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Review on Fractional Differential Equations and their Applications

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***Abstract*—With the research done in past three decades, the subject of fractional calculus has gained much importance due to the application in diverse field in science and engineering. The fractional derivatives and integrals enable the description of the memory and hereditary properties. Hence there is growing need to find the solutions behaviour of these fractional differential equations.**

In this paper we reviewed the literature about the basics, standard approaches, analytical and approximate methods to the problem of fractional differential equations, while discussing about the basic properties including the rules for their compositions and the conditions for the equivalence of various definitions.

I. INTRODUCTION

RACTIONAL differential equation is a generalization of Fordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation.

Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multi poles, neuron modelling encompassing different branches of physics, chemistry and biological sciences. There have been many excellent books and monographs available on this field. Most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus was given by many.

Many physical processes appear to exhibit fractional order behaviour that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on. The results have been obtained by using fixed point theorems.

Fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. The concept of fractional calculus(fractional derivatives and fractional integral) is not new. In 1695 *L'Hospital* asked the question as to the meaning of $d^n y/dx^n$ if $n = 1/2$; that is " what if n is fractional?". *Leibniz* replied that "$d^{1/2}x$ will be equal to $x\sqrt{dx} : x$".

II. BASIC DEFINITIONS

This section is devoted to review three important definition of fractional derivative and give some examples of fractional differential equations equipped by them

i. Riemann-Liouville definition:

The popular definition of fractional derivative is this one:

$$ {}_{\alpha}D_t^{\alpha} f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \left(\frac{d}{dt}\right)^n \int_{\alpha}^{t} \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} \qquad (n-1 \le \alpha < n) \qquad (1)$$

This operator has the following important properties:
For a function f

$$ {}_{\alpha}D_t^{\alpha}\ {}_{\alpha}D_t^{\beta} f(x) = {}_{\alpha}D_t^{\alpha+\beta} \qquad (2)$$

By using of this definition, V. V. Anh and R. Mcvinish considered fractional differential equations of the general form

$$ \left(A_n D^{\beta_n} + \cdots + A_1 D^{\beta_1} + A_0 D^{\beta_0}\right) X(t) = \dot{L}(t) \qquad (3)$$

$$ \beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0, \qquad n \ge 1$$

where $\dot{L}$ is Levy noise.
Fractional differential equations in terms of the Riemann-Liouville derivatives require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function.
For example, in the following initial value problem (where $n-1 < \alpha < n$):

$$ {}_0D_t^{\alpha} f(t) + af(t) = h(t) \qquad (t > 0) \qquad (4)$$

$$ \left[\ {}_0D_t^{\alpha} f(t)\right]_{t\to 0} = b_k \qquad (k = 1,2,\ldots,n) \qquad (5)$$

ii. Grunwald-Letnikove:

This is another joined definition which is sometimes useful.

$$ {}_0D_t^{\alpha} = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j f(t-jh) \qquad (6) $$

iii. M. Caputo (1967):

The second popular definition is

$$ {}_{\alpha}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_{\alpha}^{n} \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} $$

$$ (n-1 \le \alpha < n) \qquad (7) $$

As an example of fractional differential equations of Caputo derivative is

$$ D^{\alpha} y(t) = a(t)y(t) + f(t) + \int_0^t K(t,s)F\big(y(s)\big)ds. $$

$$ t\epsilon[0,1] \qquad (8) $$

where α is a parameter describing the order of the fractional derivative, and $F(y(x))$ is a nonlinear continuous function. Such kind of equations arise in the mathematical modelling of various physical phenomena, such as heat conduction in materials with memory.

