APPLICATIONS OF FRACTIONAL CALCULUS IN PHYSICS

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ποταμοῖς τοῖς αὐτοῖς ἐμβαίνομέν τε καὶ οὐκ ἐμβαίνομεν,
εἶμέν τε καὶ οὐκ εἶμεν.
Ἡράκλειτος

Preface

Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. One reason could be that, until recently, the basic facts were not readily accessible even in the mathematical literature. This book intends to increase the accessibility of fractional calculus by combining an introduction to the mathematics with a review of selected recent applications in physics.

Many applications of fractional calculus amount to replacing the time derivative in an evolution equation with a derivative of fractional order. This is not merely a phenomenological procedure providing an additional fit parameter. Rather the chapters of this book illustrate that fractional derivatives seem to arise generally and universally, and for deep mathematical reasons. One central theme of this book is the fact that fractional derivatives arise as the infinitesimal generators of a class of translation invariant convolution semigroups. These semigroups appear universally as attractors for coarse graining procedures or scale changes. They are parametrized by a number in the unit interval corresponding to the order of the fractional derivative.

Despite their common theme all chapters are self contained and can be read independently of the rest. Editing has been kept to a minimum in order to preserve the diverse style and levels of formalization in the different areas of application. Its diversity shows that the field is still evolving and workers have not even agreed on a common notation for fractional integrals and derivatives.

Given the long mathematical history of fractional calculus it is appropriate that the book begins with a mathematical introduction to fractional calculus. Chapter I provides such an introduction, and reviews also mathematical applications to special functions, Euler, Bernoulli, and Stirling numbers. Chapter II discusses fractional evolution equations and their emergence from coarse graining. It stresses the general importance of fractional semigroups for applications in physics, and gives explicit solutions for some fractional differential equations. Chapter III continues the mathematical discussion of fractional semigroups and their infinitesimal generators from a functional analytic point of view. Chapters IV and V review phenomenological and physical arguments for the general importance of fractional derivatives. The arguments are based mainly on the ubiquity of long time memory in nonequilibrium processes and on the behaviour of trajectories in chaotic Hamiltonian systems. Polymer science applications of fractional calculus are discussed in Chapters VI and VII . Chapter VI focusses on surface interacting polymers and the decimation transformation of random walk models. Chapter VII discusses the Rouse model and rheological constitutive modelling. Applications to relaxation and diffusion models for biophysical phenomena are presented in Chapter VIII. Finally the last chapter (IX) reviews a somewhat unorthodox application in which fractional calculus is used to generalize the Ehrenfest classification of phase transitions in equilibrium thermodynamics.

Let me conclude this preface by wishing all readers the joy and excitement that I felt many times when wandering and wondering in the fields of fractional calculus and its applications. Last but not least it is a pleasant task to thank Marc Lätzel, Martin Ottmann and Marlies Parsons for their help with typesetting the manuscript.

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CHAPTER I

AN INTRODUCTION TO FRACTIONAL CALCULUS

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Contents

1 Various Approaches to the Fractional Calculus

1.1 Some history

As to the history of fractional calculus, already in 1695 L'Hospital raised the question as to the meaning of $d^n y/dx^n$ if $n = 1/2$, that is "what if n is fractional?". "This is an apparent paradox from which, one day, useful consequences will be drawn", Leibniz replied, together with "$d^{1/2}x$ will be equal to $x\sqrt{dx: x}$". S. F. Lacroix [100] was the first to mention in some two pages a derivative of arbitrary order in a 700 page text book of 1819. Thus for $y = x^a$, $a \in \mathbb{R}_+$, he showed that

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(a+1/2)} x^{a-1/2}. \tag{1.1}$$

In particular he had $(d/dx)^{1/2}x = 2\sqrt{x/\pi}$ (the same result as by the present day Riemann–Liouville definition below).

Although the name "fractional calculus" is actually a misnomer, the designation "integration and differentiation of arbitrary order" being more appropriate, one usually sticks to "fractional calculus", a terminology in use since the days of L'Hospital.

J. B. J. Fourier, who in 1822 derived an integral representation for $f(x)$,

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha)\, d\alpha \int_{\mathbb{R}} \cos p(x-\alpha)\, dp,$$

obtained (formally) the derivative version

$$\frac{d^\nu}{dx^\nu} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\alpha)\, d\alpha \int_{\mathbb{R}} p^\nu \cos\{p(x-\alpha) + \frac{\nu\pi}{2}\}\, dp,$$

where "the number ν will be regarded as any quantity whatever, positive or negative".

It is usually claimed that Abel resolved in 1823 the integral equation arising from the brachistochrone problem, namely

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{g(u)}{(x-u)^{1-\alpha}}\, du = f(x), \qquad 0 < \alpha < 1 \tag{1.2}$$

with the solution

$$g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(u)}{(x-u)^\alpha}\, du. \tag{1.3}$$

As J. Lützen [107, p. 314] first showed, Abel never solved the problem by fractional calculus but merely showed how the solution, found by other means, could be written as a fractional derivative. Lützen also briefly summarized what Abel actually did. Liouville [103], however, did solve the integral equation (1.2) in 1832.

Perhaps the first serious attempt to give a logical definition of a fractional derivative is due to Liouville; he published nine papers on the subject between 1832 and 1837, the last in the field in 1855. They grew out of Liouville's early work on electromagnetism. There is further work of George Peacock (1833), D. F. Gregory (1841), Augustus de Morgan (1842), P. Kelland (1846), William Center (1848). Especially basic is Riemann's student paper of 1847 [139].

Liouville[a] started in 1832 with the well known result $D^n e^{ax} = a^n e^{ax}$ where $D = d/dx$, $n \in \mathbb{N}$, and extended it at first in the particular case $\nu = 1/2$, $a = 2$, and then to arbitrary order $\nu \in \mathbb{R}_+$ by

$$D^\nu e^{ax} = a^\nu e^{ax}. \tag{1.4}$$

He assumed the series representation for $f(x)$ as $f(x) = \sum_{k=0}^\infty c_k e^{a_k x}$ and defined the derivative of arbitrary order ν by

$$D^\nu f(x) = \sum_{k=0}^\infty c_k a_k^\nu e^{a_k x}. \tag{1.5}$$

Whereas this was Liouville's first approach, his second method was applied to the explicit function x^{-a}. He considered the integral $I = \int_0^\infty u^{a-1} e^{-xu}\, du$. Substituting $xu = t$ gives the result $I = x^{-a} \int_0^\infty t^{a-1} e^{-t}\, dt = x^{-a}\Gamma(a)$ (for $\Re e\ a > 0$). Operating on both sides of $x^{-a} = I/\Gamma(a)$ with D^ν, he obtained, using $D^\nu(e^{-xu}) = (-1)^\nu u^\nu e^{-xu}$,

$$D^\nu x^{-a} = (-1)^\nu \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}. \tag{1.6}$$

Liouville used the latter in his investigations of potential theory.

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[a]The most serious and detailed examination of the work of Liouville is that presented by J. Lützen [107, pp. 303–349]

Since the ordinary differential equation $d^n y/dx^n = 0$ of n–th order has the complementary (general) solution $y = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1}$, Liouville considered that the fractional order equation $d^\alpha y/dx^\alpha = 0$, $\alpha \in \mathbb{R}_+$, should have a suitable corresponding complementary solution, too. In this respect Riemann added a $\psi(x)$ to (1.8) below as the complementary function (shown to be of indeterminate nature by Cayley in 1880). For details on complementary functions see [127].

As partly indicated above, among the mathematicians spearheading research in the broad area of fractional calculus until 1941 were S.F. Lacroix, J.B.J. Fourier, N.H. Abel, J. Liouville, A. De Morgan, B. Riemann, Hj. Holmgren, K. Grünwald, A.V. Letnikov, N.Ya. Sonine, J. Hadamard, G.H. Hardy, H. Weyl, M. Riesz, H.T. Davis, A. Marchaud, J.E. Littlewood, E.L. Post, E.R. Love, B.Sz.-Nagy, A. Erdélyi and H. Kober.

Fractional calculus has developed especially intensively since 1974 when the first international conference in the field took place. It was organized by Bertram Ross [144] and took place at the University of New Haven, Connecticut in 1974. It had an exceptional turnout of 94 mathematicians; the proceedings contain 26 papers by the experts of the time. It was followed by the conferences conducted by Adam Mc Bride and Garry Roach [115] (University of Strathclyde, Glasgow, Scotland) of 1984, by Katsuyuki Nishimoto [125] (Nihon University, Tokyo, Japan) of 1989, and by Peter Rusev, Ivan Dimovski and Virginia Kiryakova [150] (Varna, Bulgaria) of 1996. In the period 1975 to the present, about 600 papers have been published relating to fractional calculus.

Samko et al in their encyclopedic volume [153, p. xxxvi] state and we cite: "We pay tribute to investigators of recent decades by citing the names of mathematicians who have made a valuable scientific contribution to fractional calculus development from 1941 until the present [1990]. These are M.A. Al-Bassam, L.S. Bosanquet, P.L. Butzer, M.M. Dzherbashyan, A. Erdélyi, T.M. Flett, Ch. Fox, S.G. Gindikin, S.L. Kalla, I.A. Kipriyanov, H. Kober, P.I. Lizorkin, E.R. Love, A.C. McBride, M. Mikolás, S.M. Nikol'skii, K. Nishimoto, I.I. Ogievetskii, R.O. O'Neil, T.J. Osler, S. Owa, B. Ross, M. Saigo, I.N. Sneddon, H.M. Srivastava, A.F. Timan, U. Westphal, A. Zygmund and others". To this list must of course be added the names of the authors of Samko et al [153] and many other mathematicians, particularly those of the younger generation.

Books especially devoted to fractional calculus include K.B. Oldham and J. Spanier [133], S.G. Samko, A.A. Kilbas and O.I. Marichev [153], V.S. Kiryakova [91], K.S. Miller and B. Ross [121], B. Rubin [147].

Books containing a chapter or sections dealing with certain aspects of fractional calculus include H.T. Davis [37], A. Zygmund [181], M.M.Dzherbashyan [45], I.N. Sneddon [159], P.L. Butzer and R.J. Nessel [25], P.L. Butzer and W. Trebels [28], G.O. Okikiolu [132], S. Fenyö and H.W. Stolle [55], H.M. Srivastava and H.L. Manocha [162], R. Gorenflo and S. Vessella [65].

There also exist two journals devoted especially to fractional calculus, namely the one edited by K. Nishimoto [126], and the one recently founded by V. Kiryakova [92].

For an historical survey of the field until 1975 one may consult Oldham-Spanier [133, pp. 1–15] as well as Ross [145].

1.2 Basic definitions; Riemann–Liouville and Weyl approaches

Let us consider some of the starting points for a discussion of classical fractional calculus; we will also introduce notations.

One development begins with a generalization of repeated integration. Thus if f is locally integrable on (a, ∞), then the n–fold iterated integral is given by

$$\begin{aligned} {}_aI_x^n f(x) &:= \int_a^x du_1 \int_a^{u_1} du_2 \ldots \int_a^{u_{n-1}} f(u_n)\, du_n \\ &= \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)\, du \end{aligned} \tag{1.7}$$

for almost all x with $-\infty \leq a < x < \infty$ and $n \in \mathbb{N}$. Writing $(n-1)! = \Gamma(n)$, an immediate generalization is the *integral* of f of *fractional order* $\alpha > 0$,

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u)\, du \quad \text{(right hand)}, \tag{1.8}$$

and similarly for $-\infty < x < b \leq \infty$

$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (u-x)^{\alpha-1} f(u)\, du \quad \text{(left hand)}, \tag{1.9}$$

both being defined for suitable f. The subscripts in I denote the terminals of integration (in the given order). Note the kernel $(u-x)^{\alpha-1}$ for (1.9).

Observe that (1.8) for $\alpha = n$ can be shown to be (see e. g. [145, pp. 7–10]) the unique solution of the initial value problem

$$ y^{(n)}(x) = f(x),\ y(a) = y'(a) = \ldots = y^{(n-1)}(a) = 0. \tag{1.10}$$

When $a = -\infty$ equation (1.8) is equivalent to Liouville's definition, and when $a = 0$ we have Riemann's definition (without the complementary function). One generally speaks of ${}_aI_x^\alpha f$ as the *Riemann–Liouville* fractional integral of order α of f, a terminology introduced by Holmgren (1863/64). On the other hand, one usually refers to

$$ {}_xW_\infty^\alpha f(x) = {}_xI_\infty^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (u - x)^{\alpha-1} f(u)\, du \tag{1.11}$$

$$ {}_{-\infty}W_x^\alpha f(x) = {}_{-\infty}I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - u)^{\alpha-1} f(u) du \tag{1.12}$$

as *Weyl* fractional integrals of order α, they being defined for suitable f.

The right and left hand fractional integrals ${}_aI_x^\alpha f(x)$ and ${}_xI_b^\alpha f(x)$ are related via the Parseval equality (fractional integration by parts) which we give for convenience for $a = 0$ and $b = \infty$:

$$\int_0^\infty f(x)\, ({}_0I_x^\alpha g)\, (x)\, dx = \int_0^\infty ({}_xW_\infty^\alpha f)\, (x) g(x)\, dx. \tag{1.13}$$

The following properties are stated for right handed fractional integrals (with obvious changes in the case of left handed integrals).

Concerning existence of fractional integrals, let $f \in L^1_{\text{loc}}(a, \infty)$. Then, if $a > -\infty$, ${}_aI_x^\alpha f(x)$ is finite almost everywhere on (a, ∞) and belongs to $L^1_{\text{loc}}(a, \infty)$. If $a = -\infty$, it is assumed that f behaves at $-\infty$ such that the integral (1.8) converges. Under these assumptions the fractional integrals satisfy the additive index law (or semigroup property)

$$ {}_aI_x^\alpha\ {}_aI_x^\beta f = {}_aI_x^{\alpha+\beta} f \quad (\alpha, \beta > 0). \tag{1.14}$$

Indeed, by Dirichlet's formula concerning the change of the order of integration, we have

$$\begin{aligned} {}_aI_x^\alpha\ {}_aI_x^\beta f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - u)^{\alpha-1}\, du \frac{1}{\Gamma(\beta)} \int_a^u (u - t)^{\beta-1} f(t)\, dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(t)\, dt \int_t^x (x - u)^{\alpha-1} (u - t)^{\beta-1}\, du. \end{aligned}$$

The second integral on the right equals, under the substitution $y = \frac{u-t}{x-t}$,

$$(x-t)^{\alpha+\beta-1} \int_0^1 (1-y)^{\alpha-1} y^{\beta-1}\, dy \;=\; B(\alpha,\beta)(x-t)^{\alpha+\beta-1}$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-t)^{\alpha+\beta-1}, \quad (1.15)$$

$B(\alpha,\beta)$ being the Beta–function (see (1.29)). When this is substituted into the above the result follows. In particular, we have

$$_aI_x^{n+\alpha} f = {}_aI_x^n \; {}_aI_x^\alpha f \quad (n \in \mathbb{N}, \alpha > 0) \tag{1.16}$$

which implies by n-fold differentiation

$$\frac{d^n}{dx^n} \; {}_aI_x^{n+\alpha} f(x) = {}_aI_x^\alpha f(x) \quad (n \in \mathbb{N}, \alpha > 0)$$

for almost all x.

The above results also hold for complex parameters α, if the condition $\alpha > 0$ is replaced by $\Re e\ \alpha > 0$. Then the operation $_aI_x^\alpha$ may be considered as a holomorphic function of α for $\Re e\ \alpha > 0$ which can be extended to the whole complex plane by analytic continuation, if f is sufficiently smooth.

To understand and establish this fact, we assume, for convenience, that f is an infinitely differentiable function defined on $\mathbb{R}$ with compact support contained in $[a,\infty)$, if $a > -\infty$, implying that $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \ldots$. Then for any fixed $x > a$ the integral in (1.8) is a holomorphic function of α for $\Re e\ \alpha > 0$. Now, integration by parts n-times yields

$$_aI_x^\alpha f(x) = {}_aI_x^{n+\alpha} f^{(n)}(x) \quad (\Re e\ \alpha > 0, n \in \mathbb{N}). \tag{1.17}$$

Applying the semigroup property (1.16) to the expression on the right in (1.17) and differentiating the result n-times with respect to x, we obtain

$$\frac{d^n}{dx^n} \; {}_aI_x^\alpha f(x) = {}_aI_x^\alpha f^{(n)}(x) \quad (\Re e\ \alpha > 0, n \in \mathbb{N}), \tag{1.18}$$

showing that under the hypotheses assumed on f the operations of integration of fractional order α and differentiation of integral order n commute.

Returning to formula (1.17) we now realize that its right-hand side is a holomorphic function of α in the wider domain $\{\alpha \in \mathbb{C}; \Re e\ \alpha > -n\}$ and even equals there $\frac{d^n}{dx^n}\ {}_aI_x^{n+\alpha} f(x)$, by (1.18). Thus we can extend $_aI_x^\alpha f(x)$ to the domain $\{\alpha \in \mathbb{C}; \Re e\ \alpha \leq 0\}$ analytically, defining for $\alpha \in \mathbb{C}$ with $\Re e\ \alpha \leq 0$

$$_aI_x^\alpha f(x) := {}_aI_x^{n+\alpha} f^{(n)}(x) = \frac{d^n}{dx^n} \; {}_aI_x^{n+\alpha} f(x) \tag{1.19}$$

with any integer $n > -\Re e\ \alpha$. In particular, we obtain

$$ {}_aI_x^0 f(x) = f(x), \ {}_aI_x^{-n} f(x) = f^{(n)}(x) \quad (n \in \mathbb{N}). $$

Moreover, it is clear by complex function theory arguments that the semigroup property (1.14) remains valid for all $\alpha, \beta \in \mathbb{C}$.

The very elegant method of analytic continuation developed by M. Riesz and his school is restricted to a rather small class of functions (even if the assumptions are weakened somewhat which is, indeed, possible). But the expressions occurring in formula (1.19) are meaningful for much more general classes of functions and thus give rise to the following definitions of fractional derivatives which go back to Liouville.

Let α be a complex number with $\Re e\ \alpha > 0$ and $n = [\Re e\ \alpha] + 1$, where $[\Re e\ \alpha]$ denotes the integral part of $\Re e\ \alpha$. Then the right-handed fractional derivative of order α is defined by

$$ {}_aD_x^\alpha f(x) = \frac{d^n}{dx^n}\ {}_aI_x^{n-\alpha} f(x) \quad (n = [\Re e\ \alpha] + 1) \tag{1.20} $$

for any $f \in L^1_{\mathrm{loc}}(a, \infty)$ for which the expression on the right exists.

One can unify the definitions of integrals and derivatives of arbitrary order α, $\Re e\ \alpha \neq 0$, (equivalently) by, for $n \in \mathbb{N}$,

$$ {}_aD_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u)\, du & (\Re e\ \alpha < 0) \\ \left(\frac{d}{dx}\right)^n\ {}_aI_x^{n-\alpha} f(x) & (\Re e\ \alpha > 0; n-1 \le \Re e\ \alpha < n); \end{cases} \tag{1.21} $$

one often speaks of a *differintegral* of f of order α in this respect. This process is also referred to as *fractional integro-differentiation.*

Note that the left-handed fractional derivative of order α is defined by

$$ {}_xD_b^\alpha f(x) = (-1)^n \frac{d^n}{dx^n}\ {}_xI_b^{n-\alpha} f(x) \quad (n = [\Re e\ \alpha] + 1). \tag{1.22} $$

The fractional derivative of *purely imaginary order* $\alpha = i\theta$, $\theta \neq 0$, is defined by

$$ {}_aD_x^{i\theta} f(x) = \frac{1}{\Gamma(1-i\theta)} \frac{d}{dx} \int_a^x \frac{f(u)}{(x-u)^{i\theta}}\, du, $$

and the associated integral of order $\alpha = i\theta$ by

$$ {}_aI_x^{i\theta} f(x) = \frac{d}{dx}\, {}_aI_x^{1+i\theta} f(x) = \frac{1}{\Gamma(1+i\theta)} \frac{d}{dx} \int_a^x (x-u)^{i\theta} f(u)\, du $$

(since the fractional integral (1.8) diverges for $\alpha = i\theta$). Then the definition of fractional integro-differentiation for all $\alpha \in \mathbb{C}$ is completed by introducing the identity operator ${}_aD_x^0 f := {}_aI_x^0 f = f$ for $\alpha = 0$.

Let us also observe that the fractional operators are linear:

$$ {}_aD_x^\alpha [c_1 f_1(x) + c_2 f_2(x)] = c_1\, {}_aD_x^\alpha f_1(x) + c_2\, {}_aD_x^\alpha f_2(x), $$

c_1, c_2 being constants.

Concerning sufficient conditions for the existence of the fractional derivatives (1.20) and their relation to (1.26), let us consider the case $0 < \Re e\, \alpha < 1$, if $a > -\infty$. Suppose that f is absolutely continuous on the finite interval $[a, b]$, in notation $f \in AC[a, b]$, meaning that f is differentiable almost everywhere on (a, b) with $f' \in L^1(a, b)$ and has the representation on $[a, b]$

$$ f(x) = \int_a^x f'(u) du + f(a) = {}_aI_x^1 f'(x) + f(a). $$

Substituting this in ${}_aI_x^{1-\alpha} f(x)$ and noting that, by the semigroup property, the operators $I^{1-\alpha}$ and I^1 commute, we obtain

$$ {}_aI_x^{1-\alpha} f(x) = {}_aI_x^1\, {}_aI_x^{1-\alpha} f'(x) + \frac{f(a)}{\Gamma(2-\alpha)} (x-a)^{1-\alpha}. $$

By differentiating with respect to x this implies

$$ {}_aD_x^\alpha f(x) = \frac{d}{dx}\, {}_aI_x^{1-\alpha} f(x) = {}_aI_x^{1-\alpha} f'(x) + \frac{f(a)}{\Gamma(1-\alpha)} (x-a)^{-\alpha} \qquad (1.23) $$

which shows that, in general, the operators ${}_aI_x^{1-\alpha}$ and $\frac{d}{dx}$ do not commute.

By Hölder's inequality one easily derives from (1.23) that ${}_aD_x^\alpha f \in L^r(a, b)$ for $1 \le r < 1/\Re e\, \alpha$.

Formula (1.23) may be extended to complex α with $\Re e\, \alpha \ge 1$. The results are summarized in Proposition 1.1 below. Beforehand we introduce the following notation: For $n \in \mathbb{N}$, $AC^{n-1}[a, b]$ denotes the set of $(n-1)$-times differentiable functions f on $[a, b]$ such that $f, f', \dots, f^{(n-1)}$ are absolutely continuous on $[a, b]$. Note that $AC^0[a, b]$ equals $AC[a, b]$.

