Math 115A HW 6

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November 7, 2021

1. Textbook 2.4 #1
   (a) F - $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$
   (b) T - Fact for invertible transformations
   (c) T - Fact for transformations
   (d) F - Can be isomorphic with F^6, not F^5
   (e) T - $\mathcal{P}_n(F)$ can only be onto and one-to-one with the set of polynomials with the same dimension.
   (f) F - BA must be square and equal I as well for the two matrices to be inverses of each other.
   (g) T - Fact for invertible matrices
   (h) T - Fact for invertible transformations and their representative matrices
   (i) T - Fact for invertible matrices

2. Textbook 2.4 #2

   (a) No - maps to a higher-dimensional vector space are not onto. Not onto $\implies$ not invertible.
   (b) No - maps to a higher-dimensional vector space are not onto. Not onto $\implies$ not invertible.
   (c) Yes - the transformation can be represented by the matrix

$$\begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$$

   Since the columns are linearly independent, $[T]_\beta$ (in which β is the basis for $\mathbb{R}^3$) is invertible and thus T is invertible.

   (d) No - maps to a lower-dimensional vector space are not one-to-one. Not one-to-one $\implies$ not invertible.
   (e) No - maps to a lower-dimensional vector space are not one-to-one. Not one-to-one $\implies$ not invertible.
   (f) Yes - the transformation can be represented by transforming the basis vectors with

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   Since these vectors are linearly independent, the transformation is invertible.

3. (a) Suppose $f : X \to Y$ and $g : Y \to Z$ are invertible functions. Show $g \circ f$ is invertible by showing $f^{-1} \circ g^{-1}$ is an inverse for $g \circ f$.

$$g \circ f : X \to Z \text{ (def. of function composition)}$$
$$\text{Let } f(x) = y,\ g(y) = z.\ (g \circ f)(x) = z$$
$$g \circ f \text{ is invertible if } \exists h : Z \to X \text{ s.t.}$$
$$(g \circ f) \circ h = 1z,\ h \circ (g \circ f) = 1x$$

$$f^{-1}(y) = x,\ g^{-1}(z) = y$$
$$\text{Let } h = f^{-1} \circ g^{-1}$$
$$h(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$$

$$((g \circ f) \circ h)(z) = g(f(f^{-1}(g^{-1}(z))))$$
$$= g(f(f^{-1}(y))) = g(f(x)) = g(y) = z$$
$$(h \circ (g \circ f))(x) = f^{-1}(g^{-1}(g(f(x))))$$
$$= f^{-1}(g^{-1}(g(y))) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$$

Since $(g \circ f) \circ (f^{-1} \circ g^{-1}) = 1z$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = 1x$, $(g \circ f)$ is invertible.

(b) A, B are invertible matrices. $B \cdot A$ is invertible with an inverse of $A^{-1} \cdot B^{-1}$.

A linear transformation $T : V \to W$ can be represented by a matrix $[T]_\beta^\gamma$ over vector spaces V and W with bases β and γ, respectively. The result of a transformation on a vector $v \in V$, $T(v)$, would be found by multiplying $[T]_\beta^\gamma \cdot v$.

Two invertible transformations $T : V \to W$ and $U : W \to Z$ over vector spaces V, W, and Z with bases β, γ, and δ represented by matrices $[T]_\beta^\gamma$ and $[U]_\gamma^\delta$ have inverses T^{-1} and U^{-1} represented by matrices $([T]_\beta^\gamma)^{-1}$ and $([U]_\gamma^\delta)^{-1}$.

Let $A = [T]_\beta^\gamma$ and $B = [U]_\gamma^\delta$. Since both matrices are invertible, the matrices $B \cdot A$ and $A^{-1} \cdot B^{-1}$ exist. Using the logic from part (a), we can say that the transformation $U \circ T$ has an inverse $T^{-1} \circ U^{-1}$. $U \circ T$ is represented by the matrix $[U]_\gamma^\delta \cdot [T]_\beta^\gamma$ (or $B \cdot A$) while $T^{-1} \circ U^{-1}$ is represented by $([T]_\beta^\gamma)^{-1} \cdot ([U]_\gamma^\delta)^{-1}$ (or $A^{-1} \cdot B^{-1}$).

