Math 115A HW 3

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1. Textbook 1.6 #3

   (a) Since there exist values $a_1, a_2, a_3 \in \mathbb{R} : a_1(-1-x+2x^2) + a_2(2+x-2x^2) + a_3(1-2x+4x^2) = 0$, the set is not linearly independent. (Let $a_1 = 5, a_2 = 3, a_3 = -1$ and the linear combination will equal 0.) Since the set is not linearly independent, it is not a basis for $\mathcal{P}_2(\mathbb{R})$.

   (b) There do not exist any elements $a_1, a_2, a_3 \in \mathbb{R} : a_1(1+2x+x^2) + a_2(3+x^2) + a_3(x+x^2) = 0$ other than $a_1 = a_2 = a_3 = 0$. Since the set is linearly independent and has 3 elements, and $\dim(\mathcal{P}_2(\mathbb{R})) = 3$, the set is a basis for $\mathcal{P}_2(\mathbb{R})$.

2. Textbook 1.6 #8

$$u_1 \text{ can be in the basis.}$$
$$u_2 = -3u_1, \therefore u_2 \notin \text{ the basis.}$$
$$u_3 \text{ is not a multiple of } u_1\text{, thus } \{u_1, u_3\} \text{ is linearly independent and } u_3 \text{ can be in the basis.}$$
$$\text{To find if } \{u_1, u_3, u_4\} \text{ is linearly independent, we can use write this as a system of linear equations.}$$
$$2x_1 + 3x_2 = 2$$
$$-3x_1 + -2x_2 = -8$$
$$4x_1 + 7x_2 = 2 \text{ add 4th eqn.}$$
$$-5x_1 - 9x_2 = -2$$
$$2x_1 + x_2 = 6$$

$$2x_1 + 3x_2 = 2$$
$$--3x_1 + 2x_2 = -8$$
$$-x_1 - 2x_2 = 0$$
$$-5x_1 - 9x_2 = -2$$
$$2x_1 + x_2 = 6 \text{ subtract 1st eqn.}$$

$$2x_1 + 3x_2 = 2$$
$$-3x_1 - 2x_2 = -8$$
$$-x_1 - 2x_2 = 0$$
$$-5x_1 - 9x_2 = -2$$
$$-2x_2 = 4$$

$$2x_1 + 3x_2 = 2 \text{ subtract 3 times 5th eqn.}$$
$$-3x_1 - 2x_2 = -8$$
$$-x_1 - 2x_2 = 0 \text{ add 2 times 5th eqn.}$$
$$-5x_1 - 9x_2 = -2$$
$$x_2 = -2$$

$$2x_1 = 8$$
$$-3x_1 - 2x_2 = -8$$
$$-x_1 = -4$$
$$-5x_1 - 9x_2 = -2$$
$$x_2 = -2$$

$$x_1 = 4$$
$$-3x_1 - 2x_2 = -8 \text{ add 3 times 1st eqn.}$$
$$-5x_1 - 9x_2 = -2 \text{ add 5 times 1st eqn.}$$
$$x_2 = -2$$

$$x_1 = 4$$
$$-2x_2 = 4$$
$$-9x_2 = 18$$
$$x_2 = -2$$

$$x_1 = 4$$
$$x_2 = -2$$

By solving the system of equations, we can deduce that $4u_1 - 2u_2 = u_4$, therefore $4_u1 - 2u_2 - u_4 = 0$. Thus, $\{u_1, u_3, u_4\}$ is not linearly independent. Now we will test if u_5 can be written as a linear combination of the other vectors.

$$2x_1 + 3x_2 = -1$$
$$-3x_1 - 2x_2 = 1$$
$$4x_1 + 7x_2 = 2 \text{ minus 2 times 1st eqn.}$$
$$-5x_1 - 9x_2 = 1$$
$$2x_1 + x_2 = -3 \text{ minus 1st eqn.}$$

$$2x_1 + 3x_2 = -1$$
$$-3x_1 - 2x_2 = 1$$
$$x_2 = 4$$
$$-5x_1 - 9x_2 = 1$$
$$-5x_2 = -1$$

x_2 cannot both be 4 and $\frac{1}{5}$. Thus, there are no solutions to the system of linear equations and u_1, u_3, and u_5 are linearly independent. Now we test if u_6 is linearly independent with the other vectors.