Proposition 1.1. *a) If $f \in AC[a,b]$, f being given just in the (finite) interval $[a,b]$, then ${}_aD_x^\alpha f$, ${}_xD_b^\alpha f$ exist a. e. for $0 < \Re e\ \alpha < 1$. Moreover, ${}_aD_x^\alpha f \in L^r(a,b)$ for $1 \le r < 1/\Re e\ \alpha$ with*

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left\{ \frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(u)}{(x-u)^\alpha}\, du \right\}. \tag{1.24} $$

Correspondingly for ${}_xD_b^\alpha f$.

b) If $f \in AC^{n-1}[a,b]$, $n = [\Re e\ \alpha] + 1$, then ${}_xD_a^\alpha f$ exists a. e. for $\Re e\ \alpha \ge 0$ and has the representation

$$ {}_aD_x^\alpha f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(u)}{(x-u)^{\alpha-n+1}}\, du. \tag{1.25} $$

An alternative way to define a fractional derivative of order α, also due to Liouville, is

$$ {}_a\overline{D}_x^\alpha f(x) := {}_aI_x^{n-\alpha} f^{(n)}(x) \quad (n = [\Re e\ \alpha] + 1). \tag{1.26} $$

Obviously, f has to be n-times differentiable in order that the right-hand side of (1.26) exists. The relation between the two fractional derivatives (1.20) and (1.26) is given by formula (1.25) above, namely

$$ {}_aD_x^\alpha f(x) = {}_a\overline{D}_x^\alpha f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha}, $$

holding under the assumptions stated in Proposition 1.1 b). Definition (1.20) is commonly used in mathematical circles, while definition (1.26) is often preferable in problems of physical interest when initial conditions are expressed in terms of integer derivatives and Laplace transform methods are applied; see (5.27) for the Laplace transform of ${}_0\overline{D}_x^\alpha f$. The use of definition (1.26) in this context goes back to Caputo [32].

For a new real function space C_α, $0 \le \alpha \le 1$, larger than the space of continuous functions enabling one to study fractional order continuity, derivability and integrability, see Bonilla et al [10].

There are many results concerned with two-weight problems for various types of fractional integrals (and transforms). Many arose especially after the appearance of the monographs Kokilashvili-Krbec [95], Genebashvili et al [60]. Let us indicate two of these. Let $L^p_w(0,\infty)$ be the class of Lebesgue measurable functions on $\mathbb{R}_+$ for which $\|f\|_{L^p_w} = \left(\int_0^\infty |f(u)|^p w(u)du\right)^{1/p} < +\infty$, where $w(x) > 0$ a.e. and is locally integrable on $\mathbb{R}_+$. For simplicity take $J^\alpha f(x) = \int_0^x (x-u)^{\alpha-1} f(u)du$. There holds:

Let $1 < p \leq q < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$, and $p' = p/(p-1)$. Then the operator J^α is bounded from $L^p(0,\infty)$ to $L^q_w(0,\infty)$, i.e.,

$$\|J^\alpha f\|_{L^q_w} \leq A\|f\|_{L^p} \tag{1.27}$$

for a constant A independent of f, iff

$$B = \sup_{t>0} B(t) = \sup_{t>0} \left(\int_t^\infty u^{(\alpha-1)q} w(u)du \right)^{1/q} t^{1/p'} < \infty. \tag{1.28}$$

Moreover, if A is the best constant in (1.27), then $A \approx B$. Further, this operator J^α is compact from $L^p(0,\infty)$ to $L^q_w(0,\infty)$ iff condition (1.28) holds as well as the condition $\lim_{t\to 0} B(t) = \lim_{t\to\infty} B(t) = 0$.

The more general problem of boundedness and compactness of J^α from $L^p_v(0,\infty)$ (with $v \not\equiv 1$) to $L^q_w(0,\infty)$ has also been solved, the formulation being more difficult. The two-weight problem for the Weyl operator ${}_xW^\alpha_\infty f(x)$ is also a closed problem. There exist similar results for the Erdélyi-Kober operators, operators with power-logarithmic kernels, for Riesz potentials, etc. For this growing field see e.g. A. Meshki [118], H.P. Heinig [70] and [96], [97], and the extensive literature cited there.

1.3 Three examples

Let us compare the foregoing with the classical calculus. For this purpose let us first evaluate the fractional derivative ${}_0D^\alpha_x f(x)$ for the function $f(x) = x^b$ for $b > -1$. In view of the evaluation of the beta integral, valid for all $p > 0$ and $q > 0$ (or $\Re e\, p > 0$, $\Re e\, q > 0$), namely

$$B(p,q) := \int_0^1 u^{p-1}(1-u)^{q-1}\, du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{1.29}$$

we have for $m-1 \le \alpha < m$, $m \in \mathbb{N}$,

$$
\begin{aligned}
{}_0D_x^\alpha x^b &= \left(\frac{d}{dx}\right)^m \left\{ \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-u)^{m-\alpha-1} u^b \, du \right\} \\
&= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \left\{ x^{m-\alpha+b} \int_0^1 (1-v)^{m-\alpha-1} v^b \, dv \right\} \\
&= \frac{1}{\Gamma(m-\alpha)} \frac{\Gamma(b+1)\Gamma(m-\alpha)}{\Gamma(b+1+m-\alpha)} \left(\frac{d}{dx}\right)^m x^{m-\alpha+b} \\
&= \frac{\Gamma(b+1)}{\Gamma(b+1+m-\alpha)} x^{b-\alpha} \frac{\Gamma(m-\alpha+b+1)}{\Gamma(m-\alpha+b-m+1)} \\
&= \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)} x^{b-\alpha},
\end{aligned}
\tag{1.30}
$$

where we made use of, with $p \in \mathbb{R}\backslash\{-1,-2,-3,\dots\}$,

$$
\left(\frac{d}{dx}\right)^m x^p = \frac{\Gamma(p+1)}{\Gamma(p-m+1)} x^{p-m} = p(p-1)\dots(p-m+1)x^{p-m} \quad (m \in \mathbb{N}).
\tag{1.31}
$$

For $b=0$ we obtain ${}_0D_x^\alpha c = cx^{-\alpha}/\Gamma(1-\alpha)$ for any $\alpha > 0$. Thus fractional differentiation of a constant c is zero only for positive integral values of $\alpha = n \in \mathbb{N}$ (recall $\Gamma(1-n)=\infty$). On the other hand, for any $\alpha > 0$, ${}_0D_x^\alpha f(x) \equiv 0$ if $f(x) = x^{\alpha-k}$, $k = 1, 2, \dots, 1+[\alpha]$.

Let us consider a second example, namely ${}_0I_x^\alpha f$ for $f(x) = \log x$. Indeed, under the substitution $u = x(1-v)$,

$$
\begin{aligned}
{}_0I_x^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} \log u \, du \qquad (1.32) \\
&= \frac{(\log x) x^\alpha}{\Gamma(\alpha)} \int_0^1 v^{\alpha-1} \, dv + \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 v^{\alpha-1} \log(1-v) \, dv \\
&= \frac{x^\alpha}{\Gamma(\alpha+1)} \log x - \frac{x^\alpha}{\Gamma(\alpha)\alpha} \int_0^1 \log(1-v) \, d(1-v^\alpha) \\
&= \frac{x^\alpha}{\Gamma(\alpha+1)} \left\{ \log x - (1-v^\alpha)\log(1-v)\Big|_0^1 - \int_0^1 \frac{1-v^\alpha}{1-v} \, dv \right\}
\end{aligned}
$$

by integration by parts. Noting

$$\int_0^1 \frac{v^x - v^y}{1-v}\,dv = \psi(y+1) - \psi(x+1) \quad (\Re e\, x, \Re e\, y > -1),$$

where the psi function, defined by $\psi(x) = [\Gamma(x)]^{-1}(d/dx)\Gamma(x)$, obeys the recursion $\psi(x+1) - \psi(x) = x^{-1}$ with $-\psi(1) = \gamma = 0.5772157\ldots$, this yields

$${}_0I_x^\alpha \log x = \frac{x^\alpha}{\Gamma(\alpha+1)}\{\log x - \psi(\alpha+1) + \psi(1)\}. \tag{1.33}$$

Thus, for $m - 1 \le \Re e\, \alpha < m$,

$$\begin{aligned} {}_0D_x^\alpha \log x &= \left(\frac{d}{dx}\right)^m {}_0I_x^{m-\alpha} \log x \qquad (1.34) \\ &= \left(\frac{d}{dx}\right)^m \frac{x^{m-\alpha}}{\Gamma(m-\alpha+1)}\left[\log x - \psi(m-\alpha+1) + \psi(1)\right], \end{aligned}$$

where classical termwise differentiation is to be applied. Combining the result with (1.33) we obtain for all $\alpha \in \mathbb{C}$,

$${}_0D_x^\alpha \log x = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\left\{\log x - \psi(1-\alpha) - \gamma\right\}, \tag{1.35}$$

where in case $\alpha = n \in \mathbb{N}$ the expression on the right has to be interpreted as the limit for $\alpha \to n$. In fact, from the limit $\lim_{\alpha \to n} \psi(1-\alpha)/\Gamma(1-\alpha) = (-1)^{-n}\Gamma(n)$, the rule $(d/dx)^n \log x = -\Gamma(n)(-x)^{-n}$ readily follows. However for $\alpha = -n \in \mathbb{N}$ there holds the classical result

$${}_0D_x^{-n} \log x = {}_0I_x^n \log x = \frac{x^n}{n!}\left\{\log x - \sum_{j=1}^n \frac{1}{j}\right\},$$

also observing that $\psi(n+1) - \psi(1) = \sum_{j=1}^n j^{-1}$, which follows from the recursion $\psi(x+1) - \psi(x) = x^{-1}$.

As to our third example, take Weyl's definition, for $m-1 \le \alpha < m$, $m \in \mathbb{N}$,

$${}_xD_\infty^\alpha f(x) = (-1)^m \left(\frac{d}{dx}\right)^m {}_xW_\infty^{m-\alpha} f(x) \tag{1.36}$$

for which, for the example $f(x) = e^{-px}$, $p > 0$, under the substitution $u - x = y/p$,

$$\begin{aligned} {}_xW_\infty^\alpha e^{-px} &= \frac{1}{\Gamma(\alpha)} \int_x^\infty (u-x)^{\alpha-1} e^{-pu}\, du \\ &= \frac{e^{-px}}{\Gamma(\alpha) p^\alpha} \int_0^\infty y^{\alpha-1} e^{-y}\, dy = \frac{e^{-px}}{p^\alpha} \qquad (\alpha > 0). \end{aligned} \tag{1.37}$$

This yields for $p > 0$,

$${}_xD_\infty^\alpha e^{-px} = (-1)^m \left(\frac{d}{dx}\right)^m p^{-(m-\alpha)} e^{-px} = p^\alpha e^{-px} \quad (m - 1 \leq \alpha < m). \tag{1.38}$$

1.4 Approaches by Hadamard, by contour integration and other methods

The approach, due to J. Hadamard (1892), is to consider fractional differentiation of an analytic function via differentiation of its Taylor series $f(z) = \sum_{k=0}^\infty c_k (z - z_0)^k$ formally "α times", thus

$${}_{z_0}D_z^\alpha f(z) = \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} c_k (z - z_0)^{k-\alpha} \quad (c_k = f^{(k)}(z_0)/k!). \tag{1.39}$$

This approach is discussed in articles by Gaer-Rubel [59], Lavoie, Tremblay and Osler [101], Ross [145].

If z is complex, the natural extension of the Riemann-Liouville definition of the fractional integral is

$${}_aI_z^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z - \zeta)^{\alpha-1} f(\zeta) d\zeta \tag{1.40}$$

where the path of integration is along the line from $\zeta = a$ to $\zeta = z$ in the complex ζ-plane. The multivalued function $(z - \zeta)^{\alpha-1}$ is defined by $(z-\zeta)^{\alpha-1} = \exp[(\alpha - 1)\log(z - \zeta)]$ with $\log(z - \zeta) = \log|z - \zeta| + i \arg(z - \zeta)$. To obtain a single-valued branch for the integrand of (1.40) it is appropriate to choose $\arg(z - \zeta) = \arg(z - a)$ for all ζ on the line from a to z, where $\arg(z - a)$ is fixed by $-\pi < \arg(z - a) \leq \pi$.

An important method of defining a derivative of arbitrary order for holomorphic functions is given through a generalization of Cauchy's integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int\limits_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad (z \in \mathrm{Int}\ \mathcal{C}; n \in \mathbb{N}_0),$$

where f is holomorphic in a simply connected region $\mathcal{D}$ in the complex ζ-plane, and $\mathcal{C}$ is a piecewise smooth Jordan curve in $\mathcal{D}$. N.Y. Sonine (1872), P.A. Nekrassov (1888) et al. and more recently T.J. Osler [134] defined a fractional derivative of order α of $f(z)$ by

$$_aD_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int\limits_{\mathcal{C}(a,z+)} \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}}\, d\zeta. \tag{1.41}$$

Here the integral curve $\mathcal{C}(a, z+)$ in $\mathcal{D}$ is a closed contour starting at $\zeta = a$, encircling $\zeta = z$ once in the positive sense and returning to $\zeta = a$. For nonintegral α, the integrand has a branch line which is chosen as the ray beginning at $\zeta = z$ and passing through $\zeta = a$. Thus, this branch line cuts the integration contour in its beginning and ending point, but nowhere else. For the function $(\zeta - z)^{-\alpha-1}$ its principal branch is taken; it is that continuous range of the function for which $\arg(\zeta - z)$ is zero when $\zeta - z$ is real and positive.

Note that the generalized Cauchy integral formula (1.41) is defined for all $\alpha \in \mathbb{C}$; for negative integers $-n\,(n \in \mathbb{N})$ it is to be understood as the limit for $\alpha \to -n$. If $\Re e\ \alpha < 0$, then (1.41) coincides with the Riemann-Liouville integral $_aI_z^{-\alpha} f(z)$.

The Cauchy type complex contour integral approach is especially useful in the study of special functions. See e.g. Lavoie-Osler-Tremblay [102].

There exist further approaches to fractional integration by Erdélyi and Kober [51,93], R.K. Saxena [154], G.O. Okikiolu [131], Kalla-Saxena [90], K. Nishimoto [124], J. Cossar [36], S. Ruscheweyh [149], etc.

2 Leibniz Rule and Applications; Semigroups of Operators

2.1 Fractional Leibniz rule for functions

The classical Leibniz rule for the n-fold derivative of a product of two functions f, g as a sum of products of operations performed on each function is given by, provided f and g are n-times differentiable at z,

$$D_z^n[f(z)g(z)] = \sum_{k=0}^{n} \binom{n}{k} D_z^{n-k} f(z) \cdot D_z^k g(z), \tag{2.1}$$

where $D_z^n = (d/dz)^n$. This rule can be extended to fractional values of α: replacing n by α we have for holomorphic functions f, g

$${}_aD_z^\alpha[f(z)g(z)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} \, {}_aD_z^{\alpha-k} f(z) \cdot D_z^k g(z) \quad (\alpha \in \mathbb{C}) \tag{2.2}$$

a result which basically goes back to Liouville (1832), where ${}_aD_z^\alpha$ is now understood in the sense of Osler [134]. This formula suffers from the apparent drawback that the interchange of $f(z)$ and $g(z)$ on the right side of (2.2) is not obvious. An interesting generalization of this rule without the drawback, due to Y.Watanabe (1931) [170] and Osler [134] (1970), is

$${}_aD_z^\alpha[f(z)g(z)] = \sum_{k=-\infty}^{\infty} \binom{\alpha}{k+\mu} \, {}_aD_z^{\alpha-k-\mu} f(z) \cdot \, {}_aD_z^{k+\mu} g(z), \tag{2.3}$$

where μ is arbitrary, rational, irrational, or a complex number. The special case $\mu = 0$ reduces to (2.2).

Observe that if $\Re e\ \alpha < 0$, then formula (2.2) is actually the counterpart of Leibniz rule for fractional integrals. There also exists a *symmetrical* fractional Leibniz rule in the form (due to Osler [137])

$${}_aD_z^\alpha[f(z)g(z)] = \sum_{k=-\infty}^{\infty} c \binom{\alpha}{ck+\mu} \, {}_aD_z^{\alpha-ck-\mu} f(z) \cdot \, {}_aD_z^{ck+\mu} g(z), \tag{2.4}$$

where $\alpha, \mu \in \mathbb{C}$ for which $\binom{\alpha}{ck+\mu}$ is well-defined, and $0 < c \leq 1$; it yields (2.3) for $c = 1$, and reduces to (2.2) for $c = 1$ and $\mu = 0$.

As special cases of (2.3), there holds for $f(z) = 1$, and $g(z)$ renamed as $f(z)$,

$${}_aD_z^\alpha f(z) = \frac{\Gamma(\alpha+1)\sin((\alpha-\mu)\pi)}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \frac{(z-a)^{k+\mu-\alpha} \, {}_aD_z^{\mu+k} f(z)}{(\alpha-\mu-k)\Gamma(\mu+k+1)} \tag{2.5}$$

for $\Re e\ \alpha > -1$, $\alpha - \mu \notin \mathbb{Z}$, noting $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ for $z \notin \mathbb{Z}$.

A further result, also due to Osler [134], is the case $\alpha = 0$ of (2.3), namely

$$f(z)g(z) = \frac{\sin \mu\pi}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \frac{{}_aD_z^{-(\mu+k)} f(z) \cdot {}_aD_z^{\mu+k} g(z)}{\mu+k} \qquad (\mu \notin \mathbb{Z}). \quad (2.6)$$

Note that (2.3) has under suitable conditions the interesting integral analogue

$$ {}_aD_z^{\alpha}[f(z)g(z)] = \int_{-\infty}^{\infty} \binom{\alpha}{\tau+\mu} {}_aD_z^{\alpha-\mu-\tau} f(z) \cdot {}_aD_z^{\tau+\mu} g(z)\, d\tau, \qquad (2.7)$$

where $\alpha, \mu \in \mathbb{C}\backslash\mathbb{Z}^-$. It assumes an elegant form for $\mu = 0$ (see Osler [136]). By setting $\alpha = 0$ formula (2.7) readily reduces to (see [87, p. 16])

$$f(z)g(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \pi(\mu+\tau)}{\mu+\tau} {}_aD_z^{-\mu-\tau} f(z) \cdot {}_aD_z^{\mu+\tau} g(z)\, d\tau \qquad (\mu \notin \mathbb{Z}^-). \qquad (2.8)$$

Recently Kalia and his coworkers deduced from the Leibniz rule (2.4) a number of interesting expansion formulae associated with the Gamma function $\Gamma(z)$, the *Psi function* $\psi(z) := \Gamma'(z)/\Gamma(z)$, with the *incomplete Gamma function* $\gamma(a, z)$, defined by

$$\gamma(a, z) = \int_0^a t^{z-1} e^{-t}\, dt \qquad (\Re e\ z > 0), \qquad (2.9)$$

as well as with the *entire incomplete Gamma function* $\gamma^*(a, z)$, defined by

$$a^z \gamma^*(a, z) = \frac{\gamma(a, z)}{\Gamma(z)} \qquad (|\arg(z)| \le \pi - \epsilon; 0 < \epsilon < \pi). \qquad (2.10)$$

One is the well-known result (see e.g. Magnus, Oberhettinger and Soni [108]) concerning the psi function (recall (1.33))

$$\psi(\beta - \alpha + 1) - \psi(\beta + 1) = \sum_{k=1}^{\infty} \frac{(-\alpha)_k}{k(\beta - \alpha + 1)_k} \qquad (2.11)$$

valid for $\Re e\ \beta > -1$ and $\beta - \alpha + 1 \notin \mathbb{Z}^-$. Here $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$ for $k \in \mathbb{N}$.

A newer expansion, one for the entire incomplete Gamma function, is given for $\Re e\, \beta > -1; \alpha \notin \mathbb{Z}^-, \alpha - \nu \notin \mathbb{Z}$ by

$$\gamma^*(\beta - \alpha, z) = \frac{\sin \pi(\alpha - \nu)}{\pi} \Gamma(\alpha + 1) \sum_{k=-\infty}^{\infty} \frac{(-1)^k \gamma^*(\beta - \nu - k, z)}{(\alpha - \nu - k)\Gamma(\nu + k + 1)}. \quad (2.12)$$

The special case of (2.12) when $\beta = 0$, which easily reduces to

$$\gamma(\alpha, z) = -\frac{\sin \pi(\alpha + \nu) \cdot \sin(\pi\nu)}{\pi \sin \pi\alpha} \sum_{k=-\infty}^{\infty} \frac{z^{\alpha+\nu+k}}{\alpha + \nu + k} \gamma(-\nu - k, z) \quad (2.13)$$

with $\alpha, \alpha + \nu \notin \mathbb{Z}$, was given earlier by [134, p. 671, Entry 18]. The particular case $\nu = 0$ of (2.12) gives

$$\gamma^*(\beta - \alpha, z) = \frac{\sin \pi\alpha}{\pi} \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(\alpha - k)\, k!} \gamma^*(\beta - k, z) \quad (2.14)$$

valid for $\Re e\, \beta > -1; \alpha \notin \mathbb{Z}$.

An interesting application of the Leibniz rule to hypergeometric functions is given by the identity

$$_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \quad (2.15)$$

valid for $\Re e\ c > \Re e(a + b)$, $c \notin Z_0^-$, which follows directly from the integral representation (which may be established by the Leibniz rule – see [121, p. 114])

$$_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 u^{a-1}(1 - u)^{c-a-1} \cdot (1 - uz)^{-b} du \quad (2.16)$$

$$(\Re e\ c > \Re e\ a > 0,\ |z| < 1),$$

or by applying the Leibniz rule for fractional integrals to the product of $f(x) = x^\mu$ and $g(x) = x^\lambda$, $\lambda, \mu \geq 0$, yielding

$$\frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + \mu + \nu + 1)} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} {}_2F_1(-\lambda, \nu, \mu + \nu + 1; 1).$$

The more conventional notation $a = -\lambda$, $b = \nu$, $c = \mu + \nu + 1$ then gives (2.15).

Observe that formula (2.15) is actually a particular case of the Shannon sampling theorem of signal analysis (see [19]). A further one is given by ([121, p. 77]), namely

$$ {}_2F_1(a,b,c;z) = (1-z)^{-b}{}_2F_1(c-a,b,c;\frac{z}{z-1}). $$

Let us finally add an unusual integral analogue of the fractional version of Taylor's theorem (see Osler [135], [87, p. 297–8])

$$ f(z) = c\sum_{k=-\infty}^{\infty} \frac{{}_aD_w^{ck+\mu} f(w)}{\Gamma(ck+\mu+1)}(z-w)^{ck+\mu} \qquad (0 < c \leq 1; \mu \in \mathbb{C}) $$

in the form

$$ f(z) = \int_{-\infty}^{\infty} \frac{{}_aD_w^{t+\mu} f(w)}{\Gamma(t+\mu+1)}(z-w)^{t+\mu}\,dt \qquad (\mu \in \mathbb{C}; |z-w| = |w-a|). \quad (2.17) $$

For fractional versions of Taylor's formula with integral remainder terms as well as with those of Lagrange type one may consult M.M. Dzherbashyan et al. [47] and J. Trujillo et al. [168] and the literature cited there.

2.2 Fractional Landau-Kallman-Rota-Hille inequalities for operators

There exists an interesting inequality due to E. Landau (1913) which has been generalized in recent years in many directions. It is customary to write it in the form

$$ |f'(x)|^2 \leq 4\ \max_{x\in[0,1]}|f(x)| \cdot \max_{x\in[0,1]}|f''(x)| $$

provided f, f' and f'' are continuous on $[0,1]$. The inequality is also valid for the spaces $C[0,\infty]$ and $L^p(0,\infty)$, $L^p(-\infty,\infty)$ for any $1 \leq p \leq \infty$. In some cases the constant "4" may be different.

