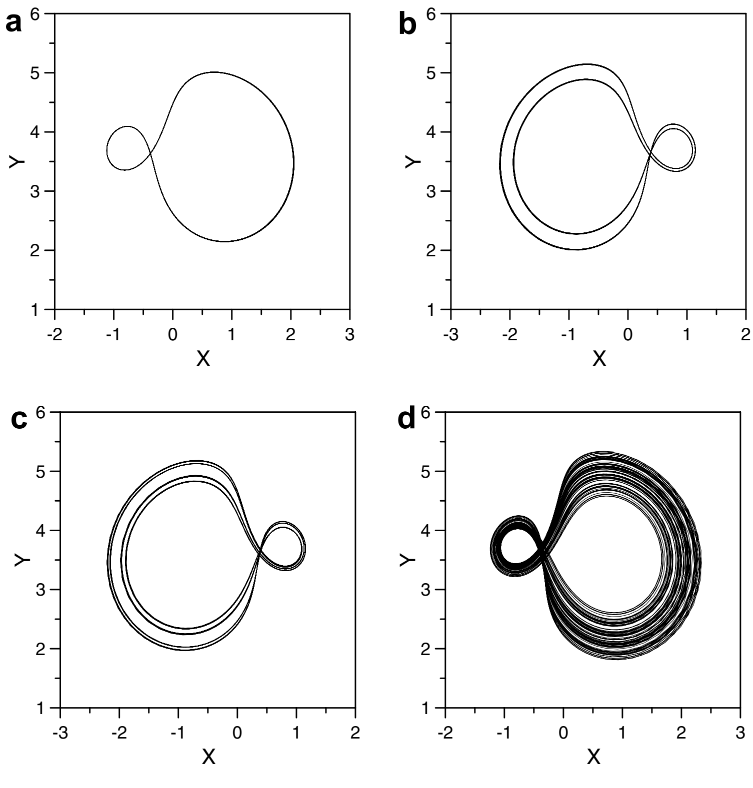

Fig. 7. Phase diagrams showing the route to chaos via period doubling for system (2) at $q_1 = q_2 = 1$ and (a) $q_3 = 0.83$, (b) $q_3 = 0.84$, (c) $q_3 = 0.845$, and (d) $q_3 = 0.86$.

appears. At $q_3 = 0.2$, the system degenerates to a fixed point. The lowest order at which this case generates chaotic motion was 2.35.

Both the intermittency and period-doubling routes to chaos were found in this case. Fig. 6a–d shows the time histories when $q_3 = 0.28$, 0.30, 0.32, and 0.35, respectively. The periodic behavior just before the onset of intermittency is shown for $q_3 = 0.28$, which corresponds to a stable cycle as reflected by the periodic oscillations. As q_3 further increases, the system response consists of long stretches of oscillations that appear to be regular, but this regular behavior is intermittently interrupted by chaotic bursts at irregular intervals. With an increase in q_3, the regular phase between two consecutive bursts becomes shorter and shorter and more and more difficult to recognize, as shown in Fig. 6c. As q_3 is increased further, eventually regular phases disappear and the response becomes fully chaotic at $q_3 = 0.35$ where the largest LE is $\lambda_{\max} = 0.102$.

As q_3 increases to around 0.83, a different route to chaos was found. Fig. 7a–d shows that the system can display period-1, period-2, period-4, and chaotic motions at $q_3 = 0.83$, 0.84, 0.845, and 0.86, respectively. Thus, Fig. 7 identifies a period doubling route to chaos.

4. Conclusions

A fractional model of finance is proposed as a generalization of an integer-order model that has recently been reported. It is an attempt to examine the two most attractive characteristics, memory and chaos, in simulations of financial systems. The fact that financial variables possess long memories makes fractional modeling appropriate for dynamic behaviors in financial systems. Two typical routes to chaos – period doubling and intermittency – are found in fractional financial systems. Results show that chaos exists in fractional-order financial system with orders less than 3. In this study, the lowest order at which this system yielded chaos was 2.35. It is hoped that the idea of fractional nonlinear systems, which simultaneously possess memory and chaos, might offer greater insights towards understanding the complex behavior of financial systems.

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