Math 115A Discussion 1

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September 2021

I. Background for proofs.

II. Examples of fields.

1. Prove that $\mathbb{Q}[\sqrt{2}]$ is a field.

   (a) FS4. Additive inverse and multiplicative inverse
   Proof 1: Additive Inverse

$$\forall a, b, a + \sqrt{2}b \in \mathbb{Q}[\sqrt{2}],\ a + \sqrt{2}b - a - \sqrt{2}b = 0$$

$$\text{Since } -a \in \mathbb{Q} \text{ and } -b \in \mathbb{Q}, -a - \sqrt{2}b \in \mathbb{Q}[\sqrt{2}].$$

$$\text{Since } -a - \sqrt{2}b \in \mathbb{Q}[\sqrt{2}], a + \sqrt{2}b \in \mathbb{Q}[\sqrt{2}] \text{ has an additive inverse.}$$

   Proof 2: Multiplicative Inverse

$$\forall a, b, a + \sqrt{2}b \in \mathbb{Q}[\sqrt{2}],\ a + \sqrt{2}b * \frac{1}{a - \sqrt{2}b} = 1$$

$$\frac{1}{a - \sqrt{2}b} * \frac{a - \sqrt{2}b}{a - \sqrt{2}b} = \frac{a - \sqrt{2}b}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} + \frac{(-1)\sqrt{2}b}{a^2 + 2b^2}$$

$$\frac{a}{a^2 + 2b^2} \in \mathbb{Q} \text{ and } \frac{(-1)b}{a^2 - 2b^2} \in \mathbb{Q}, \text{ therefore}$$

$$\frac{a}{a^2 - 2b^2} + \frac{(-1)\sqrt{2}b}{a^2 + 2b^2} \in \mathbb{Q}[\sqrt{2}]$$

$$\text{Since } \frac{1}{a - \sqrt{2}b} \in \mathbb{Q}[\sqrt{2}],\ a + \sqrt{2}b \text{ has a multiplicative inverse } \forall a, b$$

III. Computational problems.

1. Transposed matrices

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix},\ A^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 74 & 10,802 & 13 \\ 0 & 1 & 81 \end{pmatrix},\ B^T = \begin{pmatrix} 74 & 0 \\ 10,802 & 1 \\ 13 & 81 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix},\ C^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

2. Matrix sum $A + B$ and matrix product AB

(a) Polynomial entries

$$A+B=\begin{pmatrix} x+3x^3+1 & 2x^2+2 \\ 6-5x & 9+x \end{pmatrix}$$

$$AB=\begin{pmatrix} 3x^3+10x^2+x & 2x^3+14x^2+2 \\ -15x^4+3x^3-5x^2+x+10 & -8x+16 \end{pmatrix}$$

(b) Entries in $\mathbb{Z}/2\mathbb{Z}$

$$A+B=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$AB=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$