Moreover, these equations are encountered in combined condition, convection and radiation problems. Initial conditions for the Caputo derivatives are expressed in terms of initial values of integer order derivatives. It is known that for zero initial conditions the Riemann-Liouvillee, Gr¨unwald-Letnikov and Caputo fractional derivatives coincide. This allows a numerical solution of initial value problems for differential equations of non integer order independently of the chosen definition of the fractional derivative. For this reason, many authors either resort to Caputo derivatives, or use the Riemann-Liouvillee derivatives but avoid the problem of initial values of fractional derivatives by treating only the case of zero initial conditions.

iv. K. S. Miller, B. Ross (1993):

They used differential operator D as

$$ D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_n} f(t), $$
$$ \bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \qquad (9) $$

which $D^{\alpha i}$ is Riemann-Liouville or Caputo definitions.
Initial conditions of fractional differential equations in terms of Caputo derivatives are expressed in terms of initial values of order derivatives.

v. Caputo Fractional derivative

The Caputo Fractional derivative is defined by

$$ \frac{1}{\Gamma(n-a)} \int_0^x (x-t)^{n-\alpha-1} f^n(t)dt \qquad (10) $$

where $n \epsilon N$, and $f^n(t)$ is n th order derivative of $f(t)$.
Caputo Fractional derivative is upgrade form of Riemann Liouville Fractional integral .It is used for differential equations.

vi. Mittag-Leffler function

In mathematics the Mittag-Leffler function $E_{\alpha,\beta}$ is a special function which depends on two complex parameters α and β. It may be defined by the following series when the real part of α is strictly positive:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (11) $$

vii. Hypergeometric function

The hypergeometric function is defined for $|z| < 1$ by the power series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (12) $$

It is undefined (or infinite) if c equals a non-positive integer. Here $(q)_n$ is the (rising) pochhemer symbol, which is defined by:

$$ (q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)\dots\dots(q+n-1) & n > 0 \end{cases} \qquad (13) $$

viii. Hadamard type Fractional integral

The Hadamard type Fractional integral is defined by

$$ \frac{1}{\Gamma\alpha} \int_0^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t} \qquad (14) $$

where $\alpha \in (0,\infty)$.

$$ F(s) = M\{f(t), s\} = \int_0^{\infty} t^{s-1} f(t)dt $$

ix. Fourier transform

The Fourier transform of a function $f(t)$, of a real variable $t \in (-\infty, \infty)$ is defined by

$$ F(s) = F\{f(t), s\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)dt. \qquad (15) $$

Where $f(t)$is sectionally continuous differentiable and absolutely integrable. Also $f(t)$ is said to be inverse Fourier transform of F(s) and it can be written as

$$ f(t) = F^{-1}\{F(s), t\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} F(s)ds \qquad (16) $$

x. Laplace transform

The Laplace transform of a function $f(t)$ of a variable $t \in (0,\infty)$ is defined by

$$ F(s) = L\{f(t), s\} = \int_0^{\infty} e^{-st} f(t)dt \qquad (17) $$

where f(t) is sectionally continuous in every finite interval and is of exponential order. Also f(t)is said to be inverse Laplace transform of F(s) and it can be written as

$$f(t) = L^{-1}\{F(s), t\} \qquad (18)$$

xi. Mellin transform

The Mellin transform of a function $f(t)$ of a variable $t \in (0, \infty)$ is defined by

$$F(s) = M\{f(t), s\} = \int_0^{\infty} t^{s-1} f(t) dt \qquad (19)$$

If the integral converges.

xii. Grunwald Letnikov Fractional derivative

The Grunwald Letnikov derivative is written as

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\frac{x-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(x-mh) \qquad (20)$$

where $\binom{n}{m}$ is binomial coefficient.

This expression can be generalized for Fractional values of n = α, where

$$D^{n} f(x) = \lim_{h \to 0} \frac{1}{h^{n}} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x-mh) \qquad (21)$$

where a is lower limit of differentiation

III. Numerical Methods for the Solution of Ordinary and Partial Fractional Differential Equations

When working with problems stemming from "real-world' applications, it is only rarely possible to evaluate the solution of a given Fractional differential equation in closed form, and even if such an analytic solution is available, it is typically too complicated to be used in practice. Therefore it is indispensable to have a number of numerical algorithms at hand so that one is able to compute numerical solutions with a sufficient accuracy in reasonable time.

