Math 115A HW 7

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November 14, 2021

Question 1

(a) If A is similar to B, $\det(A) = \det(B)$.

Proof.

$$B = Q^{-1}AQ \text{ (defn. similarity)}$$
$$\therefore \det(B) = \det(Q^{-1}AQ)$$
$$= \det(Q^{-1})\det(A)\det(Q)$$
$$= \det(Q^{-1})\det(Q)\det(A) \text{ (associativity of real numbers)}$$
$$\det(Q^{-1} \cdot \det(Q) = 1 \text{ (property of inverses)}$$
$$= 1 \cdot \det(A) = \det(A)$$
$$\therefore \det(B) = \det(A) \quad \square$$

(b) If A is similar to B, $\det(A - tI_n) = \det(B - tI_n)$

Proof.

$$B = Q^{-1}AQ \text{ (defn. similarity)}$$
$$B - tI_n = Q^{-1}AQ - tI_n$$
$$\det(B - tI_n) = \det(Q^{-1}AQ - tI_n)$$
$$\text{Claim: } tI_n = Q^{-1}tI_nQ$$
$$I_nQ = Q$$
$$\therefore Q^{-1}tI_nQ = Q^{-1}tQ$$
$$= tQ^{-1}Q \text{ (associativity of scalars)} Q^{-1}Q = I_n$$
$$\therefore Q^{-1}tI_nQ = tI_n$$
$$\therefore \det(Q^{-1}AQ - tI_n) = \det(Q^{-1}AQ - Q^{-1}tI_nQ)$$
$$\det(B - tI_n) = \det(Q^{-1}AQ - Q^{-1}tI_nQ) = \det(Q^{-1}(A - tI_n)Q)$$
$$= \det(Q^{-1})\det(A - tI_n)\det(Q) = \det(Q^{-1})\det(Q)\det(A - tI_n) \text{ (associativity of real numbers)}$$
$$\det(Q^{-1})\det(Q) = 1 \text{ (property of inverses)}$$
$$\therefore \det(Q^{-1})\det(Q)\det(A - tI_n) = \det(A - tI_n)$$
$$\therefore \det(B - tI_n) = \det(A - tI_n) \quad \square$$

(c) The characteristic polynomial does not depend on the choice of matrix representation.

Proof.

Let β and γ be bases for V.

$[T]_\beta$ is similar to $[T]_\gamma$ (transformation, change of basis)

The characteristic polynomial of T is $\det(A - tI_n)$ in which A is a matrix representation of T.

Let $A = [T]_\beta$. Thus, the characteristic polynomial is $\det([T]_\beta - tI_n)$

Since $[T]_\beta$ and $[T]_\gamma$ are similar, $\det([T]_\beta - tI_n) = \det([T]_\gamma - tI_n)$.

$\therefore$ the characteristic polynomial does not depend on matrix representation. $\square$

5.1 #1

(a) F - Several linear operators (such as $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$) over n-dimensional vectors spaces have less eigenvalues than n.

(b) T - If x is an eigenvector of A, $Ax = \lambda x$. x can be scaled by any constant c such that $A \cdot cx = \lambda cx$. Thus, A has infinite eigenvectors.

(c) T - Certain rotation matrices do not scale any vectors.

(d) F - Eigenvalues can be zero (there are a few in this HW).

(e) F - Part (b) says that there can be infinite scalar multiples of eigenvectors, which are linearly dependent.

(f) F - A linear operator with eigenvalues ± 1 does not have an eigenvalue $-1 + 1 - 0$.

(g) F - Let $T(x_1, ...) = c(x_1, ...)$ be the transformation over F^∞. c is an eigenvalue of T despite T being a linear operator over an infinite-dimensional vector space.

(h) T - This is true for linear transformations over F^n, therefore it must be true for matrices.

(i) T - Characteristic polynomials of similar matrices are the same, therefore the eigenvalues must be the same.

(j) F - The eigenvectors are not necessarily the same since they are in different bases.

