Properties and Applications of the Caputo Fractional Operator

Master Thesis

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submitted by

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Hereby I declare that the present master thesis is written by myself and no other sources than the references are used.

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Contents

Introduction

The fractional calculus is a theory of integrals and derivatives of arbitrary real or even complex order. It is a generalization of the classical calculus and therefore preserves many of the basic properties As an intensively developing area of the calculus during the last couple decades it offers tremendous new features for research and thus becomes more and more in use in various applications.

- **Historical Development**

The beginning of the fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1695, where the notation for differentiation of non-integer order $1/2$ is discussed. In addition, Leibniz writes: "Thus it follows that $d^{1/2}$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn" (see Miller and Ross [7], p. 1).

Nowadays, not only fractions but also arbitrary real and even complex numbers are considered as order of differentiation. Nevertheless, the name "fractional calculus" is kept for the general theory.

A lot of contributions to the theory of fractional calculus up to the middle of the 20-th century, of famous mathematicians are known: Laplace (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1837), Riemann (1847), Grünwald (1867-1872), Letnikov (1868-1872), Heaviside (1892-1912), Weyl (1917), Erdélyi (1939-1965) and many others (see Gorenflo and Mainardi [6]). However, this topic is a matter of particular interest just the last thirty years. The first specialized conference on fractional calculus and its applications in 1974 at the University of New Haven, USA, initiates the up-to-date books of Oldham and Spanier ([9], 1974) (the first monograph in the field), Samko, Kilbas, and Marichev ([12], 1987), Miller and Ross ([7], 1993), Podlubny ([10], 1999), etc.

- **The Fractional Calculus**

Fractional calculus (see Podlubny [10]) is a name for the theory of integrals and derivatives of arbitrary order (called fractional integrals and derivatives), which unifies and generalizes the integer-order differentiation and n-fold integration. In other words, fractional derivatives and integrals can be considered as an "interpolation" of the infinite sequence

$$\cdots \,, \quad \int_a^t \int_a^{\tau_1} f(\tau_2)d\tau_2 d\tau_1 \,, \quad \int_a^t f(\tau_1)d\tau_1 \,, \quad f(t) \,, \quad \frac{df(t)}{dt} \,, \quad \frac{d^2 f(t)}{dt^2} \,, \quad \cdots$$

of the classical n-fold integrals and n-fold derivatives.

- **Geometric and Physical Interpretation**

Integer-order derivatives and integrals have clear physical interpretation and are used for describing different concepts in classical physics. For example, the position of a moving object can be represented as a function of time, the object's velocity is then the first derivative of the function, the acceleration is the second derivative and so on. Fractional derivatives and integrals, being generalization of the classical derivatives and integrals, are expected to have even broader meaning. Unfortunately, there is no such result in the literature until now.

Some authors (Moshrefi-Torbati and Hammond [8]) consider the fractional operators as linear filters and also seek the geometrical interpretation of the fractional operators in the fractal geometry, of which classical geometry is a subclass. The fractal Cantor's set and a domino ladder network (series of resistors and capacitors that can be connected in different configurations) are used as illustration. The conventional physics and geometry are restricted to rigid boundaries and integer dimensions. Functions and processes that fall between discrete dimensions cannot be described. An example of it is a Cantor's set that has a dimension between that of a line and a point. But by means of fractal geometry the properties of any system with non-integer dimension can be interpreted geometrically.

Another author (Podlubny [11]) provides a physical interpretation of the fractional integration in terms of two different time scales, namely, the homogeneous, equably flowing scale and the inhomogeneous time scale.

- **Application of Fractional Calculus**

The first application of a semi-derivative (derivative of order $1/2$) is done by Abel in 1823 (see Miller and Ross [7], Oldham and Spanier [9]). This application of fractional calculus is in relation with the solution of the integral equation for the tautochrone problem. That problem deals with the determination of the shape of the curve such that the time of descent of a frictionless point mass sliding down along the curve under the action of gravity is independent of the starting point.

The last decades prove that derivatives and integrals of arbitrary order are very convenient for describing properties of real materials, e.g., polymers (Podlubny [10]). The new fractional-order models are more satisfying than former integer-order ones. Fractional derivatives are an excellent tool for describing the memory and hereditary properties of various materials and processes while in integer-order models such effects are neglected.

The fractional calculus finds also applications in different fields of science (see Gorenflo and Mainardi [6]), including theory of fractals, numerical analysis, physics, engineering, biology, economics and finance. Some problems of viscoelasticity are formulated and solved by M. Caputo (see Podlubny [10]) with his own definition of fractional differentiation. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, where for the description of the controlled system and the controller fractional differential equations are used.

- **The Problem**

Integer-order derivative and integral are uniquely determined in the classical analysis. It is also the same for the fractional integral, since in the literature (Miller and Ross [7], Oldham and Spanier [9], Podlubny [10], Samko, Kilbas, and Marichev [12]) an unique definition is used (see Subsection 1.2). However, for the fractional derivative the situation is more complicated. There are many different definitions, which do not coincide in general. Possibly it is due to the fact, that the different authors try to preserve different properties of the classical integer-order derivative.

- **The Continuous and the Discrete Approaches**

There are two main approaches to the fractional calculus (see Gorenflo and Mainardi [6]), namely, the continuous and the discrete approaches. The continuous approach is based on the Riemann-Liouville fractional integral, which has the Cauchy integral formula ((1.7), Oldham and Spanier [9], p. 38) as a starting point. The discrete approach is based on the Grünwald-Letnikov fractional derivative. As a generalization of the fact that ordinary derivatives are limits of difference quotients, the Grünwald-Letnikov fractional derivative is defined as a limit of a fractional-order backward difference (Podlubny [10], p.52). More about the Grünwald-Letnikov approach can be found in the paper of Gorenflo and Mainardi [6] and in the book of Podlubny [10]. This survey is focused on the continuous approach, which also has different branches (see Subsection 1.3 and Subsection 1.4).

- **Motivation and Description of the Survey**

This survey studies the two mostly used definitions for fractional differentiation, namely, the Riemann-Liouville and Caputo fractional operators. The emphasis is given to the Caputo operator. The definition of Riemann-Liouville plays an important role mainly in the development of the theory of fractional derivatives and integrals and for its application in pure mathematics (solution of integer-order differential equations, definitions of new function classes and so on) (see Podlubny [10]). However, applied problems require proper definitions of fractional derivatives which can provide initial conditions with clear physical interpretation for the differential equations of fractional order. This makes the Caputo fractional derivative (see Subsection 1.4 and Subsection 1.3) more suitable to be applied. Since it is usually not discussed in the literature, the following master thesis contains results mainly on the properties of the Caputo fractional derivative. Proofs of some of the results, published in the paper of Diethelm, Ford, Freed, and Luchko [3] are given and some new results are presented in Section 3, Section 4, and Section 5.

This master thesis consists of 5 sections.

The special functions as Gamma and Beta functions, the complementary error function, Mittag-Leffler function, and the confluent hypergeometric function are introduced in Section 1. These functions are most frequently used in the fractional calculus and especially in solving fractional differential equations. The Riemann-Liouville and the Caputo definitions of the fractional derivatives are also given in the same section.

The Caputo fractional derivative is discussed in more details in Section 2. A comparison with the Riemann-Liouville fractional derivative is made and a proof of an important relation formula that appears in the paper of Gorenflo and Mainardi [6] without a proof, between these two derivatives, is deduced. As a corollary, the Leibniz rule for the Caputo operator is also derived.

Some interesting examples of Caputo fractional derivatives of arbitrary constant, power, exponential, sine and cosine functions are studied in Section 3. Proofs for the general formulas are provided that cannot be found in the literature. Some special cases are visualized by drawing graphs. All the results are summarized in a special chart.

Fractional initial value problems with Caputo derivatives are discussed in Section 4. These are fractional ordinary differential equations with classical initial conditions, which are not open discussed until now. A general solution using the Laplace transform method is obtained in this survey. The convergence of the solution is also studied and some interesting new properties are discovered. Particular examples are calculated and illustrated by their graphs.

The main aim in Section 5 is the introduction of the C-Laguerre functions as a generalization of the classical Laguerre polynomials by means of the Caputo fractional operator. Since thus generalized functions are unknown so far, some of the fundamental properties are examined and compared with the classical case. The main results are summarized at the end of the section.

Appendix A contains a table of Caputo derivatives of some particular functions. Appendix B consists of the original programs created by Matlab for the visualization of the results in this survey. There are 12 entries in the references, published on the topic up to 2003.

The figures are numbered with two numbers. The first one is for the number of the section, the second one is for the number of the figure in the section. The same rule holds for the formulas as well as for definitions, theorems, lemmas, corollaries, comments and propositions, which have common numbering. The tables are numbered separately.

- **Acknowledgements**

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1 Basic Definitions

Some special functions, important for the fractional calculus, as Gamma and Beta functions, the complementary error function, Mittag-Leffler function, and the confluent hypergeometric function are summarized in this section (see Gradshteyn and Ryzhik [4], Samko, Kilbas and Marichev [12], Podlubny [10]).

Furthermore, fractional integration as well as Riemann-Liouville and Caputo fractional differentiation operators are introduced (see Podlubny [10], Gorenflo and Mainardi [6]).

1.1 Special Functions

- **The Gamma Function**

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$, i. e., $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt\,, \quad \operatorname{Re} z > 0\,.$$

By analytic continuation the function is extended to the whole complex plane except for the points $0,-1,-2,-3,\ldots$, where it has simple poles. Thus, $\Gamma : \mathbb{C} \setminus \{0,-1,-2,\ldots\} \longrightarrow \mathbb{C}$. Fig. 1.1 presents the graph of the function.

Some of the most important properties are

$$\begin{aligned} \Gamma(1) = \Gamma(2) &= 1\,, \\ \Gamma(z+1) &= z\Gamma(z)\,, \\ \Gamma(n) &= (n-1)!\,,\ n \in \mathbb{N}, \\ \Gamma(1/2) &= \sqrt{\pi}\,, \\ \Gamma(n+1/2) &= \frac{\sqrt{\pi}}{2^n}\,(2n-1)!!\,,\ n \in \mathbb{N}. \end{aligned} \tag{1.1}$$

The Gamma function is studied by many mathematicians. There is a long list of well-known properties (see, for example, Gradshteyn and Ryzhik [4], pp. 933-938) but in this survey formulas (1.1) are sufficient.

- **The Beta Function**

The Beta function is defined by the integral

$$B(z,\, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt\,, \quad \operatorname{Re} z > 0\,, \quad \operatorname{Re} w > 0\,.$$

In addition, $B(z,\, w)$ is used sometimes for convenience to replace a combination of Gamma functions. This relation between the Gamma and Beta function (Gradshteyn

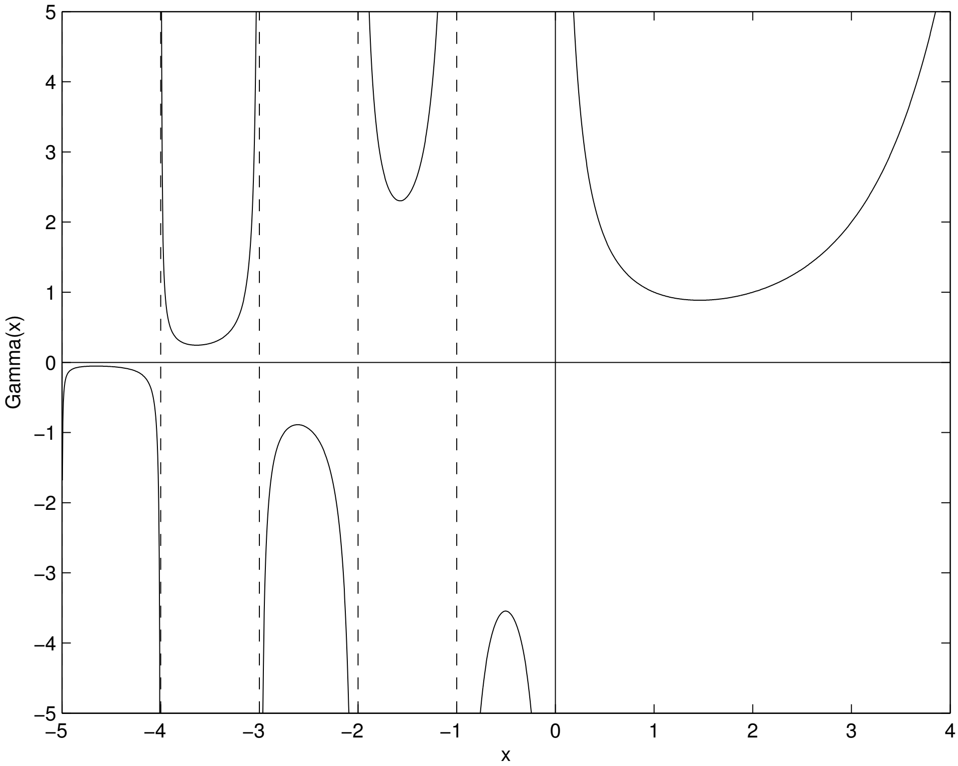

Figure 1.1: The Gamma function for real argument

and Ryzhik [4], p. 950),

$$B(z,\, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{1.2}$$

is used later on.

Equation (1.2) provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the Gamma function.

It should also be mentioned that the Beta function is symmetric, i. e.,

$$B(z,\, w) = B(w,\, z)\ .$$

- **The Complementary Error Function (erfc)**

The complementary error function (see http://mathworld.wolfram.com/Erfc.html and Podlubny [10], p. 18) is an entire function, defined as

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt.$$

The graph of the function is presented in Fig. 1.2.

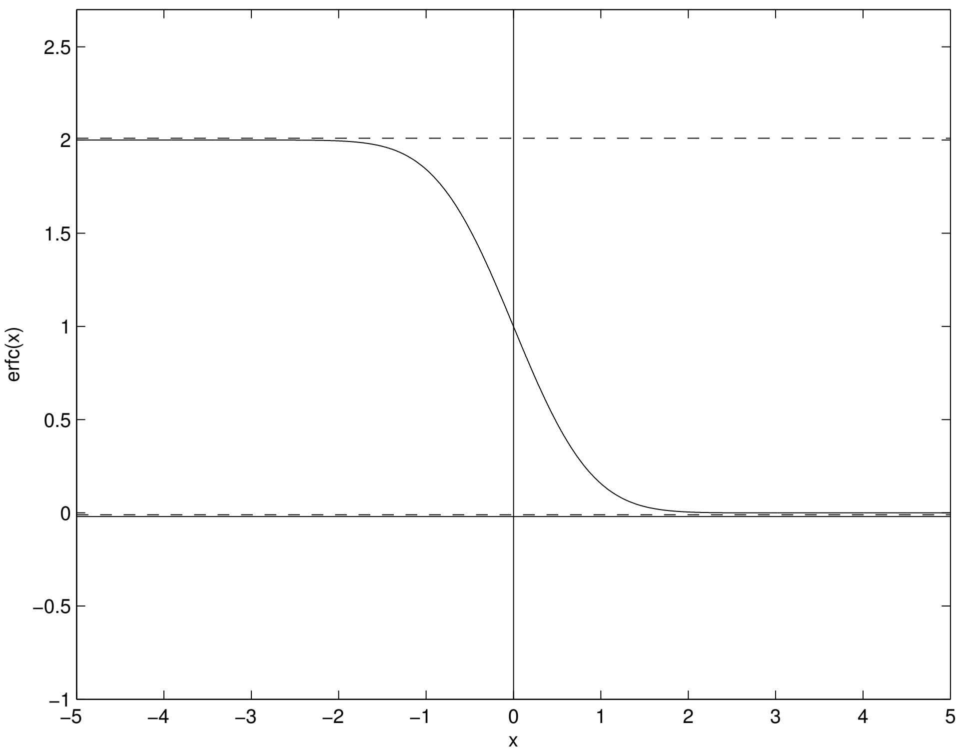

Figure 1.2: The Complementary error function

Special values of the complementary error function are

$$\begin{aligned}
\mathrm{erfc}\,(-\infty) &= 2\,, \\
\mathrm{erfc}\,(\,0\,) &= 1\,, \\
\mathrm{erfc}\,(+\infty) &= 0\,.
\end{aligned}$$

The following relations are interesting to be mentioned

$$\begin{aligned}
\mathrm{erfc}(-x) &= 2 - \mathrm{erfc}(x)\,, \\
\int_0^\infty \mathrm{erfc}(x)\,dx &= \frac{1}{\sqrt{\pi}}\,, \\
\int_0^\infty \mathrm{erfc}^2(x)\,dx &= \frac{2-\sqrt{2}}{\sqrt{\pi}}\,.
\end{aligned}$$

- **The Mittag-Leffler Function**

While the Gamma function is a generalization of the factorial function, the Mittag-Leffler function is a generalization of the exponential function, first introduced as a one-parameter function by the series (Podlubny [10], p. 16)

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0,\ \alpha \in \mathbb{R},\ z \in \mathbb{C}.$$

Later, the two-parameter generalization is introduced by Agarwal (see Fig. 1.3)

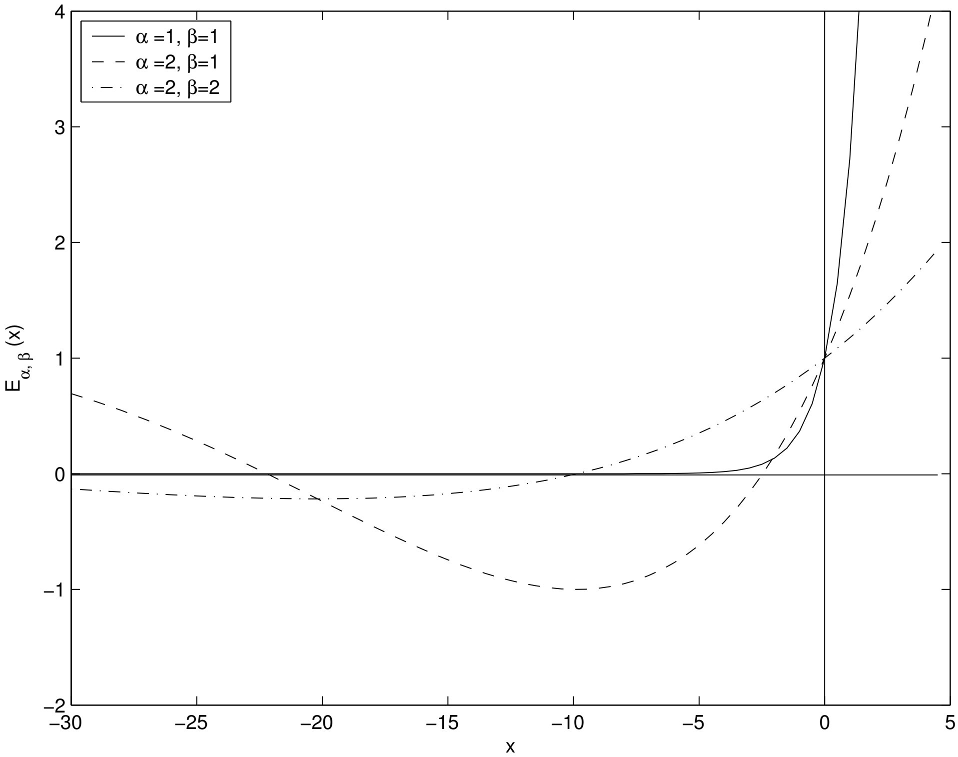

Figure 1.3: Examples of the two-parameter function of Mittag-Leffler type

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}\ , \quad \alpha,\ \beta > 0,\ \alpha,\ \beta \in \mathbb{R},\ z \in \mathbb{C}\ , \tag{1.3}$$

which is of great importance for the fractional calculus. It is called two-parameter function of Mittag-Leffler type.