3.1. The Approximation of Fractional Differential and Integral Operators

3.2. Direct Methods for Fractional Ordinary Differential Equations

3.3. Methods for Fractional Ordinary Differential Equations

3.4. Linear Multistep Methods

3.5. Methods for Fractional Ordinary Differential Equations

3.6. Linear Multistep Methods

3.7. Other Methods

3.8. Methods for Terminal Value Problems

3.9. Numerical methods for Multi Term Fractional Differential Equations and Multi Order Fractional differential systems

3.10. The Extension to Fractional Partial Differential Equations

3.1. The Approximation of Fractional Differential and Integral Operators

Let us recall from the definitions, the Fractional integral in the sense of Riemann Liouvillee, defined in equation

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \quad . \qquad (1.3.1)$$

Obviously, J^{α} is an integral operator. The Riemann Liouville Fractional derivative

$$RL\ D^{\alpha} f(x) = \frac{d^n}{dx^n} J^{n-\alpha} f(x), \qquad n = \lceil \alpha \rceil \qquad (1.3.2)$$

Where $\lceil \bullet \rceil$ denotes the ceiling function that rounds up to the nearest integer, is at first sight a combination of a classical differential operator and an integral operator, but for suitable functions f it can also be written as a pure integral operator, namely

$$RL\ D^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} f(t) dt \ . \qquad (1.3.4)$$

a. Methods based on quadrature theory

It is evident from the representations (22) and (24) that all Fractional derivatives and integrals that we are interested in can be interpreted as integral operators in some sense. Therefore it seems to be very natural that the first approximation methods for such operators that we shall look at are based on principles from quadrature theory i.e. the theory of numerical integration.

b. Grunwald–Letnikov methods

Another very obvious approach for the discretization of Fractional differential and integral operators is based on a straightforward generalization of concepts form classical calculus to the Fractional case. Specifically, it is well known that an integer order derivative can be written as a differential quotient viz.

$$D^{n} f(x) = \lim_{h \to 0} \frac{1}{h^{n}} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} f(x-kh). \qquad (1.3.5)$$

c. Lubich's Fractional linear multistep methods

None of the methods described so far has exhibited very fast convergence. To overcome this difficulty we now describe a class of methods introduced by Lubich in a series of papers in the 1980s [81-84]

The starting point for Lubich's approach is a classical linear multistep method for first order initial value problems of the form

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0 \qquad (1.3.6)$$

3.2. Direct Methods for Fractional Ordinary Differential Equations

We shall now apply the approximation methods introduced above in the context of the numerical solution of Fractional differential equations. In this and the following two sections, we will look at ordinary differential equations of Fractional order;

For the purpose of exposition of the methods for solving ordinary Fractional differential equations, we believe a distinction of cases to be useful. Specifically, we shall classify the numerical methods under consideration into direct and indirect methods

a. **The basic idea**

So let us now start our investigations by introducing the direct methods. They are characterized by the fact that we take the given initial value problem

$$ {}^{C}D^{\alpha} y(x) = f(x, y(x)),$$
$$ y^{(k)}(0) = y_0^{(k)} \qquad (k = 0, 1, \ldots\ldots, \lceil \alpha \rceil - 1) \qquad (1.3.7)$$

and apply one of the approximation algorithms of 1.3.1 directly to the operator ${}^{C}D^{\alpha}$. Since we shall restrict our attention to linear schemes, i.e., methods of the form

$$ {}^{C}D^{\alpha} y(x_n) \approx \sum_{k=0}^{n} a_{k,n}\, y(x_k) \qquad (1.3.8)$$

b. Quadrative based direct methods

We begin by looking at the method obtained by using a discretization of the differential operator based on quadrature theory, this leads to an algorithm of the form

$$ h^{-\alpha} \sum_{j=0}^{k} A_{j,k} y_j = f(x_k . y_k) \qquad (k = 1,2,\ldots\ldots N) \qquad (1.3.9)$$