$$2x_1 + 3x_2 - x_3 = 0 \text{ minus 5th eqn.}$$
$$-3x_1 - 2x_2 + x_3 = -3$$
$$4x_1 + 7x_2 + 2x_3 = -18 \text{ minus 2 times 5th eqn.}$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$2x_2 + 2x_3 = -12 \text{ divide by 2}$$
$$-3x_1 - 2x_2 + x_3 = -3$$
$$x_2 + 4x_3 = -18$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$x_2 + x_3 = -6$$
$$-3x_1 - 2x_2 + x_3 = -3$$
$$x_2 + 4x_3 = -18 \text{ subtract 1st eqn.}$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$x_2 + x_3 = -6$$
$$-3x_1 - 2x_2 + x_3 = -3$$
$$3x_3 = -12 \text{ divide by 3}$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$x_2 + x_3 = -6 \text{ subtract 3rd eqn.}$$
$$-3x_1 - 2x_2 + x_3 = -3 \text{ subtract 5th eqn.}$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$x_2 = -2$$
$$-5x_1 - 3x_2 + 4x_3 = -15 \text{ subtract 4th eqn.}$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$x_2 = -2$$
$$6x_2 + 3x_3 = -24 \text{ divide by 3}$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12$$

$$x_2 = -2$$
$$2x_2 + x_3 = -8 \text{ cancels out}$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 + x_2 - 3x_3 = 12 \text{ substitute } x_2 \text{ and } x_3$$

$$x_2 = -2$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 - 2 - 3(-4) = 12$$

$$x_2 = -2$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9$$
$$2x_1 = 2 \text{ divide by 2}$$

$$x_2 = -2$$
$$x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 9 \text{ substitute x's}$$
$$x_1 = 1$$

$$x_2 = -2$$
$$x_3 = -4$$
$$-5 - 9(-2) + (-4) = 9 \text{ cancels out}$$
$$x_1 = 1$$

There is a non-trivial solution for $a_1u_1 + a_3u_3 + a_5u_5 = a_6u_6$, therefore $\{u_1, u_3, u_5, u_6\}$ is not linearly independent. Now we will test if u_7 is linearly independent with the vectors.

$$2x_1 + 3x_2 - x_3 = 1 \text{ minus 5th eqn.}$$
$$-3x_1 - 2x_2 + x_3 = 0 \text{ minus 5th eqn.}$$
$$4x_1 + 7x_2 + 2x_3 = -2 \text{ minus 2 times 1st eqn.}$$
$$-5x_1 - 9x_2 + x_3 = 3$$
$$2x_1 + x_2 - 3x_3 = -2$$

$$2x_2 + 2x_3 = 3 \text{ minus 2 times 3rd eqn.}$$
$$-5x_1 - 3x_2 + 4x_3 = 2 \text{ minus 4th eqn.}$$
$$x_2 + 4x_3 = -4$$
$$-5x_1 - 9x_2 + x_3 = 3$$
$$2x_1 + x_2 - 3x_3 = -2$$

$$-6x_3 = 11$$
$$6x_2 + 3x_3 = -1 \text{ plus 0.5 times 1st eqn.}$$
$$x_2 + 4x_3 = -4 \text{ plus 2/3 times 1st eqn.}$$
$$-5x_1 - 9x_2 + x_3 = 3$$
$$2x_1 + x_2 - 3x_3 = -2$$

$$-6x_3 = 11$$
$$6x_2 = 4.5$$
$$x_2 = \frac{10}{3}$$
$$-5x_1 - 9x_2 + x_3 = 3$$
$$2x_1 + x_2 - 3x_3 = -2$$

$$-6x_3 = 11$$
$$x_2 = 0.75$$
$$x_2 = \frac{10}{3}$$
$$-5x_1 - 9x_2 + x_3 = 3$$
$$2x_1 + x_2 - 3x_3 = -2$$

x_2 cannot equal both 0.75 and 10/3, therefore $\{u_1, u_3, u_5, u_7\}$ is linearly independent. Now we will test if u_8 is linearly independent with the vectors.