Some of its interesting properties are (Podlubny [10], pp. 17-18)

$$\begin{aligned}
E_{1,1}(z) &= e^z\,, \\
E_{2,1}(z^2) &= \cosh(z)\,, \\
E_{2,2}(z^2) &= \frac{\sinh(z)}{z}\,, \\
E_{\alpha,1}(z) &= E_\alpha(z)\,, \\
E_{1/2,1}(z) &= e^{z^2}\operatorname{erfc}(-z)\,,
\end{aligned} \tag{1.4}$$

where erfc(z) is the complementary error function (see above).

- **The Confluent Hypergeometric Function**

The function

$$ {}_1F_1(a,\, b;\, z) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{z^k}{k!} \tag{1.5}$$

is called the confluent hypergeometric or Kummer function. The series converges (see Miller and Ross [7], p. 304) for $a,\, b,\, z \in \mathbb{C}$, $-b \notin \mathbb{N}_0$, $|z| < \infty$.

Comment 1.1. The function ${}_1F_1(a,\, b;\, z)$ is a generalization of the exponential function, since for $a = b$ from the definition it follows

$$ {}_1F_1(a,\, a;\, z) = \frac{\Gamma(a)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a+k)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z\,. $$

Compare it also with the corresponding property (1.4) of the Mittag-Leffler function.

Other useful properties of the confluent hypergeometric function (Gradshteyn and Ryzhik [4], p. 1058) are

$$\begin{aligned}
{}_1F_1(a,\, b;\, 0) &= 1\,, \\
\frac{d}{dz}\,{}_1F_1(a,\, b;\, z) &= \frac{a}{b}\,{}_1F_1(a+1,\, b+1;\, z)\,.
\end{aligned} \tag{1.6}$$

1.2 Fractional Integration

Cauchy's formula for repeated integration (Oldham and Spanier [9], p. 38 and Podlubny [10], p. 64)

$$J^n f(t) := \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau)\, d\tau \cdots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_a^t f(\tau)(t-\tau)^{n-1}\, d\tau \,, \qquad (1.7)$$

holds for $n \in \mathbb{N}$, $a, t \in \mathbb{R}$, $t > a$. If n is substituted by a positive real number α and $(n-1)!$ by its generalization $\Gamma(\alpha)$, a formula for fractional integration is obtained.

Definition 1.2. Suppose that $\alpha > 0$, $t > a$, $\alpha, a, t \in \mathbb{R}$. Then the fractional operator

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1}\, d\tau \qquad (1.8)$$

is referred to as Riemann-Liouville fractional integral of order α.

- **Properties**

By convention

$$J^0 f(t) := f(t) \,,$$

i. e., $J^0 := I$ is the identity operator.

Another property is the linearity

$$J^\alpha(\lambda f(t) + g(t)) = \lambda J^\alpha f(t) + J^\alpha g(t) \,, \quad \alpha \in \mathbb{R}_+ \,,\ \lambda \in \mathbb{C}.$$

If $f(t)$ is continuous for $t \geq 0$ the following equalities hold (see Podlubny [10], pp. 65-67)

$$\lim_{\alpha \to 0} J^\alpha f(t) = f(t) \,,$$
$$J^\alpha(J^\beta f(t)) = J^\beta(J^\alpha f(t)) = J^{\alpha+\beta} f(t) \,, \quad \alpha, \beta \in \mathbb{R}_+ \,,\ \lambda \in \mathbb{C} \,.$$

1.3 The Riemann-Liouville Fractional Differential Operator

After the introduction of the fractional integration operator it is reasonable to define also the fractional differentiation operator. There are different definitions, which do not coincide in general. This survey regards two of them, namely, the Riemann-Liouville and the Caputo fractional operator (see Gorenflo and Mainardi [6], Podlubny [10]).

Definition 1.3. Suppose that $\alpha > 0$, $t > a$, $\alpha, a, t \in \mathbb{R}$. Then

$$D^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau \ , & n-1 < \alpha < n \in \mathbb{N} \ , \\ \dfrac{d^n}{dt^n} f(t) \ , & \alpha = n \in \mathbb{N} \ , \end{cases} \tag{1.9}$$

is called the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order α.

Comment 1.4. It should be mentioned that the operator (1.9) is the left-inverse operator of the fractional integral (1.8) (Gorenflo and Mainardi [6]), i. e.,

$$D^{\alpha} J^{\alpha} = I \ .$$

By convention it is defined

$$D^0 f(t) := f(t) \ , \quad \text{i. e.,} \quad D^0 := I.$$

The main properties of the Riemann-Liouville operator (1.9) are treated in Section 2, together with the corresponding properties of the Caputo fractional differential operator, which are described in more details.

1.4 The Caputo Fractional Differential Operator

In this subsection an alternative operator to the Riemann-Liouville operator (1.9) is considered (see Caputo [1]).

Definition 1.5. Suppose that $\alpha > 0$, $t > a$, $\alpha, a, t \in \mathbb{R}$. The fractional operator

$$D_*^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau \ , & n-1 < \alpha < n \in \mathbb{N} \ , \\ \dfrac{d^n}{dt^n} f(t) \ , & \alpha = n \in \mathbb{N} \ , \end{cases} \tag{1.10}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α. This operator is introduced by the Italian mathematician Caputo in 1967 (Caputo [1]).

- **Example**

Let $a = 0$, $\alpha = 1/2$, $(n = 1)$, $f(t) = t$. Then, applying formula (1.10) we get

$$D_*^{1/2} t = \frac{1}{\Gamma(1/2)} \int_0^t \frac{1}{(t-\tau)^{1/2}} \, d\tau .$$

Taking into account the properties of the Gamma function (1.1) and using the substitution $u := (t-\tau)^{1/2}$ the final result for the Caputo fractional derivative of the function $f(t) = t$ is obtained as

$$\begin{aligned} D_*^{1/2} t &= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \, d(t-\tau) \\ &= -\frac{1}{\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{1}{u} \, du^2 \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2u}{u} \, du \\ &= \frac{2}{\sqrt{\pi}} (\sqrt{t} - 0) \, . \end{aligned}$$

Thus, it holds

$$D_*^{1/2} t = \frac{2\sqrt{t}}{\sqrt{\pi}} \, . \tag{1.11}$$

The same result follows from the general formula for the Caputo derivative of the power function (3.3) (see also Subsection 3.2, Examples).

Further examples are given in Appendix A (p. 58), where formulas as well as particular derivatives of some functions are summarized in a special chart.

- **Motivation**

To illustrate the main advantage of using the Caputo operator, consider the following initial value problems

$$\begin{aligned} D^\alpha y(t) - \lambda y(t) &= 0 \, , \qquad t > 0 \, , \quad n-1 < \alpha < n \, , \\ \left[D^{\alpha-k-1} y(t) \right]_{t=0} &= b_k \, , \qquad k = 0, \ldots, n-1 \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} D_*^\alpha y(t) - \lambda y(t) &= 0 \, , \qquad t > 0 \, , \quad n-1 < \alpha < n \, , \\ y^{(k)}(0) &= b_k \, , \qquad k = 0, \ldots, n-1 \, . \end{aligned} \tag{1.13}$$

In (1.12) the Riemann-Liouville fractional differentiation operator is applicable. In this case, also in the initial conditions fractional derivatives are required. Such initial value problems can successfully be solved theoretically, but their solutions are practically useless, because there is no clear physical interpretation of this type of initial conditions (Podlubny [10], p. 78). On the contrary, in (1.13) where the Caputo fractional differentiation operator is applicable, standard initial conditions in terms of derivatives of integer order are involved. These initial conditions have clear physical interpretation as an initial position $y(a)$ at the point a (where y is the unknown function), the initial velocity $y'(a)$, initial acceleration $y''(a)$ and so on.

On the other hand, the Caputo fractional derivative is more restrictive, as it can be seen from (1.9) and (1.10), since it requires the existence of the n-th derivative of the function. Fortunately, most functions that appear in applications fulfill this requirement. Later, whenever the Caputo operator is used, it is assumed that this condition is satisfied.

Comment 1.6. Unlike the Riemann-Liouville case, for which $\lim\limits_{\alpha \to n-1} D^\alpha f(t) = f^{(n-1)}(t)$, for the Caputo case $\lim\limits_{\alpha \to n-1} D_*^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$ as it is proved in Subsection 2.1. Nevertheless, it is still reasonable to set the 0-th Caputo derivative to be the identity operator, i. e., $D_*^0 := I$. Motivation for this replacement can be found in Section 2 and in Section 3.

Comment 1.7. In this master thesis the case $a = 0$ is considered only. For simplicity, since $t > a = 0$ the right-hand side limit 0^+ appears always as 0.

Comment 1.8. Sometimes the symbols ${}_aD_t^\alpha f(t)$ and ${}_a^CD_t^\alpha f(t)$ are used for the Riemann-Liouville and Caputo fractional derivatives respectively (Podlubny [10]). a and t are called terminals (lower and upper correspondingly), but since in this master thesis only $a = 0$ is considered (see Comment 1.7), the symbols $D^\alpha f(t)$ and $D_*^\alpha f(t)$ are adopted.

Comment 1.9. Taking $a = -\infty$ and requiring reasonable behavior of the function and its derivatives for $t \to -\infty$, one and the same formula is obtained for both Riemann-Liouville and Caputo fractional derivatives (Podlubny [10], p. 80), namely,

$$
{}_{-\infty}D_t^\alpha f(t) = {}_{-\infty}^{\;\;\,C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau \, .
$$

Putting $a = -\infty$ can be interpreted physically as setting the starting time of the physical process to $-\infty$. Therefore transient effects cannot be investigated. On the contrary, the steady-state dynamical processes can be studied and for these processes the Riemann-Liouville and Caputo derivatives are interchangeable.

2 The Caputo Fractional Derivative

In this section the fundamental properties of the Riemann-Liouville and Caputo fractional derivatives are discussed in a parallel (Podlubny [10]). A comparison between them is given and all the results are summarized in a table. A useful formula (given without a proof in the paper of Gorenflo and Mainardi [6]), relating the two derivatives is proved. As a corollary the Leibniz rule for the Caputo operator is derived.

2.1 Main Properties

Consider the class of functions $f(t)$, continuous and integrable in every finite interval $(0, x)$, $x \in \mathbb{R}$. Suppose also (see Podlubny [10], p. 63) that these functions may have an integrable singularity of order $r < 1$ at the point $t = 0$, i. e.,

$$\lim_{t \to 0} t^r f(t) = const \neq 0.$$

This is the class of functions, for which (1.8) and (1.9) in Subsection 1.2 and Subsection 1.3 are well-defined. As already mentioned in Subsection 1.4, for the Caputo fractional differential operator (1.10) the integrability of the n-th derivative of the function is additionally required. Later on, all the functions in this survey are considered to be in the corresponding class.

Comment 2.1. The operator D^n, $n \in \mathbb{N}$ used in the following sections is the standard integer-order differentiation operator, i. e., $D^n = \dfrac{d^n}{dt^n}$.

- **Representation**

Lemma 2.2. *Let $n - 1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $D_*^\alpha f(t)$ exists. Then*

$$D_*^\alpha f(t) = J^{n-\alpha} D^n f(t)\ . \tag{2.1}$$

This means that the Caputo fractional operator is equivalent to $(n - \alpha)$-fold integration after n-th order differentiation. Equation (2.1) follows immediately from (1.10) (see also Gorenflo and Mainardi [6]).

The Riemann-Liouville fractional derivative is equivalent to the composition of the same operators ($(n - \alpha)$-fold integration and n-th order differentiation) but in reverse order, i. e.,

$$D^\alpha f(t) = D^n J^{n-\alpha} f(t)\,. \tag{2.2}$$

From (2.1) and (2.2), since $J^{n-\alpha}D^n \neq D^n J^{n-\alpha}$ it follows the result.

Proposition 2.3. *In general the two operators, Riemann-Liouville and Caputo, do not coincide, i. e.,*

$$D^\alpha f(t) \neq D_*^\alpha f(t)\,.$$

For a class of functions the two operators are identical (see Comment 2.25).

- **Interpolation**

Lemma 2.4. *Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and $f(t)$ be such that $D_*^\alpha f(t)$ exists. Then the following properties for the Caputo operator hold*

$$\begin{aligned}\lim_{\alpha\to n} D_*^\alpha f(t) &= f^{(n)}(t)\,,\\ \lim_{\alpha\to n-1} D_*^\alpha f(t) &= f^{(n-1)}(t)-f^{(n-1)}(0)\,.\end{aligned} \tag{2.3}$$

Proof. The proof uses integration by parts (Podlubny [10], p. 79).

$$\begin{aligned}D_*^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,d\tau\\ &= \frac{1}{\Gamma(n-\alpha)}\left(-f^{(n)}(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\bigg|_{\tau=0}^{t} - \int_0^t -f^{(n+1)}(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\,d\tau\right)\\ &= \frac{1}{\Gamma(n-\alpha+1)}\left(f^{(n)}(0)t^{n-\alpha} + \int_0^t f^{(n+1)}(\tau)(t-\tau)^{n-\alpha}\,d\tau\right)\,.\end{aligned}$$

Now, by taking the limit for $\alpha\to n$ and $\alpha\to n-1$, respectively, it follows

$$\lim_{\alpha\to n} D_*^\alpha f(t) = \left(f^{(n)}(0)+f^{(n)}(\tau)\right)\Big|_{\tau=0}^{t} = f^{(n)}(t)$$

and

$$\begin{aligned}\lim_{\alpha\to n-1} D_*^\alpha f(t) &= \left(f^{(n)}(0)t+f^{(n)}(\tau)(t-\tau)\right)\Big|_{\tau=0}^{t} - \int_0^t -f^{(n)}(\tau)\,d\tau\\ &= f^{(n-1)}(\tau)\Big|_{\tau=0}^{t}\\ &= f^{(n-1)}(t)-f^{(n-1)}(0)\,. \qquad \square\end{aligned}$$

For the Riemann-Liouville fractional differential operator the corresponding interpolation property reads

$$\begin{aligned}\lim_{\alpha\to n} D^\alpha f(t) &= f^{(n)}(t)\,,\\ \lim_{\alpha\to n-1} D^\alpha f(t) &= f^{(n-1)}(t)\,.\end{aligned}$$

- **Linearity**

Lemma 2.5. *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha, \lambda \in \mathbb{C}$ and the functions $f(t)$ and $g(t)$ be such that both $D_*^\alpha f(t)$ and $D_*^\alpha g(t)$ exist. The Caputo fractional derivative is a linear operator, i. e.,*

$$D_*^\alpha(\lambda f(t) + g(t)) = \lambda D_*^\alpha f(t) + D_*^\alpha g(t) \ . \tag{2.4}$$

Proof. The proof follows straightforwardly from the definition (formula (1.8)) of fractional integration and the fact that the integral and the classical integer-order derivative are linear operators. □

Similarly, the Riemann-Liouville operator satisfies

$$D^\alpha(\lambda f(t) + g(t)) = \lambda D^\alpha f(t) + D^\alpha g(t) \ .$$

- **Non-commutation**

Lemma 2.6. *Suppose that $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and the functions $f(t)$ is such that $D_*^\alpha f(t)$ exists. Then in general*

$$D_*^\alpha D^m f(t) = D_*^{\alpha+m} f(t) \neq D^m D_*^\alpha f(t) \ . \tag{2.5}$$

Corollary 2.7. *Suppose that $n-1 < \alpha < n$, $\beta = \alpha-(n-1)$, $(0 < \beta < 1)$, $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ and the functions $f(t)$ is such that $D_*^\alpha f(t)$ exists. Then*

$$D_*^\alpha f(t) = D_*^\beta D^{n-1} f(t) \ .$$

Proof. Substitute β for α and $n-1$ for m in (2.5). Then

$$D_*^\beta D^{n-1} f(t) = D_*^{\beta+n-1} f(t) = D_*^{\alpha-(n-1)+n-1} f(t) = D_*^\alpha f(t) \ . \quad \square$$

Comment 2.8. To find the Caputo fractional derivative of arbitrary order α, $(n-1 < \alpha < n)$ of a function $f(t)$, it is sufficient to find the Caputo derivative of order $\beta = \alpha-(n-1)$ of the $(n-1)$-th derivative of the function. Notice that $\alpha-(n-1)$ is a real number between 0 and 1. Hence, studying the behavior of the Caputo derivative of order $\beta \in (0, 1)$ is sufficient for finding the Caputo derivatives of arbitrary order. Nevertheless, for completeness, later on in this survey the general results for the Caputo fractional derivative are given.