(k) F - This is true for eigenvectors corresponding to the same eigenvalue, but not for all eigenvectors of a linear operator.

5.1 #2

(a)

$$\beta = \{e_1, e_2\}$$
$$T(e_1) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2e_1 + 5e_2,\ T(e_2) = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = -1e_1 + 3e_2$$
$$[T]_\beta = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$$
$$\det \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} = 11$$
$$f_A(t) = \det \begin{pmatrix} 2-t & -1 \\ 5 & 3-t \end{pmatrix}$$
$$f_A(t) = (2-t)(3-t) + 5 = t^2 - 5t + 11$$

(b)

$$\beta = \{e_1, e_2, e_3\}$$
$$T(e_1) = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = 1e_1 - 2e_2 + 4e_3,\ T(e_2) = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = -3e_1 + 1e_2 + 0e_3,\ T(e_3) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 2e_1 + 1e_2 - 1e_3$$
$$[T]_\beta = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 1 & 1 \\ 4 & 0 & -1 \end{pmatrix}$$
$$\det \begin{pmatrix} 1 & -3 & 2 \\ -2 & 1 & 1 \\ 4 & 0 & -1 \end{pmatrix} = -15$$
$$f_A(t) = \det \begin{pmatrix} 1-t & -3 & 2 \\ -2 & 1-t & 1 \\ 4 & 0 & -1-t \end{pmatrix}$$
$$f_A(t) = -t^3 + t^2 + 15t - 15$$

(c)

$$\beta = \{1, x, x^2, x^3\}$$
$$T(1) = 1(1) + -1x + 1x^2 - 0x^3,\ T(x) = 0(1) + 1x + 1x^2 - 0x^3$$
$$T(x^2) = -1(1) + 0x + 0x^2 - 1x^3,\ T(x^3) = 0(1) + 1x - 1x^2 - 0x^3$$
$$[T]_\beta = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
$$\det \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -2$$
$$f_A(t) = \begin{pmatrix} 1-t & 0 & -1 & 0 \\ -1 & 1-t & 0 & 1 \\ 1 & 1 & 0-t & -1 \\ 0 & 0 & -1 & 0-t \end{pmatrix}$$
$$f_A(t) = t^4 - 2t^3 + t^2 + t - 2$$

(d)

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1E_{11} + 0 + 0 + 0$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = 0E_{11} - 1E_{12} + 2E_{21} + 0E_{22}$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = 0E_{11} 2E_{12} - 1E_{21} + 0E_{22}$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0 + 0 + 1E_{11}$$

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -3$$

$$f_A(t) = \det \begin{pmatrix} 1-t & 0 & 0 & 0 \\ 0 & -1-t & 2 & 0 \\ 0 & 2 & -1-t & 0 \\ 0 & 0 & 0 & 1-t \end{pmatrix}$$

$$f_A(t) = t^4 - 6t^2 + 8t - 3$$

5.1 #3

(β_i refers to the i-th vector in β)

(a)

$$T(\beta_1) = \begin{pmatrix} -2 \\ -3 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$T(\beta_2) = \begin{pmatrix} -2 \\ -4 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$[T]_\beta = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

Since $[T]_\beta$ is not diagonal, β is not an eigenbasis.

(b)

$$T(\beta_1) = -6 - 8x = -2(3 + 4x) + 0(2 + 3x)$$
$$T(\beta_2) = -6 - 9x = 0(3 + 4x) - 3(2 + 3x)$$
$$[T]_\beta = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

Since $[T]_\beta$ is diagonal, β is an eigenbasis.

(f)

$$T(\beta_1) = \begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix} = -3\beta_1 + 0\beta_2 + 0\beta_3 + 0\beta_3$$
$$T(\beta_2) = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = 0\beta_1 + 1\beta_2 + 0\beta_3 + 0\beta_4$$
$$T(\beta_3) = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = 0\beta_1 + 0\beta_2 + 1\beta_3 + 0\beta_4$$
$$T(\beta_4) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = 0\beta_1 + 0\beta_2 + 0\beta_3 + 1\beta_4$$
$$[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $[T]_\beta$ is diagonal, β is an eigenbasis.