Now E. Hille [82] generalized this inequality by replacing the differential operators (d/dx) and $(d/dx)^2$ by semi-group generators A and their powers of arbitrary integral orders in the form

$$ ||A^k f||^n \leq C_{n,k}^n ||f||^{n-k} ||A^n f||^k, \qquad (2.18) $$

where n and k are integers with $1 \leq k < n$, $C_{n,k}$ being a constant independent of f.

As to the concepts involved above and below, a family $\{T(t); t \geq 0\}$ of operators mapping a Banach space X (with norm $\|\cdot\|_X$) into itself, satisfying the functional equation $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$, $T(0) = I$ (identity operator), together with the strong continuity property

$$\lim_{t \to 0+} \|T(t)f - f\|_X = 0 \quad (f \in X),$$

is said to be a (C_0)-*semigroup* of operators. Its *infinitesimal generator* A is defined by $Af = s - \lim_{h\to 0+} h^{-1}[T(h) - I]f = T'(0+)f$ for all $f \in X$ for which this limit exists, so for all $f \in D(A)$.

The fractional power $(-A)^\alpha$ of order $\alpha > 0$ can be defined by

$$(-A)^\alpha f = \lim_{\varepsilon \to 0+} \frac{1}{C_{\alpha,m}} \int_\varepsilon^\infty \frac{[I - T(t)]^m f}{t^{\alpha+1}} dt \tag{2.19}$$

for each $f \in X$ for which the limit exists in the X-norm. Here $0 < \alpha < m$, $C_{\alpha,m}$ is the constant of (3.24), and so (2.19) can be regarded as an abstract generalization of the Marchaud fractional derivative of (3.22). Alternatively this power may be defined by

$$(-A)^\alpha f = \lim_{t \to 0+} \frac{[I - T(t)]^\alpha f}{t^\alpha} \tag{2.20}$$

which is motivated by the Liouville-Grünwald fractional derivative defined in (3.1).

The fractional version of (2.18), which is due to Trebels and Westphal [167], reads: *Let $\{T(t);\ t \geq 0\}$ be a uniformly bounded* $(\|T(t)\| \leq M)(C_0)$-*semigroup of operators, A its infinitesimal generator. If $f \in D((-A)^\gamma)$ and $0 < \alpha < \gamma$, then*

$$\|(-A)^\alpha f\|^\gamma \leq C_{\gamma,\alpha}^\gamma \|f\|^{\gamma-\alpha} \|(-A)^\gamma f\|^\alpha, \tag{2.21}$$

where $C_{\gamma,\alpha}$ is a certain constant depending on γ and α. Applying (2.21) with $\alpha = 1$, $\gamma = 2$ and $M = 1$ yields

$$\|Af\|^2 \leq \left(\frac{2}{\log 2}\right)^2 \|f\| \|A^2 f\|$$

with $C_{2,1} = (2/\log 2) \sim 2.885$ instead of the constant 2 (which is better).

As a further application one may deduce Bernstein's inequality for *fractional* derivatives from the classical one for integers. The latter reads

$$\|\sum_{k=-n}^{n} (ik)^r c_k e^{ikx}\| \leq n^r \|t_n(x)\| \qquad (r = 1, 2, \ldots), \tag{2.22}$$

where $t_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ is a trigonometric polynomial of degree n, the norm being either $C_{2\pi}$ or $L^p_{2\pi}$, $1 \le p < \infty$.

As to the fractional version, take the particular *translation* semigroup $T(t)f(x) = f(x-t)$ which has infinitesimal generator $A = (-d/dx)$. Then for fractional α

$$(-A)^\alpha t_n(x) = t_n^{(\alpha)}(x) = \sum_{k=-n}^{n} (ik)^\alpha c_k e^{ikx}$$

which is just the Riemann-Liouville or Liouville-Grünwald derivative of order α. Thus, by the foregoing theorem and (2.22) we have for $0 < \alpha < r$,

$$\|t_n^{(\alpha)}\|^r \le C^r_{r,\alpha} \|t_n\|^{r-\alpha} \|t_n^{(r)}\|^\alpha \le C^r_{r,\alpha} n^{r\alpha} \|t_n\|^r,$$

which gives

$$\|t_n^{(\alpha)}\| \le C_{r,\alpha} n^\alpha \|t_n\|.$$

Replacing $L^p_{2\pi}$ by $L^p(-\infty, +\infty)$ and t_n by entire functions of exponential type $\le n$, one arrives at results of Lizorkin [104]; see also Junggeburth-Scherer-Trebels [86], as well as Section 3.4.

Let us finally remark that the above argument may be extended to groups of operators using the analysis of Westphal [173, II]. Inequalities estimating intermediate derivatives via a higher derivative and the function itself are also referred to as Hadamard (1914) - Kolmogorov (1939) type inequalities (also in the more general instance when the terms of the inequalities are taken with respect to different norms). The literature in this respect is quite large; see e.g. [153, p. 313] and the literature cited there. It includes papers by Hardy, Landau and Littlewood (1935), S.P. Geisberg (1965), R.J. Hughes (1977), G.G. Magaril-Il'yaev and V.M. Tikhomirov (1981). The case for Marchaud-type derivatives is quite popular. For a recent and excellent account of all aspects of the classical Landau-Kolmogorov inequality see S. Bagdasarov [4], also M.K. Kwong and A. Zettl [99].

2.3 The behaviour of semigroup operators at zero and infinity with rates

Fractional integration may be used to investigate methods of summation of series, integrals and functions (see e.g. [153, pp. 276 ff, 304, 314 f] and the extended literature cited there). Thus for a function $\varphi : \mathbb{R}_+ \to \mathbb{C}$ one says $\varphi(t) \to L$ as $t \to 0+$ in the sense of Cesàro (C, α)-summability provided $(C^\alpha \varphi)(t) := \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \varphi(u) du \to L$ as $t \to 0+$.

For semigroup operators, apart from the strong limit s-$\lim_{t\to 0+} T(t)f = f$ for $f \in X$ one is interested in the limit of the Cesàro means

$$C_T^\alpha(t)f := \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}T(u)f\,du \quad (\alpha > 0) \tag{2.23}$$

for $t \to 0+$ as well as for $t \to \infty$.

Concerning the *approximation* theoretical behaviour of $\{T(t); t \geq 0\}$ and $\{C_T^\alpha(t); t \geq 0\}$ for $t \to 0+$, it is connected as follows:

$$\begin{aligned} & \|T(t)f - f\|_X = \begin{cases} \mathcal{O}(t) \\ o(t) \end{cases} \\ \iff \quad & \|C_T^\alpha(t)f - f\|_X = \begin{cases} \mathcal{O}(t) \\ o(t) \end{cases} \\ \iff \quad & \begin{cases} f \in D(A) \\ f \in N(A) \end{cases} \end{aligned} \tag{2.24}$$

provided the Banach space X is reflexive. Otherwise $D(A)$ above has to be replaced by the completion of $D(A)$ relative to X, namely $\widetilde{D(A)}^X$ (for this concept see [15] and the literature cited there). Above, $N(A) = \{f \in D(A); Af = \theta\}$ is the null space of A.

As to the *ergodic* theoretical behaviour of $\{T(t); t \geq 0\}$ for $t \to \infty$,

$$s - \lim_{t\to\infty} C_T^\alpha(t)f = Pf \tag{2.25}$$

for each $f \in X$ and $\alpha \geq 1$, where P is the linear, bounded projection of X onto the kernel $N(A)$ parallel to $\overline{R(A)}$, the closure of the range $R(A)$. The result is true provided $\{T(t); t \geq 0\}$ is uniformly bounded and X is reflexive.

As to the *rate* of approximation of $C_T^\alpha(t)f$ to Pf for $t \to \infty$, the study of which was initiated in [29], there holds for $\alpha \geq 1$,

$$\begin{aligned} & \|t^{-1}\int_0^t T(u)f\,du - Pf\|_X = \begin{cases} \mathcal{O}(1/t) \\ o(1/t) \end{cases} \\ \iff \quad & \|\alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}T(u)f - Pf\| = \begin{cases} \mathcal{O}(1/t) \\ o(1/t) \end{cases} \\ \iff \quad & \begin{cases} f \in D(B) \\ f \in N(B) = N(A), \text{i.e.}, Pf = f. \end{cases} \end{aligned} \tag{2.26}$$

Above, the operator B is an appropriate extension of the inverse of the generator, A^{-1}, restricted to $\overline{R(A)}$. [More precisely, B is the closed, densely defined operator mapping $f = Ag + Pf \in R(A) \oplus N(A)$ onto $g \in D(A) \cap N(P)$ with $Pg = \theta$.]

There is a most surprising connection between the approximation and ergodic theoretical behaviour of the semigroup $\{T(t); t \geq 0\}$, thus between the approximation of $T(t)f$ to f for $t \to 0+$ and of $(1/t)\int_0^t T_B(u)f du$ to Pf for $t \to \infty$, where $\{T_B(t);\, t \geq 0\}$ is the semigroup generated by B.

It was raised as an open problem in a different setting (astronomy) on the occasion of a colloquium talk given at Aachen in the early sixties by Prof. R. Kurth and established in the doctoral thesis of A. Gessinger [62].

In this respect, first note that the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$ of A can be represented as the Laplace transform (see Section 5.3) of $\{T(t); t \geq 0\}$ for each $\Re e\, \lambda > 0$ and $f \in X$ in the form

$$R(\lambda; A)f = \int_0^\infty e^{-\lambda t} T(t) f dt.$$

The Abelian mean ergodic theorem in this frame states that for each $f \in X$ the strong Abel limit

$$s - \lim_{\lambda \to \infty} \lambda R(\lambda; A) f = f.$$

The fundamental connection between $R(\lambda; A)$ and the resolvent $R(\lambda; B)$ of B is

$$\lambda R(\lambda; A)f - Pf = f - \lambda^{-1} R(\lambda^{-1}; B)f \quad (\Re e\, \lambda > 0; f \in X).$$

Further, if the operator A generates a holomorphic semigroup $\{T_A(t); t \geq 0\}$ (see [12, p. 15] or [15] for this concept) so does the operator B, to be denoted by $\{T_B(t); t \geq 0\}$, and conversely. On top, the latter semigroup associated with B can be given explicitly via

$$T_B(t) = f + Pf - \sqrt{t} \int_0^\infty \frac{J_1(2\sqrt{tu})}{\sqrt{u}} T_A(u) f du,$$

$J_1(u)$ being the Bessel function of first kind.

The fundamental connection between $T_A(t)$ and $T_B(t)$ announced above, now reads:

Theorem 2.1. *If $\{T_A(t); t \geq 0\}$ is a holomorphic semigroup with generator A acting on the reflexive Banach space X, then:*

$$\|T_A(t)f - f\|_X = \begin{cases} \mathcal{O}(t) \\ o(t) \end{cases} \quad (t \to 0+)$$

$$\Longleftrightarrow \quad \|t^{-1}\int_0^t T_B(u)f du - Pf\|_X = \begin{cases} \mathcal{O}(t^{-1}) \\ o(t^{-1}) \end{cases} \quad (t \to \infty)$$

$$\Longleftrightarrow \quad \|\lambda R(\lambda; A)f - f\|_X = \begin{cases} \mathcal{O}(\lambda^{-1}) \\ o(\lambda^{-1}) \end{cases} \quad (\lambda \to \infty)$$

$$\Longleftrightarrow \quad \|\lambda R(\lambda; B)f - Pf\|_X = \begin{cases} \mathcal{O}(\lambda) \\ o(\lambda) \end{cases} \quad (\lambda \to 0+)$$

$$\Longleftrightarrow \quad \begin{cases} f \in D(A) \\ f \in N(A) = N(B). \end{cases}$$

The foregoing matter should be of special interest in the physical applications sketched in Section 4.7 since often semigroup and ergodic theory methods are applied.

As an application to Theorem 2.1 consider the Weierstrass semigroup

$$[W_A(t)f](x) = \sum_{k=-\infty}^{\infty} e^{-tk^2} f^\wedge(k)e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\vartheta_3(u,t)du \qquad (2.27)$$

with $x \in \mathbb{R}$ and $t > 0$ where $\vartheta_3(x,t) = \sum_{k=-\infty}^{\infty} e^{-tk^2}e^{ikx}$ is Jacobi's theta function, and

$$f^\wedge(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-iku}du$$

are the Fourier coefficients of $f \in L^2_{2\pi}$. It is well-known that the infinitesimal generator is given by $[Af](x) = f''(x)$ with domain $D(A) = \{f \in L^2_{2\pi}; f \in AC^1_{2\pi}, f'' \in L^2_{2\pi}\}$. The associated operator B turns out to be (see [15])

$$[Bf](x) = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} k^{-2}f^\wedge(k)e^{ikx} \quad (x \in \mathbb{R}) \qquad (2.28)$$

for which $\|Bf\|_{L^2_{2\pi}} \le \|f\|_{L^2_{2\pi}}$, so that B is bounded. Since $\{W_A(t); t \ge 0\}$ is holomorphic, the operator B also generates a holomorphic semigroup, namely $\{W_B(t); t \ge 0\}$, given by

$$[W_B(t)f](x) = f^\wedge(0) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} f^\wedge(k) e^{-tk^{-2}} e^{ikx} \quad (x, t \in \mathbb{R}), \tag{2.29}$$

where the exponent $-tk^2$ of (2.27) is now $-tk^{-2}$. The projector is given by $Pf = f^\wedge(0)$, and the resolvent by (see [16])

$$\begin{aligned} \lambda[R(\lambda, B)f](x) &= f(x) + f^\wedge(0) - \lambda^{-1}[R(\lambda^{-1}; A)f](x) \\ &= f^\wedge(0) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \lambda(\lambda + k^{-2})^{-1} f^\wedge(k) e^{ikx}. \end{aligned}$$

Corollary 2.2. *The following assertions are equivalent for $f \in L^2_{2\pi}$ and any $\alpha > 0$, $\beta > 0$:*

(i) $\|W_A(t)f - f\|_{L^2_{2\pi}} = \left\{ \begin{array}{l} \mathcal{O}(t) \\ o(t) \end{array} \right. \quad (t \to 0+),$

(ii) $\|C^\alpha_{W_A}(t)f - f\|_{L^2_{2\pi}} = \left\{ \begin{array}{l} \mathcal{O}(t) \\ o(t) \end{array} \right. \quad (t \to 0+),$

(iii) $\|t^{-1}\int_0^t W_B(u) f du - f^\wedge(0)\|_{L^2_{2\pi}} = \left\{ \begin{array}{l} \mathcal{O}(t^{-1}) \\ o(t^{-1}) \end{array} \right. \quad (t \to \infty),$

(iv) $\|C^{1+\beta}_{W_B}(t)f - f^\wedge(0)\|_{L^2_{2\pi}} = \left\{ \begin{array}{l} \mathcal{O}(t^{-1}) \\ o(t^{-1}) \end{array} \right. \quad (t \to \infty),$

(v) $\|\lambda R(\lambda; A)f - f\|_{L^2_{2\pi}} = \left\{ \begin{array}{l} \mathcal{O}(\lambda^{-1}) \\ o(\lambda^{-1}) \end{array} \right. \quad (\lambda \to \infty),$

(vi) $\|\lambda R(\lambda; B)f - f^\wedge(0)\|_{L^2_{2\pi}} = \left\{ \begin{array}{l} \mathcal{O}(\lambda) \\ o(\lambda) \end{array} \right. \quad (\lambda \to 0+),$

(vii) $\left\{ \begin{array}{l} f \in D(A) = \{f \in L^2_{2\pi}; f \in AC^1_{2\pi}, f'' \in L^2_{2\pi}\} \\ f(x) = f^\wedge(0) \ a.e. \end{array} \right.$

Observe that $w(x,t) := [W_A(t)f](x)$ of (2.27) is the solution of the heat equation $(\partial/\partial t)w(x,t) = (\partial^2/\partial x^2)w(x,t)$, $-\pi \le x \le \pi$, $t > 0$ with the boundary conditions $w(-\pi, t) = w(\pi, t)$, $(\partial/\partial x)w(-\pi, t) = (\partial/\partial x)w(\pi, t)$

and the initial condition $\lim_{t\to 0+} w(x,t) = f(x)$. Then assertions (i), (ii) and (iii) of Corollary 2.2 state that either $W_A(t)f(x)$ or the arithmetic mean $t^{-1}\int_0^t W_A(u)f du$ of the solution $W_A(t)f$ (having generator A) tends to the initial value $f(x)$ with the rate $\mathcal{O}(t)$ as $t \to 0+$ iff the mean $t^{-1}\int_0^t W_B(u)f du$ associated with the generator B (namely (2.29), (2.28)) tends to $f^\wedge(0) = (1/2\pi)\int_{-\pi}^{\pi} f(u)du$ with the rate $\mathcal{O}(t^{-1})$ as $t \to \infty$.

Let us consider now Abel summability of a uniformly bounded (C_0)-semigroup $\{T(t);\, t \geq 0\}$ with generator A acting on a Banach space X, the integral order powers of the resolvent $R(\lambda; A)$ now being extended to the fractional case $\gamma > 0$, thus

$$R^\gamma(\lambda; A)f = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-\lambda t}T(t)f dt \quad (f \in X). \tag{2.30}$$

In this respect H. Komatsu [98] showed that

$$s - \lim_{\lambda\to 0+} \lambda^\gamma R^\gamma(\lambda; A)f \text{ exists } \Leftrightarrow f \in X_0,$$

where $X_0 = N(A) \oplus \overline{R(A)}$. If so, the limit is equal to Pf.

The mean ergodic theorem for $\lambda^\gamma R^\gamma(\lambda; A)f$ with rates, first established by U. Westphal [176], states that

Theorem 2.3. *There holds for $f \in X_0$, $\gamma > 0$ and $\lambda \to 0+$*

$$\|\lambda^\gamma R^\gamma(\lambda; A)f - Pf\|_X = \begin{cases} \mathcal{O}(\lambda^\gamma) \\ o(\lambda^\gamma) \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} f \in \widetilde{D(B^\gamma)}^{X_0} \quad \textit{or } f \in D(B^\gamma) \textit{ if } X \textit{ is reflexive} \\ f \in N(A) \end{cases}. \tag{2.31}$$

As to the above, $(-A)^\gamma$ is defined either by (2.19) or (2.20). Further, observe that $R^\gamma(\lambda; A)f \in D((-A)^\gamma)$ for $f \in X$, and

$$(-A)^\gamma R^\gamma(\lambda; A)f = [I - \lambda R(\lambda; A)]^\gamma f.$$

Then the operator B^γ is defined by $B^\gamma f = g$ where g is the unique element in $D((-A)^\gamma) \cap \overline{R(A)}$ satisfying $(-A)^\gamma g = f - Pf$. Note that B^γ is closed with $\overline{D(B^\gamma)} = X_0$, and $N(B^\gamma) = N(A)$.

It is interesting to observe that the rate of approximation in (2.31) increases with the exponent γ of the resolvent $R^\gamma(\lambda; A)$. This is not the case with the order α of the Cesàro means of (2.23).

Let us now apply Theorem 2.3 to the (C_0)-semigroup of translations on $L^2(0,\infty)$, namely $[T_A(t)f](x) = f(x-t)$ if $x > t \geq 0$, and $[T_A(t)f](x) = 0$, if $0 < x < t$ with $[Af](x) = -f'(x)$, domain $D(A) = \{f \in L^2(0,\infty); f \in AC_{\text{loc}}, f' \in L^2(0,\infty)\}$, and $\|T_A(t)\| \leq 1$, $N(A) = \{0\}$, $X_0 = L^2(0,\infty)$, the projector P being the null operator. Now for $f \in L^2(0,\infty)$, $\gamma > 0$,

$$[[I - T(t)]^\gamma f](x) = \sum_{j=0}^{[x/t]} (-1)^j \binom{\gamma}{j} f(x - jt),$$

so that $(-A)^\gamma f$ is the strong fractional Grünwald-Liouville derivative of f of order γ. In terms of the Laplace transform $(-A)^\gamma$ is characterized by (see Section 5.3) $D((-A)^\gamma) = \{f \in L^2(0,\infty); \exists g \in L^2(0,\infty)$ such that $s^\gamma f_{\mathcal{L}}^{\wedge}(s) = g_{\mathcal{L}}^{\wedge}(s), \Re e\ s > 0\}$, $[(-A)^\gamma f]_{\mathcal{L}}^{\wedge}(s) = s^\gamma f_{\mathcal{L}}^{\wedge}(s)\quad, \Re e\ s > 0$.

The operators $R^\gamma(\lambda; A)$ and their Laplace transforms are given for $f \in L^2(0,\infty)$, $\Re e\ s > 0$ by

$$\begin{aligned} [R^\gamma(\lambda; A)f](x) &= \frac{1}{\Gamma(\gamma)} \int_0^x t^{\gamma-1} e^{-\lambda t} f(x-t)\,dt \\ [R^\gamma(\lambda; A)f]_{\mathcal{L}}^{\wedge}(s) &= f_{\mathcal{L}}^{\wedge}(s)/(\lambda + s)^\gamma. \end{aligned}$$

Again, using the Laplace transform (cf. Section 5.3), it follows that $B^\gamma f = {}_0I_x^\gamma f$ and $D(B^\gamma) = \{f \in L^2(0,\infty);\ {}_0I_x^\gamma f \in L^2(0,\infty)\}$. This leads to the following application involving fractional integrals of the type ${}_0I_x^\gamma e^{\lambda x} f(x)$ with the exponential weight $e^{\lambda x}$, studied e.g. in [153, pp. 195–199].

Corollary 2.4. *Let $\gamma > 0$ and $f \in L^2(0,\infty)$. One has for $\lambda \to 0+$:*

a) $\left\| \frac{1}{\Gamma(\gamma)} \int_0^x t^{\gamma-1} e^{-\lambda t} f(x-t)\,dt \right\|_{L^2} = o(1)$ iff $f = 0$;

b) $\left\| \frac{1}{\Gamma(\gamma)} \int_0^x t^{\gamma-1} e^{-\lambda t} f(x-t)\,dt \right\|_{L^2} = \mathcal{O}(1)$

$\Leftrightarrow f \in D(B^\gamma) \Leftrightarrow {}_0I_x^\gamma f \in L^2(0,\infty)$.

Observe that $(1/\Gamma(\gamma)) \int_0^x t^{\gamma-1} e^{-\lambda t} f(x-t)\,dt = e^{-\lambda x}\ {}_0I_x^\gamma e^{\lambda x} f(x)$.