Since $A_{k.k} = 1/\Gamma(2-\alpha)$, we may rewrite this identify in the form

$$ y_k = \Gamma(2-\alpha) h^{\alpha} f(x_k . y_k) - \Gamma(2-\alpha) \sum_{j=0}^{k-1} A_{j.k} y_j \qquad (1.3.10)$$

The usual procedure for the computation of the numerical solution is to calculate y_1 first, then y_2, then y_3, etc., until y_N is reached. In this procedure, the currently computed value is the only unknown in the equation; all other data have already been calculated in previous steps and hence are known. It is then natural to ask whether Equation (30) can be solved for y_k the following result states that the answer is positive provided that the step size is sufficiently small.

3.3. Indirect Methods for Fractional Ordinary Differential Equations

a. The basic idea

In contrast to the direct methods that we had introduced in the previous section, indirect methods are constructed not by applying a discretization directly to the Fractional differential equation, but by first performing some analytical manipulation on the initial value problem and by then applying the numerical method to the equation obtained as the result of this analytical operation. The most common idea in this context is to apply the Riemann-Liouville integral operator J^{α} to the initial value problem (1.27), thus creating the nonlinear and weakly singular Volterra integral equation of the second kind.

$$ y(x) = \sum_{k=0}^{[\alpha]-1} \frac{y_0^{(k)}}{k!} x^k + J^{\alpha}[f(\cdot, y(\cdot))](x). \qquad (1.3.11)$$

b. An Adams type predictor corrector method

It is based on the approximation of the integral operator in Equation (1.27) by the product trapezoidal method introduced in ξ 1.3.1.1 this leads to the formula

$$ y_k = \sum_{j=0}^{[\alpha]-1} \frac{x_k^j}{j!} y_0^{(j)} + h^{\alpha} \sum_{j=0}^{k} \alpha_{jk} f(x_j, y_j) \text{ For k = 1, 2…N} \qquad (1.3.12)$$

In this limit case $\alpha \to 1$ this method reduced to the classical second order Adams Moulton formula; hence we shall call our method a Fractional Adams Moulton formula

c. The Cao-Burrage Abdullah approach

Stating from the initial value problem.

$$ {}^{c}D^{\alpha}{}_{y}(x) = f(x, y(x)), \quad y(0) = y_0 \qquad (1.3.13)$$

With $0 < \alpha < 1$, they decided not to apply the integral operator of order α but a differential operator of order $1-\alpha$ to the differential equation. Interestingly, it is necessary here not to use a Caputo operator but a Riemann-Liouville, operator, thus using the formal relation obtaining the problem

$$ y'(x) = RLD^{1-\alpha}\, {}^{C}D^{\alpha}{}_{y}(x) = RLD^{1-\alpha}[f(\cdot, y(\cdot))](x) \qquad (1.3.14)$$

Still augmented by the originally given initial condition $y(0) = y_0$.

3.4. Linear Multistep Methods

A particularly well understood class of methods is based on using the linear multistep methods described in Subsection 1.3.1.3 above for the discretization of the Fractional differential and integral operators arising in our equations.

3.5. Other Methods

Some authors have also suggested other approaches than those described so far. Most of these methods have not gained a substantial amount of attention. The primary exception is probably the so called Adomian decomposition that is usually traced back to Adomian's books even though its roots can actually be found in a series of much older papers by Perron. The idea of the method is to write the differential equation in the abstract form

$$M(x, y(x)) = g(x) \qquad (1.3.15)$$

Where g is a given function, y is the unknown solution and M is a suitable operator.