In general, the Riemann-Liouville operator is also non-commutative and satisfies

$$D^m D^\alpha f(t) = D^{\alpha+m} f(t) \neq D^\alpha D^m f(t) \ , \quad n-1 < \alpha < n, \ m, n \in \mathbb{N}, \ \alpha \in \mathbb{R} \ . \tag{2.6}$$

Comment 2.9. The inequalities in equations (2.5) and (2.6) become equalities under the following additional conditions (Podlubny [10], p. 81)

$$\begin{aligned} f^{(s)}(0) &= 0, \ s = n, n+1, \ldots, m \ , \quad \text{for } D_*^\alpha \ \text{ and} \\ f^{(s)}(0) &= 0, \ s = 0, \ 1, \ 2, \ \ldots, \ m \ , \quad \text{for } D^\alpha \ . \end{aligned}$$

It should be noticed, that in the case of Caputo derivative there are no restrictions on the values $f^{(s)}(0)$, $s = 0, \ 1, \ 2 \ldots, n-1$. For example, for $m = 3$, $n = 2$ the function

$$f(t) = a_0 + a_1\, t + a_4\, t^4 + a_5\, t^5 + \cdots$$

satisfies the condition for Caputo but doesn't satisfy the condition for Riemann-Liouville.

2.2 The Laplace Transform

In this subsection the Laplace transform is discussed. First, a general definition is given (see Greenberg [5]), then the Laplace transforms of the two-parameter function of Mittag-Leffler type, the Riemann-Liouville and the Caputo fractional derivatives are studied (see Podlubny [10]). The results are used later in Section 4 for solving differential equations.

Definition 2.10. If the function

$$F(s) := L\{f(t); \ s\} := \int_0^\infty e^{-st} f(t)\, dt \ , \quad s \in \mathbb{C} \ , \tag{2.7}$$

exists, it is called the Laplace transform of f(t).

Comment 2.11. (Sufficient conditions for existence of the Laplace transform)
Let the function $f(t)$ be (see Greenberg [5], p. 103)

(i) piecewise smooth over every finite interval in $[0, \ \infty)$ and

(ii) of exponential order α, i. e.,
there exist constants $M > 0$ and $T > 0$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t > T$.

then the Laplace transform $L\{f(t); \ s\}$ of $f(t)$ exists.

Comment 2.12. The Laplace transform is most applicable to initial value problems on semi-infinite domains.

Comment 2.13. (The inverse Laplace transform)
The original function $f(t)$ can be restored from its Laplace transform $F(s)$ using the inverse Laplace transform

$$f(t) = L^{-1}\{F(s);\, t\} := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds\,, \quad c = Re(s) > c_0\,, \tag{2.8}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral (2.7). The integral in (2.8) is also called Bromwich integral.

Lemma 2.14. (Basic properties of the Laplace transform)
Suppose that $f(t)$ and $g(t)$ are two functions, which are equal to zero for $t < 0$ and for which the Laplace transforms $F(s)$ and $G(s)$ exist. The following statements hold (see Greenberg [5], pp. 105-115*):*

(a) *The Laplace transform and its inverse are linear operators, i. e., suppose that $\lambda \in \mathbb{R}$, then*

$$\begin{aligned} L\{\lambda f(t) + g(t);\, s\} &= \lambda L\{f(t);\, s\} + L\{g(t);\, s\} = \lambda F(s) + G(s)\,, \\ L^{-1}\{\lambda F(s) + G(s);\, t\} &= \lambda L^{-1}\{F(s);\, t\} + L^{-1}\{G(s);\, t\} = \lambda f(t) + g(t)\,. \end{aligned} \tag{2.9}$$

(b) *For the Laplace transform of the convolution of $f(t)$ and $g(t)$ it follows*

$$L\{f(t) * g(t);\, s\} = F(s)\, G(s)\,, \tag{2.10}$$

where the convolution is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau = \int_0^t f(\tau) g(t-\tau)\, d\tau\,.$$

(c) *The limit of the function $sF(s)$ for $s \to \infty$ is given by*

$$\lim_{s\to\infty} s\, F(s) = f(0)\,. \tag{2.11}$$

(d) *The Laplace transform of the $n-th$ derivative ($n \in \mathbb{N}$) of $f(t)$ is given by*

$$L\{f^{(n)}(t);\, s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)\,. \tag{2.12}$$

Lemma 2.15. (Laplace transforms of the basic fractional operators)
*Suppose that $p > 0$ and $F(s)$ is the Laplace transform of $f(t)$. Then the following statements hold (*Podlubny [10], pp. 104-105, p. 21*):*

(a) *The Laplace transform of the fractional integral of order α* (1.8) *is given by*

$$L\{J^\alpha f(t);\, s\} = s^{-\alpha} F(s) \,. \tag{2.13}$$

(b) *The Laplace transform of the Riemann-Liouville fractional differential operator of order α* (1.9) *is given by*

$$\begin{aligned} L\{D^\alpha f(t);\, s\} &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \left[D^{\alpha-k-1} f(t)\right]_{t=0} \\ &= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{n-k-1} \left[D^k J^{n-\alpha} f(t)\right]_{t=0} \,, \quad n-1 < \alpha < n \,. \end{aligned} \tag{2.14}$$

(c) *Let $\alpha,\, \beta,\, \lambda \in \mathbb{R}$, $\alpha,\, \beta > 0$, $p \in \mathbb{N}$. Then the Laplace transform of the two-parameter function of Mittag-Leffler type* (1.3) *is given by*

$$L\{t^{\alpha p+\beta-1} E^{(p)}_{\alpha,\beta}(\pm\lambda t^\alpha);\, s\} = \frac{p!\, s^{\alpha-\beta}}{(s^\alpha \mp \lambda)^{p+1}} \,, \quad Re(s) > |\lambda|^{1/\alpha} \,. \tag{2.15}$$

Of great interest in this thesis is the Laplace transform of the Caputo fractional derivative of $f(t)$. The following statement is proved.

Theorem 2.16. *Suppose that $p > 0$ and $F(s)$ is the Laplace transform of $f(t)$. Then the Laplace transform of the Caputo fractional differential operator of order α* (1.10) *is given by*

$$L\{D^\alpha_* f(t);\, s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \,, \quad n-1 < \alpha < n \,. \tag{2.16}$$

Proof. To show the validity of (2.16) consider first equation (2.1)

$$D^\alpha_* f(t) = J^{n-\alpha} D^n f(t) \,.$$

Let $g(t) := D^n f(t)$. Then (2.1) becomes

$$D^\alpha_* f(t) = J^{n-\alpha} g(t) \,. \tag{2.17}$$

Using the Laplace transform of the fractional integral (2.13) of order $n-\alpha$ of $g(t)$ and equation (2.17) (Podlubny [10], p. 106), the Laplace transform of the Caputo fractional operator can be written as

$$L\{D_*^\alpha f(t);\, s\} = L\{J^{n-\alpha} g(t);\, s\} = s^{-(n-\alpha)} G(s)\ , \tag{2.18}$$

where $G(s) = L\{g(t);\, s\}$ can be expressed using (2.12) in the following way

$$G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)\ . \tag{2.19}$$

Finally, substituting (2.19) in (2.18), the statement of the theorem

$$L\{D_*^\alpha f(t);\, s\} = s^{-(n-\alpha)} \left(s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \right) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$$

is proved. □

Comment 2.17. Suppose that both Laplace transforms of the Riemann-Liouville and of the Caputo fractional derivatives exist for the function $f(t)$. For Riemann-Liouville in equation (2.14) initial values of the fractional integral $J^{n-\alpha}$ and of its integer derivatives of order $k = 1, 2, \ldots, n-1$ are required (see also Gorenflo and Mainardi [6]). For Caputo in equation (2.16) only the initial values of the function and its integer derivatives of order $k = 1, 2, \ldots, n-1$ are required.

Comment 2.18. The Laplace transform of the Caputo fractional derivative (2.16) is a generalization of the Laplace transform of the integer-order derivative (2.12), where n is replaced by α. The same does not hold for the Riemann-Liouville case. This property is an important advantage of the Caputo operator over the Riemann-Liouville operator.

2.3 Comparison with the Riemann-Liouville Operator

In this subsection a comparison between the Caputo and Riemann-Liouville fractional derivatives is given. The results from Subsection 2.1 and Subsection 2.2 (Gorenflo and Mainardi [6], Podlubny [10]) are summarized in Table 1.

- **General observations**

As it is already known from the previous the Riemann-Liouville (1.9) and the Caputo (1.10) fractional differential operators do not coincide (see Proposition 2.3).

Comment 2.19. Let $f(t)$ be a function for which both $D^\alpha f(t)$ and $D_*^\alpha f(t)$ exist and $n-1 < \alpha < n \in \mathbb{N}$. Then in general it holds

$$D^\alpha f(t) \neq D_*^\alpha f(t) \ .$$

In some sense they are reverse to each other, since they can be represented as a composition of the same operators but in a reverse order (see the representation property in Table 1). Considering $n-1 < \alpha < n$, $n \in \mathbb{N}$, in the interpolation property there are also some differences for the values of the parameter $\alpha \to n-1$, although for $\alpha \to n$ the result for both operators is the same.

Proposition 2.20. *Let $n-1 < \alpha < n$. Then it holds*

$$\lim_{\alpha \to n} D^\alpha f(t) = \lim_{\alpha \to n} D_*^\alpha f(t) = f^{(n)}(t) \ .$$

Comment 2.21. Concerning the commutation (see Comment 2.9) for functions $f(t)$ such that $f^{(s)}(0) = 0, \ s = 0, 1, 2 \ldots, m$ each of the two fractional derivatives commutes with the m th order derivative ($m \in \mathbb{N}$), namely,

$$D^m D^\alpha f(t) = D^{\alpha+m} f(t) = D^\alpha D^m f(t)$$

and

$$D_*^\alpha D^m f(t) = D_*^{\alpha+m} f(t) = D^m D_*^\alpha f(t) \ .$$

In relation to that, another similarity between the Caputo and the Riemann-Liouville fractional derivatives is given in the following statement.

Proposition 2.22. *Let the function $f(t)$ be such that $f^{(s)}(0) = 0, \ s = 0, 1, 2 \ldots, n-1$. Then the Riemann-Liouville and the Caputo fractional derivatives coincide (see* Comment 2.25 *in* Subsection 2.4*), i. e.,*

$$D_*^\alpha f(t) = D^\alpha f(t) \ .$$

Property	Riemann-Liouville	Caputo
Representation	$D^\alpha f(t) = D^n J^{n-\alpha} f(t)$	$D_*^\alpha f(t) = J^{n-\alpha} D^n f(t)$
Interpolation	$\lim_{\alpha \to n} D^\alpha f(t) = f^{(n)}(t)$ $\lim_{\alpha \to n-1} D^\alpha f(t) = f^{(n-1)}(t)$	$\lim_{\alpha \to n} D_*^\alpha f(t) = f^{(n)}(t)$ $\lim_{\alpha \to n-1} D_*^\alpha f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$
Linearity	$D^\alpha(\lambda f(t) + g(t)) = \lambda D^\alpha f(t) + D^\alpha g(t)$	$D_*^\alpha(\lambda f(t) + g(t)) = \lambda D_*^\alpha f(t) + D_*^\alpha g(t)$
Non-commutation	$D^m D^\alpha f(t) = D^{\alpha+m} f(t) \neq D^\alpha D^m f(t)$	$D_*^\alpha D^m f(t) = D_*^{\alpha+m} f(t) \neq D^m D_*^\alpha f(t)$
Laplace transform	$L\{D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \left[D^{\alpha-k-1} f(t) \right]_{t=0}$	$L\{D_*^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$
Leibniz rule	$D^\alpha(f(t)\, g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} f(t) \right) g^{(k)}(t)$	$D_*^\alpha(f(t)\, g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} f(t) \right) g^{(k)}(t)$ $- \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(t) g(t))^{(k)}(0) \right)$
$f(t) = c = const$	$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha} \neq 0, \quad c = const$	$D_*^\alpha c = 0, \quad c = const$

Table 1: Comparison between Riemann-Liouville and Caputo

Comment 2.23. It should also be mentioned that in both cases Riemann-Liouville and Caputo only derivatives of order β in the interval $(0,1)$ can be considered, since (see Comment 2.8 and formulas (2.5) and (2.6)) for $n-1<\alpha<n$

$$D_*^\alpha f(t) = D_*^{\alpha-(n-1)}\, D^{n-1} f(t)\ ,$$

$$D^\alpha f(t) = D^{n-1}\, D^{\alpha-(n-1)} f(t)\ ,$$

where $\beta = \alpha-(n-1) \in (0,1)$ and D^{n-1} is the classical integer-order derivative.

- **The constant function**

One of the most impressing inconformities between the two operators is the differentiation of the constant function. For Caputo it holds

$$D_*^\alpha\, c = 0, \quad c = const\ ,$$

whereas for Riemann-Liouville

$$D^\alpha c = \frac{c}{\Gamma(1-\alpha)}\, t^{-\alpha} \neq 0, \quad c = const\ .$$

2.4 Relation with the Riemann-Liouville Operator

The central point in this subsection is the following statement (Gorenflo and Mainardi [6]) for which a proof is derived.

Theorem 2.24. *Let* $t>0$, $\alpha\in\mathbb{R}$, $n-1<\alpha<n\in\mathbb{N}$. *Then the following relation between the Riemann-Liouville* (1.9) *and the Caputo* (1.10) *operators holds*

$$D_*^\alpha\, f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0)\ . \tag{2.20}$$

Proof. The well-known Taylor series expansion about the point 0 reads

$$\begin{aligned} f(t) &= f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \cdots + \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1}\ , \end{aligned}$$

where, considering also (1.7)

$$R_{n-1} = \int_0^t \frac{f^{(n)}(\tau)(t-\tau)^{n-1}}{(n-1)!}\, d\tau = \frac{1}{\Gamma(n)} \int_0^t f^{(n)}(\tau)(t-\tau)^{n-1}\, d\tau = J^n f^{(n)}(t)\ .$$

Now, using the linearity property of the Riemann-Liouville fractional derivative, the Riemann-Liouville fractional derivative of the power function (see (3.2)), the properties of the fractional integral and representation formula (2.1)

$$\begin{aligned}
D^\alpha f(t) &= D^\alpha \left(\sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0) + R_{n-1} \right) \\
&= \sum_{k=0}^{n-1} \frac{D^\alpha t^k}{\Gamma(k+1)} f^{(k)}(0) + D^\alpha R_{n-1} \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} \frac{t^{k-\alpha}}{\Gamma(k+1)} f^{(k)}(0) + D^\alpha J^n f^{(n)}(t) \\
&= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + J^{n-\alpha} f^{(n)}(t) \\
&= \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) + D_*^\alpha f(t) \; .
\end{aligned}$$

This means that

$$D_*^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0) \; . \qquad \square$$

Comment 2.25. This formula implies that the Caputo and the Riemann-Liouville fractional operator coincide if and only if $f(t)$ together with its first $n-1$ derivatives vanish at $t = 0$.

Corollary 2.26. *The following relation between the Riemann-Liouville and Caputo fractional derivatives holds*

$$D_*^\alpha f(t) = D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right) \; .$$

Proof. This formula is proved (see also Gorenflo and Mainardi [6]) using relation (2.20), the Riemann-Liouville fractional derivative of the power function (3.2) and the linearity property of the Riemann-Liouville operator, i. e.,

$$\begin{aligned}
D_*^\alpha f(t) &= D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0) \\
&= D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{D^\alpha t^k}{\Gamma(k+1)} f^{(k)}(0) \\
&= D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right) \; . \qquad \square
\end{aligned}$$

The Leibniz rule for the Caputo derivative is not discussed in the literature. In this survey it is derived as a corollary from Theorem 2.24. Later in Section 5 it is used for proving some properties of the C-Laguerre functions.

Corollary 2.27. (**Leibniz Rule**) *Let* $t > 0\,,\ \alpha \in \mathbb{R}\,,\ n-1 < \alpha < n \in \mathbb{N}$. *If* $f(\tau)$ *and* $g(\tau)$ *and all its derivatives are continuous in* $[0,\, t]$ *then the following holds*

$$D_*^\alpha(f(t)\,g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} f(t)\right) g^{(k)}(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(t)g(t))^{(k)}(0)\right). \tag{2.21}$$

Proof. Applying consecutively relation (2.20) and the Leibniz Rule for Riemann-Liouville (Podlubny [10], p. 96)

$$D^\alpha(f(t)\,g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} f(t)\right) g^{(k)}(t)\,,$$

then the Leibniz rule for the Caputo derivative is obtained

$$\begin{aligned} D_*^\alpha\,(f(t)\,g(t)) &= D^\alpha(f(t)\,g(t)) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(t)\,g(t))^{(k)}(0)\right) \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} f(t)\right) g^{(k)}(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((f(t)g(t))^{(k)}(0)\right). \end{aligned}$$

□

3 Examples of Fractional Derivatives

In this section examples of fractional derivatives are discussed, e.g., the constant, the power and the exponential function, as well as the sine and cosine function. The Caputo fractional derivatives of these functions are studied (see also Diethelm, Ford, Freed, and Luchko [3]), some proofs not found in the literature are proposed and a comparison with the Riemann-Liouville fractional derivative (see Podlubny [10]) is given.