3 Liouville-Grünwald, Marchaud and Riesz Fractional Derivatives

3.1 Liouville-Grünwald derivatives and their chief properties

A. K. Grünwald[b] (1867) [66] and A.V. Letnikov (1868) developed an approach to fractional differentiation for which the definition of the fractional derivative $D^\alpha f(x)$ is the limit of a fractional difference quotient, thus

$$D^\alpha f(x) = \lim_{h\to 0} \frac{\Delta_h^\alpha f(x)}{h^\alpha}, \tag{3.1}$$

$\Delta_h^\alpha f(x)$ being the right-handed difference of fractional order

$$\Delta_h^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x-jh) \tag{3.2}$$

which coincides with the classical r-th right handed difference $\Delta_h^r f(x)$ for $\alpha = r \in \mathbb{N}$ (noting that $\binom{r}{j} = 0$ for $j \geq r+1$). While the arguments of Grünwald were rather formal, Letnikov's were rigorous. They showed in particular that $D^{-\alpha} f$ coincides with Liouville's expression (1.8) for $\alpha > 0$ and sufficiently good functions f. In fact, Liouville (1832) already formulated a definition of the form (3.1), however only for functions which are expressible as sums of exponentials. Although the foregoing approach was also developed further by E. L. Post (1930), it does not seem to have received general attention until the work of Westphal (1974) and Butzer and Westphal (1975) (see however also the earlier work of K. F. Moppert (1953), N. Stuloff (1951), and M. Mikolás (1963) — see [120] 1975). Observe that the difference (3.2) is generally not defined for $\alpha < 0$. Thus for the example $f(x) \equiv 1$, the sum

$$\sum_{j=0}^{r} (-1)^j \binom{\alpha}{j} = \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(r+1-\alpha)}{\Gamma(r+1)} \tag{3.3}$$

diverges for $r \to \infty$ if $\alpha < 0$. In this section we confine the matter to periodic functions, treating definition (3.1) in the norm-topologies of $C_{2\pi}$ and $L_{2\pi}^p$, $1 \leq p < \infty$. (Note that the series (3.2) always converges with respect to these norms.) Thus this approach is a global and not a pointwise one; it enables one

[b]According to Lavoie-Osler-Tremblay [101], the Grünwald approach is "the most difficult, yet in some ways the most natural approach to a representation for fractional differentiation". This approach is carried out in the frame of 2π-periodic functions below. The Butzer-Westphal treatment [30] via fractional differences was also incorporated into Samko et al [153, pp. 371–381].

to present a fully rigorous and thorough approach to fractional calculus using elementary means of Fourier analysis. It extends the matter in the integral case (i. e., $\alpha = r$) as considered in [25] (Chapter 10); see also [11] (1961).

The background to this approach is best seen by transfering definition (3.1) into the setting of Fourier expansions of 2π-periodic functions. If

$$f(x) \sim \sum_{k=-\infty}^{\infty} f^\wedge(k) e^{ikx},$$

$$f^\wedge(k) = [f(\cdot)]^\wedge(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iku} f(u)\, du \qquad (k \in \mathbb{Z})$$

defining the finite Fourier transform of f (or Fourier-coefficients of f), one is led to introduce the fractional derivative $D^\alpha f$ for $f \in C_{2\pi}$ or $L^p_{2\pi}$ by

$$D^\alpha f(x) \sim \sum_{k=-\infty}^{\infty} f^\wedge(k)(ik)^\alpha e^{ikx} \tag{3.4}$$

(noting that $(d/dx)^r e^{ikx} = (ik)^r e^{ikx}$ for $\alpha = r$) or, equivalently, by

$$[D^\alpha f(\cdot)]^\wedge(k) = (ik)^\alpha f^\wedge(k) \qquad (k \in \mathbb{Z}). \tag{3.5}$$

To connect definition (3.1) with this set-up, instead of inserting the factor $(ik)^\alpha$ one could also take an approximation for it, i. e. ,

$$(ik)^\alpha = \lim_{h\to 0} \left(\frac{1 - e^{-ikh}}{h}\right)^\alpha, \tag{3.6}$$

suggesting as a definition

$$D^\alpha f(x) \sim \lim_{h\to 0} \sum_{k=-\infty}^{\infty} \left(\frac{1 - e^{-ikh}}{ikh}\right)^\alpha f^\wedge(k)(ik)^\alpha e^{ikx}. \tag{3.7}$$

This is indeed reasonable since (see [30, p. 126])

$$[\Delta_h^\alpha f(\cdot)]^\wedge(k) = (1 - e^{-ikh})^\alpha f^\wedge(k) \quad (k \in \mathbb{Z}). \tag{3.8}$$

Returning to the classical instance $\alpha = r \in \mathbb{N}$, it was Riemann who (in the case $r = 2$) first defined $D^r f$ by

$$\lim_{h\to 0} \frac{\Delta_h^r f(x)}{h^r} = D^r f(x)$$

in the pointwise sense. If the r-th ordinary derivative, defined recursively by $f^{(r)}(x) = \lim_{h\to 0}[f^{(r-1)}(x+h) - f^{(r-1)}(x)]/h$, exists at a point $x = x_0$, so does the pointwise (Riemann) derivative $D^r f$ and both are equal. But the converse is not necessarily valid. However, the existence of $D^r f$ in the uniform norm is equivalent to the fact that the r-th ordinary derivative $f^{(r)}(x)$ exists for *all* x and is continuous. A corresponding result holds with respect to the $L^p_{2\pi}$-norm. This again suggests that the fractional derivative $D^\alpha f$ can be handled by considering the limit in norm. If this is the case, then

$$\lim_{h\to 0} h^{-\alpha}[\Delta_h^\alpha f(\cdot)]^\wedge(k) = [D^\alpha f(\cdot)]^\wedge(k) \qquad (k \in \mathbb{Z}),$$

or $(ik)^\alpha f^\wedge(k) = [D^\alpha f(\cdot)]^\wedge(k)$ by (3.8) and (3.6), giving the connection with (3.5).

Thus, the α-th strong Liouville-Grünwald derivative $D^\alpha f$ of f will be that function g in $C_{2\pi}$ or $L^p_{2\pi}$, $1 \le p < \infty$, respectively, for which the limit

$$\lim_{h\to 0+} \|h^{-\alpha}\Delta_h^\alpha f - g\| = 0 \tag{3.9}$$

exists. For simplicity we shall briefly write $D^\alpha f \in C_{2\pi}$ or $D^\alpha f \in L^p_{2\pi}$, respectively.

To show how definition (3.9) is correctly related to formula (3.4) we shall prove that $((1-e^{-ikh})/(ikh))^\alpha$ is the finite Fourier transform of some function $\chi_\alpha(x;h)$ which is 2π-periodic, integrable, and behaves properly as h approaches zero. These results are stated in Props. 3.1 through 3.3 below.

As to the fractional integral, since an r-fold indefinite integration of e^{ikx} yields $e^{ikx}/(ik)^r$, one is led to introduce it by, following H. Weyl [177],

$$I^\alpha f(x) \sim \sum_{k=-\infty}^{\infty}{}' f^\wedge(k)e^{ikx}/(ik)^\alpha \qquad (\alpha > 0), \tag{3.10}$$

(the dash indicating that the term $k = 0$ is omitted), or equivalently by

$$[I^\alpha f(\cdot)]^\wedge(k) = \begin{cases} (ik)^{-\alpha} f^\wedge(k), & k = \pm 1, \pm 2, \dots \\ 0, & k = 0. \end{cases} \tag{3.11}$$

To give the formal definition (3.10) a more correct interpretation, one customarily defines for $f \in L^p_{2\pi}, 1 \le p < \infty$ (or $C_{2\pi}$) and $\alpha > 0$,

$$I^\alpha f(x) = (f * \psi_\alpha)(x) := \frac{1}{2\pi}\int_0^{2\pi} f(x-u)\psi_\alpha(u)\,du \tag{3.12}$$

$$\psi_\alpha(x) := \sum_{k=-\infty}^{\infty}{}' \frac{e^{ikx}}{(ik)^\alpha} = 2\sum_{k=1}^{\infty} \frac{\cos(kx - \alpha\pi/2)}{k^\alpha} \tag{3.13}$$

the series being convergent for all $x \in (0, 2\pi)$, $\alpha > 0$, uniformly so on $\epsilon \le x \le 2\pi - \epsilon$, $\epsilon > 0$, so that it represents the Fourier series of ψ_α. Thus

$$[\psi_\alpha(\cdot)]^\wedge(k) = \begin{cases} (ik)^{-\alpha}, & k = \pm1, \pm2, \dots \\ 0, & k = 0, \end{cases}$$

where $(ik)^\alpha = |k|^\alpha \exp(\frac{\alpha\pi i}{2} \text{sign } k)$ for $k \in \mathbb{Z}$.

Hence the convolution and uniqueness theorems give that definition (3.12) is consistent with (3.10) for any $\alpha > 0$, noting again that (3.11) holds. Since $\psi_\alpha \in L^1_{2\pi}$ (see [25, p. 426]), $I^\alpha f$ belongs to $L^p_{2\pi}$ or $C_{2\pi}$ if f does so, since the right side of (3.12), which is called a *Weyl*[c] *fractional integral* of order α, is a convolution product.

3.2 A crucial proposition; basic theorems

The function $\chi_\alpha(x; h)$ mentioned after Definition (3.9) will depend upon a basic function, namely

$$p_\alpha(x) := \Delta_1^\alpha k_\alpha(x) = \begin{cases} [\Gamma(\alpha)]^{-1} \sum_{0 \le j < x} (-1)^j \binom{\alpha}{j} (x-j)^{\alpha-1}, & 0 < x < \infty \\ 0, & -\infty < x < 0, \end{cases} \tag{3.14}$$

where $k_\alpha(x) = [\Gamma(\alpha)]^{-1} x_+^{\alpha-1}$. It will be needed several times in this chapter. Some of its properties are collected in the following proposition, $\mathcal{F}[p_\alpha(\cdot)](v)$ denoting its Fourier transform $\int_{-\infty}^{\infty} e^{-ivu} p_\alpha(u)\, du$, $v \in \mathbb{R}$ (see Section 5.1). For references and proof see [172].

Proposition 3.1. *The function*[d] $p_\alpha(x)$ *has for* $\alpha > 0$ *the properties*

a) $p_\alpha \in L^1(\mathbb{R})$, $\int_{-\infty}^{\infty} p_\alpha(u)\, du = 1$;

b) $\mathcal{F}[p_\alpha(\cdot)](v) = \begin{cases} (iv)^{-\alpha}(1 - e^{-iv})^\alpha, & v \ne 0 \\ 1, & v = 0; \end{cases}$

c) $p_\alpha(x) \equiv 0$ *for* $x > \alpha$ *in the cases* $\alpha = 1, 2, \dots$.

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[c]In fact, Weyl only considered fractional integration in this form.

[d]The most difficult part of this proposition is the proof of p_α belonging to $L^1(\mathbb{R})$, a fact treated in Westphal [172] (see Section 3.2 of [175]).

If $\alpha > 1$, then

d) $p_\alpha(x) = \mathcal{O}(x^{-\alpha-1}) \qquad (x \to \infty)$,

e) p'_α *and* xp'_α *belong to* $L^1(\mathbb{R})$.

Now the function $\chi_\alpha(x;h)$ is defined by

$$\chi_\alpha(x;h) = \frac{2\pi}{h} \sum_{-(x/2\pi)<j<\infty} p_\alpha\left(\frac{x+2\pi j}{h}\right) \qquad (h > 0). \tag{3.15}$$

It has the properties, established in the following crucial proposition.

Proposition 3.2. *For $\alpha > 0$ and $h > 0$ one has*

a) $\chi_\alpha(\cdot;h) \in L^1_{2\pi}, \int_0^{2\pi} \chi_\alpha(u;h)\,du = 2\pi$;

b) $\|\chi_\alpha(\cdot;h)\|_{L^1_{2\pi}} \le M$, *uniformly in h;*

c) $\lim_{h\to 0+} \int_\delta^\pi |\chi_\alpha(u;h)|\,du = 0$ *for* $\delta > 0$;

d) $[\chi_\alpha(\cdot;h)]^\wedge(k) = \begin{cases} (ikh)^{-\alpha}(1-e^{-ikh})^\alpha, & k \neq 0 \\ 1, & k = 0; \end{cases}$

e) $\chi_\alpha(x;h) = \frac{\Delta_h^\alpha \psi_\alpha(x)}{h^\alpha} + 1$.

Relation e) clarifies the need for $\chi_\alpha(x;h)$. Now the latter function is a so called approximate identity. This follows especially from

Proposition 3.3. *If f belongs to $L^p_{2\pi}$, $1 \le p < \infty$ or to $C_{2\pi}$, then for any $\alpha > 0$,*

$$\lim_{h\to 0+} \|J_{h,\alpha} f - f\| = 0,$$

the norm being taken with respect to $L^p_{2\pi}$ or $C_{2\pi}$, where

$$J_{h,\alpha} f(x) := \frac{1}{2\pi} \int_0^{2\pi} \chi_\alpha(x-u;h) f(u)\,du \quad (h > 0, x \in \mathbb{R}). \tag{3.16}$$

Now to the basic results, including the monotonicity and additivity laws, and the fundamental theorem of the fractional calculus. Part a) expresses definition (3.9) in terms of the finite Fourier transform.

Theorem 3.4. *a) The following three assertions are equivalent for $f \in L^p_{2\pi}$, $1 \le p < \infty$ and $\alpha > 0$:*

(i) $D^\alpha f \in L^p_{2\pi}$;

(ii) $\exists g \in L^p_{2\pi} : (ik)^\alpha f^\wedge(k) = g^\wedge(k),\ k \in \mathbb{Z}$;

(iii) $\exists g \in L^p_{2\pi} : f(x) - f^\wedge(0) = I^\alpha g(x)$ *a. e.*

If (i) or (ii) are satisfied, then $D^\alpha f = g$.

b) There holds for $f \in L^p_{2\pi}, \alpha, \beta > 0$:

(i) (Monotonicity law)
If $D^\alpha f \in L^p_{2\pi}$, *then* $D^\beta f \in L^p_{2\pi}$ *for any* $0 < \beta < \alpha$;

(ii) (Additivity law)
$D^\alpha D^\beta f = D^{\alpha+\beta} f$ *whenever one of the two sides is meaningful;*

(iii) (Fundamental theorem of fractional calculus)

$$D^\alpha(I^\alpha)f = f - f^\wedge(0) = I^\alpha(D^\alpha f), \tag{3.17}$$

the latter equality holding if $D^\alpha f \in L^p_{2\pi}$.

c) If $f \in L^p_{2\pi}$ *for* $1 < p < \infty$, *then* $D^\alpha f \in L^p_{2\pi}$ *if and only if*

$$\|\Delta^\alpha_h f\|_{L^p_{2\pi}} = \mathcal{O}(h^\alpha) \qquad (h > 0).$$

d) If $f \in L^1_{2\pi}$, *then the following statements are equivalent for* $\alpha > 0$:

(i) $\begin{cases} D^{\alpha-1} f \in BV_{2\pi} \cap L^1_{2\pi}, & \alpha > 1 \\ f \in BV_{2\pi}, & \alpha = 1 \\ I^{1-\alpha} f \in BV_{2\pi}, & \alpha < 1; \end{cases}$

(ii) $\exists \mu \in BV_{2\pi}$ *such that* $(ik)^\alpha f^\wedge(k) = \mu^\vee(k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iku} d\mu(u)$, $k \in \mathbb{Z}$

(iii) $\|\Delta^\alpha_h f\|_{L^1_{2\pi}} = \mathcal{O}(h^\alpha) \qquad (h > 0)$;

(iv) $\exists \mu \in BV_{2\pi}$ *such that for every* $s \in C_{2\pi}$

$$\lim_{h \to 0+} \int_0^{2\pi} s(u)[h^{-\alpha}\Delta^\alpha_h f(u)]\, du = \int_0^{2\pi} s(u)\, d\mu(u).$$

Above $BV_{2\pi}$ is the space of all functions μ which are of bounded variation on every finite interval, are normalized by $\mu(x) = [\mu(x+0) + \mu(x-0)]/2$, and satisfy $\mu(x+2\pi) = \mu(x) + \mu(2\pi) - \mu(0)$ for all x.

In connection with part b) (iii) let us remark that the formulae

$$h^{-\alpha}\Delta_h^\alpha(I^\alpha f) = J_{h,\alpha}f - f^\wedge(0) = I^\alpha(h^{-\alpha}\Delta_h^\alpha f),$$

being valid for $f \in L^p_{2\pi}$, $1 \le p < \infty$, may be regarded as precursors to (3.17). Theorem 3.4 b)(iii) is actually the counterpart of the fundamental theorem of the differential and integral calculus for the fractional case, in the sense that D^α and I^α, introduced independently of each other in this section, are precisely inverse operators provided $f^\wedge(0) = 0$. Therefore, it is justified to set $D^\alpha = I^{-\alpha}$ if $\alpha < 0$. Then one has as a generalization of Part b)(ii) that if α, β are any reals, $f^\wedge(0) = 0$ and $D^\alpha D^\beta f$ exists, then $D^\alpha D^\beta f = D^{\alpha+\beta} f$.

The foregoing results were considered for $L^p_{2\pi}$-functions. Their chief counterparts are now stated for $f \in C_{2\pi}$ all together in one theorem.

Theorem 3.5. *a) If $f \in C_{2\pi}$, then $D^\alpha f \in C_{2\pi}$ if and only if $\exists g \in C_{2\pi} : (ik)^\alpha f^\wedge(k) = g^\wedge(k)$, $k \in \mathbb{Z}$.*

b) The following statements are equivalent for $f \in C_{2\pi}$, $\alpha > 0$:

(i) $\exists g \in L^\infty_{2\pi} : (ik)^\alpha f^\wedge(k) = g^\wedge(k), \quad k \in \mathbb{Z}$,

(ii) $\exists g \in L^\infty_{2\pi} : f(x) - f^\wedge(0) = I^\alpha g(x)$ *for all* x,

(iii) $\|\Delta_h^\alpha f\|_{C_{2\pi}} = \mathcal{O}(h^\alpha), \quad (h \to 0+)$,

(iv) $\exists g \in L^\infty_{2\pi}$ *such that for every* $s \in L^1_{2\pi}$

$$\lim_{h\to 0+} \int_0^{2\pi} s(u)[h^{-\alpha}\Delta_h^\alpha f(u) - g(u)]\,du = 0,$$

i.e., $D^\alpha f = g$ exists in this weak-sense,*

(v) $I^{n-\alpha} f \in AC^{n-1}_{2\pi}$ *and* $(d/dx)^n I^{n-\alpha} f \in L^\infty_{2\pi}$, *with* $n = [\alpha] + 1$.

3.3 The point-wise Liouville-Grünwald fractional derivative

The conventional way to define a fractional derivative is via point-wise convergence. The foregoing norm-convergence approach is especially reasonable in view of the remark to part b) (iii) of Theorem 3.4. The connection of Definition (3.9) with a certain point-wise version of it, at least for $\alpha \ge 1$, is given by

Theorem 3.6. *a) If $f \in L^1_{2\pi}$ and $\alpha > 1$, then for almost all x*

$$\lim_{h\to 0+} J_{h.\alpha} f(x) = f(x).$$

b) If $f \in L^p_{2\pi}$, $1 \le p < \infty$, $\alpha \ge 1$, then $D^\alpha f \in L^p_{2\pi}$ iff there exists a $g \in L^1_{2\pi}$ such that $\Delta^\alpha_h f(x) = (h^\alpha/2\pi)\int_0^{2\pi} \chi_\alpha(x-u;h)g(u)\,du$ for all $x \in \mathbb{R}$ and $h \ge 0$, and the pointwise derivative $D^\alpha f$ belongs to $L^p_{2\pi}$.

The Grünwald-Liouville approach via fractional differences in the non-periodic case was considered by Samko [151]; see Samko et al [153, pp. 382–85].

3.4 Extensions and applications of the Liouville-Grünwald calculus

The foregoing rigorous and modern approach to the Liouville-Grünwald fractional calculus was further developed by Butzer, Dyckhoff, Görlich and Stens [13], Wilmes [178], [179], Taberski [164,165], Diaz and Osler [39], V.G Ponomarenko [138], D.P. Driavnov [44]. In particular, Butzer et al [13] defined moduli of continuity of fractional order, defined generalized Lipschitz spaces via these moduli and applied the matter to the theory of best approximation by *trigonometric* polynomials.

Basic in this respect is a *Bernstein-type inequality* for trigonometric polynomials in the fractional case. If $t_n(x)$ is a trigonometric polynomial of degree $n \in \mathbb{N}$, then it reads that

$$\|D^\alpha t_n\|_{X_{2\pi}} \le B(\alpha) n^\alpha \|t_n\|_{X_{2\pi}} \quad (\alpha > 0). \tag{3.18}$$

Needed in the proofs is further a *Jackson-type inequality* (cf. [24]) in the fractional case. It states: If $D^\alpha f \in X_{2\pi}$, then

$$E_n(f; X_{2\pi}) := \inf_{t_n} \|f - t_n\|_{X_{2\pi}} \le J(\alpha) n^{-\alpha} \|D^\alpha f\|_{X_{2\pi}}.$$

A certain generalization of inequality (3.18), is a fractional version of the so-called *M. Riesz inequality.* It states that (see Wilmes [178], [179])

$$\|D^\alpha t_n\|_{X_{2\pi}} \le (n/2)^\alpha \|\Delta^\alpha_{\pi/n} t_n\|_{X_{2\pi}}.$$

Above, $B(\alpha)$ and $J(\alpha)$ are constants depending on $\alpha > 0$.

Left-handed fractional order differences (with step 1) in the complex plane were considered by Diaz and Osler [39] in the form

$$\Delta^\alpha_1 f(z) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(z+\alpha-j), \quad z \in \mathbb{C},$$

with the main result

$$\Delta_1^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_C \frac{f(u)\Gamma(u-z-\alpha)}{\Gamma(u-z+1)}du,$$

where the contour C envelopes the ray $L = \{u; u = z+\alpha-\xi,\ \xi \geq 0\}$ in the positive sense. Here f is assumed to be analytic in a domain containing L such that $|f(z)| \leq M|(-z)^{\alpha-p}|$ for some positive constants M and p. They also obtained a Leibniz formula for these differences.