3.6. Methods for Terminal Value Problems

Sometimes one is interested in the solution of a somewhat different class of problems that are occasionally known under the name terminal value problems; specifically we still want to solve a differential equation of the form

$$^{C}D^{\alpha}y(x) = f(x, y)(x)) \qquad (1.3.16)$$

3.7. Numerical methods for Multi Term Fractional Differential Equations and Multi Order Fractional differential systems

Up to this point we have only discussed so called single term Fractional differential equations, i.e. equations containing only one differential operator. Whereas these equations form appropriate models for many problems in physics and other sciences, they are insufficient in some other cases. Indeed it is sometimes necessary to use differential equations involving differential operators of more than one order. Among the most prominent examples of such a situation we mention the Bagley Torvik equation

$$AD^{2}y(x) + B^{C}D^{3/2}y(x) + Cy(x), \quad y(0) = y_0, y'(0) = y_0^{'}, \qquad (1.3.17)$$

where $A \neq 0$ and B and C are arbitrary real numbers.

3.8. The Extension to Fractional Partial Differential Equations

In this Chapter, we have dealt only with ordinary differential equations of Fractional order so far. However, in the modeling of various phenomena in finance, engineering, physics and other area the use of partial differential equations with Fractional differential operators is becoming more and more popular. A particularly important class of applications arising in this context is the so called time Fractional diffusion wave equations.

$$^{C}D_{t}^{\alpha}y(x,t) + \phi(x,t)\sum_{j=1}^{n}\frac{\partial^{2}}{\partial x_{j}^{2}}y(x,t) = f(x,t) \qquad (1.3.18)$$

for $x = (x_1, \ldots\ldots, x_n) \in G \subset R^n$ and $t \in [0, T]$. Here, the notation $^{C}D_{t}^{\alpha}y(x,t)$ indicates the partial derivative of Caputo type of order α of the function y with respect to t. It is clear that this equation unifies the classical concepts on the n-dimensional diffusion equation (heat equation) which is obtained for $\alpha = 1$ and the n-dimensional wave equation ($\alpha = 2$). In order to obtain a well posed problem, it is natural to combine equation (31) with two boundary conditions and one (if $\alpha \leq 1$) or two (if $1 < \alpha \leq 2$) initial conditions. These conditions are usually given in the same form as in the classical case, viz.

$$A(x,t)y(x,t) + B(x,t)\frac{\partial}{\partial n}y(x,t) = g(x,t) \qquad (1.3.19)$$

for $t \geq 0$ and $x \in \partial G$

(With $\partial / \partial n$ denoting the partial derivative in the direction of the outer normal of the boundary at the point x) and

$$y(x,0) = f_1(x), \quad \frac{\partial}{\partial t}y(x,0) = f_2(x) \qquad (1.3.20)$$

for $x \in G$.

IV. Specific Efficient Methods for the Solution of Ordinary and Partial Fractional Differential Equations

In the previous section we had introduced a number of numerical methods for (ordinary and partial) Fractional differential equations. Our goal now is to look more closely at some particularly important methods from this selection. In particular we want to provide some basic knowledge that allows deciding for a concrete algorithm for a specific problem at hand. Moreover we want to mention a few techniques that are generally applicable to many classes of algorithms and that allow to obtain slightly modified versions that retain the convergence behaviour but that require less computational effort.

4.1 *Methods for Ordinary Differential Equations*

We shall begin with the investigation of algorithms for ordinary Fractional differential equations.

a. Dealing with non-locality: The finite memory principle, nested meshed, and the approaches of Deng and Li

A key problem in connection with the numerical solution of Fractional differential equations is the fact that Fractional differential operators are never local. This means that, whenever we want to compute the value of the expression $^{C}D^{\alpha}y(x)$, we need to take into consideration the entire history of the function y, i.e. The function values on the complete interval [0,x]. In contrast, for a derivative of integer order it would be sufficient to have information on an arbitrarily small neighbourhood of x. This crucial difference implies, as can be seen from a close inspection of the numerical algorithms introduced in article 1.2, the computation of a numerical solution of an ordinary Fractional differential equation on a fixed interval [0,b], say, by one of the standard algorithms using a step size of h has an arithmetic complexity of $O(h^{-2})$ which is much more effort

than the $O(h^{-1})$ operation count that we can observe for differential equations of first order.