5.1 #4

(c)
i.

$$\det\begin{pmatrix} i-t & 1 \\ 2 & -i-t \end{pmatrix} = 0$$
$$(i-t)(-i-t) - 2 = 0$$
$$t^2 - 1 = 0$$
$$\lambda = \pm 1$$

ii.

$$Av = \lambda v$$
$$Av - \lambda v = 0$$
$$(A - \lambda I)v = 0$$
$$\lambda = 1$$
$$\text{Let } v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\left(\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$x_2 = (1-i)x_1$$
$$\text{eigenvector for } \lambda = 1 \ v = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$
$$\lambda = -1$$
$$\left(\begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$
$$x_2 = (-1-i)x_1$$
$$\text{eigenvector for } \lambda = -1 \ v = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$

iii.

$$\left\{\begin{pmatrix} 1 \\ 1-i \end{pmatrix}, \begin{pmatrix} 1 \\ -1-i \end{pmatrix}\right\}$$

iv.

$$Q = \begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(d)

i.

$$\det\begin{pmatrix}2-t & 0 & -1\\ 4 & 1-t & -1\\ 2 & 0 & -1-t\end{pmatrix} = -t(-1+t)^2$$

$$\lambda = 0, 1$$

ii.

$$\lambda = 0$$

$$\begin{pmatrix}2 & 0 & -1\\ 4 & 1 & -1\\ 2 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \vec{0}$$

$$\text{(solve with Gaussian elimination)} x_3 = 2x_1,\ \ x_2 = 4x_1$$

$$\text{eigenvector for } \lambda = 0\ v = \begin{pmatrix}1\\ 4\\ 2\end{pmatrix} \lambda = 1$$

$$\begin{pmatrix}1 & 0 & -1\\ 4 & 0 & -1\\ 2 & 0 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \vec{0}$$

$$\text{(solve with Gaussian elimination)} x_1 = x_3$$

$$\text{eigenvectors for } \lambda = 1\ v = \begin{pmatrix}1\\ 0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$$

iii.

$$\left\{\begin{pmatrix}1\\ 4\\ 2\end{pmatrix}, \begin{pmatrix}1\\ 0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right\}$$

iv.

$$Q = \begin{pmatrix}1 & 1 & 0\\ 4 & 0 & 1\\ 2 & 1 & 0\end{pmatrix},\ D = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

5.1 #5

(d)

$$S = \{x, 1\}$$

$$[T]_S = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$$

$$\det \begin{pmatrix} 1 - t + -6 \\ 2 + -6 - t \end{pmatrix} = t^2 + 5t + 6$$

$$\lambda = -3, -2$$

$$\begin{pmatrix} 2+3 & -6 \\ 1 & -6+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

$$2x_1 = 3x_2$$

$$\text{eigenvector for } \lambda = -2 \ v = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2+2 & -6 \\ 1 & -6+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

$$x_1 = 2x_2$$

$$\text{eigenvector for } \lambda = -3 \ v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \right\}$$

(g)

$$[T]_S = \begin{pmatrix} -1 & -2 & -4 & -8 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\det \begin{pmatrix} -1-t & -2 & -4 & -8 \\ 0 & 1-t & 2 & 6 \\ 0 & 0 & 2-t & 0 \\ 0 & 0 & 0 & 3-t \end{pmatrix} = (t+1)(t-1)(t-2)(t-3)$$

$$\lambda = -1, 1, 2, 3$$

$$\begin{pmatrix} -1+1 & -2 & -4 & -8 \\ 0 & 1+1 & 2 & 6 \\ 0 & 0 & 2+1 & 0 \\ 0 & 0 & 0 & 3+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$x_2 = x_3 = x_4 = 0$$