3.1 The Constant Function

From physical point of view it is reasonable to have the fractional derivative of a constant equal to zero. For the Riemann-Liouville operator it holds

$$D^{\alpha}c = \frac{c}{\Gamma(1-\alpha)}\, t^{-\alpha} \neq 0,\; c = const.$$

Thus, the following property is one of the advantages of the Caputo derivative over the Riemann-Liouville derivative (see Podlubny [10], p. 80).

Lemma 3.1. *For the Caputo fractional derivative it holds*

$$D_*^{\alpha}\, c = 0\ , \quad c = const.$$

Proof. As usual $0 < n-1 < \alpha < n\,,\ n \in \mathbb{N}$, which means $n \geq 1$. Applying the definition of the Caputo derivative (1.10) and since the n-th derivative $c^{(n)}$ $(n \in N,\ n \geq 1)$ of a constant equals 0, it follows

$$D_*^{\alpha}c = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{c^{(n)}}{(t-\tau)^{\alpha+1-n}}\, d\tau = 0. \quad \square$$

3.2 The Power Function

The power function needs a detailed examination because of its great importance. Recall the Taylor expansion (see Greenberg [5], p. 35)

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \cdots .$$

Furthermore, the Caputo fractional derivative is linear (see formula (2.4)). So if $D_*^{\alpha}\, t^p$ is known, then the Caputo fractional derivative for arbitrary function can be represented in the following manner

$$D_*^{\alpha}\, f(t) = D_*^{\alpha} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, t^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, D_*^{\alpha}\, t^k. \tag{3.1}$$

Theorem 3.2. *The Riemann-Liouville fractional derivative of the power function satisfies*

$$D^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}\,, \qquad n-1<\alpha<n,\ p>-1,\ p\in\mathbb{R}\,. \tag{3.2}$$

Proof. See Podlubny [10], p. 72. □

Theorem 3.3. *The Caputo fractional derivative of the power function satisfies*

$$D_*^\alpha\, t^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} = D^\alpha t^p\,, & n-1<\alpha<n,\ p>n-1,\ p\in\mathbb{R}\,, \\ 0\,, & n-1<\alpha<n,\ p\le n-1,\ p\in\mathbb{N}\,. \end{cases} \tag{3.3}$$

Proof. The proof of the second case ($D_*^\alpha\, t^p = 0$, $n-1<\alpha<n$, $p\le n-1$, $p\in\mathbb{N}$) follows the pattern of the proof of the differentiation of the constant function, since $(t^p)^{(n)}=0$ for $p\le n-1,\ p,\, n\in\mathbb{N}$.

The more interesting case is the first one. It can be proved in the following two ways
– directly, using the definition of the Caputo fractional derivative (1.10) and the properties of the Gamma and Beta functions, and
– indirectly, using the relation between the Caputo and Riemann-Liouville derivatives (2.20) as well as the Riemann-Liouville fractional derivative of the power function (3.2).

Let $n-1<\alpha<n,\ p>n-1,\ p\in\mathbb{R}$.

- The direct way reads

$$\begin{aligned} D_*^\alpha t^p &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(\tau^p)^{(n)}}{(t-\tau)^{\alpha+1-n}}\,d\tau \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\Gamma(p+1)}{\Gamma(p-n+1)}(\tau^{p-n})(t-\tau)^{n-\alpha-1}\,d\tau\,, \end{aligned}$$

and using the substitution $\tau=\lambda t\,,\quad 0\le\lambda\le 1$

$$\begin{aligned} D_*^\alpha t^p &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}\int_0^1 (\lambda\, t)^{p-n}((1-\lambda)\, t)^{n-\alpha-1}\, t\, d\lambda \\ &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\,\Gamma(p-n+1)}\, t^{p-\alpha}\int_0^1 \lambda^{p-n}(1-\lambda)^{n-\alpha-1}\,d\lambda \\ &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\,\Gamma(p-n+1)}\, t^{p-\alpha}\, B(p-n+1, n-\alpha) \\ &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\,\Gamma(p-n+1)}\, t^{p-\alpha}\,\frac{\Gamma(p-n+1)\,\Gamma(n-\alpha)}{\Gamma(p-\alpha+1)} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\, t^{p-\alpha} \quad . \end{aligned}$$

- The indirect way reads

$$D_*^\alpha t^p = D^\alpha t^p - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} (t^p)^{(k)}|_{t=0} \quad ,$$

and taking into account $(t^p)^{(k)}|_{t=0} = 0$, for $k \le n-1 < p$

$$\begin{aligned} D_*^\alpha t^p &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \,.\, 0 \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \quad . \end{aligned}$$

$\square$

Formula (3.3) matches the corresponding result, mentioned by Diethelm, Ford, Freed, and Luchko [3] without any proof.

Comment 3.4. The case $-1 < p < n-1$, $p \in \mathbb{R}$ is included in (3.2) but is not included in (3.3). There isn't any formula for the Caputo operator for this case until now.

Comment 3.5. The coefficients in (3.3) are illustrated in Fig. 3.1. When values of the parameters p (the power) and α (the order of the fractional derivative) increase, the corresponding coefficients increase very fast, which is exactly the same as in the classical case.

Comment 3.6. For $p > n-1$ the Caputo fractional derivative of the power function (3.3) is a generalization of the integer-order derivative of the power function. Recall that

$$\begin{aligned} (t^p)^{(n)} &= (p\, t^{p-1})^{(n-1)} = (p(p-1)\, t^{p-2})^{(n-2)} = \cdots = p(p-1)\cdots(p-n+1)\, t^{p-n} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)} t^{p-n}, \ n \in \mathbb{N}, \ p \in \mathbb{R}. \end{aligned}$$

Comment 3.7. For $n = 1$, i. e., $0 < \alpha < 1$ once more $D_*^\alpha t^p = D^\alpha t^p$, $p > 0$, $p \in \mathbb{R}$.

Proposition 3.8. *The Caputo fractional derivative for an arbitrary function $f(t)$ can be computed by the formula*

$$D_*^\alpha f(t) = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}.$$

Proof. Taking into account (3.1) and (3.3) the following equalities hold

$$D_*^\alpha f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} D_*^\alpha t^k = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha} = \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}. \ \square$$

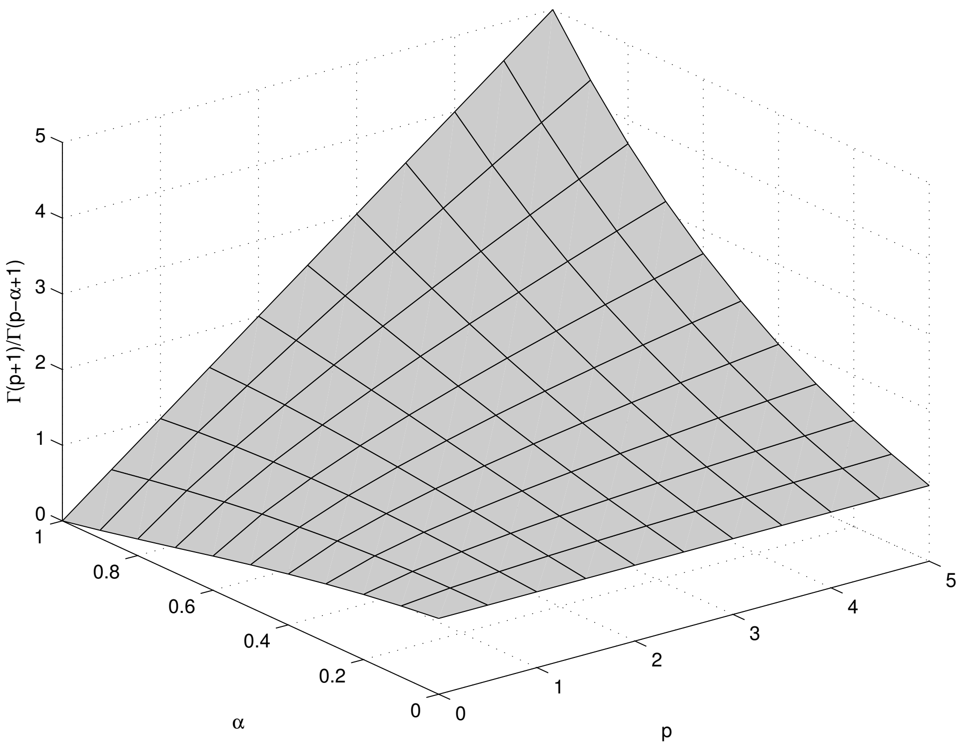

Figure 3.1: The coefficients in the derivative of the power function

- **Examples**

Suppose that $\alpha \in \mathbb{R}$ but $\alpha \notin \mathbb{N}$ (α is the order of differentiation). Two examples are considered, namely fractional derivatives of the functions t^2 and t, i. e., $p = 2$ and $p = 1$ respectively. More examples are given in Appendix A.

$\underline{p = 2}$

Here, the function t^2 is discussed. Suppose that $0 < \alpha < 1$. Following (3.3) the fractional derivative is given in the following manner

$$D_*^\alpha t^2 = \frac{\Gamma(2+1)}{\Gamma(2-\alpha+1)}\, t^{2-\alpha} = \frac{2}{\Gamma(3-\alpha)}\, t^{2-\alpha}\,,\ n-1 < \alpha < n < 3\,.$$

More special cases can be considered for fixed values of the parameter α, for example, $\alpha = 1/3$, $\alpha = 1/2$ and $\alpha = 3/4$.

For $\alpha = 1/3 \quad D_*^{1/3} t^2 = \dfrac{2}{\Gamma(3-1/3)}\, t^{2-1/3} = \dfrac{2}{\Gamma(8/3)}\, t^{5/3} \approx 1.33\, t^{5/3}\,,$

$$\text{for } \alpha = 1/2 \quad D_*^{1/2} t^2 = \frac{2}{\Gamma(3-1/2)}\, t^{2-1/2} \; = \frac{8}{3\sqrt{\pi}}\sqrt{t^3} \quad \approx 1.5\,\sqrt{t^3}\,,$$

$$\text{for } \alpha = 3/4 \quad D_*^{3/4} t^2 = \frac{2}{\Gamma(3-3/4)}\, t^{2-3/4} \; = \frac{2}{\Gamma(9/4)}\, t^{5/4} \approx 1.77\, t^{5/4}\,.$$

These results are illustrated in Fig. 3.2. The graphs of the fractional derivatives are enclosed by the graphs of the classical integer-order derivatives everywhere except for a small interval. The greater the order of the derivative, the closer is its graph to the graph of the 1[st] derivative of t^2, the smaller the order, the closer is its graph to the graph of the 0-th derivative of t^2 (see also Comment 3.9).

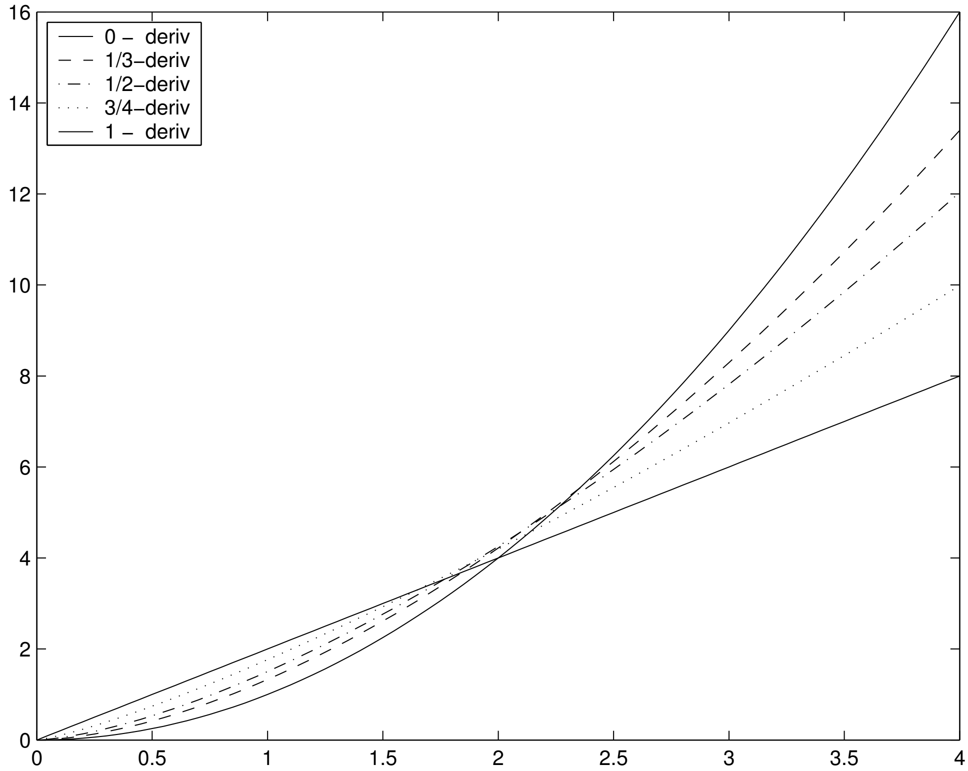

Figure 3.2: Fractional derivatives of $f(t) = t^2$

Comment 3.9. Using $\Gamma(2) = 1$, $\Gamma(3) = 2$ (see (1.1)) the following conclusion can be drawn

$$D_*^{\alpha} t^2 = \frac{2}{\Gamma(3-\alpha)}\, t^{2-\alpha} \to \begin{cases} 2t\,, & \alpha \to 1\,, \\ t^2\,, & \alpha \to 0\,, \end{cases}$$

which confirms the interpolation property (2.3) of the Caputo fractional derivative.

$\underline{p = 1}$

Now $f(t) = t^p = t$. For this case (3.3) reads

$$D_*^\alpha t = \frac{\Gamma(1+1)}{\Gamma(1-\alpha+1)}\, t^{1-\alpha} = \frac{1}{\Gamma(2-\alpha)}\, t^{1-\alpha} \, .$$

Thus, for $\alpha = 1/2$

$$D_*^{1/2} t = \frac{1}{\Gamma(2-1/2)}\, t^{1-1/2} = \frac{2\sqrt{t}}{\sqrt{\pi}} \, ,$$

which coincides with formula (1.11).

Some graphs can be seen in Fig. 3.3. Here the graphs of the fractional derivatives are enclosed between the graphs of the integer-order derivatives (except for a small interval). Although there is no convergence to the first derivative in the point 0, the interpolation property still holds, since the fractional derivatives are defined only for $t > 0$ (see Definition 1.5 and Comment 1.7).

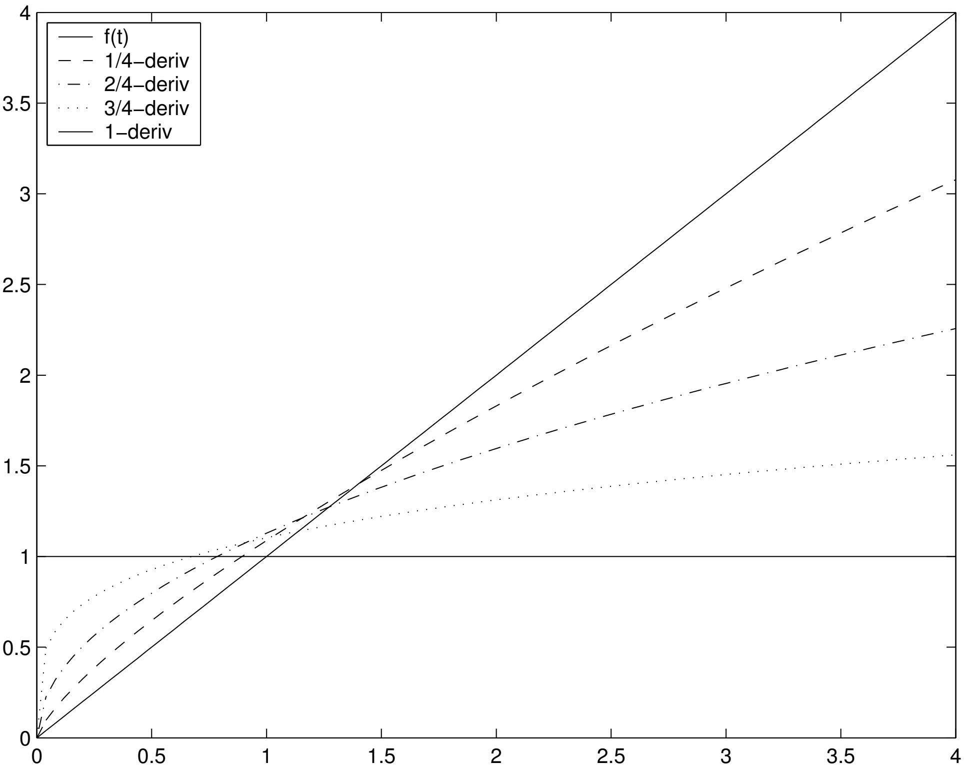

Figure 3.3: Fractional derivatives of $f(t) = t$

In Fig. 3.4 a three dimensional graph of the fractional derivatives of the function t^2 is presented. Derivatives of order between 0 and 2 are considered. The bold lines are the classical integer-order derivatives, i. e., the functions t^2, $2t$ and 2 respectively. The fractional derivatives are interpolating these classical derivatives. The same figure can also be considered as a graph of the fractional derivatives of the function t, (more precisely, of the function $2t$) of order between 0 and 1, where the corresponding order for the function t^2 is between 1 and 2, since $(t^2)' = 2t$ and $D_*^\alpha t^2 = D_*^{\alpha-1}(t^2)'$ (see Subsection 2.1).