The idea of fractional differentiation by Liouville-Grünwald allows other wide reaching generalizations as Samko et al [153, p. 446] report. Replacing the translation operator $(\tau_h f)(x) = f(x-h)$ in the definition (3.2), which can be rewritten as $\Delta_h^\alpha f(x) = (I-\tau_h)^\alpha f(x)$, by other generalized translation operators, one can obtain various forms of fractional differentiation. This idea was carried out by Butzer and Stens [26,27] for the so-called Chebyshev translation operator $\overline{\tau}_h$, defined by

$$\overline{\tau}_h f(x) = \frac{1}{2}[f(xh+\sqrt{(1-x^2)(1-h^2)}) + f(xh-\sqrt{(1-x^2)(1-h^2)})]$$

for $x, h \in [-1,1]$ with the associated fractional derivative

$$D^{(\alpha)}f = \lim_{h\to 1-}\frac{\overline{\Delta}_h^\alpha f(x)}{(1-h)^\alpha}, \quad \overline{\Delta}_h^\alpha f(x) := (-1)^{[\alpha]}\sum_{j=0}^\infty (-1)^j\binom{\alpha}{j}\overline{\tau}_h^j f(x).$$

The limit here is taken in the norm of the space $X = C[-1,1]$ or $X = L_w^p$ with norm $\|f\|_p = \{(1/\pi)\int_{-1}^1 |f(u)|^p w(u)du\}^{1/p}$, $w(x) = (1-x^2)^{-1/2}$. For a pointwise interpretation of $D^{(1)}$ in terms of ordinary derivatives, $D^{(1)}f(x) = (1-x^2)f''(x) - xf'(x)$; further $D^{(1/2)}f(x) = -\sqrt{1-x^2}(d/dx)(Hf)(x)$, the Hilbert transform of (3.36) now being defined by

$$(Hf)(x) = \lim_{r\to 1-} 2\sum_{k=1}^\infty r^k f^\wedge(k)\sin(k\arccos x),$$

where $f^\wedge(k) = (1/\pi)\int_{-1}^1 f(u)\cos(k\arccos u)w(u)du$ is the Chebyshev transform of $f \in X$. The Chebyshev integral $I^{(\alpha)}f$ of order $\alpha > 0$ is defined by the convolution product $(I^{(\alpha)}f)(x) = (1/\pi)\int_{-1}^1(\overline{\tau}_x f)(u)\psi_\alpha(u)w(u)du$, where $\psi_\alpha^\wedge(k) = (-1)^{[\alpha]}k^{-2\alpha}$, $k \in \mathbb{N}$. The basic theorem here is: *If* $f \in X$, *then* (i) $D^{(\alpha)}f \in X$ *iff* (ii) $\exists g_0 \in X$ *with* $(-1)^{[\alpha]}k^{2\alpha}f^\wedge(k) = g_0^\wedge(k)$, $k \in \mathbb{N}$, *iff* (iii) $\exists g_1 \in X$ *with* $f(x) = \left(I^{(\alpha)}g_1\right)(x) +$ const. (a.e.). *Then* $D^{(\alpha)}f(x) = g_i(x) - g_i^\wedge(0)$,

$i = 0, 1$. The fundamental theorem for fractional Chebyshev derivatives and integrals then reads: $D^{(\alpha)}(I^{(\alpha)}f) = f - f^{\wedge}(0) = I^{(\alpha)}(D^{(\alpha)}f)$, the latter equality holding if $D^{(\alpha)}f \in X$. This calculus enables one to present a simple and unified approach to the theory of best approximation by *algebraic* polynomials, which also covers the delicate behaviour on the boundary of the interval $[-1, 1]$.

Discrete fractional differences $\Delta^\alpha u_k := \sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}u_{k+j}$ for sequences $\{u_k\}_{k\geq 0}$ of real numbers, introduced by A.F. Andersen (1928), the discrete fractional bounded variation spaces

$$bv_{\alpha+1} = \{\{u_k\} \in l^\infty; \quad \sum_{j=0}^{\infty}\binom{j+\alpha}{j}|\Delta^{\alpha+1}u_j| + \lim_{j\to\infty}|u_j| < \infty\}$$

and their continuous fractional counterparts $BV_\alpha(\mathbb{R}^+)$, introduced by W. Trebels [166], play important roles in various investigations of fractional differentiation in connection with Fourier- multiplier problems, of the Hankel, Jacobi-types, etc. In W. Trebels [166], G. Gasper-W. Trebels [57], [58] and H.J. Mertens-R.J. Nessel-G. Wilmes [117] also the Cossar derivative is applied.

D. Elliott [48] and D. Delbourgo-D. Elliott [38] treat algorithms for the numerical evaluation of certain Hadamard finite-part integrals (which are indeed fractional derivatives) by making use of the Grünwald fractional differences. The algorithms in question are based on the use of Bernstein polynomials; convergence rates are included.

3.5 The Marchaud fractional derivative

Marchaud's idea (1927) [111] was to try to define a fractional derivative of order α directly via ${}_{-\infty}I_x^\alpha f(x)$ of (1.8), replacing α by $-\alpha$. This suggests defining

$$I^{-\alpha}f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty u^{-\alpha-1}f(x-u)\,du \quad (\alpha > 0). \tag{3.19}$$

However, no matter how smooth f might be, the latter integral diverges due to the singularity of $u^{-\alpha-1}$ at the origin. Hence, following J. Hadamard (1892), first considering $0 < \alpha < 1$, one takes the "finite part" of (3.19) in the sense that one subtracts from (3.19) that part which makes it diverge, namely the term $\int_\epsilon^\infty u^{-\alpha-1}f(x)\,du = \alpha^{-1}\epsilon^{-\alpha}f(x)$ with $\epsilon \to 0+$. This yields the correct interpretation of (3.19), namely

$$\lim_{\epsilon\to 0+}\frac{1}{\Gamma(-\alpha)}\int\limits_{\epsilon}^{\infty} u^{-\alpha-1}[f(x-u)-f(x)]\,du. \tag{3.20}$$

It will include the definition given by (1.21). Indeed, by partial integration one has (formally), noting $\Gamma(1-\alpha) = -\alpha\Gamma(-\alpha)$, that the limit expression (3.20) equals

$$\lim_{\epsilon\to 0+}\frac{1}{\Gamma(1-\alpha)}\int\limits_{\epsilon}^{\infty} u^{-\alpha}f'(x-u)\,du = {}_{-\infty}I_x^{1-\alpha}f'(x).$$

In order to extend equation (3.20) to the case $\alpha > 1$, one tries to replace the first difference there by a difference of higher order l of f, with step u, i. e. ,

$$\Delta_u^l f(x) := \sum_{j=0}^{l}(-1)^j\binom{l}{j}f(x-ju). \tag{3.21}$$

In fact, the *Marchaud fractional derivative* of order α, $0<\alpha<l$, (Samko et al [153, p. 116 ff.]), reads

$$D^\alpha f(x) = \frac{1}{C_{\alpha,l}}\int\limits_0^\infty \frac{\Delta_u^l f(x)}{u^{1+\alpha}}\,du \tag{3.22}$$

provided f is sufficiently smooth, where

$$C_{\alpha,l} = \int\limits_0^\infty \frac{(1-e^{-u})^l}{u^{1+\alpha}}\,du \tag{3.23}$$

having the representation

$$C_{\alpha,l} := \begin{cases} \Gamma(-\alpha)\sum_{j=1}^{l}(-1)^j\binom{l}{j}j^\alpha & (0<\alpha<l, l\notin\mathbb{N}) \\ \frac{(-1)^{\alpha+1}}{\alpha!}\sum_{j=1}^{l}(-1)^j\binom{l}{j}j^\alpha\log j & (\alpha = 1,2,\dots,l-1). \end{cases} \tag{3.24}$$

To see that (3.22) is independent of l whenever $l > \alpha$, consider only the case $0 < \alpha < 1$. Then for any $l \in \mathbb{N}$,

$$\int_{\epsilon}^{\infty} \frac{\Delta_u^l f(x)}{u^{1+\alpha}}\,du = \frac{f(x)}{\alpha\epsilon^{\alpha}} + \sum_{j=1}^{l}(-1)^j\binom{l}{j}j^{\alpha}\int_{j\epsilon}^{\infty}\frac{f(x-u)}{u^{1+\alpha}}\,du \tag{3.25}$$

$$= \sum_{j=1}^{l}(-1)^j\binom{l}{j}j^{\alpha}\int_{j\epsilon}^{\infty}\frac{f(x-u)-f(x)}{u^{1+\alpha}}\,du$$

which yields the result as $\epsilon \to 0+$, noting (3.24). It should be observed that the functions $C_{\alpha,l}$ of (3.23) are practically the Stirling functions $S(\alpha, l)$ of Section 4. 3. In fact, $S(\alpha,l) = (-1)^l C_{\alpha,l}[l!\Gamma(-\alpha)]^{-1}$.

Usually, definition (3.22) can be interpreted in a more general sense in order to include larger classes of functions f for which a Marchaud derivative is defined. In fact, introducing the truncated integrals

$$D_{\varepsilon}^{\alpha,l}f(x) = \frac{1}{C_{\alpha,l}}\int_{\varepsilon}^{\infty}\frac{\Delta_u^l f(x)}{u^{1+\alpha}}\,du \quad (\varepsilon > 0), \tag{3.26}$$

their limit as $\varepsilon \to 0+$ may be regarded with respect to different types of convergence depending on the problems under consideration. We shall treat the case where convergence takes place in the norm of L^p, $1 \le p < \infty$, for functions f defined on the whole real line or on the half-axis $\mathbb{R}_+$. Though the latter class of functions may and will be regarded as a subclass of the former by extending f to be zero on the negative real axis, nicer and more well-rounded results may be obtained for functions defined on $\mathbb{R}_+$. Moreover, the Laplace transform is an effective tool in this case, which shall be considered briefly first. The treatment below is based on [6,7] and [173, I].

If $f \in L_{\text{loc}}(\mathbb{R}_+)$, its Laplace transform is defined by

$$\mathcal{L}[f](s) = \int_0^{\infty} e^{-su}f(u)du$$

for each complex number s for which the integral exists (see Section 5.3). Usually it suffices to confine oneself to real s. The following lemma shows how the Laplace transform acts on finite differences (3.21) and truncated integrals (3.26).

Lemma 3.7. *Let $f \in L_{\text{loc}}(\mathbb{R}_+)$ such that the Laplace transform of f is absolutely convergent for each $s > 0$. Then for $0 < \alpha < l$ and $s, t, \varepsilon > 0$,* *(i)* $\mathcal{L}[t^{-\alpha}\Delta_t^l f](s) = s^\alpha \mathcal{L}[f](s) \cdot \mathcal{L}[p_{\alpha,l}](ts)$,

(ii) $\mathcal{L}[D_\varepsilon^{\alpha,l} f](s) = s^\alpha \mathcal{L}[f](s) \cdot \mathcal{L}[q_{\alpha,l}](\varepsilon s)$,

where the functions $p_{\alpha,l}$ *and* $q_{\alpha,l}$ *are defined by*

$$p_{\alpha,l}(u) = \Delta_1^l \frac{1}{\Gamma(\alpha)} u_+^{\alpha-1} = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{l} (-1)^j \binom{l}{j} (u-j)_+^{\alpha-1}$$

for $0 < \alpha \leq l$ *and*

$$q_{\alpha,l}(u) = D_1^{\alpha,l} \frac{1}{\Gamma(\alpha)} u_+^{\alpha-1} = \frac{1}{C_{\alpha,l}\Gamma(\alpha+1)u} \sum_{j=0}^{l} (-1)^j \binom{l}{j} (u-j)_+^{\alpha}$$

for $0 < \alpha < l$. *Both functions belong to* $L^1(\mathbb{R}_+)$ *and satisfy*

$$\int_0^\infty p_{\alpha,l}(u)du = \begin{cases} 0 & \text{if} \quad 0 < \alpha < l \\ 1 & \phantom{\text{if}} \quad \alpha = l \end{cases} \quad \text{and} \quad \int_0^\infty q_{\alpha,l}(u)du = 1.$$

Their Laplace transforms are given by

$$\mathcal{L}[p_{\alpha,l}](s) = s^{-\alpha}(1-e^{-s})^l \quad (s > 0)$$

and, respectively,

$$\mathcal{L}[q_{\alpha,l}](s) = \frac{s^{-\alpha}}{C_{\alpha,l}} \int_1^\infty u^{-\alpha-1}(1-e^{-su})^l du \quad (s > 0).$$

Part (ii) of Lemma 3.7 implies the following characterization of the Marchaud fractional derivative (see [6], [7]).

Theorem 3.8. *Let* $f \in L_{\mathrm{loc}}(\mathbb{R}_+)$ *such that the Laplace transform of* f *is absolutely convergent for each* $s > 0$, *and let* $g \in L^p(\mathbb{R}_+)$, $1 \leq p < \infty$. *Then the following are equivalent for* $0 < \alpha < l$*:*

(i) $s^\alpha \mathcal{L}[f](s) = \mathcal{L}[g](s) \quad (s > 0)$,

(ii) ${}_0I_x^\alpha g = f$,

(iii) $D^\alpha f = g$, *i.e.*,
for each $\varepsilon > 0$, $D_\varepsilon^{\alpha,l} f \in L^p(\mathbb{R}_+)$ *and* $\lim_{\varepsilon\to 0+} \|D_\varepsilon^{\alpha,l} f - g\|_{L^p(\mathbb{R}_+)} = 0$.

Let us remark that the equivalence of (ii) and (iii) states that ${}_0I_x^\alpha$ and D^α are inverse operations. The implication (ii) $\Longrightarrow$ (iii) may be interpreted in the sense that if $g \in L^p(\mathbb{R}_+)$, then the integral equation ${}_0I_x^\alpha g = f$ has a solution which can be represented by the Marchaud fractional derivative $D^\alpha f$. As for the analogue of this result for functions defined on the whole real axis, note that $g \in L^p(\mathbb{R})$ implies the existence of the fractional integral ${}_{-\infty}I_x^\alpha g$, in general, only if $1 \le p < 1/\alpha$. Otherwise one has to assume it expressly; cf. Theorem 3.10 below.

Also note that in the above theorem the function f itself need not belong to $L^p(\mathbb{R}_+)$. If, however, one is interested in a full theory for Marchaud fractional derivatives including, for example, the monotonicity and additivity laws, then it is appropriate to take a fixed L^p-space as a basis which implies that fractional derivatives are explained a priori only for functions belonging to the underlying space. Thus, considering now functions on the whole real line, we assume that the operator D^α has the domain

$$D(D^\alpha) := \{f \in L^p(\mathbb{R});\ \exists g \in L^p(\mathbb{R}) \text{ such that } \lim_{\varepsilon\to 0+} \|D_\varepsilon^{\alpha,l} f - g\|_{L^p(\mathbb{R})} = 0\}, \tag{3.27}$$

and for g as in (3.27) we set $D^\alpha f := g$.

Then, as a counterpart to Lemma 3.7, the following formulas may be regarded as representative for the whole theory: *For each $f \in L^p(\mathbb{R})$ and $0 < \alpha < l$ the convolution integrals*

$$(f * \frac{1}{t} p_{\alpha,l}(\frac{\cdot}{t}))(x) := \int_0^\infty f(x-u) p_{\alpha,l}(\frac{u}{t}) \frac{du}{t} \quad (t > 0)$$

and

$$(f * \frac{1}{\varepsilon} q_{\alpha,l}(\frac{\cdot}{t}))(x) := \int_0^\infty f(x-u) q_{\alpha,l}(\frac{u}{\varepsilon}) \frac{du}{\varepsilon} \quad (\varepsilon > 0)$$

belong to $D(D^\alpha)$ and satisfy

$$\begin{aligned} t^{-\alpha}\Delta_t^l f &= D^\alpha(f * \frac{1}{t} p_{\alpha,l}(\frac{\cdot}{t})), \\ D_\varepsilon^{\alpha,l} f &= D^\alpha(f * \frac{1}{\varepsilon} q_{\alpha,l}(\frac{\cdot}{\varepsilon})). \end{aligned}$$

Moreover, if $f \in D(D^\alpha)$, then

$$t^{-\alpha}\Delta_t^l f = D^\alpha f * \frac{1}{t} p_{\alpha,l}(\frac{\cdot}{t}), \tag{3.28}$$

$$D_\varepsilon^{\alpha,l} f = D^\alpha f * \frac{1}{\varepsilon} q_{\alpha,l}(\frac{\cdot}{\varepsilon}). \tag{3.29}$$

From these relations and others of similar type the following theorem is deduced. It is an application of results obtained in the more general framework of semigroups of operators; cf. [173, I], [176] and Chapter III of this volume.

Theorem 3.9. *a) For each $\alpha > 0$, D^α is a closed, densely defined linear operator in $L^p(\mathbb{R})$, $1 \le p < \infty$. There holds for $f \in L^p(\mathbb{R})$, $\alpha, \beta > 0$:*

(i) (Monotonicity law) If $f \in D(D^\alpha)$, then $f \in D(D^\beta)$ for any $0 < \beta < \alpha$ and

$$D^\beta f = \frac{1}{C_{\beta,l}} \int_0^\infty \frac{\Delta_u^l f}{u^{1+\beta}} du \quad (0 < \beta < l).$$

(ii) (Additivity law) There holds $f \in D(D^{\alpha+\beta})$ if and only if $f \in D(D^\beta)$ and $D^\beta f \in D(D^\alpha)$. In this case,

$$D^{\alpha+\beta} f = D^\alpha D^\beta f.$$

b) If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then the following statements are equivalent for $\alpha > 0$:

(i) $f \in D(D^\alpha)$,

(ii) $\|D_\varepsilon^{\alpha,l} f\|_{L^p(\mathbb{R})} = \mathcal{O}(1) \quad (\varepsilon \to 0+)$, where $0 < \alpha < l$,

(iii) there exists $g \in L^p(\mathbb{R})$ such that

$$\lim_{\delta \to 0+} \left\| \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\delta u} u^{\alpha-1} g(\cdot - u) du - f(\cdot) \right\|_{L^p(\mathbb{R})} = 0.$$

In this event, $D^\alpha f = g$.

In Part b) (iii) of Theorem 3.9 the limit in L^p-norm of the integral

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\delta u} u^{\alpha-1} g(x-u) du$$

as $\delta \to 0+$ may be considered as a generalization of the fractional integral ${}_{-\infty}I_x^\alpha g(x)$ which need not exist, if $g \in L^p(\mathbb{R})$, as was mentioned above. In this wider sense, the fractional integral is the inverse operator of the fractional derivative defined by (3.27). This is the interpretation of the equivalence of (i) and (iii) in Theorem 3.9 b) above.

On the other hand, if for $g \in L^p(\mathbb{R})$ it is assumed that the integral ${}_{-\infty}I_x^\alpha g(x)$ does exist in the sense of Lebesgue for almost all real x, then for $0 < \alpha < l$ and $t > 0$ one has

$$t^{-\alpha}\Delta_t^l \; {}_{-\infty}I_x^\alpha g(x) = \int_0^\infty g(x-u)p_{\alpha,l}(\frac{u}{t})\frac{du}{t}$$

which implies, by Fubini's theorem, for $\varepsilon > 0$,

$$D_\varepsilon^{\alpha,l} \; {}_{-\infty}I_x^\alpha g(x) = \int_0^\infty g(x-u)q_{\alpha,l}(\frac{u}{\varepsilon})\frac{du}{\varepsilon}. \tag{3.30}$$

Note the similarity of these equations with formulas (3.28) and (3.29) above, which hold under different assumptions. The integral on the right-hand side of (3.30) belongs to $L^p(\mathbb{R})$ and converges to g in norm if $\varepsilon \to 0+$. Thus, there holds the following theorem which has its roots in the paper of Marchaud [111], cf. Samko [153, p. 125/6] and Rubin [147, p. 163].

Theorem 3.10. *Let $g \in L^p(\mathbb{R})$, $1 \le p < \infty$, and assume that the fractional integral $f(x) = {}_{-\infty}I_x^\alpha g(x)$ $(\alpha > 0)$ exists in the sense of Lebesgue for almost all real x. Then for $0 < \alpha < l$ and each $\varepsilon > 0$, $D_\varepsilon^{\alpha,l} f \in L^p(\mathbb{R})$ and*

$$\lim_{\varepsilon \to 0+} ||D_\varepsilon^{\alpha,l} f - g||_{L^p(\mathbb{R})} = 0.$$

3.6 Equivalence of the Weyl and Marchaud fractional derivatives

At first to the situation for $0 < \alpha < 1$. If $f \in L_{2\pi}^1$, the Weyl fractional derivative

$$D^\alpha f(x) := \frac{d}{dx} I^{1-\alpha} f(x) = \frac{d}{dx}\left(\frac{1}{2\pi}\int_0^{2\pi} f(x-u)\psi_{1-\alpha}(u)\,du\right) \tag{3.31}$$

may, under (formal) differentiation under the integral sign and partial integration, be written as

$$\frac{1}{2\pi}\int_0^{2\pi} [f(x-u) - f(x)]\,\psi_{1-\alpha}'(u)\,du; \tag{3.32}$$

Samko et al [153, p. 352, 109] then speak of the *Weyl–Marchaud* derivative.

Denoting the truncated version of (3.32) by $\overline{D_\epsilon^{\alpha,1}}f(x)$, i.e.,

$$\overline{D_\epsilon^{\alpha,1}}f(x) = \frac{1}{2\pi}\int_\epsilon^{2\pi}[f(x-u)-f(x)]\,\psi_{1-\alpha}'(u)\,du,$$

then it can be shown for $f \in L_{2\pi}^p$, $1 \le p < \infty$, that the two truncated fractional derivatives $D_\epsilon^{\alpha,1}f(x)$ (cf. (3.26)) and $\overline{D_\epsilon^{\alpha,1}}f(x)$ converge for almost all x, and in $L_{2\pi}^p$ simultaneously, and

$$D^\alpha f(x) = \lim_{\epsilon\to 0} D_\epsilon^{\alpha,1}f(x) = \lim_{\epsilon\to 0}\overline{D_\epsilon^{\alpha,1}}f(x) =: \overline{D^\alpha}f(x), \tag{3.33}$$

so that the Marchaud derivative for periodic functions coincides with that of Weyl (-Marchaud). Moreover the $L_{2\pi}^p$-convergence of the truncated Marchaud derivative, namely that $\exists g \in L_{2\pi}^p$ such that $\|D_\epsilon^{\alpha,1}f - g\|_{L_{2\pi}^p} \to 0$ for $\epsilon \to 0$, is equivalent to the representability of $f(x)$ as a Weyl fractional integral of some $g \in L_{2\pi}^p$, namely $f(x) = f^\wedge(0) + I^\alpha g(x)$. If $1 < p < \infty$, then this is also equivalent to $\|D_\epsilon^{\alpha,1}f\|_{L_{2\pi}^p} = \mathcal{O}(1)$, $\mathcal{O}$ being independent of ϵ.

Recalling Theorem 3.4, it is also clear that the strong Liouville- Grünwald derivative and the strong Marchaud derivative coincide for periodic functions in $L_{2\pi}^p$, $1 \le p < \infty$. The foregoing results may be extended from $0 < \alpha < 1$ to any $\alpha > 0$.

Marchaud's approach has proved to be very useful in building up integral representations of numerous operators in analysis and mathematical physics, in particular in finding explicit and approximate solutions to various integral equations, arising in mechanics, electrostatics, diffraction theory, and especially in constructing explicit inverses to various fractional integrals and operators of potential type. This matter is the main topic of B. Rubin's specialized volume [147]; see also the books by V.I. Fabrikant [53,54].

3.7 Riesz derivatives on $\mathbb{R}$

Let us here consider a modified type of fractional integrals which were introduced by M. Riesz (1949) [140]. If $0 < \alpha < 1$, the integral

$$R^\alpha f(x) = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi)/2}\int_{-\infty}^{\infty}\frac{f(u)}{|x-u|^{1-\alpha}}du \tag{3.34}$$

is called the *Riesz potential* of f of order α. With (3.34) there is associated the so-called *conjugate Riesz potential* of order α defined by

$$\widetilde{R}^\alpha f(x) = \frac{1}{2\Gamma(\alpha)\sin(\alpha\pi)/2} \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-u)}{|x-u|^{1-\alpha}} f(u)du. \tag{3.35}$$

Under appropriate assumptions on f, say, for instance, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the integrals $R^\alpha f(x)$ and $\widetilde{R}^\alpha f(x)$ exist almost everywhere and belong to $L^1_{loc}(\mathbb{R})$. They are related to each other via the Hilbert transform H, namely

$$\widetilde{R}^\alpha f = R^\alpha H f,$$

where H is defined by the Cauchy principal value

$$(Hf)(x) \equiv f^\sim(x) = \lim_{\delta\to 0+} \frac{1}{\pi} \int_{|x-u|\geq\delta} \frac{f(u)}{x-u} du \tag{3.36}$$

at every point x for which this limit exists.