$$\text{eigenvector for } \lambda = -1 \ v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-1 & -2 & -4 & -8 \\ 0 & 1-1 & 2 & 6 \\ 0 & 0 & 2-1 & 0 \\ 0 & 0 & 0 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$x_1 = -x_2, x_3 = x_4 = 0$$

$$\text{eigenvector for } \lambda = 1 \ v = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+2 & -2 & -4 & -8 \\ 0 & 1+2 & 2 & 6 \\ 0 & 0 & 2+2 & 0 \\ 0 & 0 & 0 & 3+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$3x_1 = -8x_3, x_2 = 2x_3 = x_4 = 0$$

$$\text{eigenvector for } \lambda = 2 \ v = \begin{pmatrix} -8 \\ 6 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+3 & -2 & -4 & -8 \\ 0 & 1+3 & 2 & 6 \\ 0 & 0 & 2+3 & 0 \\ 0 & 0 & 0 & 3+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$2x_1 = -7x_4, x_2 = 3x_4, x_3 = 0$$

$$\text{eigenvector for } \lambda = 3 \ v = \begin{pmatrix} -7 \\ 6 \\ 0 \\ 2 \end{pmatrix}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ 6 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 6 \\ 0 \\ 2 \end{pmatrix} \right\}$$

(j)

$$[T]_S = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

$$\det\begin{pmatrix} 3-t & 0 & 0 & 1 \\ 0 & -t & 1 & 0 \\ 0 & 1 & -t & 0 \\ 1 & 0 & 0 & 3-t \end{pmatrix} = (t+1)(t-1)(t-2)(t-4)$$

$$\lambda = -1, 1, 2, 4$$

$$\begin{pmatrix} 3+1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$x_1 = x_4 = 0, -x_2 = x_3$$

$$\text{eigenvector for } \lambda = -1 \ v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$x_1 = x_4 = 0, x_2 = x_3$$

$$\text{eigenvector for } \lambda = 1 \ v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-2 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$-x_1 = x_4, x_2 = x_3 = 0$$

$$\text{eigenvector for } \lambda = 2 \ v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 3-4 & 0 & 0 & 1 \\ 0 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 0 & 3-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

$$x_1 = x_4, x_2 = x_3 = 0$$

$$\text{eigenvector for } \lambda = -1 \ v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\beta = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

5.1 #7

Proof.

$$T(v) = [T]_\beta v \text{ (def. matrix rep. of linear transformations)}$$
$$\text{Let } \lambda \text{ be an eigenvalue of } [T]_\beta.$$
$$[T]_\beta v = \lambda v \text{ (def. matrix eigenvalue)}$$
$$T(v) = [T]_\beta v = \lambda v$$
$$\therefore T(v) = \lambda v$$
$$\therefore \lambda \text{ is an eigenvalue of } T \text{ (def. transformation eigenvalue)}$$

$$\text{Let } \lambda \text{ be an eigenvalue of } T$$
$$T(v) = \lambda v \text{ (def. transformation eigenvalue)}$$
$$T(v) = \lambda v = [T]_\beta v$$
$$\therefore \lambda v = [T]_\beta v$$
$$\therefore \lambda \text{ is an eigenvalue of } [T]_\beta \text{ (def. matrix eigenvalue)} \quad \square$$

5.1 #9

(a) *Proof.*

$$\text{Let } 0 \text{ be an eigenvalue of } T.$$
$$\exists v \text{ s.t. } T(v) = 0 \cdot v \text{ (def. eigenvalue)}$$
$$0 \cdot v = \vec{0} \ \forall v \in V$$
$$\ker(T) = v \neq \{\vec{0}\}$$
$$\therefore T \text{ is not one-to-one and is not invertible} \text{Therefore, } T \text{ is not invertible if it has an eigenvalue } 0.$$

$$\text{If } 0 \text{ is not an eigenvalue of } T,\ T(v) \neq 0v,\ \ v \neq \vec{0}$$
$$0v = \vec{0} \ \forall v \in V$$
$$\therefore T(v) \neq \vec{0} \text{ unless } v = \vec{0}$$
$$\therefore \ \ker(T) = \{\vec{0}\} \implies \text{T is one-to-one.}$$
$$\therefore T \text{ can be invertible.} \quad \square$$