$$
\begin{gathered}
2x_1 + 3x_2 - x_3 + x_4 = 2 \text{ minus 5th eqn.} \\
-3x_1 - 2x_2 + x_3 \quad = -1 \text{ minus 5th eqn.} \\
4x_1 + 7x_2 + 2x_3 - 2x_4 = 1 \text{ minus 2 times 1st eqn.} \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$
\begin{gathered}
2x_2 + 2x_3 + 3x_4 = -5 \\
-5x_1 - 3x_2 + 4x_3 + 2x_4 = -8 \text{ minus 4th eqn.} \\
x_2 + 4x_3 - 4x_4 = -3 \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$
\begin{gathered}
2x_2 + 2x_3 + 3x_4 = -5 \text{ minus 2 times 3rd eqn.} \\
6x_2 + 3x_3 - x_4 = 1 \text{ minus 3 times 1st eqn.} \\
x_2 + 4x_3 - 4x_4 = -3 \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$
\begin{gathered}
-6x_3 + 11x_4 = 1 \text{ minus 2 times 2nd eqn.} \\
-3x_3 - 10x_4 = 16 \\
x_2 + 4x_3 - 4x_4 = -3 \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$
\begin{gathered}
31x_4 = -31 \\
-3x_3 - 10x_4 = 16 \\
x_2 + 4x_3 - 4x_4 = -3 \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$
\begin{gathered}
x_4 = -1 \\
-3x_3 - 10x_4 = 16 \text{ substitute } x_4 \\
x_2 + 4x_3 - 4x_4 = -3 \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$
\begin{gathered}
x_4 = -1 \\
-3x_3 + 10 = 16 \text{ substitute } x_4 \\
x_2 + 4x_3 - 4x_4 = -3 \\
-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \\
2x_1 + x_2 - 3x_3 - 2x_4 = 7
\end{gathered}
$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 + 4x_3 - 4x_4 = -3 \text{ substitute x's}$$
$$-5x_1 - 9x_2 + x_3 + 3x_4 = -9$$
$$2x_1 + x_2 - 3x_3 - 2x_4 = 7$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 - 8 + 4 = -3$$
$$-5x_1 - 9x_2 + x_3 + 3x_4 = -9$$
$$2x_1 + x_2 - 3x_3 - 2x_4 = 7$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 = 1$$
$$-5x_1 - 9x_2 + x_3 + 3x_4 = -9 \text{ substitute x's}$$
$$2x_1 + x_2 - 3x_3 - 2x_4 = 7$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 = 1$$
$$-5x_1 - 9 - 2 - 3 = -9$$
$$2x_1 + x_2 - 3x_3 - 2x_4 = 7$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 = 1$$
$$-5x_1 = 5$$
$$2x_1 + x_2 - 3x_3 - 2x_4 = 7$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 = 1$$
$$x_1 = -1$$
$$2x_1 + x_2 - 3x_3 - 2x_4 = 7 \text{ substitute x's}$$

$$x_4 = -1$$
$$x_3 = -2$$
$$x_2 = 1$$
$$x_1 = -1$$
$$-2 + 1 - 3(-2) - 2(-1) = 7\text{cancels out}$$

Since there is a solution for $a_1u_1 + a_3u_3 + a_5u_5 + a_7u_7 = a_8u_8$, $\{u_1, u_3, u_5, u_7, u_8\}$ is not linearly independent. Thus, the basis for W is $\{u_1, u_3, u_5, u_7\}$.

3. Textbook 1.6 #9

$$b_1(1,1,1,1) + b_2(0,1,1,1) + b_3(0,0,1,1) + b_4(0,0,0,1) = (a_1,a_2,a_3,a_4)$$
$$b_1(1) + b_2(0) + b_3(0) + b_4(0) = a_1;\ b_1 = a_1$$
$$(a_1,a_1,a_1,a_1) + b_2(0,1,1,1) + b_3(0,0,1,1) + b_4(0,0,0,1) = (a_1,a_2,a_3,a_4)$$
$$a_1 + b_2(1) + b_3(0) + b_4(0) = a_2;\ a_1 + b_2 = a_2$$
$$b_2 = a_2 - a_1$$
$$(a_1,a_1,a_1,a_1) + (0,a_2-a_1,a_2-a_1,a_2-a_1) + b_3(0,0,1,1) + b_4(0,0,0,1) = (a_1,a_2,a_3,a_4)$$
$$a_1 + (a_2-a_1) + b_3 = a_3;\ a_2 + b_3 = a_3$$
$$b_3 = a_3 - a_2$$
$$(a_1,a_1,a_1,a_1) + (0,a_2-a_1,a_2-a_1,a_2-a_1) + (0,0,a_3-a_2,a_3-a_2) + b_4(0,0,0,1) = (a_1,a_2,a_3,a_4)$$
$$a_1 + (a_2-a_1) + (a_3-a_2) + b_4 = a_4\ a_3 + b_4 = a_4$$
$$b_4 = a_4 - a_3$$