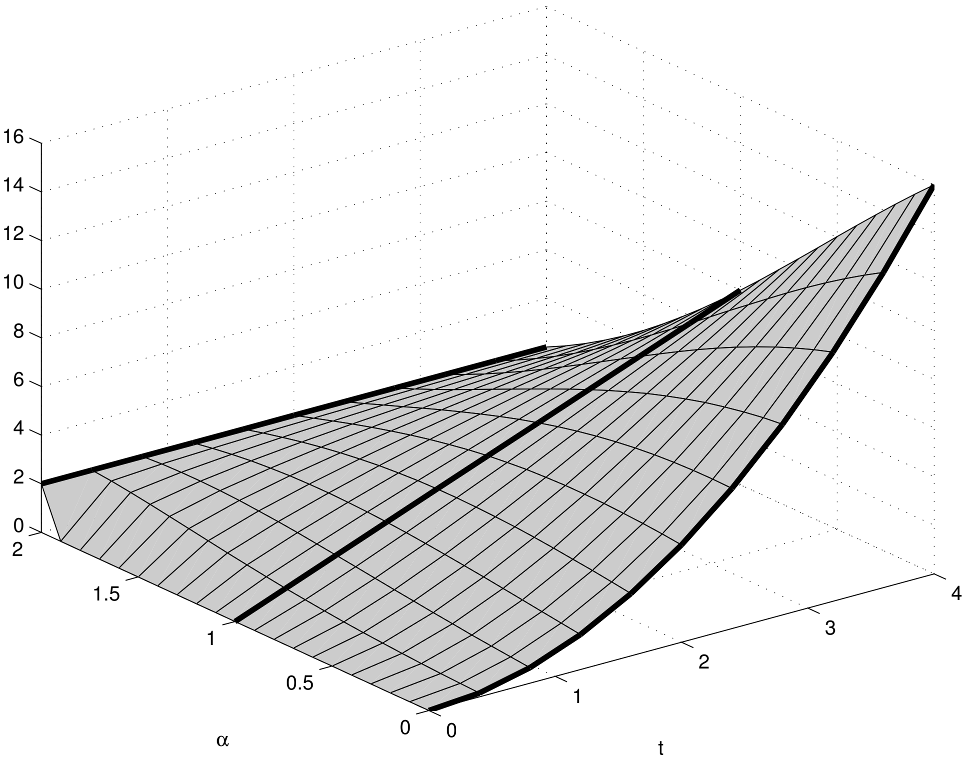

Figure 3.4: 3D representation of the fractional derivatives of $f(t) = t^2$

3.3 The Exponential Function

After discussing the fractional derivatives of the power function, we consider the exponential function $e^{\lambda t}$. The application of the Caputo operator is shown in the following statement.

Theorem 3.10. *Let $\alpha \in \mathbb{R}$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$. Then the Caputo fractional derivative of the exponential function has the form*

$$D_*^\alpha e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t) , \tag{3.4}$$

where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffler type.

Proof. To prove the theorem the relation between Caputo and Riemann-Liouville fractional derivatives (2.20) as well as the well-known Riemann-Liouville fractional derivative of the exponential function, namely,

$$D^\alpha e^{\lambda t} = t^{-\alpha} E_{1,1-\alpha}(\lambda t)$$

could be used. Then for the Caputo fractional derivative holds

$$\begin{aligned}
D_*^\alpha e^{\lambda t} &= D^\alpha e^{\lambda t} - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} (e^{\lambda t})^{(k)}(0) \\
&= t^{-\alpha} E_{1,1-\alpha}(\lambda t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} . \lambda^k \\
&= \sum_{k=0}^{\infty} \frac{(\lambda t)^k t^{-\alpha}}{\Gamma(k+1-\alpha)} - \sum_{k=0}^{n-1} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} \\
&= \sum_{k=n}^{\infty} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+n+1-\alpha)} \\
&= \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t) . \qquad \square
\end{aligned}$$

Formula (3.4) matches the corresponding result, mentioned by Diethelm, Ford, Freed, and Luchko [3] without any proof.

- **Special case**

Let $\lambda = 1$. In Fig. 3.5 and Fig. 3.6 some graphs of fractional derivatives of the function e^t are presented. The fractional derivatives are enclosed by the functions e^t and $e^t - 1$ and in general the exponential function and its derivatives have the same shape.

Inspecting (3.4), for $\lambda = 1$ the right hand side doesn't dependent on n but on $n - \alpha$. From this the following conclusion can be drawn.

Proposition 3.11. *Let $n - 1 < \alpha < n$ and $s \in \mathbb{Z}$, $s > -n$. Then*

$$D_*^\alpha e^t = D_*^{\alpha+s} e^t .$$

This means in fact that for computing the Caputo fractional derivative of the exponential function only the value after the decimal point of the order of differentiation is important.

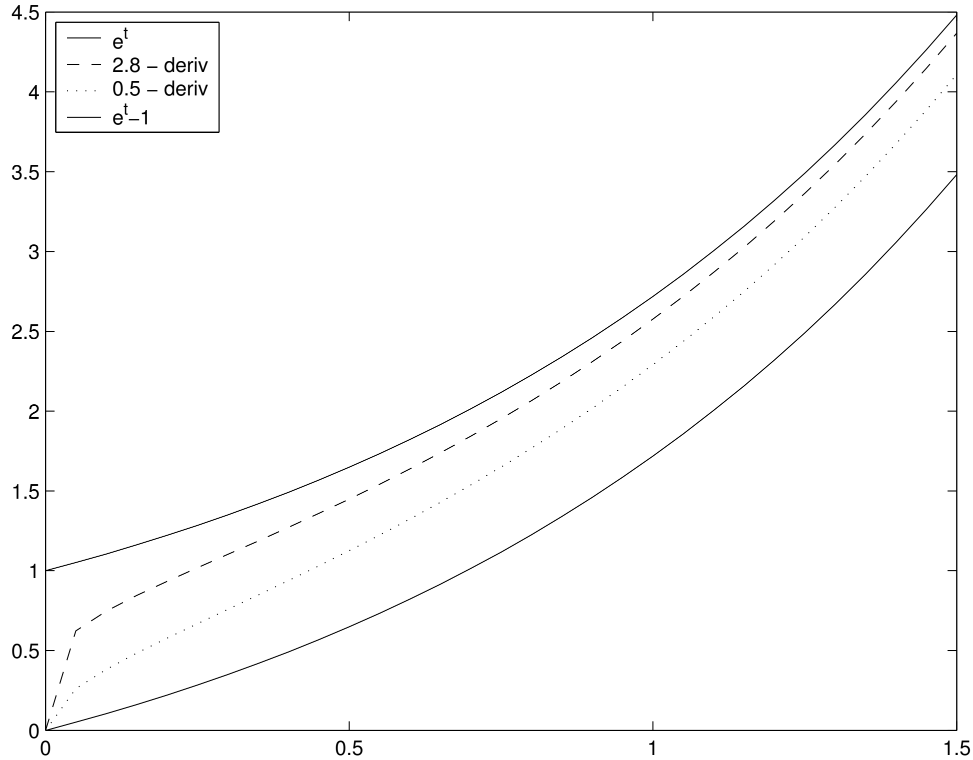

Figure 3.5: 0.5-th and 2.8-th fractional derivatives of $f(t) = e^t$ in the interval (0, 1.5]

For example, for $\alpha = 0.8$, $\alpha = 1.8$ and $\alpha = 7.8$ the same result is obtained, namely, $D_*^{0.8}\, e^t = D_*^{1.8}\, e^t = D_*^{7.8}\, e^t = \sqrt[5]{t}\, E_{1,1.2}(t)$.

In addition, the α-th order $(n-1 < \alpha < n)$ fractional derivatives of the exponential function are "moving" from $e^t - 1$ to e^t, for α taking values from $n-1$ to n. The same graphs are received for each $n \in \mathbb{N}$.

By using (3.4), the limits $\lim\limits_{\alpha \to n} D_*^\alpha\, e^t$ and $\lim\limits_{\alpha \to n-1} D_*^\alpha\, e^t$ can be evaluated exactly

$$\begin{aligned}
\lim_{\alpha \to n} D_*^\alpha\, e^t &= \lim_{\alpha \to n} t^{n-\alpha}\, E_{1,n-\alpha+1}(t) = E_{1,1}(t) = e^t = D^n\, e^t\,,\ n \in \mathbb{N}\,, \\
\lim_{\alpha \to n-1} D_*^\alpha\, e^t &= \lim_{\alpha \to n-1} t^{n-\alpha}\, E_{1,n-\alpha+1}(t) = tE_{1,2}(t) \\
&= t \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{t^{k+1}}{\Gamma(k+2)} = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(k+1)} \\
&= e^t - 1 = D^n\, e^t - D^n\, e^t|_{t=0}\,,\ n \in \mathbb{N}.
\end{aligned}$$

These limit evaluations coincide with the interpolation property (2.3) of the Caputo derivative.

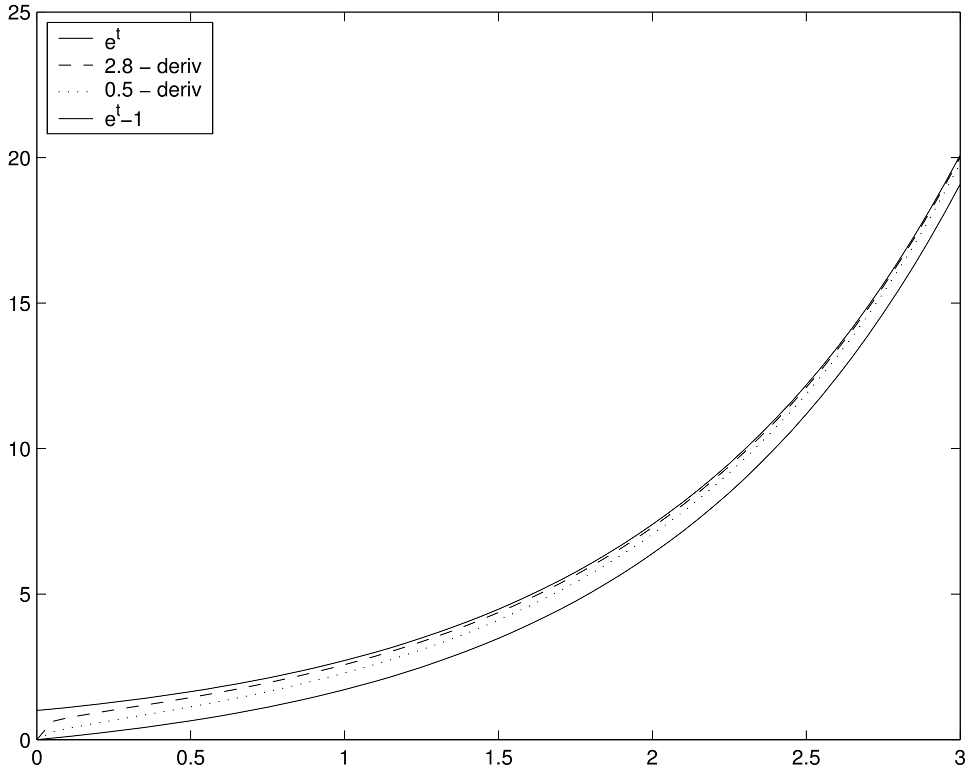

Figure 3.6: 0.5-th and 2.8-th fractional derivatives of $f(t) = e^t$ in the interval (0, 3]

In Fig. 3.7 a three-dimensional graph of the fractional derivatives of the function e^t is presented. The left bold line is the function e^t itself (which coincides with all its integer-order derivatives); the right bold line is the function $e^t - 1$. In this figure the above results can be observed.

3.4 Other Frequently Used Functions

Another two functions that appear very often are the sine and the cosine functions. The behavior of the Caputo derivative applied to each of them is discussed in this subsection. Similar representations are given by Diethelm, Ford, Freed, and Luchko [3], which are also mentioned here for comparison.

Theorem 3.12. *Let $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $n-1 < \alpha < n$. Then*

$$D_*^\alpha \sin \lambda t = -\frac{1}{2} i (i\lambda)^n t^{n-\alpha} \ \left(E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \right) . \tag{3.5}$$

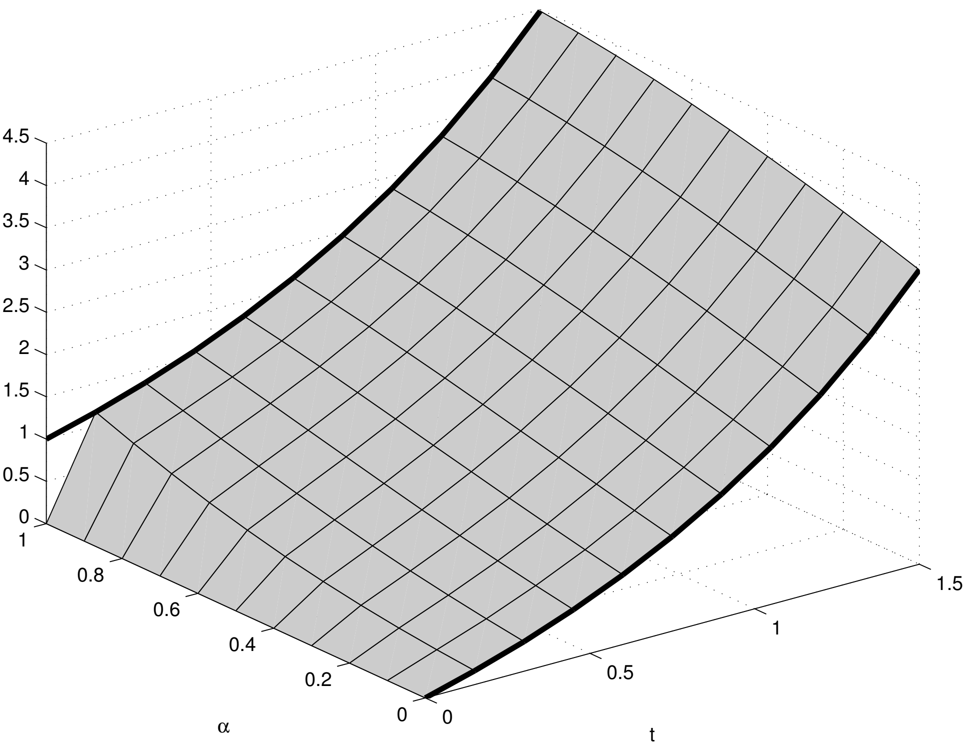

Figure 3.7: 3D representation of the fractional derivatives of $f(t) = e^t$

Proof. The following representation of the sine function is used

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}.$$

Now, using the linearity property (2.4) of the Caputo fractional derivative and formula (3.4) for the exponential function it can be shown that

$$\begin{aligned}
D_*^\alpha \sin \lambda t &= D_*^\alpha \frac{e^{i\lambda t} - e^{-i\lambda t}}{2i} \\
&= \frac{1}{2i}\left(D_*^\alpha e^{i\lambda t} - D_*^\alpha e^{-i\lambda t}\right) \\
&= \frac{1}{2i}\left((i\lambda)^n t^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) - (-i\lambda)^n t^{n-\alpha} E_{1,n-\alpha+1}(-i\lambda t)\right) \\
&= -\frac{1}{2} i (i\lambda)^n t^{n-\alpha} \left(E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t)\right). \quad \square
\end{aligned}$$

In the same manner a formula for the Caputo derivative of the cosine function is received. The corresponding representation is

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

The corresponding result for the cosine function is formulated in the following statement.

Theorem 3.13. *Let $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $n-1 < \alpha < n$. Then*

$$D_*^\alpha \cos \lambda t = \frac{1}{2}(i\lambda)^n t^{n-\alpha} \ \left(E_{1,\,n-\alpha+1}(i\lambda t) + (-1)^n E_{1,\,n-\alpha+1}(-i\lambda t)\right) \ . \tag{3.6}$$

There exist representations of the Caputo derivative of the sine and cosine function in terms of the confluent hypergeometric function (Diethelm, Ford, Freed, and Luchko [3]).

Let λ, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $n-1 < \alpha < n$. Then

$$D_*^\alpha \sin \lambda t = \begin{cases} -\dfrac{i\lambda^n (-1)^{n/2} t^{n-\alpha}}{2\Gamma(n-\alpha+1)} \left[{}_1F_1(1,\, n-\alpha+1;\, i\lambda t) - {}_1F_1(1,\, n-\alpha+1;\, -i\lambda t)\right] , \\ \hfill \text{if } n \text{ is even} , \\ \dfrac{\lambda^n (-1)^{(n-1)/2} t^{n-\alpha}}{2\Gamma(n-\alpha+1)} \left[{}_1F_1(1,\, n-\alpha+1;\, i\lambda t) + {}_1F_1(1,\, n-\alpha+1;\, -i\lambda t)\right] , \\ \hfill \text{if } n \text{ is odd} . \end{cases} \tag{3.7}$$

and

$$D_*^\alpha \cos \lambda t = \begin{cases} \dfrac{\lambda^n (-1)^{n/2} t^{n-\alpha}}{2\Gamma(n-\alpha+1)} \left[{}_1F_1(1,\, n-\alpha+1;\, i\lambda t) + {}_1F_1(1,\, n-\alpha+1;\, -i\lambda t)\right] , \\ \hfill \text{if } n \text{ is even} , \\ \dfrac{i\lambda^n (-1)^{(n-1)/2} t^{n-\alpha}}{2\Gamma(n-\alpha+1)} \left[{}_1F_1(1,\, n-\alpha+1;\, i\lambda t) - {}_1F_1(1,\, n-\alpha+1;\, -i\lambda t)\right] , \\ \hfill \text{if } n \text{ is odd} . \end{cases} \tag{3.8}$$

In this survey the representations (3.5) and (3.6) are preferred, since the Mittag-Leffler function is considered as a basic function of the fractional calculus.