Now, we must verify that the inverse of $B \cdot A$ is indeed $A^{-1} \cdot B^{-1}$. Multiplication of both matrices will yield the identity matrix if $B \cdot A$ is invertible.

$$(B \cdot A) \cdot (A^{-1} \cdot B^{-1}) = B \cdot A \cdot A^{-1} \cdot B^{-1}$$
$$= B \cdot (A \cdot A^{-1}) \cdot B^{-1} = B \cdot I_n \cdot B^{-1}$$
$$\text{(associative matrix multiplication)}$$
$$B \cdot I_n = B \therefore B \cdot I_n \cdot B^{-1} = B \cdot B^{-1}$$
$$= I_n$$

$$(A^{-1} \cdot B^{-1}) \cdot (B \cdot A) = A^{-1} \cdot B^{-1} \cdot B \cdot A$$
$$= A^{-1} \cdot (B^{-1} \cdot B) \cdot A$$
$$\text{(associative matrix multiplication)}$$
$$A^{-1} \cdot I_n = A^{-1} \therefore A^{-1} \cdot I_n \cdot A = A^{-1} \cdot A$$
$$= I_n$$

4. Textbook 2.4 #6

If A is invertible and $AB = 0$, then $B = 0$
Proof.

$$\exists A^{-1} \text{ s.t. } A^{-1}A = I \text{ (def. invertibility)}$$
$$IC = C \ \forall C \in \mathrm{Mat}(F)$$
$$0C = 0 \ \forall C \in \mathrm{Mat}(F)$$
$$AB = 0$$
$$A^{-1}AB = A^{-1}0$$
$$(A^{-1}A)B = A^{-1}0$$
$$IB = 0, \ \therefore B = 0 \quad \square$$

5. Textbook 2.4 #14

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$$

$$\text{Let } T : V \to F^3$$

$$T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, a+b, c)$$

$$T^{-1}(a, a+b, c) = \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}$$

$$T \text{ is invertible.}$$

$$T\left(\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & d+e \\ 0 & f \end{pmatrix}\right) = T\begin{pmatrix} a+d & a+d+b+e \\ 0 & c+f \end{pmatrix} = (a+d, a+d+b+e, c+f)$$

$$T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} + T\begin{pmatrix} d & d+e \\ 0 & f \end{pmatrix} = (a, a+b, c) + (d, d+e, f) = (a+d, a+d+b+e, c+f)$$

$$\lambda T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = \lambda(a, a+b, c) = (\lambda a, \lambda a + \lambda b, \lambda c)$$

$$T\begin{pmatrix} \lambda a & \lambda(a+b) \\ 0 & \lambda c \end{pmatrix} = (\lambda a, \lambda a + \lambda b, \lambda c)$$

$$T(v+w) = T(v) + T(w) \text{ and } \lambda T(v) = T(\lambda v), \text{ thus } T \text{ is linear.}$$

Since T is linear and invertible, T is an isomorphism from V to F^3.

6. Textbook 2.5 #1

   (a) F - The jth column is $[x'_j]_\beta$
   (b) T - Fact for change of coordinate matrices
   (c) T - $[T]_\beta = Q[T]_{\beta'}Q^{-1}$
   (d) F - $A, B \in \mathrm{Mat}_{n\times n}(F)$ are similar if $B = Q^{-1}AQ$ for some $Q \in \mathrm{Mat}_{n\times n}(F)$
   (e) T - There is a change of coordinate matrix Q from β to γ and vice-versa. The procedure for such, $[T]_\gamma = Q[T]_\beta Q^{-1}$ is the definition of similarity.

7. Textbook 2.5 #2

   (a)
$$(a_1, a_2) = a_1e_1 + a_2e_2$$
$$(b_1, b_2) = b_1e_1 + b_2e_2$$
$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

   (b)
$$(0, 10) = 4(-1, 3) + 2(2, -1)$$
$$(5, 0) = 1(-1, 3) + 3(2, -1)$$
$$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

   (c)
$$e_1 = 3(2, 5) + 5(-1, -3)$$
$$e_2 = -1(2, 5) + -2(-1, -3)$$
$$\begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$$

   #3

   (d) Solved using systems of linear equations.
$$x^2 + x + 4 = 2(x^2 - x + 1) + 3(x + 1) + -1(x^2 + 1)$$
$$4x^2 - 3x + 2 = 1(x^2 - x + 1) + -2(x + 1) + 3(x^2 + 1)$$
$$2x^2 + 3 = 1(x^2 - x + 1) + 1(x + 1) + 1(x^2 + 1)$$
$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$$

8. Textbook 2.5 #6

(a)

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{1 \cdot 2 - 1 \cdot 1} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$AQ = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$$

$$[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$$

(c)

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\det(Q)} \begin{pmatrix} 0-1 & -(2-1) & 1-0 \\ -(2-1) & 2-1 & -(1-1) \\ 1-0 & -(1-1) & 0-1 \end{pmatrix}^T = \frac{1}{\det(Q)} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^T = \frac{1}{\det(Q)} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det(Q) = 1(0-1) - 1(2-1) + 1(1-0) = -1$$

$$Q^{-1} = -1 \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$AQ = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

$$Q^{-1}AQ = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$$

$$[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$$

9. Textbook 2.5 #7

(a)

$$\beta = \{(1,m),(-m,1)\}$$

$$[I]_\beta^s = Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

$$T(1,m) = 1(1,m) + 0(-m,1)$$

$$T(-m,1) = (m,-1) = 0(1,m) + -1(-m,1)$$

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[I]_s^\beta = Q = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$

$$[T]_s = Q^{-1}[T]_\beta Q$$

$$[T]_\beta Q = \frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}$$

$$Q^{-1}[T]_\beta Q = \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

$$[T]_s = \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix} = \begin{pmatrix} \frac{-m^2+1}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{pmatrix}$$

$$T(x,y) = \begin{pmatrix} \frac{-m^2+1}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\frac{-xm^2+x+2ym}{m^2+1}, \frac{2xm+ym^2-y}{m^2+1})$$