$$(a_1,a_1,a_1,a_1) + (0,a_2-a_1,a_2-a_1,a_2-a_1) + (0,0,a_3-a_2,a_3-a_2) + (0,0,0,a_4-a_3)$$
$$(a_1+0+0+0, a_1+a_2-a_1+0+0, a_1+a_2-a_1+a_3-a_2+0, a_1+a_2-a_1+a_3-a_2+a_4-a_3)$$
$$= (a_1,a_2,a_3,a_4)$$

$$\therefore \forall a \in F^4, (a_1,a_2,a_3,a_4) = \text{ the linear combination } a_1u_1 + (a_2-a_1)u_2 + (a_3-a_2)u_3 + (a_4-a_3)u_4$$

4. Textbook 1.6 #16
The basis for $\mathrm{Mat}_{n\times n}(F)$ consists of the span of the sum of $n \times n$ matrices with a 1 in every value. Thus, we must find the amount of matrices in which there is a 1 in every value of the upper triangle of a $n \times n$ matrix. The upper triangle consists of all values in the diagonal and above. For any $n \times n$ matrix, there are n values in the diagonal. In the diagonal above, there are $n-1$ values. In the diagonal above this one, there are $n-2$ values. This goes all the way to the upper-right-most part of the matrix in which there is just 1 value. The sum $n + (n-1) + (n-2) + ... + 1$ is simply the sum of all integers from 1 to n. This can be represented by the well-known sum of n integers, $\frac{n(n+1)}{2}$. Since we now know that there are $\frac{n(n+1)}{2}$ values in the upper triangle and that the basis of upper triangles must be the sum of the spans of all matrices with only a 1 in each respective position of the upper triangle, there are $\frac{n(n+1)}{2}$ vectors within the basis. Since the dimension of a vector space is defined as the number of vectors in the basis of the vector space, the dimension of all upper triangular matrices equals $\frac{n(n+1)}{2}$.

5. Textbook 1.6 #21
A vector space is infinite-dimensional if and only if it contains an infinite linearly independent subset.
Proof.
The basis of an n-dimensional vector space V is defined as a subset of V with n linearly independent vectors in which any vector $v \in V$ is a linear combination of such vectors. Thus, a $n = \infty$-dimensional vector space has a basis with $n = \infty$ linearly independent vectors. Since a basis of a vector space is a subset of the vector space, an $n = \infty$-dimensional vector space contains a subset of infinite linearly independent vectors. □

6. Textbook 1.6 #31
W_1 and W_2 are subspaces of vector space V with respective dimensions m and n where $m \geq n$.

   (a) $\dim(W_1 \cap W_2) \leq n$
   Proof.
   $W_1 \cap W_2$ is the set of vectors within both W_1 and W_2, therefore all of the vectors must be within W_2. Since all of the vectors must be within W_2, there are n vectors within W_2, and $n \leq m$, there must be $\leq n$ vectors within $W_1 \cap W_2$. The dimension of any vector space is the number of vectors in the basis of that vector space (which is a subset of said vector space). Therefore, the dimension

of $W_1 \cap W_2$ is the number of vectors in the basis generating $W_1 \cap W_2$. The basis is a subset of $W_1 \cap W_2$, therefore there must be $\leq n$ vectors in the basis. By the definition of dimension, there are $\leq n$ vectors in the basis of $W_1 \cap W_2$, thus $\dim(W_1 \cap W_2) \leq n$. $\square$

(b) $\dim(W_1 + W_2) \leq m + n$
Proof.

$$\begin{gathered}\text{The basis of } W_1 \text{ has } m \text{ elements by the def. of dimension.}\\ \text{The basis of } W_2 \text{ has } n \text{ elements by the def. of dimension.}\\ \text{Elements in } W_1 + W_2 \text{ could be written as a linear combination of unique elements of}\\ \text{their combined bases. Their combined basis has } m+n \text{ elements if their intersection is only}\\ \text{the zero vector; it has less than } m+n \text{ elements if their intersection is more}\\ \text{than the zero vector.}\\ \therefore \dim(W_1 + W_2) \leq m + n \quad \square\end{gathered}$$

7. Textbook 1.6 #33

(a) $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.
If $V = W_1 \oplus W_2$, $W_1 \cap W_2 = \{\vec{0}\}$.

$$\begin{gathered}\beta_1 \subset W_1 \text{ and } \beta_2 \subset W_2\\ \therefore \text{ the only possible intersection could be } \{\vec{0}\}.\\ \text{However, } \vec{0} \notin \beta_1, \beta_2 \text{ since any subset } S \cup \vec{0} \text{ is not linearly independent and}\\ \text{bases must be linearly independent.}\\ \therefore \beta_1 \cap \beta_2 = \varnothing\end{gathered}$$

Elements in $W_1 + W_2$ can be represented as a linear combination of elements in each respective basis. V is the direct sum of W_1 and W_2, therefore elements of V can be written as a linear combination of elements in $\beta_1 \cup \beta_2$. This is the definition of a basis, thus $\beta_1 \cup \beta_2$ is a basis for V.

