Math 115A HW 5

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1. Prove Theorem 2.10.

   (a) $T \circ (U_1 + U_2) = T \circ U_1 + T \circ U_2$

   $$\text{Let } v \in V$$
   $$T \circ (U_1 + U_2)(v) = T \circ (U_1(v) + U_2(v))$$
   $$T(U_1(v) + U_2(v)) = T(U_1(v)) + T(U_2(v)) \text{ (linearity)}$$
   $$\therefore T \circ U_1 + T \circ U_2 = T \circ (U_1 + U_2)$$

   (b) $T \circ (U_1 \circ U_2) = (T \circ U_1) \circ U_2$

   $$\text{Let } v \in V$$
   $$T \circ (U_1 \circ U_2)(v) = T((U_1 \circ U_2)(v)) = T(U_1(U_2(v)))$$
   $$((T \circ U_1) \circ U_2)(v) = (T \circ U_1)(U_2(v)) = T(U_1(U_2(v)))$$
   $$\text{Since both equal } T(U_1(U_2(v))),\ T \circ (U_1 \circ U_2) = (T \circ U_1) \circ U_2$$

   (c) Given $I_V(v) = v$, $I_V \circ T = T \circ I_V = T$

   $$\text{Let } v \in V$$
   $$(I_v \circ T)(v) = I_v(T(v)) = T(v) \text{ (def. of } I_v)$$
   $$(T \circ I_v)(v) = T(I_v(v)) = T(v) \text{ (def. of } I_v)$$
   $$\text{Since both equal } T(v),\ I_V \circ T = T \circ I_V = T$$

   (d) $(aU_1) \circ U_2 = a(U_1 \circ U_2)$

   $$\text{Let } v \in V$$
   $$((aU_1) \circ U_2)(v) = aU_1(U_2(v))$$
   $$(a(U_1 \circ U_2))(v) = a((U_1 \circ U_2)(v)) \text{ (linearity)}$$
   $$a((U_1 \circ U_2)(v)) = a(U_1(U_2(v))$$
   $$\text{Since both equal } a(U_1(U_2(v))),\ (aU_1) \circ U_2 = a(U_1 \circ U_2)$$

2. $(\lambda A) \cdot B = A \cdot (\lambda B)$
   Proof.

   $$(AB)_{ij} = \sum_{l=1}^{m} A_{il} B_{lj},\ 1 \le i \le n,\ 1 \le j \le k$$
   $$\text{(Def. of matrix multiplication)}$$
   $$((\lambda A)B)_{ij} = \sum_{l=1}^{m} (\lambda A_{il}) B_{lj}$$
   $$= \sum_{l=1}^{m} \lambda (A_{il}) B_{lj} \text{ (Def. scalar multiplication)}$$
   $$= \lambda \sum_{l=1}^{m} (A_{il}) B_{lj} \text{ (Distributive mult. of field elements)}$$

$$(A(\lambda B))_{ij} = \sum_{l=1}^{m} A_{il}(\lambda B_{lj})$$
$$= \sum_{l=1}^{m} A_{il}\lambda(B_{lj}) \text{ (Def. scalar multiplication)}$$
$$= \lambda\sum_{l=1}^{m} A_{il}(B_{lj}) \text{ (Distributive mult. of field elements)}$$

Since both $((\lambda A)\cdot B)_{ij}$ and $(A\cdot(\lambda B))_{ij}$ equal $\lambda\sum_{l=1}^{m} A_{il}B_{lj}$, $(\lambda A)\cdot B = A\cdot(\lambda B)$. □

3. Textbook 2.2 #9
   T is linear.
   Proof.

$$\text{Let } y, z \in V : y = y_1 + y_2 i,\ \ z = z_1 + z_2 i$$
$$T(y+z) = T(y_1 + y_2 i + z_1 + z_2 i) = T(y_1 + z_1 + i(y_2+z_2)) = y_1 + z_1 - i(y_2+z_2)$$
$$T(y) + T(z) = T(y_1 + y_2 i) + T(z_1 + z_2 i) = y_1 - y_2 i + z_1 - z_2 i = y_1 + z_1 - i(y_2 + z_2)$$
$$\therefore T(y+z) = T(y) + T(z)$$

$$T(\lambda z) = T(\lambda(z_1 + z_2 i)) = T(\lambda z_1 + \lambda z_2 i) = \lambda z_1 - \lambda z_2 i$$
$$\lambda T(z) = \lambda T(z_1 + z_2 i) = \lambda(z_1 - z_2 i) = \lambda z_1 - \lambda z_2 i \text{ (dist. multiplication over a field)}$$
$$\therefore T(\lambda z) = \lambda T(z)$$

