Math 115A HW 2

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1. Textbook 1.4 #2

(a)

$$\begin{aligned} \text{I. } 2x_1 - 2x_2 - 3x_3 \qquad &= -2 \\ \text{II. } 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ \text{III. } x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 - 2x_3 - x_4 &= -3 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7; -3(\text{I}) \\ 2x_1 - 2x_2 - 3x_3 \qquad &= -2; -2(\text{I}) \end{aligned}$$

$$\begin{aligned} x_1 - x_2 - 2x_3 - x_4 &= -3 \\ 4x_3 + 8x_4 &= 16; = 4(\text{III}) \\ x_3 + 2x_4 &= 4 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 \qquad + 3x_4 &= 5 \\ x_3 + 2x_4 &= 4 \end{aligned}$$

$$\begin{aligned} x_1 &= 5 + x_2 - 3x_4 \\ x_3 &= 4 - 2x_4 \end{aligned}$$

Let $x_2 = r$, $x_4 = s$

$$(x_1, x_2, x_3, x_4) = (2r - 3s + 5, r, -2s + 4, s)$$

Infinitely many solutions

(b)

$$\begin{aligned} \text{I. } 3x_1 + 7x_2 + 4x_3 &= 10 \\ \text{II. } x_1 - 2x_2 + x_3 &= 3 \\ \text{III. } 2x_1 - x_2 - 2x_3 &= 6 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 2x_3 &= 6;\ -2(\text{I}) \\ 3x_1 - 7x_2 + 4x_3 &= 10;\ -3(\text{I}) \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 3x_2 - 4x_3 &= 0; +3(\text{III}) \\ -x_2 + x_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= 6 \\ -x_3 &= 3; \\ -x_2 &= 4 \end{aligned}$$

$$x_1 = -2$$
$$x_3 = -3;$$
$$x_2 = -4$$
$$(x_1, x_2, x_3) = (-2, -4, -3)$$

2. **Spans and sums.**

   (a) If $S_1, S_2 \subset V$, $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$
   By the definition of span, the span of $S_1 \cup S_2$ is the set of all linear combinations of elements in S_1 and S_2. This can be defined as

   $$a_1 s_1 + a_2 s_2 + ... + a_n s_n \ \forall s_i \in S_1 \cup S_2, \forall a_i \in F$$

   The span of S_1 can be defined as

   $$b_1 v_1 + b_2 v_2 + ... + b_n v_n \ \forall v_i \in S_1$$

   while the span of S_2 can be defined as

   $$c_1 u_1 + c_2 u_2 + ... + c_n u_n \ \forall u_i \in S_1$$

   The sum of the two spans is

   $$b_1 v_1 + b_2 v_2 + ... + b_n v_n + c_1 u_1 + c_2 u_2 + ... + c_n u_n$$

   Since b and c are arbitrary values, they can be represented as a_i. The set consisting of v_i and u_i comprises $S_1 \cup S_2$. Such elements can be denoted as $s_i \in S_1 \cup S_2$.

   $$\therefore b_1 v_1 + b_2 v_2 + ... + b_n v_n + c_1 u_1 + c_2 u_2 + ... + c_n u_n = a_1 s_1 + ... a_n s_n$$

   This is true because both sides are all linear combinations of elements in S_1 and S_2. Thus,t he sum of the span of two subsets S_1 and S_2 in V is equal to the span of the union of the two subsets. $\square$

   (b) $\text{span}(W_1 \cup W_2) = W_1 + W_2$ for two subspaces W_1 and $W_2 \subset V$, a vector space
   Proof.
   Since the span of the union of two subsets is equal to the sum of the spans of two subsets, this can be written as
   $$\text{span}(W_1 \cup W_2) = \text{span}(W_1) + \text{span}(W_2)$$
   W_1 and W_2 are both subspaces, therefore they are both closed under addition and scalar multiplication. This means
   $$a_1 w_1 + ... + a_n w_n \in W_1 \text{ or } W_2$$
   The span of W_1 or W_2 would be defined as the set of all linear combinations of elements in W_1 or W_2, respectively. These are all contained within W_1 and W_2 since they are both subspaces, therefore

   $$\text{span}(W) = W \text{ if W is a subspace} \therefore \text{span}(W_1 \cup W_2) = \text{span}(W_1) + \text{span}(W_2) = W_1 + W_2$$