All the main results from Section 3 are summarized in Table 2. The conditions for which these formulas hold are not given in the table for simplicity but can be found within Section 3.

Function	$f(t)$	Caputo derivative $D_*^\alpha f(t)$
Constant function	$f(t) = c = const$	$D_*^\alpha c = 0$
Power function	$f(t) = t^p$	$D_*^\alpha t^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} = D^\alpha t^p \ , & n-1 < \alpha < n,\ p > n-1,\ p \in \mathbb{R} \ , \\ 0 \ , & n-1 < \alpha < n,\ p \leq n-1,\ p \in \mathbb{N} \end{cases}$
Exponential function	$f(t) = e^{\lambda t}$	$D_*^\alpha e^{\lambda t} = \sum_{k=0}^{\infty} \dfrac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t)$
Sine function	$f(t) = \sin \lambda t$	$D_*^\alpha \sin \lambda t = -\dfrac{1}{2} i\, (i\lambda)^n\, t^{n-\alpha}\ \left(E_{1,\, n-\alpha+1}(i\lambda t) - (-1)^n E_{1,\, n-\alpha+1}(-i\lambda t)\right)$
Cosine function	$f(t) = \cos \lambda t$	$D_*^\alpha \cos \lambda t = \dfrac{1}{2} (i\lambda)^n\, t^{n-\alpha}\ \left(E_{1,\, n-\alpha+1}(i\lambda t) + (-1)^n E_{1,\, n-\alpha+1}(-i\lambda t)\right)$

Table 2: Caputo derivatives of the most used functions

4 Fractional Ordinary Differential Equations

In this section fractional initial value problems are studied. These are fractional ordinary differential equations with classical initial conditions (see Podlubny [10], Subsection 1.4). Using the Laplace transform a general solution for a general initial value problem is given. Then the convergence of this solution for the order of differentiation α tending from below and from above to an integer number to the solution of classical initial value problems is established. Finally, some particular examples are given, including a simplified initial value problem and the fractional damped simple harmonic oscillator.

4.1 The Initial Value Problem (IVP)

- **The Problem**

The linear initial value problem reads

$$\begin{aligned} D_*^\alpha y(t) - \lambda y(t) &= 0 \ , \qquad t > 0 \ , \quad n-1 < \alpha < n \ , \\ y^{(k)}(0) &= b_k \ , \qquad b_k \in \mathbb{R} \ , \quad k = 0, \ldots, n-1 \ . \end{aligned} \tag{4.1}$$

- **The Solution**

Theorem 4.1. *The solution of problem* (4.1) *is given by*

$$y(t) = \sum_{k=0}^{n-1} b_k \, t^k \, E_{\alpha,k+1}(\lambda t^\alpha) \ , \tag{4.2}$$

where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffler type.

Proof. To solve problem (4.1) the Laplace transform method is used (Subsection 2.2). Applying the Laplace transform to the fractional differential equation in (4.1) it becomes (see (2.16))

$$s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0) - \lambda Y(s) = 0 \ , \tag{4.3}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $L\{-\lambda y(t); s\} = -\lambda Y(s)$ since Laplace transform is linear (see Lemma 2.14 (a)). Equation (4.3) can be solved with respect to $Y(s)$ as follows

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha - \lambda} y^{(k)}(0) \ . \tag{4.4}$$

Substituting the initial conditions from (4.1) into (4.4) it follows

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha - \lambda} b_k \; .$$

Considering (2.15) for the Laplace transform of the two-parameter function of Mittag-Leffler type as well as the linearity property (Lemma 2.14 (a)) it follows

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha - \lambda} b_k = \sum_{k=0}^{n-1} L\{t^k E_{\alpha,\,k+1}(\lambda t^\alpha);\, s\}\, b_k = L\{\sum_{k=0}^{n-1} b_k\, t^k E_{\alpha,\,k+1}(\lambda t^\alpha);\, s\} \; .$$

Then using the inverse Laplace transform $y(t)$ can be found as

$$y(t) = y(t,\,\alpha) = \sum_{k=0}^{n-1} b_k\, t^k\, E_{\alpha,k+1}(\lambda t^\alpha) \; . \qquad \square$$

The proof of Theorem 4.1 is analogous to the corresponding result of Podlubny [10], p. 140, for the initial value problem with the Riemann-Liouville fractional derivative.

4.2 Convergence of the Solution

We consider the limits $1 < \alpha < 2 : \lim\limits_{\alpha \to 2} y(t,\,\alpha)$ and $2 < \alpha < 3 : \lim\limits_{\alpha \to 2} y(t,\,\alpha)$ and prove that these limits are solutions of classical initial value problems. This illustrates once more that fractional derivatives have the classical derivatives as special cases.

The aim of this subsection is to solve problems (4.5) and (4.8) and to compare their solutions with the solution (4.2) of the fractional initial value problem (4.1).

- Consider the second-order initial value problem with integer-order derivatives

$$\begin{aligned} y''(t) - \lambda y(t) &= 0 \;, \qquad t > 0 \;, \\ y^{(k)}(0) &= b_k \;, \qquad b_k \in \mathbb{R} \;, \quad k = 0,\,1 \;. \end{aligned} \tag{4.5}$$

To solve problem (4.5), consider first its characteristic equation $a^2 - \lambda = 0$. Its roots are $\pm\sqrt{\lambda}$, so the solution of the differential equation has the form

$$y(t) = c_1\, e^{\sqrt{\lambda} t} + c_2\, e^{-\sqrt{\lambda} t} \;,$$

where $c_1,\, c_2 \in \mathbb{R}$ are constants. Using the initial conditions, the solution of the classical problem (4.5) is

$$\begin{aligned} y(t) &= \frac{\sqrt{\lambda} b_0 + b_1}{2\sqrt{\lambda}} e^{\sqrt{\lambda} t} + \frac{\sqrt{\lambda} b_0 - b_1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda} t} \\ &= b_0 \cosh(\sqrt{\lambda} t) + \frac{1}{\sqrt{\lambda}} b_1 \sinh(\sqrt{\lambda} t) \; . \end{aligned} \tag{4.6}$$

Lemma 4.2. *The solution of the fractional initial value problem* (4.1) *with* $n = 2$ *converges for* $\alpha \to 2$ *to the solution of the classical problem* (4.5).

Proof. Consider the case $n = 2 :\ \alpha \to 2,\ 1 < \alpha < 2$. The solution (4.2) of problem (4.1) satisfies

$$\begin{aligned} \lim_{\alpha \nearrow 2} y(t,\, \alpha) &= \lim_{\alpha \nearrow 2} \sum_{k=0}^{n-1} b_k\, t^k\, E_{\alpha,k+1}(\lambda t^\alpha) \\ &= \lim_{\alpha \nearrow 2} b_0\, E_{\alpha,1}(\lambda t^\alpha) + b_1\, t\, E_{\alpha,2}(\lambda t^\alpha) \\ &= b_0\, E_{2,1}(\lambda t^2) + b_1\, t\, E_{2,2}(\lambda t^2) \\ &= b_0 \cosh(\sqrt{\lambda} t) + \frac{1}{\sqrt{\lambda}} b_1 \sinh(\sqrt{\lambda} t)\ , \end{aligned} \tag{4.7}$$

where the absolute convergence and properties (1.4) of the two-parameter function of Mittag-Leffler type are used.

The proof of the lemma is completed by comparing equations (4.6) and (4.7), where in both cases the same holds. $\square$

Comment 4.3. The solution of the classical initial value problem (4.5) of second order coincides with the solution of the fractional initial value problem for values of the order α such that $\alpha \to 2,\ 1 < \alpha < 2$. This result confirms the convergence statement of this subsection as well as the general idea of the fractional derivatives, which is a generalization of the classical integer-order derivatives.

- For $\alpha \to 2,\ 2 < \alpha < 3$ another convergence result (Lemma 4.4) holds. Consider problem (4.5) but instead of $y''(t)$ take $y''(t) - b_2$, where b_2 is the same constant as b_2 in (4.1), i. e., $b_2 = y''(0)$. Thus, the initial value problem

$$\begin{aligned} y''(t) - \lambda y(t) &= b_2 \quad , \qquad t > 0, \quad b_2 \in \mathbb{R}\ , \\ y^{(k)}(0) &= b_k \quad , \qquad b_k \in \mathbb{R}\ , \quad k = 0,\, 1 \end{aligned} \tag{4.8}$$

is received. The motivation for considering this new integer-order initial value problem is the behavior of the Caputo fractional derivative for values of the order α such that $n - 1 < \alpha < n$, $\alpha \to n - 1$, namely (see Subsection 2.1),

$$\lim_{\alpha \to n-1} D_*^\alpha y(t) = y^{(n-1)}(t) - y^{(n-1)}(0)$$

and the fact that in (4.1), for $2 < \alpha < 3$, i. e., $n = 3$

$$y''(0) = b_2.$$

The solution $y(t)$ of the differential equation in (4.8) is the sum of the general solution $y_0(t)$ of the homogeneous equation $y''(t) - \lambda y(t) = 0$ and a particular solution $z(t)$ of the non-homogeneous equation $y''(t) - \lambda y(t) = b_2$. For example, $y(t)$ can be represented as

$$y(t) = y_0(t) + z(t) = c_1\, e^{\sqrt{\lambda} t} + c_2\, e^{-\sqrt{\lambda} t} - \frac{b_2}{\lambda}\,.$$

The constants c_1 and c_2 can be computed from the initial conditions in (4.8). Finally, the solution of the initial value problem (4.8) is

$$y(t) = \frac{1}{2}\left(b_0 + \frac{b_2}{\lambda} + \frac{b_1}{\sqrt{\lambda}}\right) e^{\sqrt{\lambda} t} + \frac{1}{2}\left(b_0 + \frac{b_2}{\lambda} - \frac{b_1}{\sqrt{\lambda}}\right) e^{-\sqrt{\lambda} t} - \frac{b_2}{\lambda}. \tag{4.9}$$

Lemma 4.4. *The solution of the fractional initial value problem* (4.1) *with* $n = 3$ *converges for* $\alpha \to 2$ *to the solution of the classical problem* (4.8).

Proof. Consider the case $n = 3 :\ \alpha \to 2,\ 2 < \alpha < 3$. The solution (4.2) of problem (4.1) satisfies

$$\begin{aligned}
\lim_{\alpha \searrow 2} y(t,\, \alpha) &= \lim_{\alpha \searrow 2} \sum_{k=0}^{n-1} b_k\, t^k\, E_{\alpha,k+1}(\lambda t^\alpha) \\
&= \lim_{\alpha \searrow 2} b_0\, E_{\alpha,1}(\lambda t^\alpha) + b_1\, t\, E_{\alpha,2}(\lambda t^\alpha) + b_2\, t^2\, E_{\alpha,3}(\lambda t^\alpha) \\
&= b_0\, E_{2,1}(\lambda t^2) + b_1\, t\, E_{2,2}(\lambda t^2) + b_2\, t^2\, E_{2,3}(\lambda t^2)\,.
\end{aligned} \tag{4.10}$$

Taking into account that

$$\begin{aligned}
\lambda t^2\, E_{2,3}(\lambda t^2) &= \lambda t^2 \sum_{k=0}^{\infty} \frac{(\lambda t^2)^k}{\Gamma(2k+3)} &&= \sum_{k=0}^{\infty} \frac{(\lambda t^2)^{k+1}}{\Gamma(2(k+1)+1)} \\
&= \sum_{k=1}^{\infty} \frac{(\lambda t^2)^k}{\Gamma(2k+1)} &&= \sum_{k=0}^{\infty} \frac{(\lambda t^2)^k}{\Gamma(2k+1)} - 1 \\
&= E_{2,1}(\lambda t^2) - 1\,,
\end{aligned}$$

then (4.10) can be rewritten as

$$\begin{aligned}
\lim_{\alpha \searrow 2} y(t,\, \alpha) &= b_0\, E_{2,1}(\lambda t^2) + b_1\, t\, E_{2,2}(\lambda t^2) + b_2\, t^2\, E_{2,3}(\lambda t^2) \\
&= b_0\, E_{2,1}(\lambda t^2) + b_1\, t\, E_{2,2}(\lambda t^2) + \frac{b_2}{\lambda}(E_{2,1}(\lambda t^2) - 1) \\
&= (b_0 + \frac{b_2}{\lambda})\, E_{2,1}(\lambda t^2) + b_1\, t\, E_{2,2}(\lambda t^2) - \frac{b_2}{\lambda}\,.
\end{aligned}$$

and using properties (1.4) of the Mittag-Leffler function it follows

$$\begin{aligned}
\lim_{\alpha \searrow 2} y(t,\,\alpha) &= (b_0 + \frac{b_2}{\lambda}) \cosh(\sqrt{\lambda} t) + b_1\, t\, \frac{\sinh(\sqrt{\lambda} t)}{\sqrt{\lambda} t} - \frac{b_2}{\lambda} \\
&= (b_0 + \frac{b_2}{\lambda}) \frac{e^{\sqrt{\lambda} t} + e^{-\sqrt{\lambda} t}}{2} + \frac{b_1}{\sqrt{\lambda}} \frac{e^{\sqrt{\lambda} t} - e^{-\sqrt{\lambda} t}}{2} - \frac{b_2}{\lambda} \\
&= \frac{1}{2} \left(b_0 + \frac{b_2}{\lambda} + \frac{b_1}{\sqrt{\lambda}} \right) e^{\sqrt{\lambda} t} + \frac{1}{2} \left(b_0 + \frac{b_2}{\lambda} - \frac{b_1}{\sqrt{\lambda}} \right) e^{-\sqrt{\lambda} t} - \frac{b_2}{\lambda}\,,
\end{aligned}$$

which coincides with (4.9). Thus, the second main result for this subsection is proved. □

- **Summary**

In Subsections 4.1 and 4.2 the following results for the initial value problem (4.1) have been shown

- the general solution is given by the formula (4.2),
- for $\alpha \to 2$, $1 < \alpha < 2$, the solution of (4.1) converges to the solution of the integer-order initial value problem (4.5),
- for $\alpha \to 2$, $2 < \alpha < 3$, the solution of (4.1) converges to the solution of the integer-order initial value problem (4.8).

Comment 4.5. Unlike the case $\alpha \to 2$, $1 < \alpha < 2$, where the solution of (4.1) converges to the solution of (4.5), in the case $\alpha \to 2$, $2 < \alpha < 3$ there is convergence to the solution of (4.8). This fact is in accordance with the interpolation property of the Caputo fractional operator (see formula (2.3)).

4.3 Special Cases

- **The particular problem**

In order to visualize formula (4.2) and the convergence result from Subsection 4.2, consider the following special case. Let $n = 1,\ \lambda = 1,\ b_0 = 1$. Problem (4.1) then has the particular form

$$\begin{aligned} D_*^\alpha y(t) &= y(t)\ , \\ y(0) &= 1\ . \end{aligned} \tag{4.11}$$

- **The solution**

Its solution is a special case of the general solution (4.2), i. e.,

$$y(t) = \sum_{k=0}^{n-1} b_k\, t^k\, E_{\alpha,k+1}(\lambda t^\alpha) = E_{\alpha,1}(t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{\alpha\, k}}{\Gamma(\alpha k + 1)}\ .$$

For the particular value of the order of differentiation $\alpha = 1/2$,

$$y(t) = E_{\alpha,1}(t^\alpha) = E_{1/2,1}(\sqrt{t}) = e^t\, \mathrm{erfc}(-\sqrt{t}) = e^t\, \frac{2}{\sqrt{\pi}} \int_{-\sqrt{t}}^{\infty} e^{-t^2}\, dt\ ,$$

where for the last equality formula (1.4) is used.

- **The classical case**

The corresponding classical initial value problem of first order reads

$$\begin{aligned} y'(t) &= y(t)\ , \\ y(0) &= 1\ . \end{aligned} \tag{4.12}$$

Its solution is well-known and is the exponential function e^t.

- **Visualization**

The solutions of the fractional (4.11) and the classical (4.12) initial value problems are illustrated in Fig. 4.1 An interesting result (besides the convergence) that can be seen from the figure, is that the shape of the solution is preserved.