Since $T(y+z) = T(y) + T(z)$ and $T(\lambda z) = \lambda T(z)$, T is linear. □

Find $[T]_\beta$.

$$\beta = \{1, i\}$$
$$T(1) = (1)1 + (-0)i$$
$$T(i) = (0)1 + (-1)i$$
$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. The i-th column of A is $A\cdot e_i$
   Proof.

$$(A\cdot e_i)_{j1} = \sum_{l=1}^{n} A_{jl}e_{lj} \text{ (Def. matrix multiplication)}$$
$$(A\cdot e_i)_{11} = \sum_{l=1}^{n} A_{1l}e_{l1}$$
$$= 0(A_{11}) + ... + 1(A_{1i}) + ... + 0(A_{1n}) = A_{1i}$$
$$(A\cdot e_i)_{21} = \sum_{l=1}^{n} A_{2l}e_{l1}$$
$$= 0(A_{21}) + ... + 1(A_{2i}) + ... + 0(A_{2n}) = A_{2i}$$
$$\vdots$$
$$(A\cdot e_i)_{k1} = \sum_{l=1}^{n} A_{kl}e_{l1}$$
$$= 0(A_{k1}) + ... + 1(A_{ki}) + ... + 0(A_{kn}) = A_{ki}$$
$$A\cdot e_i = (A_{1i}, A_{2i}, ..., A_{mi})^T \text{ (this is the } i\text{-th column of } A)$$
$$\therefore A\cdot e_i = \text{ the } i\text{-th column of } A. \quad \square$$

5. $D : \text{Poly}(\mathbb{R}) \to \text{Poly}(\mathbb{R})$ by $D(f(x)) = f'(x)$

(a) D is linear.
Proof.

$$\text{Let } f, g \in \text{Poly}(\mathbb{R})$$
$$D(f(x)) = f'(x)$$
$$D(g(x)) = g'(x)$$
$$D(f(x)) + D(g(x)) = f'(x) + g'(x)$$

$$D(f(x) + g(x)) = \frac{d}{dx}(f(x) + g(x))$$
$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\therefore D(f(x)) + D(g(x)) = D(f(x) + g(x)) \forall f, g \in \text{Poly}(\mathbb{R})$$

$$\text{Let } f \in \text{Poly}(\mathbb{R}),\ \lambda \in \mathbb{R}$$
$$\lambda D(f(x)) = \lambda \cdot f'(x)$$

$$D(\lambda f(x)) = \frac{d}{dx}(\lambda f(x))$$
$$\frac{d}{dx}(\lambda f(x)) = \lambda \cdot f'(x)$$
$$\therefore \lambda D(f(x)) = D(\lambda f(x)) \forall f, g \in \text{Poly}(\mathbb{R})$$

Since $D(f(x)) + D(g(x)) = D(f(x) + g(x)) \forall f, g \in \text{Poly}(\mathbb{R})$ and $\lambda D(f(x)) = D(\lambda f(x)) \forall f, g \in \text{Poly}(\mathbb{R})$, D is a linear transformation. □

(b) For $n = 3$, find $[D]_{\beta_2}^{\beta_3}$ w.r.t. bases

$$\beta_3 = \{1, x, x^2, x^3\},\ \beta_2 = \{1, x, x^2\}$$

$$D(1) = 0 = (0)1 + (0)x + (0)x^2$$
$$D(x) = 1 = (1)1 + (0)x + (0)x^2$$
$$D(x^2) = 2x = (0)1 + (2)x + (0)x^2$$
$$D(x^3) = 3x^2 = (0)1 + (0)x + (3)x^2$$

$$[D]_{\beta_2}^{\beta_3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(c) Find $[D]_{\alpha}^{\gamma}$ where

$$\alpha = \{x + 2, x - 2, x^2 + x, x^3 + x^2\},\ \gamma = \{1, 2x, x^2 + 1\}$$

$$D(x + 2) = 2 = (2)1 + (0)2x + (0)(x^2 + 1)$$
$$D(x - 2) = -2 = (-2)1 + (0)2x + (0)(x^2 + 1)$$
$$D(x^2 + x) = 2x + 1 = (1)1 + (1)2x + (0)(x^2 + 1)$$
$$D(x^3 + x^2) = 3x^2 + 2x = (-3)1 + (1)2x + (3)(x^2 + 1)$$

$$[D]_{\alpha}^{\gamma} = \begin{pmatrix} 2 & -2 & 1 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