We shall look at the three best known and most frequently used concepts and discuss their merits.

The first of three methods is the finite memory principle (sometimes also denoted as short memory principle). Basically, one defines a fixed value T > 0, the so-called memory length, and modifies the Caputo derivatives

$$ {}_c D^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} y^{(n)}(t)dt $$

(Where $n = \lceil \alpha \rceil$ of the function y if x > T. (No. Changes are necessary for $x \le T$) In the case x > T the length of the range of integration, i.e. The length of the actual memory used in the definition of the Caputo operator, is greater than the prescribed memory length T, and it grows even larger as x grows. The idea of the finite memory principle is then not to use the complete memory but only a portion of length T of it. Thus one uses

$$ \frac{1}{\Gamma(n-\alpha)} \int_{x-T}^x (x-t)^{n-\alpha-1} y^{(n)}(t)dt $$

Instead of ${}^C D^{\alpha} y(x)$ if x > T. In mathematical terms, we increase the lower terminal point of the interval of integration, i.e. We choose to take into account the right part of [0,*x*] and to ignore the contribution of the left part of this interval. This choice has been because of the monotonicity property of the kernel $(x-t)^{n-\alpha-1}$, the exponent of this expression is negative, and so the contribution of the left part is likely to be much smaller than the contribution of the right part, thus this choice minimizes the error introduced by the scheme. In this way one always needs to approximate an operator that only takes into account function values from the interval [*x*-T,*x*] Whose length never exceeds T. Hence, this very simple strategy allows reducing the arithmetic complexity of standard numerical methods to the $O(h^{-1})$ count that we know from algorithms for first order equations.

b. Parallelization of algorithms

Another straight forward approach to tackle the computational complexity introduced by the non-locality of the Fractional differential operators is based on using parallel computers. We shall explain this approach, first discussed in [38] on the basis of the Adams-Bashforth-Moulton method in its standard form. However, the basic principle can also be applied to modifications for this algorithm like the nested mesh concept described above or even to many completely different classes of numerical methods.

c. When and when not to use Fractional linear multistep formulas

Traditionally, the Fractional linear multistep formulas (in particular, the Fractional backward differentiation formulas) proposed by Lubich that we had introduced in Subsection 1.2.1.3 and in Section 1.2.4 have been considered to be very good methods for the numerical solutions of Fractional differential equations due to the fact that they combine a conceptual simplicity, a straight forward method to compute their coefficients from the corresponding coefficients of well known methods for first order differential equations and a high order of convergence. High order Fractional linear multistep methods are a very useful concept in theory but as long as the problem of finding an accurate scheme for the numerical computation of the starting weights is unsolved, they cannot be reliably computed in practice except for a small set of the differential equation under consideration happens to be in this set.

d. The use of series expansions

An essential feature of Fractional multistep formulas is that they consist of two components, the convolution quadrature and the starting quadrature. The main problem of these formulas that we had found in the previous sub-section was that it is extremely difficult to compute the starting quadrature with a sufficient accuracy. No such problems are associated with the convolution quadrature. Thus one might be tempted to find some approach that avoids the use of the starting quadrature. In order to construct such an approach it is useful to recall the background that has leaded us to introducing the starting quadrature in the first place. In fact the reason was that we wanted to have a high order of convergence for a most general class of equations, and it is known that the solutions of such equations are typically not smooth at the origin; rather they have an asymptotic expansion in powers of the form $x^{j+k\alpha}$ with $j,k \in N$ .The introduction of the starting quadrature was an attempt to make the complete formula exact for linear combinations of such powers with small j and k, thus eliminating all low order terms in the asymptotic expansion of the error with respect to the step size.

e. The generalized Adams methods as an efficient tool for multi-order Fractional differential equations

In Section 3.7 we had dealt with the basics of multi term equations, i.e. equations involving more than one Fractional derivative. In particular we had constructed a number of possible methods to convert such equations into systems of single-order equations. In some of these approaches, all differential equations of the resulting system were of Fractional order in other approaching we had a mixture of Fractional order and integer – order equations. Of course, the reformulation of the given multi-term equation in the form of a system of single term equations alone is not sufficient to obtain a numerical solution. Rather we need to solve the system that we have created by a suitable numerical method.