Moreover, provided f behaves well enough, then the following additivity laws are satisfied: If $\alpha > 0$, $\beta > 0$ such that $\alpha + \beta < 1$, then

$$\begin{aligned} R^\alpha R^\beta f &= R^{\alpha+\beta} f, \\ \widetilde{R}^\alpha \widetilde{R}^\beta f &= -R^{\alpha+\beta} f. \end{aligned}$$

The Riesz potential as well as its conjugate counterpart can be extended to all complex α with $Re\ \alpha \geq 0$ by distributional methods. See Samko et al [153, §25.2], where the matter is treated even in the multidimensional case.

To find an appropriate Riesz derivative of fractional order, let us look very formally at the Fourier transforms of $R^\alpha f$ and $\widetilde{R}^\alpha f$. (For notation and some basic properties of the Fourier transform, compare Section 5.1.) Since for $0 < \alpha < 1$

$$\int_{-\infty}^{\infty} \frac{e^{ivu}}{|u|^{1-\alpha}} du = 2|v|^{-\alpha} \int_0^{\infty} \frac{\cos u}{u^{1-\alpha}} du = 2|v|^{-\alpha}\Gamma(\alpha)\cos\frac{\pi\alpha}{2},$$

we have for $v \neq 0$, by the Fourier transform of convolutions,

$$[R^\alpha f]^\wedge(v) = |v|^{-\alpha} f^\wedge(v), \quad [\widetilde{R}^\alpha f]^\wedge(v) = (-i\ \operatorname{sgn} v)|v|^{-\alpha} f^\wedge(v),$$

and by the Fourier transform of derivatives,

$$\left[\frac{d}{dx} R^{1-\alpha} f(x)\right]^\wedge (v) = (iv)|v|^{\alpha-1} f^\wedge(v) = (i\ \operatorname{sgn} v)|v|^\alpha f^\wedge(v),$$

$$\left[\frac{d}{dx}\widetilde{R}^{1-\alpha}f(x)\right]^{\wedge}(v) = (iv)(-i\ \operatorname{sgn} v)|v|^{\alpha-1}f^{\wedge}(v) = |v|^{\alpha}f^{\wedge}(v). \tag{3.37}$$

Thus, comparing the last four formulas, one is led to take $\frac{d}{dx}\widetilde{R}^{1-\alpha}f(x)$ as a candidate for the α-th Riesz derivative, as this expression inverts, at least formally, the Riesz potential $R^{\alpha}f(x)$, while $\frac{d}{dx}R^{1-\alpha}f(x)$ plays the corresponding role with respect to the conjugate Riesz potential $\widetilde{R}^{\alpha}f(x)$. We want to follow up only the first of these two cases; the other may be treated in an analogous manner. Writing $\frac{d}{dx}\widetilde{R}^{1-\alpha}f(x)$ as the limit

$$\lim_{h\to 0}\frac{\widetilde{R}^{1-\alpha}f(x+h)-\widetilde{R}^{1-\alpha}f(x)}{h},$$

we can represent the numerator of the above difference quotient as the convolution integral

$$(f * n_{h,1-\alpha})(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-u)n_{h,1-\alpha}(u)du, \tag{3.38}$$

where the kernel

$$n_{h,1-\alpha}(x) = \frac{\sqrt{2\pi}}{2\Gamma(1-\alpha)\cos(\alpha\pi)/2}\left\{\frac{\operatorname{sgn}(x+h)}{|x+h|^{\alpha}} - \frac{\operatorname{sgn} x}{|x|^{\alpha}}\right\}$$

belongs to $L^1(\mathbb{R})$ and has the Fourier transform

$$n^{\wedge}_{h,1-\alpha}(v) = (-i\ \operatorname{sgn} v)(e^{ihv}-1)|v|^{\alpha-1} \quad (v \neq 0).$$

Thus, for every $1 \le p < \infty$, the integral (3.38) exists as a function in $L^p(\mathbb{R})$, whenever $f \in L^p(\mathbb{R})$. This gives rise to the following definition of a fractional Riesz derivative in the sense of the norm topology of L^p-function spaces:
Let $f \in L^p(\mathbb{R})$, $1 \le p < \infty$. If $0 < \alpha < 1$ we define the α-th *strong Riesz derivative* $D^{\{\alpha\}}f$ of f by

$$\lim_{h\to 0}\|h^{-1}(f * n_{h,1-\alpha}) - D^{\{\alpha\}}f\|_p = 0 \tag{3.39}$$

whenever this limit exists. If $\alpha = 1$, (3.39) is replaced by

$$\lim_{h\to 0}\|h^{-1}[f^{\sim}(\cdot+h) - f^{\sim}(\cdot)] - D^{\{1\}}f\|_p = 0, \tag{3.40}$$

while for $\alpha > 1$ we proceed inductively. If $l < \alpha \leq l+1$, l being a positive integer, then we set

$$D^{\{\alpha\}}f := D^{\{\alpha-l\}}D^{\{l\}}f. \tag{3.41}$$

With this definition of an α-th strong Riesz derivative, formula (3.37) receives the following correct interpretation:
A function $f \in L^p(\mathbb{R})$, $1 \leq p \leq 2$, has a *strong Riesz derivative of order* $\alpha > 0$ if and only if the function $v \mapsto |v|^\alpha f^\wedge(v)$ is the Fourier transform of some function $g \in L^p(\mathbb{R})$. If so, g equals $D^{\{\alpha\}}f$.

Our principal aim in this section is, however, to characterize the strong Riesz derivative by an integral of Marchaud type. Indeed, making an attempt to define a Riesz potential of negative order one is led to use the method of Hadamard's finite part; for $0 < \alpha < 2$ it means to consider the integral

$$D_\varepsilon^{\{\alpha\}}f(x) = \frac{1}{K_{\alpha,2}} \int_{|u|\geq\varepsilon} \frac{f(x-u)-f(x)}{|u|^{1+\alpha}}\,du \tag{3.42}$$

as $\varepsilon \to 0+$, where

$$K_{\alpha,2} = \begin{cases} 2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2} & \text{if} \quad 0<\alpha<2, \quad \alpha \neq 1 \\ -\pi & \phantom{\text{if}} \quad \alpha = 1. \end{cases}$$

By a substitution, (3.42) may be rewritten as

$$D_\varepsilon^{\{\alpha\}}f(x) = \frac{1}{K_{\alpha,2}} \int_\varepsilon^\infty \frac{f(x+u)-2f(x)+f(x-u)}{u^{1+\alpha}}\,du$$

with a central difference of f in the numerator of the integrand. This suggests for arbitrary $\alpha > 0$ the following regularizations

$$D_\varepsilon^{\{\alpha\}}f(x) = \frac{1}{K_{\alpha,2j}} \int_\varepsilon^\infty \frac{\overline{\Delta}_u^{2j} f(x)}{u^{1+\alpha}}\,du \quad (0 < \alpha < 2j)$$

(more precisely $D_\varepsilon^{\{\alpha,2j\}}f$) as $\varepsilon \to 0+$, where the central difference of f of even order $2j$ is given by

$$\overline{\Delta}_u^{2j} f(x) = \sum_{k=0}^{2j} (-1)^k \binom{2j}{k} f(x+(j-k)u)$$

and

$$K_{\alpha,2j} = (-1)^j 2^{2j-\alpha} \int_0^\infty \frac{\sin^{2j} u}{u^{1+\alpha}} du.$$

The characterization of strong Riesz derivatives given in the next theorem is valid in L^p-spaces for all real $p \geq 1$; see Butzer-Trebels [28] and Butzer-Nessel [25], Sec. 11 (for $1 \leq p \leq 2$) and Sec. 13.2.4, 13.2.5 (for $p > 2$).

Theorem 3.11. *Let $\alpha > 0$. For $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, the following assertions are equivalent:*

(i) f has an α-th strong Riesz derivative,

(ii) there exists $g \in L^p(\mathbb{R})$ such that

$$\lim_{\varepsilon \to o+} \left\| \frac{1}{K_{\alpha,2j}} \int_\varepsilon^\infty \frac{\overline{\Delta}_u^{2j} f}{u^{1+\alpha}} du - g \right\|_p = 0, \tag{3.43}$$

where j is a positive integer chosen so that $0 < \alpha < 2j$.

In this case, $D^{\{\alpha\}} f = g$.

The above equivalence may be established by Fourier transform methods. In case of L^p-spaces, $1 \leq p \leq 2$, one may argue directly via the characterization of the strong Riesz derivative in terms of the Fourier transform which was mentioned subsequent to definitions (3.39) - (3.41), while for $p > 2$ dual methods are appropriate.

To give an idea how the Fourier transform acts on the Marchaud approximants $D_\varepsilon^{\{\alpha\}} f$, let us consider the case $0 < \alpha < 2$ more precisely. If $f \in L^p(\mathbb{R})$, $1 \leq p \leq 2$, and $0 < \alpha < 2$, then we have for almost all $v \in \mathbb{R}$

$$\left[D_\varepsilon^{\{\alpha\}} f\right]^\wedge (v) = \frac{2}{K_{\alpha,2}} |v|^\alpha f^\wedge(v) \int_{|\varepsilon v|}^\infty \frac{\cos u - 1}{u^{1+\alpha}} du,$$

which implies

$$\lim_{\varepsilon \to 0+} [D_\varepsilon^{\{\alpha\}} f]^\wedge(v) = |v|^\alpha f^\wedge(v). \tag{3.44}$$

On the other hand, there exists a function $k_\alpha \in L^1(\mathbb{R})$ satisfying $\int_{-\infty}^\infty k_\alpha(u) du = \sqrt{2\pi}$, such that

$$\frac{2}{K_{\alpha,2}} \int_{|v|}^\infty \frac{\cos u - 1}{u^{1+\alpha}} du = k_\alpha^\wedge(v) \qquad (v \in \mathbb{R})$$

(see Sunouchi [158] and [173, II]). Hence,

$$\left[D_\varepsilon^{\{\alpha\}} f\right]^\wedge (v) = |v|^\alpha f^\wedge(v) k_\alpha^\wedge(\varepsilon v). \tag{3.45}$$

Now, formulas (3.44) and (3.45) imply, by standard Fourier transform arguments, the equivalence of the statements (ii) and (iii) in the next theorem (for the case $0 < \alpha < 2$). For a different method of proof generalizing one due to Salem-Zygmund we refer to Butzer-Trebels [28], see also [25, p. 414]. Theorem 3.12 summarizes all results of this section for the spaces $L^p(\mathbb{R})$, $1 \le p \le 2$.

Theorem 3.12. *Let $\alpha > 0$. For $f \in L^p(\mathbb{R})$, $1 \le p \le 2$, the following assertions are equivalent:*

(i) f has an α-th strong Riesz derivative,

(ii) there exists $g \in L^p(\mathbb{R})$ such that $|v|^\alpha f^\wedge(v) = g^\wedge(v)$,

(iii) there exists $g \in L^p(\mathbb{R})$ such that eq. (3.43) holds for $0 < \alpha < 2j$,

(iv) if $1 < p \le 2$,

$$\left\| \frac{1}{K_{\alpha,2j}} \int_\varepsilon^\infty \frac{\overline{\Delta}_u^{2j} f}{u^{1+\alpha}} du \right\|_p = \mathcal{O}(1) \quad (\varepsilon \to 0+),$$

where $0 < \alpha < 2j$.

For the counterpart of (iv) in the space $L^1(\mathbb{R})$ we have the following characterization.

Proposition 3.13. *Let $\alpha > 0$. For $f \in L^1(\mathbb{R})$ the following assertions are equivalent:*

(i) There exists a function μ of bounded variation over $\mathbb{R}$ such that its Fourier-Stieltjes transform

$$\mu^\vee(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-ivu} d\mu(u)$$

satisfies for almost all $v \in \mathbb{R}$

$$\mu^\vee(v) = |v|^\alpha f^\wedge(v).$$

(ii) If j is a positive integer such that $0 < \alpha < 2j$, then

$$\left\| \frac{1}{K_{\alpha,2j}} \int_\varepsilon^\infty \frac{\overline{\Delta}_u^{2j} f}{u^{1+\alpha}} du \right\|_1 = \mathcal{O}(1) \quad (\varepsilon \to 0+).$$

4 Various Applications

4.1 Integral representations of special functions

Fractional calculus may be used to construct non-trivial integral representations of special functions. One example is (2.16). Another is Poisson's integral representation of the Bessel function

$$J_\nu(z) = (\frac{z}{2})^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}(\frac{z}{2})^{2k},$$

which is a solution of Bessel's equation $z^2D^2w + zDw + (z^2-\nu^2)w = 0$, namely (see [121, p. 302, 114, 99, 89])

$$J_\nu(z) = \frac{2}{\Gamma(1/2)\Gamma(\nu+1/2)}(\frac{z}{2})^\nu \int_0^1 (1-u^2)^{\nu-1/2}\cos zu\, du \quad (\Re e\, \nu > -1/2).$$

As a third example, the Legendre *function* $P_\alpha(x)$ of the first kind and degree α may be expressed as a fractional derivative. It is given in terms of hypergeometric functions as

$$P_\alpha(x) = {}_2F_1(\alpha+1, -\alpha, 1; \frac{1}{2}(1-x)), \quad |1-x| < 2. \tag{4.1}$$

If α is a nonnegative integer n, then $P_\alpha(x)$ becomes the Legendre polynomial $P_n(x)$ which may be represented by the classical Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2-1)^n.$$

To obtain a similar expression for arbitrary α note that

$$x^\alpha(1-x)^\alpha = \frac{1}{\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(k-\alpha)}{k!} x^{k+\alpha} \quad |x| < 1.$$

Differentiating termwise and using (recall (1.30))

$${}_0D_x^\alpha x^{k+\alpha} = \frac{\Gamma(\alpha+1+k)}{\Gamma(k+1)} x^k,$$

we obtain

$$\begin{aligned} {}_0D_x^\alpha x^\alpha(1-x)^\alpha &= \frac{1}{\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(k-\alpha)}{k!} \frac{\Gamma(\alpha+1+k)}{\Gamma(k+1)} x^k \\ &= \Gamma(\alpha+1)\, {}_2F_1(\alpha+1, -\alpha, 1; x). \end{aligned}$$

Thus recalling (4.1),

$$P_\alpha(1-2x) = \frac{1}{\Gamma(\alpha+1)} {}_0D_x^\alpha[x^\alpha(1-x)^\alpha].$$

By the change of variable $t = 1-2x$, this becomes

$$P_\alpha(t) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\ {}_tD_1^\alpha(1-t^2)^\alpha.$$

For further applications of fractional calculus to special functions see e.g. L.M.B.C. Campos [31], R.N. Kalia [87], Lavoie-Tremblay-Osler [102], V. Kiryakova [91].

4.2 Stirling functions of the first kind

The classical Stirling numbers, introduced by James Stirling in his "Methodus Differentialis" in 1730, which are said to be "as important as Bernoulli's, or even more so" (see Jordan ([85, 1959, p. 143])), play a major role in a variety of branches of mathematics, such as combinatorial theory, finite difference calculus, numerical analysis, interpolation theory and number theory. Those of the first kind, $s(n,k)$, can be defined via their exponential generating function

$$\frac{(\log(1+z))^k}{k!} = \sum_{n=k}^{\infty} \frac{s(n,k)}{n!} z^n \qquad (|z|<1,\ k \in \mathbb{N}_0) \tag{4.2}$$

or via their (horizontal) generating function

$$[z]_n = \sum_{k=0}^{n} s(n,k) z^k \qquad (z \in \mathbb{C},\ n \in \mathbb{N}_0) \tag{4.3}$$

where $[z]_n = z(z-1)\dots(z-n+1)$, and with the convention $s(n,0) = \delta_{n,0}$ (Kronecker's delta); see Riordan [141], Comtet [35].

The latter gives a natural possibility to define "Stirling numbers of fractional order" $s(\alpha,k)$ with $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$. In fact, these "Stirling functions", as one may call them, which were introduced by Butzer, Hauss and Schmidt [22], may be defined via the generating function

$$[z]_\alpha := \frac{\Gamma(z+1)}{\Gamma(z-\alpha+1)} = \sum_{k=0}^{\infty} s(\alpha,k) z^k \qquad (|z|<1, \alpha \in \mathbb{C}). \tag{4.4}$$

Theorem 4.1. *For $\alpha \in \mathbb{C}$ and $k > \Re e\ \alpha > -\infty$, $k \in \mathbb{N}$ there holds the integral representation*

$$s(\alpha, k) = \frac{1}{\Gamma(-\alpha)k!} \int_{0+}^{1-} \frac{(\log u)^k}{(1-u)^{\alpha+1}}\, du. \tag{4.5}$$

For a proof see [67, pp. 117-121].

This integral calls to mind the Riemann–Liouville derivative of order α with $\Re e\ \alpha > 0$ of the function $f(x) = (\log x)^k/k!$ taken at $x = 1$. In fact, equation (4.5) in this respect reads for $k > \Re e\ \alpha$, $k \in \mathbb{N}_0$,

$$s(\alpha, k) = \frac{1}{k!}\ {}_0\mathcal{D}_x^\alpha (\log x)^k \Big|_{x=1}. \tag{4.6}$$

The proof of (4.6) for $\Re e\ \alpha < 0$ is immediate from (4.5) and Definition (1.21). However, for $k > \Re e\ \alpha > 0$ it is rather long and technical (see [67, pp. 123-128]).

Formula (4.6) is the fractional counterpart of, which follows from (4.2),

$$s(n, k) = \frac{1}{k!} \left(\frac{d}{dz}\right)^n (\log(1+z))^k|_{z=0}. \tag{4.7}$$

Observe that the results (1.30) and (1.35) enable one to evaluate certain Stirling functions. Thus $s(\alpha, 0) = {}_0\mathcal{D}_x^\alpha 1|_{x=1} = x^{-\alpha}/\Gamma(1-\alpha)|_{x=1} = 1/\Gamma(1-\alpha)$, $s(1/2, 1) = \log 4/\sqrt{\pi}$, $s(-1/2, 1) = 2(\log 4 - 2)/\sqrt{\pi}$ (cf. Hauss [67, p. 107f]),

$$s(\alpha, 1) = \frac{\psi(1) - \psi(1-\alpha)}{\Gamma(1-\alpha)} = \frac{\alpha}{\Gamma(1-\alpha)} \sum_{j=1}^{\infty} \frac{1}{j(j-\alpha)} \quad (\alpha \notin \mathbb{N}), \tag{4.8}$$

$$\begin{aligned} s(\alpha, 2) &= \frac{1}{2\Gamma(1-\alpha)} \left\{ [\psi(1) - \psi(1-\alpha)]^2 + \psi'(1) - \psi'(1-\alpha) \right\} \\ &= \frac{-\alpha}{\Gamma(1-\alpha)} \sum_{k=2}^{\infty} \left(\sum_{j=1}^{k-1} \frac{1}{j} \right) \frac{1}{k(k-\alpha)} \quad (\alpha \in \mathbb{R}\backslash\mathbb{N}). \end{aligned}$$

Observe that the representation (4.5) may also be associated with the Weyl-fractional derivative. In fact, under the substitution $u = (1+t)^{-1}$ the integral (4.5) turns into

$$s(\alpha,k) = \frac{(-1)^k}{\Gamma(-\alpha)k!}\int_0^\infty [\log(1+u)]^k u^{-1-\alpha}(1+u)^{\alpha-1}\,du$$

for $k > \Re e\,\alpha$, $k \in \mathbb{N}$. For $\Re e\,\alpha < 0$ it follows immediately that

$$s(\alpha,k) = {}_x\mathcal{D}_\infty^\alpha\{(1+x)^{\alpha-1}[\log(1+x)]^k(-1)^k/k!\}\Big|_{x=0}.$$

However a proof for such a representation for $k > \Re e\,\alpha > 0$ seems to be quite difficult.

Whether it is possible to evaluate the fractional order derivatives of the $s(\alpha,k)$ is an open question. In any case, the $s(\alpha,k)$ are arbitrarily often continuously differentiable in α and one has for $\alpha \in \mathbb{R}\backslash\mathbb{N}$ and $k \in \mathbb{N}_0$,

$$(d/d\alpha)s(\alpha,k) = \sum_{j=0}^k \frac{\psi^{(j)}(1-\alpha)}{j!}s(\alpha,k-j).$$

The $s(\alpha,k)$ are connected with the fundamental *Riemann zeta function* $\zeta(s) = \sum_{j=1}^\infty j^{-s}, \Re e\; s > 1$, in a striking fashion (see [19] and the literature cited there). Thus, for $m \in \mathbb{N}$,

$$\zeta(m+1) = \lim_{\alpha\to 0}\Gamma(-\alpha)(-1)^m s(\alpha,m) = \lim_{\alpha\to 0}(-1)^m\Gamma(-\alpha)\;{}_0I_x^{-\alpha}\;\frac{(\log x)^m}{m!}\Big|_{x=1},$$

as well as

$$\zeta(m+1) = \lim_{\alpha\to 0}(-1)^{m+1}(d/d\alpha)s(\alpha,m).$$

4.3 Stirling functions of the second kind

The Stirling numbers $S(n,k)$ of the second kind could be defined via their exponential generating function

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^\infty S(n,k)\frac{x^n}{n!} \qquad (x \in \mathbb{R}; k \in \mathbb{N}_0) \tag{4.9}$$

or, equivalently, by

$$S(n,k) = \left(\frac{d}{dx}\right)^n \frac{(e^x-1)^k}{k!}\Big|_{x=0} \qquad (k,n \in \mathbb{N}_0), \tag{4.10}$$

or by the horizontal generating function

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k$$

with the associated

$$S(n,k) = \frac{1}{k!}\Delta^k x^n \Big|_{x=0} \qquad (k, n \in \mathbb{N}_0),$$

Δ being the forward (left-handed) difference operator, namely $\Delta f(x) := f(x+1) - f(x)$, $\Delta^{j+1} f(x) = \Delta(\Delta^j f)(x)$.

In analogy with the third definition, the *Stirling functions* $S(\alpha, k)$ of *second kind*, of fractional order $\alpha \in \mathbb{R}_+$, introduced in Butzer and Hauss [20], are defined by

$$S(\alpha,k) := \frac{1}{k!}\Delta^k x^\alpha \Big|_{x=0} \qquad (\alpha \in \mathbb{R}_+; k \in \mathbb{N}_0). \tag{4.11}$$

They may also be expressed by a finite sum as $S(\alpha, k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j} j^\alpha$. There exists a real integral representation for $S(\alpha, k)$, namely there holds for $0 < \alpha < k$, $k \in \mathbb{N}$, (see [67, p. 71])

$$S(\alpha,k) = \frac{1}{k!\Gamma(-\alpha)} \int_0^\infty u^{-1-\alpha}(e^{-u} - 1)^k \, du. \tag{4.12}$$

Now this integral looks formally like the *Weyl fractional derivative* of order $\alpha(> 0!)$ of the function $f(x) = (e^{-x} - 1)^k/k!$, taken at $x = 0$. As a matter of fact,

Theorem 4.2. *For $k \in \mathbb{N}$, any $\alpha > 0$ there holds*

$$S(\alpha,k) = \frac{1}{k!}\, {}_xD_\infty^\alpha \left\{(e^{-x} - 1)^k - (-1)^k\right\} \Big|_{x=0}. \tag{4.13}$$

As to the proof, recalling (1.38), for $p = j$,

$$\begin{aligned} {}_xD_\infty^\alpha \left\{(e^{-x} - 1)^k - (-1)^k\right\} \Big|_{x=0} &= {}_xD_\infty^\alpha \left(\sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} e^{-jx} \right) \Big|_{x=0} \\ &= \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} j^\alpha = k! S(\alpha,k). \end{aligned} \tag{4.14}$$

Observe that (4.13) is actually the counterpart of (4.10) in the fractional instance, since by (4.13), for $n \in \mathbb{N}$,

$$\begin{aligned} S(n,k) &= \frac{1}{k!}\,{}_xD^n_\infty \left\{(e^{-x}-1)^k - (-1)^k\right\}\Big|_{x=0} \\ &= \frac{(-1)^n}{k!}\left(\frac{d}{dx}\right)^n \left\{(e^{-x}-1)^k\right\}\Big|_{x=0} \end{aligned} \tag{4.15}$$

provided x in (4.10) is replaced by $-x$ (due to the fact that the integral (1.37) converges only for $p > 0$).