(b) *Proof.*

$$\text{Let } \lambda \text{ be an eigenvalue of } T.$$
$$T(v) = \lambda v \text{ (def. eigenvalue)}$$
$$v = T^{-1}(\lambda v)$$
$$\text{Let } w = \lambda v.\ v = \lambda^{-1} w$$
$$v = T^{-1}(w)$$
$$\lambda^{-1} w = T^{-1}(w)$$
$$\text{By the def. of an eigenvalue, } \lambda^{-1} \text{ is an eigenvalue of } T^{-1}.$$

$$\text{Let } \lambda^{-1} \text{ be an eigenvalue of } T^{-1}$$
$$T^{-1}(v) = \lambda^{-1} v$$
$$v = T(\lambda^{-1} v)$$
$$\text{Let } v = \lambda w \ (\lambda w \in V)$$
$$\lambda w = T(\lambda^{-1} \cdot \lambda w)$$
$$\lambda w = T(w)$$
$$\text{By the def. of an eigenvalue, } \lambda \text{ is an eigenvalue of } T. \quad \square$$

(c) A is invertible if and only if 0 is not an eigenvalue of A
Proof.

$$\text{Let } 0 \text{ be an eigenvalue of } A.$$
$$Av = 0v$$
$$\det(A - 0I) = 0$$
$$\det(A) = 0$$
$$\therefore A \text{ is not invertible.} \quad \square$$

λ is an eigenvalue of $A \iff \lambda^-1$ is an eigenvalue of A^-1.
Proof.

$$Ax = \lambda x$$
$$A^{-1}Ax = A^{-1}\lambda x$$
$$x = \lambda A^{-1}x$$
$$\lambda^{-1}x = A - 1x$$
$$\therefore \lambda^-1 \text{ is an eigenvalue of } A^{-1}. \quad \square$$

5.1 #12

(a) *Proof.*

$$A = Q^{-1}\lambda IQ \text{ (def. similarity)}$$
$$IQ = Q$$
$$A = Q^{-1}\lambda Q$$
$$A = \lambda Q^{-1}Q \text{ (associativity of scalars)}$$
$$Q^{-1}Q = I \ \forall Q \in \text{ set of invertible matrices}$$
$$A = \lambda I \quad \square$$

(b)

If a diagonalizable matrix A has only 1 eigenvalue,

$$\det(A - tI) = \prod_{i}^{n}(a_i - t)$$

This can be true only if A is a scalar matrix i.e.

$$A = \begin{pmatrix} a & ... & 0 \\ 0 & a & \vdots \\ 0 & ... & a \end{pmatrix}$$

(c) *Proof.*

$$\det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} = 1 - t$$
$$\lambda = 1$$

Since A has only 1 eigenvalue, and is not a scalar matrix, A is not diagonalizable. $\square$

5.1 #18

(a)

$$\text{Let } \lambda \text{ be an eigenvalue of } T.$$
$$T(A) = \lambda A$$
$$\lambda A = A^T$$
$$T(\lambda A) = \lambda \cdot \lambda A = (A^T)^T$$
$$\lambda^2 A = A$$
$$\lambda = \pm 1$$

(b)

For $\lambda = 1$, $A = A^T$. The eigenvectors for $\lambda = 1$ are symmetric matrices.
For $\lambda = -1$, $-A = A^T$. The eigenvectors for $\lambda = -1$ are anti-symmetric matrices.

(c)

$$[T]_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1+1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$
$$x_2 = -x_3, x_1 = x_4 = 0$$
$$\text{eigenvectors for } \lambda = -1 \ v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1-1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$
$$x_2 = x_3$$
$$\text{eigenvectors for } \lambda = 1 \ v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

(d)

$$\beta = \{E_{11}, E_{nn}, \text{basis of skew-symmetric matrices}, \text{basis of anti-symmetric matrices}\}$$