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Chaos and hyperchaos in the fractional-order Rössler equations

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Abstract

The dynamics of fractional-order systems have attracted increasing attentions in recent years. In this paper, we numerically study the chaotic behaviors in the fractional-order Rössler equations. We found that chaotic behaviors exist in the fractional-order Rössler equation with orders less than 3, and hyperchaos exists in the fractional-order Rössler hyperchaotic equation with order less than 4. The lowest orders we found for chaos and hyperchaos to exist in such systems are 2.4 and 3.8, respectively. Period doubling routes to chaos in the fractional-order Rössler equation are also found.

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The theory of derivatives of fractional order, i.e., non-integer order, goes back to Leibniz's note in his list to L'Hopital, dated 30 September 1695, in which the meaning of derivative of order one half was discussed [1]. Although fractional calculus has a 300-year-old history, its applications to physics and engineering are just a recent focus of interest. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [2], dielectric polarization [3], electrode–electrolyte polarization [4], and electromagnetic waves [5]. More recently, many investigations are devoted to

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control (see for example, Ref. [6]) and dynamics [7–14] of fractional-order dynamical systems. In Ref. [7], it is shown that the fractional-order Chua's circuit of order as low as 2.7 can produce a chaotic attractor. In Ref. [8], it is shown that nonautonomous Duffing systems of order less than 2 can still behave in a chaotic manner. In Ref. [9], the fractional-order Wien bridge oscillator is studied, where it is shown that limit cycle can be generated for any fractional order, with a proper value of the amplifier gain. In Ref. [10], chaotic behaviors of the fractional order "jerk" model is studied, in which chaotic attractor is generated with the system orders as low as 2.1, and in Ref. [11] chaos control of this fractional-order chaotic system is investigated. In Ref. [12], chaotic behavior of the fractional-order Lorenz system is studied, but unfortunately, the results presented in this paper are not correct [15]. In Ref. [13], bifurcation and chaotic dynamics of fractional-order cellular neural networks are studied. In Ref. [14], chaotic behaviors are found in the fractional-order Chen system [16] of system order as low as 2.1. In Ref. [17], the synchronization of fractional-order chaotic systems are investigated. In Ref. [18], a broad review of existing models of fractional kinetics and their connection to dynamical models, phase space topology, and other characteristics of chaos are presented.

In this paper, the fractional-order generalizations of the well-known Rössler equation [19] and Rössler hyperchaos equation [20] are studied. We found that chaotic behavior exists in the fractional-order Rössler equation of order as low as 2.4, and hyperchaos exists in the fractional-order Rössler hyperchaos equation of order as low as 3.8. To our knowledge, this is the first report of hyperchaos in fractional-order dynamical systems. Like many other studies of dynamics in fractional-order systems, we investigate the dynamics of the fractional Rössler equations through numerical simulations.

There are several definitions of fractional derivatives [1]. Perhaps the best known is the Riemann–Liouville definition, which is given by

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \, \mathrm{d}\tau \,, \tag{1}$$

where $\Gamma(\cdot)$ is the gamma function and $n-1 \leqslant \alpha < n$. Upon considering all the initial conditions to be zero, the Laplace transform of the Riemann–Liouville fractional derivative is

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\} \,.$$

Thus, the fractional integral operator of order "α" can be represented by the transfer function $F(s) = 1/s^\alpha$ in the frequency domain.

The standard definition of fractional differintegral does not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer order operators. In the following simulations, we use the approximation method proposed in Ref. [21], which was also adopted in Refs. [7,10,13,14]. In Table 1 of Ref. [7], approximations for $1/s^q$ with $q = 0.1$–0.9 in steps 0.1 were given with errors of approximately 2 dB. We will mainly use these approximations in the following simulations.