(b) If $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V, $V = W_1 \oplus W_2$.
Since $\beta_1 \cup \beta_2$ is a basis for V, any vector within V can be represented as a linear combination of elements within $\beta_1 \cup \beta_2$. $\therefore$ every vector within V is in $\text{span}(\beta_1) \cup \text{span}(\beta_2)$. $\text{span}(\beta_1)$, $\text{span}(\beta_2)$ is equal to W_1, W_2, thus $W_1 + W_2 = V$. Since W_1 and W_2 are disjoint, their intersection is only $\{\vec{0}\}$ and $V = W_1 \oplus W_2$.

8. Textbook 2.1 #1
(a) T (b) F (c) F (d) T
(e) F (f) F (g) T (h) F

9. Textbook 2.1 #18

$$\forall v = (x, y) \in \mathbb{R}^2,\ T(v) = (x + y, -(x + y))$$

10. (a) S and T are linear transformations $\implies S + T : V \to W$ is linear.
For elements $v, w \in V$ and $\lambda \in F$

$$\begin{gathered}(S+T)(v+w) = S(v+w) + T(v+w) = S(v) + S(w) + T(v) + T(w)\\ (S+T)(v) + (S+T)(w) = S(v) + T(v) + S(w) + T(w) = S(v) + S(w) + T(v) + T(w)\\ \text{(by commutative addition over a VS)}\\ \lambda \cdot (S+T)(v) = (\lambda S + \lambda T)(v) = \lambda S(v) + \lambda T(v)\\ (S+T)(\lambda v) = S(\lambda v) + T(\lambda v) = \lambda S(v) + \lambda T(v)\\ \text{Since } (S+T)(v+w) = (S+T)(v) + (S+T)(w) \text{ and } (S+T)(\lambda v) = \lambda(S+T)(v),\\ (S+T) \text{ is a linear transformation.}\end{gathered}$$

(b) $\lambda \in F$ and T is a transformation from V to $W \implies \lambda T : V \to$ is also linear.

$$\lambda T(v+w) = \lambda(T(v+w)) = \lambda(T(v)+T(w)) = \lambda T(v) + \lambda T(w)$$
$$c\lambda T(v) = c \cdot \lambda \cdot T(v)$$
$$\lambda T(cv) = \lambda \cdot c \cdot T(v) = c \cdot \lambda \cdot T(v)$$
$$\text{(by commutative multiplication over a VS)}$$
$$\text{Since } \lambda T(v+w) = \lambda T(v) + \lambda T(w) \text{ and } \lambda T(cv) = c\lambda T(v),$$
$$\lambda T \text{ is a linear transformation.}$$

(c) The function that takes all elements of V to the zero vector in W is a linear transformation. Let $T(v) = \vec{0}\ \forall v \in V$.

$$T(v+w) = \vec{0} \text{ (by def. of T)}$$
$$T(v) + T(w) = \vec{0} + \vec{0} \text{ (by def. of T) } = \vec{0}$$
$$T(\lambda v) = \vec{0} \text{ (by def. of T)}$$
$$\lambda T(v) = \lambda \cdot \vec{0} = \vec{0}$$
$$\text{(scalar times zero vector is the zero vector)}$$
$$\text{Since } T(v+w) = T(v) + T(w) \text{ and } T(\lambda v) = \lambda T(v),\ \text{T is a linear transformation.}$$

(d) Is the set of linear transformations from V to W a vector space over F?
Yes. $S+T$ (addition) and λT are linear transformations, therefore the set of linear transformations may be a vector space.

(e) What is the null space of $S+T$? What is the range of $S+T$? The null space of $S+T$ is:

$$(\ker(S) \cap \ker(T)) \cup \{v \in V : S(v) + T(v) = \vec{0}\}$$

The range of $S+T$ is:

$$\mathrm{im}(T) \cup \mathrm{im}(S)$$