6. $L_A(\vec{x}) = A \cdot x$

(a) Compute $[L_A]_{\beta_3}^{\beta_2}$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 8 & 9 \end{pmatrix}$$

(b) Compute $[L_A]_{\gamma_3}^{\gamma_2}$

$$L_A \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix} = \frac{17}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + -\frac{3}{2} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$
$$L_A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 18 \end{pmatrix} = 9 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + -2 \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$
$$L_A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \frac{9}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + -\frac{3}{2} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$[L_A]_{\gamma_3}^{\gamma_2} = \begin{pmatrix} \frac{17}{2} & 9 & \frac{9}{2} \\ -\frac{3}{2} & -2 & -\frac{3}{2} \end{pmatrix}$$

7. Textbook 2.3 #3

$$\beta = \{1, x, x^2\}, \ \gamma = \{e_1, e_2, e_3\}$$

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$

$$U(1 + 0x + 0x^2) = (1 + 0, 0, 1 - 0) = (1, 0, 1) = (1)e_1 + (0)e_2 + (1)e_3$$
$$U(0 + 1x + 0x^2) = (0 + 1, 0, 0 - 1) = (1, 0, -1) = (1)e_1 + (0)e_2 + (-1)e_3$$
$$U(0 + 0x + 1x^2) = (0 + 0, 1, 0 - 0) = (0, 1, 0) = (0)e_1 + (1)e_2 + (0)e_3$$
$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$T(1 + 0x + 0x^2) = 0(3 + x) + 2(1) = 2 = (2)1 + (0)x + (0)x^2$$
$$T(0 + 1x + 0x^2) = 1(3 + x) + 2(x) = 3 + 3x = (3)1 + (3)x + (0)x^2$$
$$T(0 + 0x + 1x^2) = 2x(3 + x) + 2(x^2) = 6x + 4x^2 = (0)1 + (6)x + (4)x^2$$
$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

$$U(T(1 + 0x + 0x^2)) = U(2) = (2, 0, 2) = (2)e_1 + (0)e_2 + (2)e_3$$
$$U(T(0 + 1x + 0x^2)) = U(3 + 3x) = (6, 0, 0) = (6)e_1 + (0)e_2 + (0)e_3$$
$$U(T(0 + 0x + 1x^2)) = U(6x + 4x^2) = (6, 4, -6) = (6)e_1 + (4)e_2 + (-6)e_3$$
$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & 6 \end{pmatrix}$$

Verification with Theorem 2.11:

$$[U]_\beta^\gamma \cdot [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & 6 \end{pmatrix} \tag{1}$$

(b) Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$.

$$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$U(h(x)) = U(3 - 2x + x^2) = (1, 1, 5)$$
$$[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

Verification with Theorem 2.14:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

8. Textbook 2.3 #16

(a) $\mathrm{Im}(T) \cap \ker(T) = \{0\}$
Proof.

$$\mathrm{rank}(T) = \mathrm{rank}(T^2)$$
$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V)$$
$$\mathrm{rank}(T^2) + \mathrm{nullity}(T) = \dim(V)$$
$$\mathrm{rank}(T^2) + \mathrm{nullity}(T^2) = \dim(V) \text{ (since } T : V \to V)$$
$$\therefore \ \mathrm{nullity}(T) = \mathrm{nullity}(T^2) \ \text{ and } \ \ker(T) = \ker(T^2)$$
$$\text{Let } v \in \mathrm{Im}(T) \cap \ker(T).\ \exists w \in V : T(w) = v$$
$$\text{Since } \ \in \ker(T),\ T(v) = 0$$
$$T(T(w)) = 0 \ \therefore \ T^2(w) = 0$$
$$w \in \ker(T^2) \text{ (def. of nullity)}$$
$$w \in \ker(T) \text{ (since } \ker(T) = \ker(T^2))$$
$$T(w) = 0 \therefore \ v \textit{ must} \text{ equal } 0 \text{ since } T(w) = v.$$
$$\therefore \mathrm{Im}(T) \cap \ker(T) = \{0\}. \quad \square$$

Since $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V)$ and $\mathrm{Im}(T) \cap \ker(T) = \{0\}$, $V = \mathrm{Im}(T) \oplus \ker(T)$