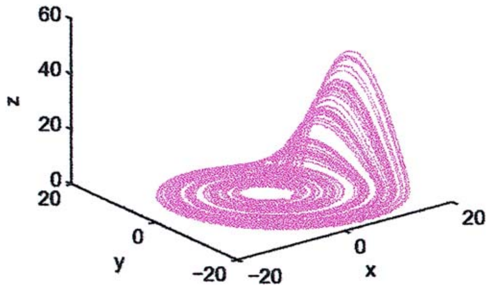

Fig. 1. Chaotic attractor of the fractional-order Rössler equation with $\alpha = 0.9$ and $a = 0.4$.

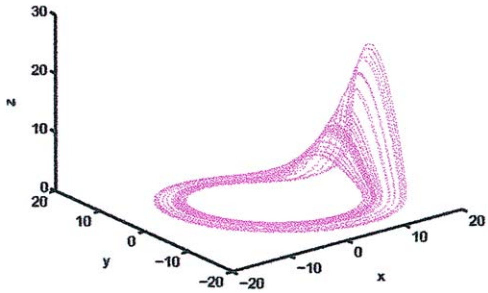

Fig. 2. Chaotic attractor of the fractional-order Rössler equation with $\alpha = 0.8$ and $a = 0.63$.

Now, consider a fractional-order generalization of the Rössler system. Here, the conventional derivative is replaced by a fractional derivative, as follows:

$$\frac{d^\alpha x}{dt^\alpha} = -(y + z) ,$$

$$\frac{d^\alpha y}{dt^\alpha} = x + ay ,$$

$$\frac{d^\alpha z}{dt^\alpha} = 0.2 + z(x - 10) , \tag{2}$$

where system parameter a is allowed to be varied, and α is the fractional order. The case of fractional-order Rössler equations with different fractional orders in different variables (i.e., x, y, z have different fractional orders) can be treated similarly. Therefore, we only consider the case where all variables have the same fractional order α in this paper. When $\alpha = 1$, Eq. (2) is equivalent to the classical integer-order Rössler equation, which is chaotic when $a = 0.15$.

Simulations were performed using $\alpha = 0.7, 0.8, 0.9$. The simulation results demonstrate that chaos indeed exist in the fractional-order Rössler equation with order less than 3. When $\alpha = 0.9$ and 0.8, chaotic attractors are found and the phase portraits are shown in Fig. 1 ($a = 0.4$) and Fig. 2 ($a = 0.63$), respectively. We can see that the chaotic attractors of the fractional-order system are similar to that of the integer-order Rössler