Let us finally observe that fractional order Stirling numbers $S(\alpha, k)$ have been considered indirectly, at least, in Westphal [173, p. 76], in her theory of fractional powers of infinitesimal generators of semigroup operators. They are related to the normalizing factor $C_{\alpha,k}$ –recall Section 3.5 – of the Marchaud fractional order derivative (3.22), (3.23), first treated by Marchaud. Also the proof of (4.12) in [20] is modelled upon [173].

4.4 Euler functions

The classical *Euler polynomials* $E_n(x)$ can be defined in terms of their exponential generating function by

$$\frac{2e^{xw}}{e^w+1} = \sum_{n=0}^{\infty} E_n(x)\frac{w^n}{n!} \quad (x \in \mathbb{R}; |w| < \pi) \tag{4.16}$$

or, equivalently, by

$$E_n(x) = \left(\frac{d}{dw}\right)^n \left(\frac{2e^{xw}}{e^w+1}\right)\Big|_{w=0}. \tag{4.17}$$

These polynomials were extended to *Euler functions* $E_\alpha(z)$ with complex indices $\alpha \in \mathbb{C}$ and $\Re e\, z > 0$ in [14] by

$$E_\alpha(z) := \frac{\Gamma(\alpha+1)}{\pi i}\int_C \frac{e^{\zeta z}}{1+e^\zeta}\zeta^{-\alpha-1}\,d\zeta, \tag{4.18}$$

C being a positively oriented loop around the negative real axis, composed of a circle C_2, of radius $0 < c < 2\pi$ around the origin together with the lower and upper edges C_1 and C_3 of the "cut" in the complex plane along $\mathbb{R}^-$. In fact, $C = C(-\infty, 0+)$ of (1.41).

These Euler functions have removable discontinuities at $\alpha \in \mathbb{Z}^-$ since the gamma function has simple poles at these points, and the integral vanishes by

Cauchy's theorem, since then the integrand is analytic. This yields (see [2]) that $E_\alpha(z)$ is an analytic function of $\alpha \in \mathbb{C}$ for $\Re e\ z > 0$. These functions can even be defined for all $z \in \mathbb{C}\backslash\mathbb{R}_0^-$; the $E_\alpha(z)$ are analytic there. There exists a Weyl-type integral representation, namely

$$E_\alpha(z) = \frac{1}{\Gamma(-\alpha)} \int\limits_{-\infty}^{0} \frac{2e^{zu}}{1+e^u}(-u)^{-\alpha-1}\, du \tag{4.19}$$

for $\Re e\ \alpha < 0$ and $\Re e\ z > 0$. This integral can be regarded as the fractional Weyl *derivative* of order α of the generating function $f_z(y) := 2e^{zy}/(1+e^y)$ at $y = 0$. Thus $E_\alpha(z) = {}_{-\infty}D_y^\alpha f_z(y)|_{y=0} = (\frac{d}{dy})^m\ {}_{-\infty}W_y^{m-\alpha} f_z(y)|_{y=0}$ holds for $m - 1 \leq \Re e\ \alpha < m$, which is the counterpart of formula (4.17). In Section 1.3 we presented three examples dealing with the evaluation of the functions x^b, $\log x$ and e^{-px} according to the Riemann-Liouville-Weyl definitions. Let us now apply the Liouville-Grünwald definition, and take the Euler functions as the example. In order to apply it we shall take the 1–antiperiodic continuation of $E_\alpha(x)$ for $x \in (0,1]$ and denote it by $\mathcal{E}_\alpha(x)$. It has the Fourier series representation

$$\mathcal{E}_\alpha(x) = 2\Gamma(\alpha+1) \sum_{k=-\infty}^{\infty} \frac{e^{(2k+1)\pi i x}}{[(2k+1)\pi i]^{\alpha+1}}$$

for $\alpha \in \mathbb{C}$, $\Re e\ \alpha > -1$. Whereas $(d/dx)E_n(x) = nE_{n-1}(x)$ for $n \in \mathbb{N}$, $(d/dx)E_\alpha(x) = \alpha E_{\alpha-1}(x)$ for $x \in \mathbb{R}_0^-$, $\Re e\ \alpha > 0$. As to the fractional order differential operator D^β for $\beta > 0$ of Definition (3.9), taken in the norm of $L_{2\pi}^1$–space, we have

$$D^\beta \mathcal{E}_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} \mathcal{E}_{\alpha-\beta}(x) \quad (x \in \mathbb{R})$$

provided $\alpha > \beta > -\infty$. As to the proof, one has for all $\nu \in \mathbb{Z}$,

$$\begin{aligned}(i\nu)^\beta [\mathcal{E}_\alpha(\cdot)]^\wedge(\nu) &= \begin{cases} \frac{2\Gamma(\alpha+1)}{[(2k+1)\pi i]^{\alpha-\beta+1}} & ,\nu = (2k+1)n \text{ for one } k \in \mathbb{Z}, \\ 0 & ,\nu \neq (2k+1)n \text{ for all } k \in \mathbb{Z} \end{cases} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} [\mathcal{E}_{\alpha-\beta}(\cdot)]^\wedge(\nu).\end{aligned}$$

An application of Theorem 3.4 then completes the proof. For details see [14].

4.5 Eulerian numbers $E(\alpha, k)$ for $\alpha \in \mathbb{R}$

The *Eulerian numbers* are well known, especially in combinatorics and discrete mathematics, but also in geometry, statistical applications and spline theory. These numbers, $E(n, k)$, defined by

$$E(n,k) := \sum_{j=0}^{k}(-1)^j \binom{n+1}{j}(k+1-j)^n \quad (k, n \in \mathbb{N}_0), \tag{4.20}$$

satisfy the recurrence formula (in n)

$$E(n,k) = (k+1)E(n-1,k) + (n-k)E(n-1,k-1) \quad (k, n \in \mathbb{N}). \tag{4.21}$$

They are also quite useful since they connect the monomials x^n with the consecutive binomial coefficients, namely

$$x^n = \sum_{k=0}^{n-1} E(n,k)\binom{x+k}{n} \qquad (x \in \mathbb{R}; n \in \mathbb{N}_0). \tag{4.22}$$

There is a natural extension to *Eulerian functions* $E(\alpha, k)$, where $n \in \mathbb{N}$ is replaced by $\alpha \in \mathbb{R}$, defined for $k \in \mathbb{N}_0$ by

$$E(\alpha,k) := \sum_{j=0}^{k}(-1)^j \binom{\alpha+1}{j}(k+1-j)^\alpha. \tag{4.23}$$

This matter was first carried out in [21]. The first four Eulerian functions are given for $\alpha \in \mathbb{R}$ by

$$\begin{aligned}
E(\alpha,0) &= 1, \\
E(\alpha,1) &= 2^\alpha - (\alpha+1), \\
E(\alpha,2) &= 3^\alpha - (\alpha+1)2^\alpha + \binom{\alpha+1}{2}, \\
E(\alpha,3) &= 4^\alpha - (\alpha+1)3^\alpha + \binom{\alpha+1}{2}2^\alpha - \binom{\alpha+1}{3}.
\end{aligned}$$

There exists a horizontal generating function for the $E(\alpha, k)$, namely

$$\sum_{k=0}^{\infty} E(\alpha,k)x^k = \frac{A_\alpha(x)}{x} \qquad (\alpha \in \mathbb{R}; |x| < 1), \tag{4.24}$$

where $A_\alpha(x)$ are the generalized Eulerian "polynomials", i. e.

$$A_\alpha(x) := (1-x)^{\alpha+1} \sum_{k=1}^{\infty} k^\alpha x^k; \tag{4.25}$$

the radius of convergence of the latter series is 1. This yields a representation of these functions in terms of derivatives, namely

$$E(\alpha,k) = \frac{1}{k!}\left(\frac{d}{dx}\right)^k \left\{\frac{A_\alpha(x)}{x}\right\}\Bigg|_{x=0} \qquad (\alpha \in \mathbb{R}; k \in \mathbb{N}_0). \tag{4.26}$$

Another such representation is that due to Worpitzky (1883) in the classical instance, namely

$$E(n,k) = \left(\frac{d}{dx}\right)^n \left(\sum_{j=0}^{k}(-1)^j \binom{n+1}{j} e^{(k-j+1)x}\right)\Bigg|_{x=0}. \tag{4.27}$$

The counterpart for the $E(\alpha,k)$ for arbitrary $\alpha \in \mathbb{R}$ can be expressed in terms of a *Weyl fractional* derivative. Indeed,

Theorem 4.3. *a) For any $\alpha < 0$, $k \in \mathbb{N}_0$ there holds*

$$E(\alpha,k) = {}_xW_\infty^{-\alpha}\left(\sum_{j=0}^{k}(-1)^j \binom{\alpha+1}{j} e^{-(k-j+1)x}\right)\Bigg|_{x=0}. \tag{4.28}$$

b) For $\alpha > 0$, $k \in \mathbb{N}_0$ one has

$$E(\alpha,k) = {}_xD_\infty^{\alpha}\left(\sum_{j=0}^{k}(-1)^j \binom{\alpha+1}{j} e^{-(k-j+1)x}\right)\Bigg|_{x=0}. \tag{4.29}$$

The proof follows basically from (1.37), (1.38) with $p = k - j + 1$. For details see [21].

The $E(\alpha,k)$ have interesting properties, asymptotic relations, and are connected with the Stirling functions of Sec. 4.3. In particular,

$$\sum_{k=1}^{m+1} k^\alpha = \sum_{k=0}^{m} \binom{\alpha+m-k+1}{\alpha+1} E(\alpha,k) \qquad (\alpha \in \mathbb{R}; m \in \mathbb{N}_0);$$

$$(m+1)^\alpha = \sum_{k=0}^{m} \binom{\alpha+m-k}{\alpha} E(\alpha,k);$$

$$\lim_{\alpha\to\infty} \frac{E(\alpha,k)}{(k+1)^\alpha} = 1, \quad \lim_{\alpha\to\infty} \frac{E(\alpha+1,k)}{E(\alpha,k)} = k+1;$$

$$\lim_{\alpha\to-\infty} \frac{E(\alpha,k)}{(-\alpha)^k} = \frac{1}{k!} \qquad (k \in \mathbb{N});$$

$$E(\alpha,k) = (-1)^{k+1} \sum_{j=1}^{k+1} (-1)^j \binom{\alpha-j}{k-j+1} j! S(\alpha,j) \qquad (\alpha > 0; k \in \mathbb{N}_0);$$

$$E(\alpha,k) = \Gamma(\alpha+1) \int_k^{k+1} p_\alpha(u)du \qquad (\alpha > 1; k \in \mathbb{N}_0);$$

$$\sum_{k=0}^{\infty} E(\alpha,k) = \Gamma(\alpha+1) \qquad (\alpha > 1).$$

The second to last result connects the Eulerian numbers with the basic function $p_\alpha(x)$ of Proposition 3.1; it yields the last one since $\int_0^\infty p_\alpha(u)du = 1$. Further, $p_\alpha(k+1) = [\Gamma(\alpha)]^{-1} E(\alpha-1,k)$ for $\alpha > 0$. The implications may be worthwhile to consider.

In any case, the foregoing connection suggests it seems to be possible to generalize the Eulerian functions $E(\alpha,k)$ to the instance when also k is allowed to be fractional in terms of the $p_\alpha(x)$ function, namely

$$E(\alpha,\beta) = \Gamma(\alpha+1) p_{\alpha+1}(\beta+1)$$

where β is real with $\beta > -1$. For the role of Eulerian numbers in mathematics see the excellent account in Hilton et al [83, pp. 217–248] and its literature.

4.6 The Bernoulli functions $B_\alpha(x)$ for $\alpha \in \mathbb{R}$

The classical Bernoulli polynomials $B_n(x)$ can be defined via their exponential generating function by

$$\frac{we^{wx}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{w^n}{n!} \quad (x \in \mathbb{R}; |w| < 2\pi), \tag{4.30}$$

or equivalently by

$$B_n(x) = \left(\frac{d}{dw}\right)^n \left(\frac{we^{wx}}{e^w - 1}\right)\bigg|_{w=0}. \tag{4.31}$$

But they can also be defined in terms of their Fourier series, at first for $0 \leq x < 1$ by $B_n(x) = \mathcal{B}_n(x)$, where $\mathcal{B}_n(x)$ is the one-periodic function

$$\mathcal{B}_n(x) := -2n! \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - n\pi/2)}{(2\pi k)^n}, \quad (x \in \mathbb{R}, n \geq 2). \tag{4.32}$$

This led Butzer et al [17] to define the *Bernoulli functions* $B_\alpha(x)$ for $\alpha \in \mathbb{C}$, firstly for $\Re e\ \alpha > 1$ by $B_\alpha(x) = \mathcal{B}_\alpha(x)$ on $0 \leq x < 1$, where

$$\mathcal{B}_\alpha(x) := -2\Gamma(\alpha+1) \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - \alpha\pi/2)}{(2\pi k)^\alpha}, \quad (x \in \mathbb{R}, \Re e\ \alpha > 1), \tag{4.33}$$

have period 1. The $B_\alpha(x)$ are then extended in a suitable way to $x \in \mathbb{R}_0^+$ and $\mathbb{R}^-$. They can also be extended (noting (4.31)) beyond the line $\Re e\ \alpha = 1$ to the whole α-plane $\mathbb{C}$ as well as to complex $x = z$, via the contour integral

$$B_\alpha(z) := \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{e^{wz}}{e^w - 1} w^{-\alpha}\, dw \quad (\alpha \in \mathbb{C}) \tag{4.34}$$

for $z \in \mathbb{C}\backslash R_0^-$, at first for $\Re e\ z > 0$, where C denotes the positively oriented loop following Defintion (4.18), and then for any $z \in \mathbb{C}\backslash\mathbb{R}_0^-$ by

$$B_\alpha(z-m) := B_\alpha(z) - \alpha \sum_{k=0}^{m-1} (z-k-1)^{\alpha-1}, \tag{4.35}$$

where $\Re e\ z \in (0,1)$, $z \notin \mathbb{R}$, $m \in \mathbb{N}$. The $B_\alpha(z)$ are now holomorphic functions of $\alpha \in \mathbb{C}$ for $z \in \mathbb{C}\backslash\mathbb{R}_0^-$.

$B_\alpha(z)$ can also be represented as a Weyl integral

$$B_\alpha(z) = \frac{1}{-\Gamma(-\alpha)} \int_{-\infty}^{0} \frac{e^{zu}}{e^u - 1} (-u)^{-\alpha}\, du \tag{4.36}$$

for $\Re e\ \alpha < 0$ and $\Re e\ z > 0$. This integral suggests that the fractional Weyl derivative of order α of the generating function $f_z(y) = (ye^{zy})/(e^y - 1)$ at $y = 0$, equals

$$B_\alpha(z) = {}_{-\infty}D_y^\alpha f_z(y)|_{y=0} = \left(\frac{d}{dy}\right)^m {}_{-\infty}W_y^{m-\alpha} f_z(y)|_{y=0} \tag{4.37}$$

for $m - 1 \leq \Re e\alpha < m$. It is the counterpart of formula (4.31) in the fractional instance.

The foregoing Bernoulli functions can be connected in an interesting way with Liouville-Grünwald derivatives. Indeed, given such a derivative $D^\alpha f \in L^p_{2\pi}$, $1 \leq p < \infty$ of order α, then the $B_\alpha(x)$ may be used to recapture $f(x)$ itself from

$$f(x) = -\frac{(2\pi)^{\alpha-1}}{\Gamma(\alpha+1)} \int_0^{2\pi} D^\alpha f(x-u) B_\alpha(u/2\pi)\, du + \frac{1}{2\pi} \int_0^{2\pi} f(u)\, du. \qquad (4.38)$$

It is due to the fact that the function $\psi_\alpha(x)$ of (3.13), which is associated with the Weyl-fractional integral, can be represented in an interesting fashion in terms of $B_\alpha(x)$ i. e. ,

$$\psi_\alpha(x) = \frac{-(2\pi)^\alpha}{\Gamma(\alpha+1)} B_\alpha\left(\frac{x}{2\pi}\right), \quad 0 < x < 2\pi. \qquad (4.39)$$

The connection (4.39) was noted in the particular case $\alpha = n \in \mathbb{N}$ by Samko et al [153]. The result (4.38) follows from Theorem 3.4.

4.7 Ordinary and partial differential equations and other applications

There exist a variety of papers dealing with applications of fractional calculus to (ordinary) differential equations of fractional order, the first discussion of which goes back at least to L.O' Shaughnessy (1918) and E.L. Post (1919). See especially the monographs by Miller-Ross [121], Samko et al [153, pp. 795–872], and the work of M. Fujiwae (1933), E. Pitcher and W.E. Sewell (1938), G.H. Hardy (1945), J.M. Barret (1954), M.A. Al-Bassam (1966), M.M. Dzherbashyan and A.B. Nersesyan (1968), L.M.B.C. Campos (1990), D. Delbosco and L. Rodino (1996). See e.g. the recent N. Hayek et al [69] and the literature cited there. An algorithm for the numerical solution of nonlinear differential equations of fractional order was developed by K. Diethelm and A.D. Freed [40,41].

As to the partial differential equations there also exist a large number of publications. For some earlier material see Oldham-Spanier [133, pp. 197–218] and for more recent results Rusev et al [150], and especially the succeeding chapters of this volume.

Thus H. Berens and U. Westphal already in 1968 [7] considered the Cauchy problem for the generalized wave equation

$$\frac{d}{dx} w(x,t) + {}_0D_t^\gamma w(x,t) = 0 \quad (x, t > 0; 0 < \gamma < 1)$$

where ${}_0D_t^{\gamma}$ is the Riemann-Liouville fractional differentiation operator applied with respect to t and suitably restricted to functions of $L^p(0,\infty)$. The solution has the form $w(x,t) = [W_{\gamma}(x)f](t)$, where $\{W_{\gamma}(x); x \geq 0\}$ is a (C_0)-semigroup in $L^p(0,\infty)$.

In the particular case $\gamma = 1/2$ it can be given explicitly as $[W_{1/2}(x)f](t) = (x/\sqrt{4\pi}) \cdot \int_0^t f(t-u)u^{-3/2}exp(-x^2/4u)du$ with $f \in L^p(0,\infty)$. The proofs are carried out by precise applications of semigroup theory and Laplace transform analysis.

This investigation of the fractional wave equation, which is discussed briefly in Samko et al [153, p. 865], seems to be unknown among mathematical physicists and physicists. Of the literature they cite, it is generally restricted to the standard works of Oldham-Spanier [133] and Ross [144]; sometimes Ross [146], McBride-Roach [115] and Nishimoto [124, Vols. I and II] are also added (see e.g. H. Beyer and S. Kempfle [8]).

Explicit solutions of fractional wave and diffusion equations were given by Schneider and Wyss [157], Hilfer [77], B.J. West et al [171] and R. Metzler and Nonnenmacher [119], in terms of Fox functions (see A.M. Mathai and R.K. Saxena [112]). F. Mainardi in a series of papers studied especially the fractional diffusion equation, based on Laplace transforms and special functions of E.M. Wright-type (cf. Kiryakova [91]; see [110], and El-Sayed [49] for a semigroup approach. Schneider [156] and Hilfer [80,81] established the relation between fractional diffusion and stochastic processes, master equations and continuous time random walks.

A review of the early contributors of the applications of fractional calculus to the theory of *viscoelasticity* has been given by R.L. Bagley and P.J. Torvik [5]. In later papers these authors and R.C. Koeller [94] show the connection between fractional calculus and Abel's integral for materials with memory and that the fractional calculus constitutive equation allows for a continuous transition from the solid state to the fluid state when the memory parameter varies from zero to one. In particular, Koeller continued the Bagley-Torvik work and Yu. N. Rabotnov's (1980) theory of hereditary solid mechanics using the method of integral equations (instead of Laplace transforms); it leads to results expressed in terms of the Mittag-Leffler function which depends on the fractional derivative parameter in question. See especially R. Gorenflo and F. Mainardi ([64]) in this respect. Chapters VII and VIII in this volume give a review of more recent work on viscoelasticity and rheology. As to statics and dynamics of polymers, Douglas [43] has emphasized the appearance of fractional integral equations for surface interacting polymers and other systems. See also Chapter VI of this volume.

Of recent increasing interest in the area of fractals and nonlinear dynamics are fractional dynamical systems. Such systems can be discussed in the frame of abstract ergodic theory as flows or semiflows on measure spaces involving fractional time derivatives. They are related to problems of time irreversibility and ergodicity breaking. For literature in this respect see e.g. R. Hilfer ([78,79]), H. Hayakawa [69].

A new class of phase transitions, called anequilibrium transitions, has recently been introduced by R. Hilfer [73]-[75]. It characterizes each phase transition by its generalized noninteger order and a slowly varying function (in the sense of E. Seneta, see e.g. Jansche [84]) Thermodynamically this characterization arises from generalizing the classification of P. Ehrenfest (1933) to noninteger orders $\lambda \geq 1$ but also $\lambda < 1$.

In the setting of Hamiltonian chaotic analysis anomalous kinetics of particles has been studied by G.M. Zaslavsky in real systems. Such systems have fractal sets of islands or tori (cylinders) in the phase space, and these sets are responsible for the anomalous kinetics. In this respect a new fractional Fokker-Planck-Kolmogorov equation describes the particle kinetics; it can be considered as a new universal scheme for those processes where chaotic dynamics meet strongly correlated ballistic dynamics similar to the Lévy flights. See [180] and its literature as well as Chapters IV and V of this volume for more information on this subject.