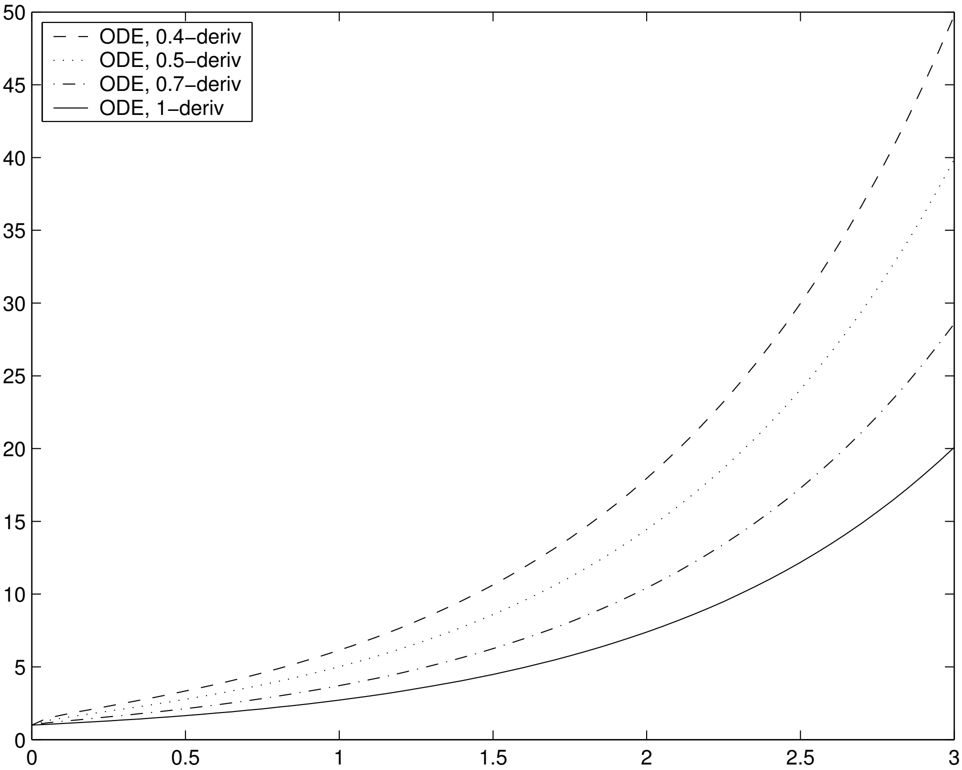

Figure 4.1: Solutions of a fractional IVPs and the corresponding classical IVP

4.4 The Fractional Damped Simple Harmonic Oscillator

In this subsection the damped simple harmonic oscillator is considered (Debnath [2]). First, the classical problem is stated and then the fractional generalization of it is given and a solution is proposed.

- **The classical damped simple harmonic oscillator**

The simple harmonic oscillator obeys the following differential equation

$$y''(t) + \omega_0^2\, y(t) = 0 \ ,$$

where ω_0 is the angular frequency of the oscillation. Adding a damping force proportional to $y'(t)$ the above equation reads

$$y''(t) + b\, y'(t) + \omega_0^2\, y(t) = 0 \ .$$

If a damped oscillator is driven by an external force $f(t)$ the equation changes to the inhomogeneous case

$$y''(t) + b\, y'(t) + \omega_0^2\, y(t) = f(t) \ .$$

- **The fractional damped simple harmonic oscillator**

The fractional initial value problem for the damped simple harmonic oscillator, obtained by substituting fractional derivative of order α $(0 < \alpha < 1)$ for $y'(t)$, and taking into account that two initial conditions are needed for a unique solution of a differential equation of second order, is described by

$$\begin{aligned} y''(t) + b\, D_*^\alpha y(t) + \omega_0^2\, y(t) &= f(t)\ ,\ 0 < \alpha < 1\ , \\ y(0) &= c_0\ , \\ y'(0) &= c_1\ , \end{aligned} \tag{4.13}$$

where b, ω_0, c_0 and c_1 are constants.

Proposition 4.6. (Debnath [2]) *The solution of problem* (4.13) *is given by*

$$y(t) = c_0\, y_0 - \frac{c_1}{\omega_0^2}\, y_0'(t) - \frac{1}{\omega_0^2} \int_0^t y_0'(t-\tau)\, f(\tau)\, d\tau\ ,$$

where $y_0(t) = L^{-1}\left\{ \dfrac{s + b\, s^{\alpha-1}}{s^2 + b\, s^\alpha + \omega_0^2};\ t \right\}$.

Proof. To solve problem (4.13) the Laplace transform is used.

After applying the Laplace transform to both sides of the given equation (see formulas (2.7), (2.12), and (2.16)) the following relation is obtained

$$s^2 Y(s) - \sum_{k=0}^{1} s^{2-k-1} y^{(k)}(0) + b\left(s^\alpha Y(s) - s^{\alpha-1} y(0)\right) + \omega_0^2 Y(s) = F(s)\ ,$$

where $Y(s)$ and $F(s)$ are the Laplace transforms of the functions $y(t)$ and $f(t)$, respectively. Solving this equation with respect to $Y(s)$ and substituting the initial conditions from (4.13) the result for the Laplace transform of the function $y(t)$ is

$$\begin{aligned} Y(s) &= \frac{s + b\, s^{\alpha-1}}{s^2 + b\, s^\alpha + \omega_0^2}\, y(0) + \frac{1}{s^2 + b\, s^\alpha + \omega_0^2}\, y'(0) + \frac{1}{s^2 + b\, s^\alpha + \omega_0^2}\, F(s) \\ &= \frac{c_0\,(s + b\, s^{\alpha-1})}{s^2 + b\, s^\alpha + \omega_0^2} + \frac{c_1}{s^2 + b\, s^\alpha + \omega_0^2} + \frac{F(s)}{s^2 + b\, s^\alpha + \omega_0^2}\ . \end{aligned} \tag{4.14}$$

The inverse Laplace transforms of the three resulting terms in (4.14) have to be found.

- Let $y_0(t) = L^{-1}\Big\{\dfrac{s + b\,s^{\alpha-1}}{s^2 + b\,s^\alpha + \omega_0^2};\ t\Big\}$, i. e., $Y_0(s) = \dfrac{s + b\,s^{\alpha-1}}{s^2 + b\,s^\alpha + \omega_0^2}$.
- Applying (2.11) $y_0(0)$ can be determined as

$$y_0(0) = \lim_{s\to\infty} s\,Y_0(s) = \lim_{s\to\infty} \frac{s\,(s + b\,s^{\alpha-1})}{s^2 + b\,s^\alpha + \omega_0^2} = 1\ . \tag{4.15}$$

Using (2.12) for finding the Laplace transform of the first-order derivative of the function $y_0(t)$ gives

$$L\{y_0'(t);\ s\} = sY_0(s) - y_0(0)\ . \tag{4.16}$$

Rewriting $\dfrac{1}{s^2 + b\,s^\alpha + \omega_0^2}$ and using (4.16) and (4.15) the following result can be obtained

$$\begin{aligned} \frac{1}{s^2 + b\,s^\alpha + \omega_0^2} &= -\frac{1}{\omega_0^2}\left(s\,\frac{s + b\,s^{\alpha-1}}{s^2 + b\,s^\alpha + \omega_0^2} - 1\right) \\ &= -\frac{1}{\omega_0^2}\big(s\,Y_0(s) - y_0(0)\big) \\ &= -\frac{1}{\omega_0^2} L\{y_0'(t);\ s\}\ , \end{aligned}$$

which means that

$$L^{-1}\Big\{\frac{1}{s^2 + b\,s^\alpha + \omega_0^2};\ t\Big\} = -\frac{1}{\omega_0^2}\,y_0'(t)\ . \tag{4.17}$$

- Taking into account formula (2.10) for the Laplace transform of a convolution of two functions

$$\begin{aligned} \frac{1}{s^2 + b\,s^\alpha + \omega_0^2}\,F(s) &= -\frac{1}{\omega_0^2} L\{y_0'(t);\ s\}\,L\{f(t);\ s\} \\ &= -\frac{1}{\omega_0^2} L\{y_0'(t) * f(t);\ s\} \\ &= -\frac{1}{\omega_0^2} L\Big\{\int_0^t y_0'(t-\tau)\,f(\tau)\,d\tau;\ s\Big\}\ . \end{aligned}$$

From the latest it follows that

$$L^{-1}\Big\{\frac{F(s)}{s^2 + b\,s^\alpha + \omega_0^2};\ t\Big\} = -\frac{1}{\omega_0^2}\int_0^t y_0'(t-\tau)\,f(\tau)\,d\tau\ . \tag{4.18}$$

Finally, taking the inverse Laplace transform of both sides of equation (4.14) and according to (4.17) and (4.18), the solution $y(t)$ of the fractional initial value problem (4.13) is given as

$$
\begin{aligned}
y(t) &= L^{-1}\{Y(s);\, t\} \\
&= L^{-1}\left\{\frac{c_0\,(s+b\,s^{\alpha-1})}{s^2+b\,s^\alpha+\omega_0^2} + \frac{c_1}{s^2+b\,s^\alpha+\omega_0^2} + \frac{F(s)}{s^2+b\,s^\alpha+\omega_0^2};\, t\right\} \\
&= c_0\,L^{-1}\left\{\frac{s+b\,s^{\alpha-1}}{s^2+b\,s^\alpha+\omega_0^2};\, t\right\} + c_1\,L^{-1}\left\{\frac{1}{s^2+b\,s^\alpha+\omega_0^2};\, t\right\} + L^{-1}\left\{\frac{F(s)}{s^2+b\,s^\alpha+\omega_0^2};\, t\right\} \\
&= c_0\,y_0 - \frac{c_1}{\omega_0^2}\,y_0'(t) - \frac{1}{\omega_0^2}\int_0^t y_0'(t-\tau)\,f(\tau)\,d\tau\ ,
\end{aligned}
$$

where $y_0(t) = L^{-1}\left\{\dfrac{s+b\,s^{\alpha-1}}{s^2+b\,s^\alpha+\omega_0^2};\, t\right\}$. $\square$

5 C-Laguerre Functions

The main idea is to generalize the classical Laguerre polynomials (see Gradshteyn and Ryzhik [4]) by means of the Caputo fractional operator. One possible way of doing this is generalizing the differential equation, which the polynomials satisfy. The solution of the new equation is then the generalization of the Laguerre polynomials. Another approach could be to generalize Rodrigues' representation.

The second approach is considered in this section. The C-Laguerre functions, unknown so far, are introduced, some of their fundamental properties are examined and compared with the corresponding properties of the classical Laguerre polynomials. The confluent hypergeometric function (Miller and Ross [7], Subsection 1.1) plays a important role in almost all of the computations.

5.1 The Laguerre Polynomials

The Laguerre Polynomials $L_n^\mu(x)$ are orthogonal polynomials that are defined by the following formulas (see Gradshteyn and Ryzhik [4], p. 1037).

Definition 5.1. Let $n \in \mathbb{N}_0$, $\mu \in \mathbb{C}$, $\mathrm{Re}\,(\mu) > -1$, $x \in \mathbb{R}$. Then

$$L_n^\mu(x) = \frac{1}{n!} e^x x^{-\mu} \frac{d^n}{dx^n}(e^{-x} x^{n+\mu}) \tag{5.1}$$

is Rodrigues' representation of the Laguerre polynomials.

Another possible definition is the series representation

$$L_n^\mu(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\mu}{n-k} \frac{x^k}{k!} \,.$$

In this section formula (5.1) is considered as the main definition.

- **Properties**

Next, some of the basic properties of the Laguerre polynomials are considered.

Lemma 5.2. *For the Laguerre polynomials it holds:*

(a) *The first derivative of a Laguerre polynomial can be represented as another Laguerre polynomial by the relation*

$$\frac{d}{dx} L_n^\mu(x) = -L_{n-1}^{\mu+1}(x) \,.$$

(b) *The Laguerre polynomials in the point* 0 *have the values*

$$L_n^\mu(0) = \binom{n+\mu}{n} .$$

(c) *A useful representation of the Laguerre polynomials is the following*

$$L_n^\mu(x) = \binom{n+\mu}{n} {}_1F_1(-n,\, \mu+1;\, x) ,$$

where ${}_1F_1(a,\, b;\, x)$ *is the confluent hypergeometric function (see* Subsection 1.1*).*

Another two important properties are the orthogonality and the relation with the Bessel function (Gradshteyn and Ryzhik [4]). The Laguerre polynomials $L_n^\mu(x)$ are orthogonal with respect to the weight function $\omega(x) = e^{-x}x^\mu$, i. e.,

$$\int_0^\infty e^{-x}x^\mu L_n^\mu(x) L_m^\mu(x)\, dx = 0 , \quad m \neq n \quad (m,\, n \in \mathbb{N}).$$

Furthermore,

$$L_n^\mu(x) = \frac{1}{n!} e^x x^{-\mu/2} \int_0^\infty e^{-t} t^{n+\mu/2} J_\mu(2\sqrt{xt}) dt ,$$

where $J_\mu(z)$ is the Bessel function

$$J_\mu(z) = \frac{z^\mu}{2^\mu} \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\mu+k+1)} , \quad |\arg z| < \pi . \tag{5.2}$$

5.2 Definition and Properties of the C-Laguerre Functions

Rodrigues' representation (5.1) is generalized in this subsection. For the generalized functions, which have the Laguerre polynomials as special cases, some of the properties of the Laguerre polynomials are preserved.

The C-Laguerre functions are defined, by taking the Caputo fractional derivative, instead of the integer-order derivative in formula (5.1) and substituting α for n and $\Gamma(\alpha+1)$ for $n!$. Thus the following definition is obtained.

Definition 5.3. Let $n \in \mathbb{N},\ \mu \in \mathbb{C},\ \mathrm{Re}\,(\mu) > 0,\ x, \alpha \in \mathbb{R},\ n-1 < \alpha < n$. Then

$$L_\alpha^\mu(x) = \frac{1}{\Gamma(\alpha+1)} e^x x^{-\mu} D_*^\alpha (e^{-x} x^{\alpha+\mu}) \tag{5.3}$$

are called C-Laguerre functions.

- **Properties**

Some analogous properties of the C-Laguerre functions to the properties of the classical Laguerre polynomials are proved. For this purpose another representation of the functions (5.3) is considered.

Proposition 5.4. *Let $n \in \mathbb{N},\ \mu \in \mathbb{C},\ Re\,(\mu) > 0,\ x, \alpha \in \mathbb{R},\ n-1 < \alpha < n$. Then the C-Laguerre functions can be represented by means of the confluent hypergeometric function in the following manner*

$$L_\alpha^\mu(x) = \left(\begin{array}{c} \alpha+\mu \\ \alpha \end{array} \right) {}_1F_1(-\alpha,\ \mu+1;\ x)\ . \tag{5.4}$$

Proof. To prove this result, first the Leibniz rule for the Caputo fractional derivative (2.21) should be applied:

$$\begin{aligned} L_\alpha^\mu(x) &= \frac{1}{\Gamma(\alpha+1)} e^x x^{-\mu} D_*^\alpha (e^{-x} x^{\alpha+\mu}) \\ &= \frac{1}{\Gamma(\alpha+1)} e^x x^{-\mu} \Big(\sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array} \right) \left(D^{\alpha-k} (x^{\alpha+\mu}) \right) (e^{-x})^{(k)} \\ &\qquad - \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)} \left((e^{-x} x^{\alpha+\mu})^{(k)}(0) \right) \Big)\ . \end{aligned}$$

Further, since $\mathrm{Re}\,(\mu) > 0$ and $n-1 < \alpha < n,\ \mathrm{Re}\,(\mu+\alpha) > n-1 \geq k$ and hence the second sum is equal to zero. Next, applying formula (3.2) for the Riemann-Liouville derivative of the power function and taking into account that the binomial coefficients with $\alpha \in \mathbb{R}$ are defined by the formula (see Samko, Kilbas, and Marichev [12], p.14)

$$\left(\begin{array}{c} \alpha \\ k \end{array} \right) = \frac{(-1)^{k-1} \alpha\, \Gamma(k-\alpha)}{\Gamma(1-\alpha)\, \Gamma(k+1)}$$

the further computation reads

$$\begin{aligned}
L_\alpha^\mu(x) &= \frac{1}{\Gamma(\alpha+1)} e^x x^{-\mu} \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \left(D^{\alpha-k}(x^{\alpha+\mu})\right) (e^{-x})^{(k)} \\
&= \frac{1}{\Gamma(\alpha+1)} e^x x^{-\mu} \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+\mu-\alpha+k+1)} x^{\alpha+\mu-\alpha+k} (e^{-x})(-1)^k \\
&= \frac{1}{\Gamma(\alpha+1)} e^x x^{-\mu} \sum_{k=0}^{\infty} \frac{(-1)^{k-1}\alpha\,\Gamma(k-\alpha)}{\Gamma(1-\alpha)\,\Gamma(k+1)} \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+k+1)} x^{\mu+k} (e^{-x})(-1)^k\,.
\end{aligned}$$

Rewriting the obtained result in a more compact form and using the properties (1.1) of the Gamma function, it holds

$$\begin{aligned}
L_\alpha^\mu(x) &= \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{-\alpha\,\Gamma(k-\alpha)}{\Gamma(1-\alpha)\,\Gamma(k+1)} \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+k+1)} x^k \\
&= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\,\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(-\alpha+k)}{\Gamma(\mu+1+k)} \frac{x^k}{k!}\,.
\end{aligned}$$

Finally, applying formula (1.5) with $a=-\alpha$ and $b=\mu+1$ and the following formula for the binomial coefficients with arbitrary $\alpha,\,\beta \in \mathbb{C}$, $(\alpha \neq -1,\,-2,\,\ldots)$ (Samko, Kilbas, and Marichev [12], p.14)

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\,\Gamma(\alpha-\beta+1)} \tag{5.5}$$

the desired result is obtained, i. e.,

$$\begin{aligned}
L_\alpha^\mu(x) &= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\,\Gamma(-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(-\alpha+k)}{\Gamma(\mu+1+k)} \frac{x^k}{k!} \\
&= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\,\Gamma(\mu+1)}\, {}_1F_1(-\alpha,\,\mu+1;\,x) \\
&= \begin{pmatrix} \alpha+\mu \\ \alpha \end{pmatrix} {}_1F_1(-\alpha,\,\mu+1;\,x)\,. \qquad \square
\end{aligned}$$

Using formula (5.4) the properties of the C-Laguerre functions can be proved directly.

$$\lim_{\alpha\to n} L_\alpha^\mu(x) = \lim_{\alpha\to n} \begin{pmatrix} \alpha+\mu \\ \alpha \end{pmatrix} {}_1F_1(-\alpha,\,\mu+1;\,x) = \begin{pmatrix} n+\mu \\ n \end{pmatrix} {}_1F_1(-n,\,\mu+1;\,x) = L_n^\mu(x)$$

and also, considering equation (1.6)

$$L_\alpha^\mu(0) = \left(\begin{array}{c} \alpha+\mu \\ \alpha \end{array} \right) {}_1F_1(-\alpha,\, \mu+1;\, 0) = \left(\begin{array}{c} \alpha+\mu \\ \alpha \end{array} \right) ,$$

thus, the following statement holds.