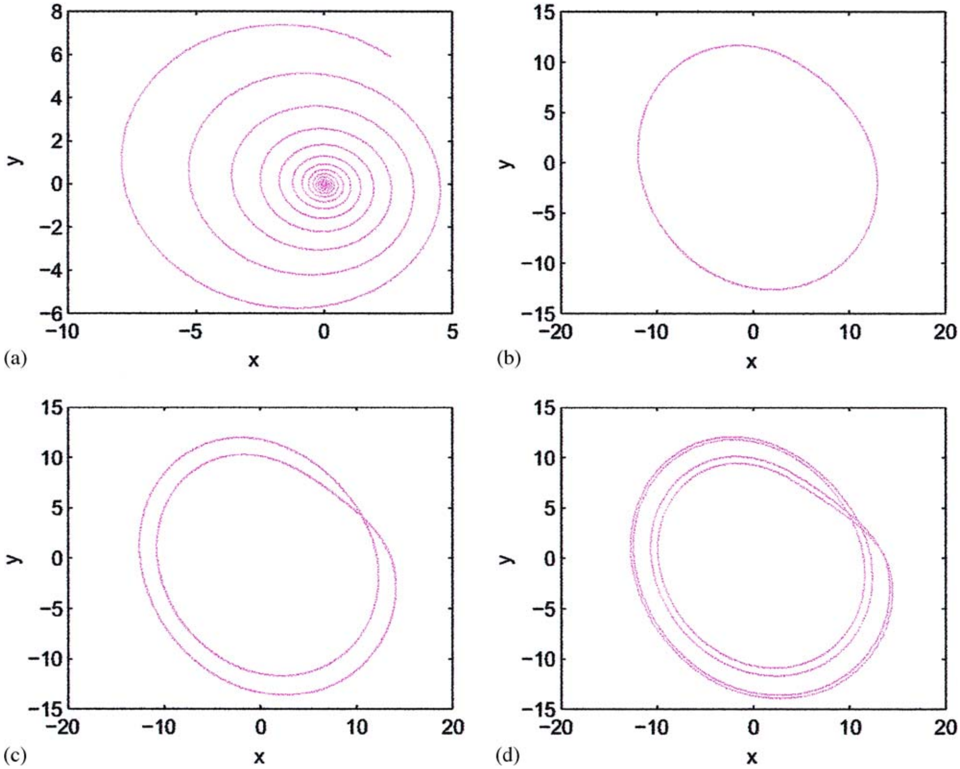

Fig. 3. Phase plots on the x–y plane when $\alpha = 0.9$: (a) $a = 0.1$; (b) $a = 0.25$; (c) $a = 0.28$; (d) $a = 0.29$.

attractor. When $\alpha = 0.7$, no chaotic behavior is found, which indicates that the lowest limit of the fractional order for this system to have chaos is $\alpha = 0.7$–0.8, so the lowest order we found to yield chaos in this system is 2.4.

We further consider the case of $\alpha = 0.9$. When $a = 0.1$, the phase plot of x–y is shown in Fig. 3(a). In this case, the system trajectory is attracted to a fixed point. When $a = 0.25, 0.28, 0.29$, the phase plot of x–y are shown in Fig. 3(b)–(d), respectively. From these figures, we can see that with the increase of the parameter value of a, a period-doubling route to chaos appears. When a passes through 0.30, the fractional-order system becomes chaotic. Similar phenomena have also been found in the case of $\alpha = 0.8$.

Next, we consider the fractional generalization of the Rössler hyperchaos equation. Again, we only consider the case of all variables having the same fractional order α. The system is given by

$$\frac{d^\alpha x}{dt^\alpha} = -(y + z)\,,$$

$$\frac{d^\alpha y}{dt^\alpha} = x + ay + w\,,$$

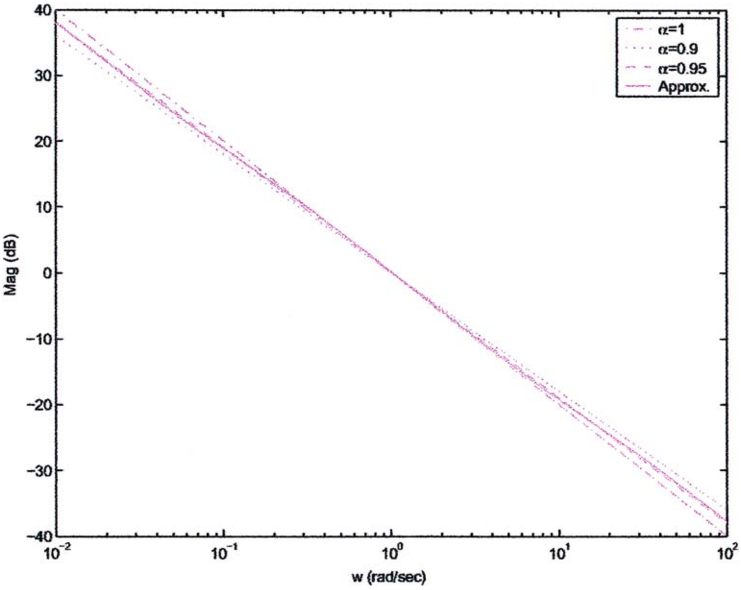

Fig. 4. Bode plot of the approximate and the actual functions.

$$\frac{d^\alpha z}{dt^\alpha} = 3 + xz ,$$

$$\frac{d^\alpha w}{dt^\alpha} = -0.5z + 0.05w , \quad (3)$$

where a is a variable parameter. When $\alpha = 1$, Eq. (3) reduces to the classical integer-order Rössler hyperchaos equation, and it has a hyperchaotic attractor when $a = 0.25$.