5 Integral Transforms and Fractional Calculus

5.1 Fourier transforms

The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$, defined by

$$\mathcal{F}[f](v) = f^{\wedge}(v) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-ivu} du, \quad v \in \mathbb{R}, \tag{5.1}$$

is a powerful tool in the analysis of operators commuting with the translation operator (see e.g. [25]). Its inverse is given by

$$f(x) = \mathcal{F}^{-1}[f^{\wedge}(v)](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f^{\wedge}(v) e^{ixv} dv$$

for almost all $x \in \mathbb{R}$, if f and $f^{\wedge}$ belong to $L^1(\mathbb{R})$. Two of the basic properties of the Fourier transform are

$$\mathcal{F}[f^{(n)}](v) = (iv)^n f^{\wedge}(v), \quad v \in \mathbb{R} \tag{5.2}$$

$$[\mathcal{F}f]^{(n)}(v) = \mathcal{F}[(-ix)^n f(x)](v) \tag{5.3}$$

valid for sufficiently smooth functions f; (5.2) holds if, for example, $f \in L^1(\mathbb{R}) \cap AC_{\text{loc}}^{n-1}(\mathbb{R})$ with $f^{(n)} \in L^1(\mathbb{R})$, while for (5.3) it is sufficient that f as well as $x^n f(x)$ belong to $L^1(\mathbb{R})$. The convolution theorem

$$\mathcal{F}[f * g](v) = \mathcal{F}[f](v)\mathcal{F}[g](v) \tag{5.4}$$

holds, for example, for $f, g \in L^1(\mathbb{R})$, where

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u)g(u)du. \tag{5.5}$$

The fractional integral ${}_{-\infty}I_x^\alpha f$ of (1.12) is such a commuting operator. The counterpart of (5.2) reads for such operators (see e.g. [147, p. 32]

Lemma 5.1. *If* $f \in L^1(\mathbb{R})$ *and* $0 < \Re e\, \alpha < 1$, *then*

$$\mathcal{F}[{}_{-\infty}I_x^\alpha f](v) = (iv)^{-\alpha} f^\wedge(v). \tag{5.6}$$

Indeed, changing the order of integration,

$$\int_{-N}^{N} [{}_{-\infty}I_x^\alpha f(x)]e^{-ivx}dx = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{-N} e^{-itv} f(t)dt \int_{-N-t}^{N-t} e^{-iyv} y^{\alpha-1} dy$$

$$+\frac{1}{\Gamma(\alpha)} \int_{-N}^{N} e^{-itv} f(t)dt \int_{0}^{N-t} e^{-iyv} y^{\alpha-1} dy.$$

But for any fixed $v \neq 0$, $0 < \Re e\, \alpha < 1$,

$$\int_{0}^{\infty} e^{-iyv} y^{\alpha-1} dy = (iv)^{-\alpha}\Gamma(\alpha). \tag{5.7}$$

Here the Fourier integral on the left is to be understood in the sense of a principal value. Hence, taking the limit above for $N \to \infty$, there results (5.6). Thus the action of fractional integration under Fourier transforms is reduced to dividing the Fourier transform by $(iv)^\alpha$.

Equation (5.6) cannot be extended to values $\Re e\, \alpha \geq 1$ immediately since the left-hand side of (5.6) may then not exist even for very smooth functions, e.g. $f \in C_0^\infty$, the space of all infinitely differentiable functions with compact support in $\mathbb{R}$. Indeed, for $\alpha = 1$, ${}_{-\infty}I_x^1 f(x) = \int_{-\infty}^x f(u)du$, so that ${}_{-\infty}I_x^1 f(x) \to$ const. as $x \to \infty$, implying that $\mathcal{F}[{}_{-\infty}I_x^1 f]$ does not exist in the usual sense (see e.g. [153, p. 139]).

If $\alpha > 1$ take a non-negative function $f \in C_0^\infty$ to be positive on some interval $[a, b]$. Then for $x > b$

$$ {}_{-\infty}I_x^\alpha f(x) \geq \frac{1}{\Gamma(\alpha)} \int_a^b (x-u)^{\alpha-1} f(u)du \geq \min_{a \leq u \leq b} f(u) \frac{(x-a)^\alpha - (x-b)^\alpha}{\Gamma(\alpha+1)}, $$

so that ${}_{-\infty}I_x^\alpha f(x) \sim Cx^{\alpha-1}/\Gamma(\alpha)$ as $x \to \infty$, implying again that $\mathcal{F}[{}_{-\infty}I_x^\alpha f]$ does not exist in the usual sense. Thus one can only expect to extend (5.6) to all α with $\Re e\, \alpha > 0$ for a special class of functions f.

For this purpose, let $S = S(\mathbb{R})$ be the space of Schwartz test functions, i.e., those $f(x)$, $x \in \mathbb{R}$ which are infinitely differentiable and rapidly decreasing together with their derivatives as $|x| \to \infty$, thus

$$ \lim_{|x| \to \infty} (1 + x^2)^m f^{(j)}(x) = 0, \qquad \text{all } m \geq 0, j \geq 0. $$

$S(\mathbb{R})$ is a topological vector space, the topology being given by the family of seminorms $\|f\|_{j,m} = \sup_{x \in \mathbb{R}} |(1+x^2)^m f^{(j)}(x)|$; it is metrizable and is complete (thus a Frechet space). Now let $\Psi = \Psi(\mathbb{R})$ be the space of functions $f \in S$ which are equal to zero at the point $x = 0$ together with their derivatives, i.e., $\Psi := \{f \in S; \quad f^{(k)}(0) = 0, \quad k \in \mathbb{N}_0.\}$ Then the Lizorkin space Φ is the space of functions in S the Fourier transforms of which belong to Ψ, thus $\Phi = \{f; \quad f \in S, \quad f^\wedge \in \Psi\}$. In fact, Φ may be characterized as the space of $f \in S$ which are orthogonal to all polynomials:

$$ \int_{\mathbb{R}} u^k f(u)du = 0, \quad k \in \mathbb{N}_0, \tag{5.8} $$

since $\int_{\mathbb{R}} e^{-iou} u^k f(u)du = i^k \sqrt{2\pi} (f^\wedge)^{(k)}(0) = 0$.

Now if f belongs to Φ, then equation (5.6) is indeed valid for all $\Re e\, \alpha \geq 0$. Let us establish it for $1 \leq \Re e\, \alpha < 2$. Indeed,

$$ \Gamma(\alpha)\mathcal{F}[{}_{-\infty}I_x^\alpha f](v) = \lim_{N \to \infty} \int_{-\infty}^N e^{-itv} dt \int_{-\infty}^t (t-u)^{\alpha-1} f(u)du $$

$$ = \lim_{N \to \infty} \int_{-\infty}^N e^{-ivu} f(u)du \int_0^{N-u} y^{\alpha-1} e^{-ivy} dy. $$

If $\Re e\, \alpha \neq 1$, integration by parts yields

$$(iv)\Gamma(\alpha)\mathcal{F}[{}_{-\infty}I_x^\alpha f](v) = \lim_{N\to\infty}\{-N^{\alpha-1}e^{-ivN}\cdot \int_{-\infty}^{N}(1-\frac{u}{N})^{\alpha-1}f(u)du+$$

$$(\alpha-1)\int_{-\infty}^{N} f(u)e^{-ivu}du\cdot \int_{0}^{N-u} y^{\alpha-2}e^{-ivy}dy\}.$$

The first term on the right tends to zero by (5.8) (noting L'Hospital's rule), while in the second one can pass to the limit directly since $\Re e\, \alpha < 2$. Noting (5.7), this yields $\Gamma(\alpha)\mathcal{F}[{}_{-\infty}I^\alpha f](v) = (\alpha-1)\Gamma(\alpha-1)(iv)^{-\alpha}f^\wedge(v)$, as required. The case $\alpha = 1$ is simple, since, by (5.8),

$$\mathcal{F}[{}_{-\infty}I_x^1 f](v) = \lim_{N\to\infty}\int_{-\infty}^{N}\frac{e^{-iuv}-e^{-ivN}}{iv}f(u)du = \frac{1}{iv}f^\wedge(v).$$

For the case $\Re e\ \alpha = 1$, $\alpha \neq 1$, i.e., $\alpha = 1+i\vartheta$, $\vartheta \neq 0$, which is quite technical (see [153, p. 149]).

As to the fractional counterpart of (5.2), we have, by Lemma 5.1 and (5.2) for sufficiently smooth functions f, for example, $f \in S$ ($n = [\Re e\ \alpha]+1$),

$$\mathcal{F}[{}_{-\infty}D_x^\alpha f](v) = \mathcal{F}[{}_{-\infty}I_x^{n-\alpha}f^{(n)}(x)](v) = (iv)^{-(n-\alpha)}(f^{(n)})^\wedge(v)$$

$$= (iv)^{-(n-\alpha)}(iv)^n f^\wedge(v) = (iv)^\alpha f^\wedge(v).$$

Let us finally remark that there exists a *fractional* Fourier transform $\mathcal{F}_\alpha$ of order α, introduced by N. Wiener (1929). It is a unitary integral transform on $L^2(\mathbb{R})$, the eigenvalues and eigenfunctions of which are, respectively, $e^{-in\alpha}$ and the normalized Hermite functions h_n ($n \in \mathbb{N}_0$). It is defined for all $\alpha \in \mathbb{R}$ by the limit in $L^2(\mathbb{R})$-norm

$$\mathcal{F}_\alpha f(\cdot) = l.i.m._{N\to\infty}\int_{-N}^{N} K_\alpha(\cdot,u)f(u)du,$$

where

$$K_\alpha(v,u) = \begin{cases} A_\alpha \exp\left(\frac{iv^2}{2}\cot\alpha\right)\exp\left(-\frac{ivu}{\sin\alpha}+\frac{iu^2}{2}\cot\alpha\right) & \alpha \notin \{l\pi; l\in\mathbb{Z}\} \\ \delta(u-v) & \alpha \in \{2\pi l; l\in\mathbb{Z}\} \\ \delta(u+v) & \alpha \in \{2\pi l+\pi; l\in\mathbb{Z}\} \end{cases}$$

and $A_\alpha = \exp[-i\left(\frac{\pi}{4}\text{sgn}\,\alpha - \frac{\alpha}{2}\right)]/\sqrt{2\pi|\sin\alpha|}$.

For $\alpha = \pi/2$ we obtain the classical Fourier transform $\mathcal{F}$ and for $\alpha = -\pi/2$ its inverse transform $\mathcal{F}^{-1}$. For $\alpha = n\pi/2$, $n \in \mathbb{Z}$, we have $\mathcal{F}^n$, the *nth* power of $\mathcal{F}$. $\mathcal{F}_\alpha$ satisfies the index law $\mathcal{F}_\alpha \mathcal{F}_\beta f = \mathcal{F}_{\alpha+\beta} f$ forall $f \in L^2(\mathbb{R})$. The inverse of $\mathcal{F}_\alpha$ is $\mathcal{F}_{-\alpha}$. This transform is of interest in sampling theory of signal analysis in case the samples are not equally spaced apart but are irregular. It is of primary interest in the case of complex valued functions. See e.g. B. Bittner [9] and especially the literature cited there. For sampling theory itself one may consult Higgins [71] and Higgins-Stens [72].

5.2 Mellin transforms

The Mellin transform of $f : \mathbb{R}^+ \to \mathbb{C}$, defined by

$$\mathcal{M}[f(u)](s) = f^\wedge_\mathcal{M}(s) = \int_0^\infty u^{s-1} f(u)du \quad (s = c + it) \tag{5.9}$$

for $f \in X_c := \{f : \mathbb{R}^+ \to \mathbb{C},\ \|f\|_{X_c} = \int_0^\infty |f(u)|u^{c-1}du < \infty\}$, for some $c \in \mathbb{R}$, has its inverse given by

$$f(x) = \mathcal{M}^{-1}[f^\wedge_\mathcal{M}(s)](x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^\wedge_\mathcal{M}(s)x^{-s}ds, \tag{5.10}$$

provided $f^\wedge_\mathcal{M} \in L^1(\{c\} \times i\mathbb{R})$. The Mellin convolution of f and g is given by

$$(f \circ g)(x) := \int_0^\infty f\left(\frac{x}{u}\right) g(u)\frac{du}{u} \tag{5.11}$$

provided it exists, and the associated convolution theorem reads for $f, g \in X_c$,

$$[f \circ g]^\wedge_\mathcal{M}(s) = [f]^\wedge_\mathcal{M}(s)[g]^\wedge_\mathcal{M}(s). \tag{5.12}$$

For the Mellin differential operator Θ_c, defined for $c \in \mathbb{R}$ by

$$\Theta_c f(x) := xf'(x) + cf(x), \quad x \in \mathbb{R}_+ \tag{5.13}$$

in case it exists, and that of order $n \in \mathbb{N}$ iteratively by $\Theta_c^1 f := \Theta_c f$, $\Theta_c^n f = \Theta_c(\Theta_c^{n-1} f)$, one has for $s = c + it$, $t \in \mathbb{R}$,

$$[\Theta_c^n f]^\wedge_\mathcal{M}(s) = (-it)^n f^\wedge_\mathcal{M}(s) \tag{5.14}$$

provided $f \in AC_{\text{loc}}^{n-1}(\mathbb{R}_+) \cap X_c$ and $\Theta_c^n f \in X_c$. See especially [23] and the literature cited there for the foregoing approach to Mellin transforms.

Whereas this is the natural operator of differentiation in the Mellin setting, for the classical *nth* order derivative $D_x^n f(x)$ we have

$$\mathcal{M}[D_x^n f](s) = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} f_{\mathcal{M}}^{\wedge}(s-n), \tag{5.15}$$

provided $f \in AC_{\text{loc}}^{n-1}(\mathbb{R}_+)$ and $D_x^k f \in X_{c+k-n}$ for $k = 0, \dots, n$. Now for the classical fractional integral on $\mathbb{R}_+$, given by

$${}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)du \tag{5.16}$$

one has

$$\mathcal{M}[{}_0I_x^\alpha f(x)](s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)} f_{\mathcal{M}}^{\wedge}(s+\alpha) \tag{5.17}$$

if $f \in X_{c+\Re e\,\alpha}$ and $c + \Re e\,\alpha < 1$. In fact, formally

$$\mathcal{M}\Big[\frac{x^\alpha}{\Gamma(\alpha)} \int_1^\infty \Big(1-\frac{1}{y}\Big)^{\alpha-1} f\big(\frac{x}{y}\big)\frac{dy}{y^2}\Big](s) = \mathcal{M}[x^\alpha \cdot (g_\alpha \circ f)(x)](s),$$

where $g_\alpha(x) = \Gamma(\alpha)^{-1}(x-1)_+^{\alpha-1}x^{-\alpha}$ (see [147, p. 43]) with $\mathcal{M}[g_\alpha(x)](s) = \Gamma(1-s)/\Gamma(1-s+\alpha)$, $\Re e\, s < 1$. Now $\mathcal{M}[x^\alpha f(x)](s) = f_{\mathcal{M}}^{\wedge}(s+\alpha)$ so that (5.17) follows since

$$\mathcal{M}[x^\alpha (g_\alpha \circ f)(x)](s) = [g_\alpha \circ f]_{\mathcal{M}}^{\wedge}(s+\alpha) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-(s+\alpha)+\alpha)} \cdot f_{\mathcal{M}}^{\wedge}(s+\alpha).$$

As to the fractional derivative on $\mathbb{R}^+$, one has by (5.15) under suitable conditions

$$\mathcal{M}[{}_0D_x^\alpha f](s) = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} \mathcal{M}[{}_0I_x^{n-\alpha} f](s-n) \quad (n-1 \le \Re e\,\alpha < n). \tag{5.18}$$

Hence by (5.17)

$$\mathcal{M}[{}_0D_x^\alpha f](s) = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} \frac{\Gamma(1-(s-\alpha))}{\Gamma(1-s+n)} f_{\mathcal{M}}^{\wedge}(s-\alpha).$$

By applying the functional equation

$$\Gamma(1-(s-\alpha)) = \frac{\pi}{\Gamma(s-\alpha)\sin\pi(s-\alpha)}$$

as well as ditto for $\alpha = n$, then

$$\begin{aligned}\mathcal{M}[{}_0D_x^\alpha f](s) &= (-1)^n\Gamma(s)\frac{\pi}{\Gamma(s-\alpha)\sin\pi(s-\alpha)}\frac{\sin\pi(s-n)}{\pi}f_{\mathcal{M}}^{\wedge}(s-\alpha) \\ &= \frac{\sin\pi s}{\sin\pi(s-\alpha)}\frac{\Gamma(s)}{\Gamma(s-\alpha)}\cdot f_{\mathcal{M}}^{\wedge}(s-\alpha).\end{aligned} \tag{5.19}$$

Sufficient conditions for the validity of (5.19) in case $n-1 \le \Re e\ \alpha < n$ are, for instance, that $c+n-\Re e\ \alpha < 1$, $f \in AC_{\mathrm{loc}}^{n-1}(\mathbb{R}_+)$ and $D_x^k f \in X_{c+k-\Re e\ \alpha}$ for $k = 0, \dots, n$. Note that (5.19) reduces to (5.15) for $\alpha = n$. Thus the factor $(\sin\pi s)/\sin\pi(s-\alpha)$ is a replacement for $(-1)^n$ in the fractional case.

Observe that the fractional integral defined by (5.16) is not the natural one connected with a possible generalization of the differential operator given via (5.14) to the fractional case in the sense that one is the inverse to the other (in the sense of Theorem 11 of [23]). What would be needed is a fractional integral which is the true counterpart of the new differential operator (5.13), at first in the particular case $\alpha = n$. In this respect see also the theory of fractional powers of some classes of operators, including Riemann-Liouville and Weyl fractional integrals, via a Mellin transform approach due to McBride [114]. Rooney [143] presented an approach via Mellin multipliers.

The fractional integral $\Gamma(\alpha)\ {}_{-\infty}I_x^\alpha f(x) = \int_0^\infty u^{\alpha-1}f(x-u)du$, $\Re e\ \alpha > 0$ is the Mellin transform of $f(x-u)$. Applying the inverse Mellin transform $\mathcal{M}^{-1}$, we deduce the following integral representation of the function $f(x)$ in terms of its fractional integral, namely

$$f(x-u) = \frac{1}{2\pi i}\int_{\Re e\ \alpha - i\infty}^{\Re e\ \alpha + i\infty} \Gamma(\alpha)\ {}_{-\infty}I_x^\alpha f(x)u^{-\alpha}d\alpha \tag{5.20}$$

provided, for fixed $x \in \mathbb{R}$ and some $c > 0$, the function $\Gamma(\alpha)_{-\infty}I_x^\alpha f(x)$ belongs to $L^1(\{c\}\times i\mathbb{R})$ with respect to α.

This formula may be regarded as another integral counterpart of the Taylor series expansion of $f(x-u)$. Further, taking $x = 0$, we have

$$f(-u) = \frac{1}{2\pi i}\int_{\Re e\ \alpha - i\infty}^{\Re e\ \alpha + i\infty} \Gamma(\alpha)\ {}_{-\infty}I_x^\alpha\ f(x)|_{x=0}\, u^{-\alpha}d\alpha \quad (u > 0).$$

This implies that $f(x)$ can be recovered from the values of its fractional integrals ${}_{-\infty}I_x^\alpha f(x)$ taken at only the one point $x = 0$ if these are known for all α on the vertical line $\Re e\, \alpha =$ const > 0.

5.3 Laplace transforms and characterizations of fractional derivatives

The Laplace transform of a locally integrable function $f : \mathbb{R}_+ \to \mathbb{C}$, defined by

$$\mathcal{L}[f](s) = f_{\mathcal{L}}^\wedge(s) = \int_0^\infty e^{-su} f(u)du \tag{5.21}$$

is another powerful tool in solving differential and integral equations of fractional analysis. In this framework it is appropriate to assume that the integral in (5.21) converges absolutely in the complex half-plane $\Re e\, s > 0$. If, in addition, f is of bounded variation in a neighbourhood of some $x > 0$, then there holds the inversion formula

$$\mathcal{L}^{-1}[f_{\mathcal{L}}^\wedge](x) = PV\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} f_{\mathcal{L}}^\wedge(s)ds = \frac{f(x+0)+f(x-0)}{2} \quad (x > 0) \tag{5.22}$$

for each fixed $\sigma > 0$. The associated convolution relation is given by

$$(f \otimes g)(x) = \int_0^x f(x-u)g(u)du \tag{5.23}$$

the transform of which is $\mathcal{L}[f \otimes g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s)$. Thus the transform of the fractional integral ${}_0I_x^\alpha f$, $\Re e\, \alpha > 0$, namely the Laplace convolution

$${}_0I_x^\alpha f(x) = \left(f \otimes \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}\right)(x),$$

is given by, noting $\mathcal{L}[x_+^{\alpha-1}/\Gamma(\alpha)](s) = s^{-\alpha}$ for $\Re e\, \alpha > 0$,

$$\mathcal{L}[{}_0I_x^\alpha f](s) = s^{-\alpha}\mathcal{L}[f](s), \tag{5.24}$$

valid for sufficiently good functions.

As to fractional differentiation, first recall that for $n \in \mathbb{N}$

$$\mathcal{L}[\frac{d^n}{dx^n} f](s) = s^n \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0+), \tag{5.25}$$

so that

$$\begin{aligned} \mathcal{L}[{}_0D_x^\alpha f](s) &= \mathcal{L}[\frac{d^n}{dx^n}\ {}_0I_x^{n-\alpha} f](s) \\ &= s^n \mathcal{L}[{}_0I_x^{n-\alpha} f](s) - \sum_{k=0}^{n-1} s^k\ \frac{d^{n-1-k}}{dx^{n-1-k}}\ {}_0I_x^{n-\alpha} f(0+) \\ &= s^\alpha \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^k\ {}_0D_x^{\alpha-1-k} f(0+) \quad (n-1 < \alpha < n). \end{aligned} \tag{5.26}$$

If, however, one would work with definition (1.26) of fractional differentiation, i.e. $\overline{{}_0D_x^\alpha} f :={}_0 I_x^{n-\alpha} f^{(n)}(x)$, then

$$\mathcal{L}[{}_0\overline{D}_x^\alpha f](s) = s^\alpha \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0+) \quad (n-1 < \alpha < n). \tag{5.27}$$

The essential difference between the expansions (5.26) and (5.27) is that the former involves the fractional derivatives ${}_0D_x^{\alpha-1-k} f(0+)$ $(k = 0, 1, .., n-1)$, whereas the latter involves only the integer derivatives $f^{(k)}(0+)$.

G. Doetsch [42, p. 163ff.] defines the fractional derivative as the solution g of the integral equation

$${}_0I_x^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} g(u) du = f(x)$$

in case it exists. Let us denote it by ${}_0^*D_x^\alpha f$. Then there holds the following connection between the three definitions: *Let $f, g \in L_{\text{loc}}(\mathbb{R}_+)$ and let their Laplace transforms converge absolutely for each s with $\Re e\ s > 0$. Then the following three assertions (i) - (iii) are equivalent for $n - 1 \le \alpha < n$:*

(i) $s^\alpha f_{\mathcal{L}}^\wedge(s) = g_{\mathcal{L}}^\wedge(s) \quad (\Re e\ s > 0)$,

(ii) $g = {}_0^*D_x^\alpha f$,

(iii) $\frac{d^k}{dx^k}[{}_0I_x^{n-\alpha}] f$ *are locally absolutely continuous on* $[0, \infty)$ *and equal to zero at* $x = 0$ *for* $k = 0, 1, \ldots, n-1$, *together with* ${}_0D_x^\alpha f = g$.

Each of these assertions is implied by the following sufficient condition:

(iv) $f^{(k)}(x)$ *are locally absolutely continuous on* $[0, \infty)$ *and equal to zero at* $x = 0$ *for* $k = 0, 1, \ldots, n-1$, *together with* ${}_0\overline{D}_x^{\alpha} f = g$.

For functions belonging to $L^p(\mathbb{R}_+)$, $1 \le p < \infty$, these concepts can be related to the Marchaud-type derivative considered in Section 3.5; cf. Theorems 3.8–3.10.

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