Proposition 5.5. *Let the conditions of Proposition* 5.4 *are fulfilled. Then*

(a) $\lim\limits_{\alpha \to n} L_\alpha^\mu(x) = L_n^\mu(x)$,

(b) $L_\alpha^\mu(0) = \left(\begin{array}{c} \alpha+\mu \\ \alpha \end{array} \right)$.

Other two properties that hold not only for the orthogonal Laguerre polynomials, but also for the C-Laguerre functions are formulated in the following result.

Proposition 5.6. *Let the conditions of Proposition* 5.4 *are fulfilled. Then*

(a) $\dfrac{d}{dx} L_\alpha^\mu(x) = -L_{\alpha-1}^{\mu+1}(x)$,

(b) $L_\alpha^\mu(x) = \dfrac{1}{\Gamma(\alpha+1)} e^x x^{-\mu/2} \displaystyle\int_0^\infty e^{-t} t^{\alpha+\mu/2} J_\mu(2\sqrt{xt})dt$,

where $J_\mu(z)$ *is the Bessel function (see formula* (5.2)*).*

Proof.

(a) To prove the equality, the representation property (5.4) and equation (1.6) are used, i.e.,

$$\begin{aligned} \frac{d}{dx} L_\alpha^\mu(x) &= \frac{d}{dx} \left(\left(\begin{array}{c} \alpha+\mu \\ \alpha \end{array} \right) {}_1F_1(-\alpha,\, \mu+1;\, x) \right) \\ &= \left(\begin{array}{c} \alpha+\mu \\ \alpha \end{array} \right) \frac{-\alpha}{\mu+1} {}_1F_1(-\alpha+1,\, \mu+2;\, x) . \end{aligned}$$

Further, using the binomial coefficient formula (5.5) and the properties of the Gamma function (1.1) the final result is obtained

$$
\begin{aligned}
\frac{d}{dx} L_\alpha^\mu(x) &= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+1)\,\Gamma(\alpha+1)} \frac{-\alpha}{\mu+1}\, {}_1F_1(-\alpha+1,\, \mu+2;\, x) \\
&= -\frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+2)\,\Gamma(\alpha)}\, {}_1F_1(-\alpha+1,\, \mu+2;\, x) \\
&= -\binom{\alpha+\mu}{\alpha-1}\, {}_1F_1(-(\alpha-1),\, (\mu+1)+1;\, x) \\
&= -L_{\alpha-1}^{\mu+1}(x)\ .
\end{aligned}
$$

(b) The formula follows directly from (5.4) and the following representation (Gradshteyn and Ryzhik [4], p. 1058)

$$
{}_1F_1(-\nu,\, \mu+1;\, z) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} e^z z^{-\mu/2} \int_0^\infty e^{-t} t^{\nu+\mu/2} J_\mu(2\sqrt{zt})\, dt\ ,
$$

$$
\left(\text{Re } (\mu+\nu+1) > 0,\ |\arg(z)| < \frac{\pi}{2}\right)
$$

of the confluent hypergeometric function by means of the Bessel function (5.2). □

- **Summary**

At the end of this section, a final summary of the properties of the orthogonal Laguerre polynomials and the C-Laguerre functions is given in Table 3.

Comment 5.7. The generalized C-Laguerre functions are functions and not polynomials. Up to now there is no idea how to define orthogonality and to identify orthogonality.

Property	Laguerre polynomials	C-Laguerre functions
Definition	$L_n^\mu(x) \;=\; \dfrac{1}{n!}\, e^x x^{-\mu}\, \dfrac{d^n}{dx^n}\,(e^{-x}x^{n+\mu})$	$L_\alpha^\mu(x) \;=\; \dfrac{1}{\Gamma(\alpha+1)}\, e^x x^{-\mu}\, D_*^\alpha\,(e^{-x}x^{\alpha+\mu})$
First derivative	$\dfrac{d}{dx}\, L_n^\mu(x) \;=\; -L_{n-1}^{\mu+1}(x)$	$\dfrac{d}{dx}\, L_\alpha^\mu(x) \;=\; -L_{\alpha-1}^{\mu+1}(x)$
Value at 0	$L_n^\mu(0) \;=\; \dbinom{n+\mu}{n}$	$L_\alpha^\mu(0) \;=\; \dbinom{\alpha+\mu}{\alpha}$
Useful representation	$L_n^\mu(x) \;=\; \dbinom{n+\mu}{n}\; {}_1F_1(-n,\, \mu+1;\, x)$	$L_\alpha^\mu(x) \;=\; \dbinom{\alpha+\mu}{\alpha}\; {}_1F_1(-\alpha,\, \mu+1;\, x)$
Relation to Bessel function	$L_n^\mu(x) \;=\; \dfrac{e^x x^{-\mu/2}}{n!} \displaystyle\int_0^\infty e^{-t} t^{n+\mu/2}\, J_\mu(2\sqrt{xt})\, dt$	$L_\alpha^\mu(x) \;=\; \dfrac{e^x x^{-\mu/2}}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty e^{-t} t^{\alpha+\mu/2}\, J_\mu(2\sqrt{xt})\, dt$

Table 3: Comparison between the Laguerre polynomials and the C-Laguerre functions

Appendix A. Table of Caputo derivatives

	$D_*^\alpha f(t)$	$D_*^{1/3} f(t)$	$D_*^{1/2} f(t)$	$D_*^{1/2} D_*^{1/2} f(t)$	$D_*^{1/2} D_*^{1/2} D_*^{1/2} f(t)$	$D_*^{1/2} D_*^{1/2} D_*^{1/2} D_*^{1/2} f(t)$
$f(t) = const$	0	0	0	0	0	0
$f(t) = t$	$\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}$	$1.1077\, t^{2/3}$	$1.1284\, t^{1/2}$	1	0	0
$f(t) = t^2$	$\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}$	$1.3293\, t^{5/3}$	$1.5045\, t^{3/2}$	$2\, t$	$2.2568\, t^{1/2}$	2
$f(t) = t^3$	$\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha}$	$1.4954\, t^{8/3}$	$1.8054\, t^{5/2}$	$3\, t^2$	$4.5135\, t^{3/2}$	$6\, t$
$f(t) = t^4$	$\frac{24}{\Gamma(5-\alpha)} t^{4-\alpha}$	$1.6314\, t^{11/3}$	$2.0633\, t^{7/2}$	$4\, t^3$	$7.2216\, t^{5/2}$	$12\, t^2$
$f(t) = t^5$	$\frac{120}{\Gamma(6-\alpha)} t^{5-\alpha}$	$1.7479\, t^{14/3}$	$2.2926\, t^{9/2}$	$5\, t^4$	$10.3166\, t^{7/2}$	$20\, t^3$
$f(t) = t^{1/2}$	$\frac{\sqrt{\pi}}{2\,\Gamma(3/2-\alpha)} t^{1/2-\alpha}$	$0.9553\, t^{1/6}$	0.8862	0	0	0
$f(t) = t^{3/2}$	$\frac{3\sqrt{\pi}}{4\,\Gamma(5/2-\alpha)} t^{3/2-\alpha}$	$1.2282\, t^{7/6}$	$1.3292\, t$	$1.5\, t^{1/2}$	1.3293	0
$f(t) = e^t$	$t^{n-\alpha}\, E_{1,n-\alpha+1}(t)$	$t^{2/3}\, E_{1,5/3}(t)$	$t^{1/2}\, E_{1,3/2}(t)$	e^t	$t^{1/2}\, E_{1,3/2}(t)$	e^t

Table 4: Caputo derivatives of particular functions

Appendix B. Matlab Programs

In this appendix the original programs for all the numerical experiments and figures in this master thesis are provided. The programs are made in Matlab and are given in the order of their appearance in the master thesis.

- **Figure 1.1**

```
function plot_gamma
% plots the gamma function

i=-5:0.005:4;
y=gamma(i);

y1=[]; x=[];
for j=-5:2:5
    x=[x,j];
    y1=[y1,-1];
end

plot (y1,x,'k--',2.*y1,x,'k--',3.*y1,x,'k--',...
        4.*y1,x,'k--',5.*y1,x,'k--',i,y,'r');
hold on;
plot(0.*y1,x,'k',x,0.*y1,'k');
xlabel('x');
ylabel('Gamma(x)');
axis ([-5 4 -5 5]);
```

- **Figure 1.2**

```
function plot_erfc
% plots the complementary error function

i=-5:0.005:5;
y=erfc(i);

y1=[]; x=[];
for j=-5:2:5
    x=[x,j];
    y1=[y1,2];
end

plot(0.*y1,x,'k',x,0.*y1-0.02,'k');
```

```
hold on;
plot (x,y1+0.01,'k--',x,0.*y1-0.01,'k--',i,y,'r');

xlabel('x');
ylabel('erfc(x)');
axis ([-5 5 -1 2.7]);
```

- **Figure 1.3**

```
function plot_mittag
% plots mittag-leffler functions

a1=1; b1=1; a2=2; b2=1; a3=2; b3=2;
x=[]; y1=[]; y2=[]; y3=[];

for i=-30:0.5:4.5;
    x=[x,i];
    y1=[y1,mittag(a1,b1,i)];
    y2=[y2,mittag(a2,b2,i)];
    y3=[y3,mittag(a3,b3,i)];
end

j=-2:1:4;

plot (x,y1,'r',x,y2,'b--',x,y3,'g-.');
hold on;
plot(x,0.*x-0.01,'k',0.*j,j,'k');

xlabel('x');
ylabel('E_{\alpha, \beta} (x)');
axis ([-30 5 -2 4]);

l1=sprintf('\\alpha =%d, \\beta=%d',a1,b1);
l2=sprintf('\\alpha =%d, \\beta=%d',a2,b2);
l3=sprintf('\\alpha =%d, \\beta=%d',a3,b3);
legend(l1,l2,l3,2);


function res=mittag(a,b,z)
% mittag(a,b,z) evaluates the mittag-leffler function
% with parameters a and b in the point z

k=sym('k');
res=double(1/gamma(b)+symsum((z.^k)/gamma(a*k+b),k,1,Inf));
```

- **Figure 3.1**

```
function gamma_coeff_3D
% 3D graph of the coefficients gamma(p+1)/gamma(p-alpha+1)

[p,alpha]=meshgrid(0:0.5:5,0:0.1:1);

y=gamma(p+1)./gamma(p-alpha+1);

colormap([0.8 0.8 0.8]);
surf(p,alpha,y);

xlabel('p');
ylabel('\alpha');
zlabel('\Gamma(p+1)/\Gamma(p-\alpha+1)');
```

- **Figure 3.2**

```
function ex1
% fractional derivatives of f(t)=t^2, i.e.,
% t^2, (t^2)^(1/3), (t^2)^(1/2), (t^2)^(3/4), (t^2)'
% y=(2/gamma(3-alpha))*(i.^(2-alpha));

i=0:0.05:4;

y1=i.^2;                          % t^2 (alpha=0)
y2=(2/gamma(8/3))*(i.^(5/3));   % alpha=1/3
y3=8*(i.^(3/2))/(3*sqrt(pi));   % alpha=1/2
y4=(2/gamma(9/4))*(i.^(5/4));   % alpha=3/4
y5=2.*i;                          % 2t (alpha=1)

plot (i,y1,'r',i,y2,'b--',i,y3,'g-.',i,y4,'k:',i,y5,'k')

legend('0 -  deriv','1/3-deriv','1/2-deriv','3/4-deriv','1 -  deriv',2)
```

- **Figure 3.3**

```
function t2(a,x)
% plots fractional derivatives of the function f(t)=t-a
% in the interval (a,x)
% the formula for the derivative of order alpha (alpha in (0,1)) is
% (1/(gamma(2-alpha))*(t-a)^(1-alpha).
% for its computation the function t is used.
```

```
% call, for example, t2(0,4)

i=a:(x-a)/100:x;
ones=i*0+1;
r1=t(1/4,a,x);
r2=t(2/4,a,x);
r3=t(3/4,a,x);

plot(i,i-a,'r-',i,r1,'b--',i,r2,'g-.',i,r3,'k:',i,ones,'r-');
legend('f(t)','1/4-deriv','2/4-deriv','3/4-deriv','1-deriv',2)
axis ([0 x 0 x])


function res=t(alpha,a,x)
% an auxiliary function for the function t2

i=a:(x-a)/100:x;
res=(1/gamma(2-alpha))*(i-a).^(1-alpha);
```

- **Figure 3.4**

```
function t2_3D(t)
% 3D-graph of the functions (t^2)^(alpha) for 0<=alpha<=2
% in the interval (0,t)
% call, for example, t2_3D(4)

[x,alpha]=meshgrid(0:t/10:t,0:0.1:2);

y=(2./gamma(3-alpha)).*(x.^(2-alpha));

colormap([0.8 0.8 0.8]);
surf(x,alpha,y);
hold on;

i=0:t/10:t;
h=plot3(i,ones(1,length(i)),2.*i,'k-',i,zeros(1,length(i)),i.^2,'k-',...
        i,2.*ones(1,length(i)),2.*ones(1,length(i)),'k-');
set(h,'LineWidth',2.8);

xlabel('t');
ylabel('\alpha');
```

- **Figure 3.5 ; Figure 3.6**

```
function fracderiv_e_plot2(alpha)
% the graph of the alpha-th and alpha+2.3-th fractional
% derivative of the function e^t. the formula is
% D*alpha(e^t)=sum_n^Inf(t^(k-alpha)/gamma(k+1-alpha)) and
% is computed by the function fracderiv_e2
% call, for example, fracderiv_e_plot2(0.5)

x=[]; y=[]; y1=[]; y2=[];

for i=0:0.05:1.5    % or i=0:0.05:3
    x=[x,i];
    y=[y,exp(i)];
    y1=[y1,fracderiv_e2(i,ceil(alpha),alpha)];
    y2=[y2,fracderiv_e2(i,ceil(alpha+2.3),alpha+2.3)];
end

plot (x,y,'k',x,y2,'b--',x,y1,'r:',x,y-1,'k')

t1=sprintf('%.1f - deriv',alpha);
t2=sprintf('%.1f - deriv',alpha+2.3);
legend('e^t',t2,t1,'e^t-1',2);


function res=fracderiv_e2(t,n,alpha)
% an auxiliary function for the function fracderiv_e_plot2

k=sym('k');
res=double(symsum(t.^(k+n-alpha)/gamma(k+n+1-alpha),k,0,Inf))
```

- **Figure 3.7**

```
function e_3D(t)
% 3D graph of the functions (e^t)^(alpha) for 0<=alpha<=1
% in the interval (0,t)
% call, for example, e_3D(1.5)

[x,alpha]=meshgrid(0:t/10:t,0:0.1:1);

n=1;
alpha=alpha+n-1;
k=sym('k');
```

```
y=double(symsum(x.^(k+n-alpha)./gamma(k+n+1-alpha),k,0,Inf));

colormap([0.8 0.8 0.8]);
surf(x,alpha,y);
hold on

i=0:t/10:t;
h=plot3(i,ones(1,length(i)),exp(i),'k-',i,zeros(1,length(i)),exp(i)-1,'k-');
set(h,'LineWidth',2.8);

xlabel('t');
ylabel('\alpha');
```

- **Figure 4.1**

```
function exdiff1(alpha,alpha1,alpha2)
% solutions of an ODE with classical and with fractional derivatives,
% alpha,alpha1,alpha2 - orders of differentiation
% call, for example, exdiff1(0.4,0.5,0.7)

i=0:0.05:3;
y1=exp(i);
y3=mittag(alpha,1,i.^(alpha));
y4=mittag(alpha1,1,i.^(alpha1));
y5=mittag(alpha2,1,i.^(alpha2));

plot (i,y3,'k--',i,y4,'b:',i,y5,'g-.',i,y1,'r')

t=sprintf('ODE, %.1f-deriv',alpha);
t1=sprintf('ODE, %.1f-deriv',alpha1);
t2=sprintf('ODE, %.1f-deriv',alpha2);
legend(t,t1,t2,'ODE, 1-deriv',2)
```

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Properties and Applications of the Caputo Fractional Operator

Contributions

of

Mariya Ishteva

- <u>**Basic Definitions**</u>

Example for the derivative $D_*^{1/2}t$, p. 14.

Figures: 1.1, 1.2, 1.3 (Graphs of special functions), pp. 8, 9, 10.

- <u>**The Caputo Fractional Derivative**</u>

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- <u>**Examples of Fractional Derivatives**</u>

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- **Fractional Ordinary Differential Equations**

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- **C-Laguerre Functions**

Definition 5.3 (C-Laguerre functions), p. 53.

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Proposition 5.5, 5.6 and their proofs (Properties of the C-Laguerre functions), pp. 55, 55.

- **Appendix A. Table of Caputo derivatives**

Table 4 (Caputo derivatives of particular functions), p. 58.

- **Appendix B. Matlab Programs**

All Matlab programs. These are the programs for the visualizations of the results in the master thesis (Figures 1.1, 1.2, 1.3, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 4.1), p. 59.

- **Open Questions**

– Formula for the Caputo derivative of the power function $D_*^\alpha t^p$ for $n-1 < \alpha < n$, $p < n-1,\ p \in \mathbb{R}$.

– Caputo derivatives of other functions: $ln(t),\ \delta(t), \ldots$.

– Generalization of formulas from the analysis for the Caputo operator: Chain rule (Caputo fractional derivative of a composite function).

– Orthogonality of functions, which generalize orthogonal polynomials.

– Numerical methods: Approximation of the Caputo fractional derivative.