When simulating Eq. (3) with α=0.9, no chaotic behavior is found. We thus consider the case of $\alpha = 0.95$. To obtain a more accurate approximation, we select the approximation error of the method presented in Ref. [21] to be 1 dB. The approximation of $1/s^{0.95}$ with error of approximately 1 dB is given by

$$\frac{1}{s^{0.95}} \approx \frac{1.2831s^2 + 18.6004s + 2.0833}{s^3 + 18.4738s^2 + 2.6574s + 0.003} .$$

This function yields a good approximation result. In Fig. 4, we show the Bode plot of the actual and the approximate transfer functions. As we can see from Fig. 4, the frequency response of the approximation function is almost superposed with that of $1/s^{0.95}$, and is separated from that of $\alpha = 0.9$ and that of $\alpha = 1.0$.

Simulations were performed using the above approximation. When $a = 0.32$, the phase plot in the x–y–z space and the phase plot on the w–y plane are shown in Figs. 5 and 6, respectively. We calculate the two largest Lyapunov exponents of this system using the well-known Wolf algorithm [22]. The values of the two largest Lyapunov exponents are $\lambda_1 \approx 0.12$ and $\lambda_2 \approx 0.04$. We know that in this case the

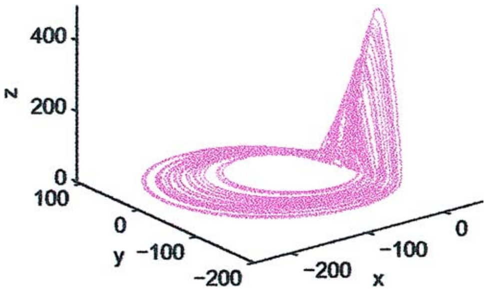

Fig. 5. Phase plot of hyperchaotic attractor in the x–y–z space.

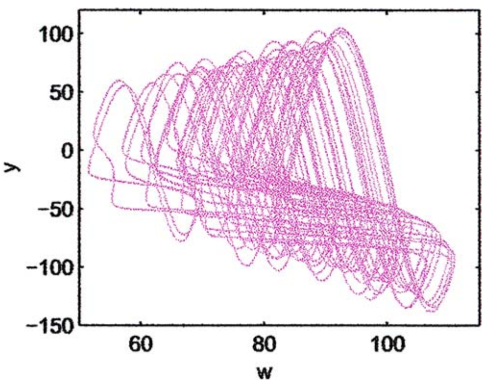

Fig. 6. Phase plot of hyperchaotic attractor on the w–y plane.

fractional-order system (3) has hyperchaos. So, the lowest limit of the fractional order for this system to have hyperchaos is $\alpha = 0.9$–0.95.

In summary, in this paper we have studied the dynamics of both the fractional-order Rössler equation and the fractional-order Rössler hyperchaos equation. We found that chaos exists in the fractional-order Rössler equation with order as low as 2.4, and hyperchaos exists in the fractional-order Rössler hyperchaos equation with order as low as 3.8. Period-doubling routes to chaos were also found in the fractional-order Rössler equations. To our knowledge, this paper is the first report of hyperchaos in fractional systems.

It is important to systematically develop some analysis methods for bifurcation and chaos in fractional-order systems from a frequency domain approach (see for example, Ref. [23]), because in the time domain fractional derivatives are still lack of physical and geometrical interpretations, despite some effort devoted to this attempt [24]. Control and synchronization of chaos and hyperchaos in fractional-order dynamical systems are also interesting topics for future studies.