(b)

$$\beta = \{(1,m),(1,-\frac{1}{m})\}$$

$$Q^{-1} = \begin{pmatrix} 1 & 1 \\ m & -\frac{1}{m} \end{pmatrix}$$

$$T(1,m) = 1(1,m) + 0(1,-\frac{1}{m})$$

$$T(1,-\frac{1}{m} = (0,0) = 0(1,m) + 0(1,-\frac{1}{m})$$

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Q = \frac{1}{-\frac{1}{m}-m}\begin{pmatrix} -\frac{1}{m} & -1 \\ -m & 1 \end{pmatrix}$$

$$[T]_\beta Q = \frac{1}{-\frac{1}{m}-m}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{m} & -1 \\ -m & 1 \end{pmatrix} = \frac{m}{-1-m^2}\begin{pmatrix} -\frac{1}{m} & -1 \\ 0 & 0 \end{pmatrix}$$

$$Q^{-1}[T]_\beta Q = \frac{m}{-1-m^2}\begin{pmatrix} 1 & 1 \\ m & -\frac{1}{m} \end{pmatrix}\begin{pmatrix} -\frac{1}{m} & -1 \\ 0 & 0 \end{pmatrix} = \frac{m}{-1-m^2}\begin{pmatrix} -\frac{1}{m} & -1 \\ -1 & -m \end{pmatrix}$$

$$[T]_s = \frac{m}{-1-m^2}\begin{pmatrix} -\frac{1}{m} & -1 \\ -1 & -m \end{pmatrix} = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{pmatrix}$$

$$T(x,y) = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\frac{x+ym}{m^2+1}, \frac{xm+ym^2}{m^2+1})$$

10. Determinants

   (a) $\det(A)$

$$A := \begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\bar{A}_{ij})$$

Using first row:

$$\det(A) = (-1)^{1+1} \cdot a \cdot \begin{vmatrix} e & f \\ y & z \end{vmatrix} + (-1)^{1+2} \cdot b \cdot \begin{vmatrix} d & f \\ x & z \end{vmatrix} + (-1)^{1+3} \cdot c \cdot \begin{vmatrix} d & e \\ x & y \end{vmatrix}$$

$$= a \cdot (ez - fy) - b \cdot (dz - fx) + c \cdot (dy - ex)$$

$$= aez - afy - bdz + bfx + cdy - cex$$

   (b) $\det(B)$

$$B := \begin{pmatrix} 1 - x & 2x^4 \\ -4x + x^6 & 7 + x^{14} \end{pmatrix}$$

Using first row:

$$\det(B) = (-1)^{1+1} \cdot (1 - x) \cdot \left|7 + x^{14}\right| + (-1)^{1+2} \cdot (2x^4) \cdot \left|-4x + x^6\right|$$

$$= (1 - x)(7 + x^{14}) - (2x^4)(-4x + x^6)$$

$$= (7 + x^{14} - 7x - x^{15}) - (-8x^5 + 2x^{10})$$

$$= 7 - 7x + 8x^5 - 2x^{10} + x^{14} - x^{15}$$

   (c) $\det(C)$

$$C := \begin{pmatrix} -3 & 4 & 0 & -1 \\ 0 & 9 & -2 & -3 \\ 1 & 1 & 1 & 1 \\ 3 & 0 & 3 & 0 \end{pmatrix}$$

Using first row:

$$\det(B) = (-1)^{1+1} \cdot -3 \cdot \begin{vmatrix} 9 & -2 & -3 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{vmatrix} + (-1)^{1+2} \cdot 4 \cdot \begin{vmatrix} 0 & -2 & -3 \\ 1 & 1 & 1 \\ 3 & 3 & 0 \end{vmatrix} + 0 + (-1)^{1+4} \cdot -1 \cdot \begin{vmatrix} 0 & 9 & -2 \\ 1 & 1 & 1 \\ 3 & 0 & 3 \end{vmatrix}$$

$$= -3 \cdot \begin{vmatrix} 9 & -2 & -3 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{vmatrix} + -4 \cdot \begin{vmatrix} 0 & -2 & -3 \\ 1 & 1 & 1 \\ 3 & 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 9 & -2 \\ 1 & 1 & 1 \\ 3 & 0 & 3 \end{vmatrix}$$

$$= -3(-36) - 4(-6) + 6$$

$$\det(B) = 138$$

11.