4.2 Methods for Partial Differential Equations

In our discussion of numerical methods for partial Fractional differential equations, we shall first concentrate on generalizations of classical problems like diffusion or wave equations obtained by replacing the integer-order time derivatives with a Fractional differential operator. In particular, the first three subsections of this section will be devoted to numerical algorithms for problems of this type. Mainly we shall describe how classical approaches can be extended to the Fractional setting, In subsection 1.4.2.4 we

shall then briefly deal with numerical methods for equations that are of Fractional order with respect to the space variables.

a. The method of lines

The first method that we shall look at explicitly, the vertical method of lines belongs to the class of semi-discrete methods. This is well known concept in the theory of classical partial differential equations of integer order where it is frequently employed for parabolic problems. We shall look at the Time-Fractional diffusion equation in one space dimension.

$$ {}^{c}D_t^{\alpha} y(x,t) = K(x,t)\frac{\partial^2}{\partial x^2} y(x,t) + f(x,t) \quad (4.2.1a)$$

For $t \in [0,T]$ and $x \in [a,b]$ say, with $0 < \alpha < 1$ and a strictly positive function K, subject to the initial condition

$$y(x,0) = y_0(x) \text{ for } x \in [a,b] \quad (4.2.1b)$$

and the boundary conditions

$$y(a,t) = r_1(t) \text{ and } y(b,t) = r_2(t) \text{ for } t \in [0,T] \quad (4.2.1c)$$

With certain given functions K, f, y_0, r_1 and r_2. From the description that we shall now give it will be evident that the method can easily be modified to handle other related problems like, for example, equations in more than one space dimension, equations with other types of boundary conditions or Time Fractional wave equations, i.e. equations like (1.4.2.1) but with $1 < \alpha < 2$ and then, of course, with a second initial condition.

b. Backward difference formulas for Time-Fractional parabolic and hyperbolic equations

For non-Fractional partial differential equations of parabolic type such as, e.g., equations of the form (1.3.2.1) with $\alpha = 1$, the finite difference methods are a standard choice. It is therefore not surprising that the Fractional generalizations of these methods are very popular tools when it comes to the numerical solution of Fractional differential equations of the form (1.3.2.1),at least with $0 < \alpha < 1$.In this context it is particularly helpful to know that many well known properties of these algorithms can be carried over from the integer-order to the fractional case. Thus we shall now look at this class of method in more detail. Specifically, a finite difference method is characterized by the fact that it discretizes the given differential equation by using finite differences to replace the derivative operators. For derivatives of second order like those appearing on the right-hand side of equation (1.3.2.1a), it is natural to choose a central difference of second order for this purpose. If a step size of h is used, then this yields an $O(h^2)$ approximation with respect to the space variable. There is no problem in transferring this idea from the case $\alpha = 1$ to the case $1 < \alpha < 2$. The first order derivative with respect to time that appears in a classical parabolic equation can be discretized in a number of ways.

c. Methods for equations with space Fractional operators

It is mathematically no problem to replace the derivatives with respect to the space variables in a partial differential equation by their Fractional order generalizations. We discuss the fully general case of a partial differential equation that is Fractional with respect to space and time. The case of an equation that has Fractional derivatives only with respect to the space variable but not with respect to the time variable is contained a fortiori as a special case in this general setting.

V. CONCLUSION

The literature about the basics defintitions, standard approaches, analytical and approximate methods to the problem of fractional differential equations has been studied, while discussing about the basic properties including the rules for their compositions and the conditions for the equivalence of various definitions has been discussed in this paper. In future the methods used for different class of fractional differential equations can be studied.

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