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References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[2] R.L. Bagley, R.A. Calico, J. Guid. Contr. Dyn. 14 (1991) 304;
R.C. Koeller, J. Appl. Mech. 51 (1984) 299;
R.C. Koeller, Acta Mech. 58 (1986) 251.
[3] H.H. Sun, A.A. Abdelwahad, B. Onaral, IEEE Trans. Autom. Control 29 (1984) 441.
[4] M. Ichise, Y. Nagayanagi, T. Kojima, J. Electroanal. Chem. 33 (1971) 253.
[5] O. Heaviside, Electromagnetic Theory, Chelsea, New York, 1971.
[6] A, Oustaloup, J. Sabatier, P. Lanusse, Fractional Calculus Appl. Anal. 2 (1999) 1;
A. Oustaloup, F. Levron, F. Nanot, B. Mathieu, IEEE Trans. CAS-I 47 (2000) 25;
Y.Q. Chen, K. Moore, IEEE Trans. CAS-I 49 (2002) 3;
T.T. Hartley, C.F. Lorenzo, Nonlinear Dyn. 29 (2002) 201;
C. Hwang, J.-F. Leu, S.-Y. Tsay, IEEE Trans. Autom. Control 47 (2002) 625;
I. Podlubny, I. Petras, B.M. Vinagre, P. O'Leary, L. Dorcak, Nonlinear Dyn. 29 (2002) 281.
[7] T.T. Hartley, C.F. Lorenzo, H.K. Qammer, IEEE Trans. CAS-I 42 (1995) 485.
[8] P. Arena, R. Caponetto, L. Fortuna, D. Porto, Proceedings of the ECCTD, Budapest, 1997, p. 1259.
[9] W. Ahmad, R. El-Khazali, A. El-Wakil, Electron. Lett. 37 (2001) 1110.
[10] W.M. Ahmad, J.C. Sprott, Chaos, Solitons Fractals 16 (2003) 339.
[11] W.M. Ahmad, W.M. Harb, Chaos, Solitons Fractals 18 (2003) 693.
[12] I. Grigorenko, E. Grigorenko, Phys. Rev. Lett. 91 (2003) 034101.
[13] P. Arena, R. Caponetto, L. Fortuna, D. Porto, Int. J. Bifur. Chaos 7 (1998) 1527;
P. Arena, L. Fortuna, D. Porto, Phys. Rev. E 61 (2000) 776.
[14] C. Li, G. Chen, Chaos, Solitons and Fractals 22 (2004) 549–554.
[15] C. Li, Private communication with I. Grigorenko, 2003.
[16] G. Chen, T. Ueta, Int. J. Bifur. Chaos 9 (1999) 1465.
[17] C. Li, X. Liao, J. Yu, Phys. Rev. E 68 (2003) 067203.
[18] G.M. Zaslavsky, Phys. Rep. 371 (2002) 461.
[19] O.E. Rössler, Phys. Lett. A 57 (1976) 397.
[20] O.E. Rössler, Phys. Lett. A 71 (1979) 155.
[21] A. Charef, H.H. Sun, Y.Y. Tsao, B. Onaral, IEEE Trans. Autom. Control 37 (1992) 1465.
[22] A. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano, Physica D 16 (1985) 285.
[23] J.L. Moiola, G. Chen, Hopf Bifurcation Analysis: A Frequency Domain Approach, World Scientific, Singapore, 1996;
R. Genesio, A. Tesi, IEE Proc. D 138 (1991) 313;
R. Genesio, A. Tesi, F. Villoresi, IEEE Trans. CAS-I 40 (1993) 819;
C. Piccardi, IEEE Trans. CAS-I 41 (1994) 315;
C. Piccardi, IEEE Trans. CAS-I 43 (1996) 1015;
F. Bonani, M. Gilli, IEEE Trans. CAS-I 46 (1999) 881.
[24] I. Podlubny, arXiv: math.CA/0110241, 2001.