3. If V is a vector space and $S_1 \subset S_2 \subset V$, then if S_2 is linearly independent, so is S_1.
   Proof.
   If S_2 is linearly independent, there are no elements in S_2 that are a linear combination of the other elements unless each coefficient equals 0.

   $$\forall s_i \in S_2,\ a_1 s_1 + ... + a_n s_n \neq 0 \text{ for values } a_i \in F \text{ unless } a_1 = a_2 = ... = 0$$

   If S_1 is a subset of S_2,
   $$s_j \in S_1 \text{ for some values } 1 \leq j \leq n$$

Thus,

$$a_1 s_1 + ... + a_j s_j + ... + a_n s_n \neq 0. \forall s_i \in S_2,\ \forall a_i \in F$$

If no sum of multiples of values within S_2 can sum to equal 0 without every coefficient equalling 0, and all of $s_j \in S_2$, there is no linear combination of s_j that can equal 0. Since s_j is all of the values in S_1, s_j is linearly independent.

4. **Linearly independent polynomials.**

   (a) The set of elements $\{x+1, x^2+x\}$ is linearly independent in Poly($\mathbb{R}$).
   Proof. If the set was linearly dependent,

   $$a_1(x+1) + a_2(x^2+x) = 0$$

   for some values $a_1, a_2 \in \mathbb{R}$. This can be rewritten as

   $$a_1(x+1) = a_2(x^2+x)$$

   There are no values $a_1, a_2 \in \mathbb{R} : a(x+1) = x^2+x$ except $a = 0 : a_1(x+1) = a_2(x^2+x)$. Thus, the set is linearly independent. □

   (b) S is a linearly independent set in Poly($\mathbb{R}$).
   The set S consists of all polynomials

   $$\{x^1 + x^{1-1}, ..., x^n + x^{n-1}\}$$

   For any value $k : 1 \leq k < n$, there does not exist a value $a \in \mathbb{R} : a(x^k + x^{k-1}) = x^n + x^{n-1}$. The degree of any term in S must change in order to make it equal another term in S, and no scalar in $\mathbb{R}$ can increase the degree.

   $$a_1(x^k + x^{k-1}) + a_2(x^n + x^{n-1}) = 0$$

   a_1 and a_2 can only equal 0 to make the equation true.

5. Textbook 1.5 #1

   (a) F - *A linearly dependent set only needs to have at least 1 of its members to be a linear combination of other vectors.*

   (b) T - *The zero vector can be obtained by multiplying any vector by the scalar 0 including itself, therefore any set containing it can have a linear combination of its values equal to the zero vector.*

   (c) F - *No vector can be multiplied or added together to obtain the absence of a vector*

   (d) F - *A linearly dependent set only needs to have at least 1 of its members to be a linear combination of other vectors. There can exist a set of linearly independent vectors within a linearly dependent set.*

   (e) T - *Proved in question 3.*

   (f) T - *This is the definition of linear independence.*

6. Textbook 1.5 #2

   (d) Linearly dependent: $-2x^3 + 3x^2 + 2x + 6 = -2(x^3 - x) + \frac{3}{2}(2x^2 + 4)$

   (g) Linearly dependent:

   $$3 \cdot \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}$$

   All of the matrices above are part of the same set.

7. **Linear independence in F^n and finite fields.**

(a) The set

$$S = \{e_1, ..., e_n\}$$

is linearly independent and spans F^n.

Proof.

Since each vector has a 1 in a different position than all of the other vectors within the set, none of them can be a linear combination of any of the other vectors.

$$a_j e_j + a_k e_k = \{(0, ..., a_j e_j, ..., a_k e_k ..., 0\} \neq \text{ any } e_i \text{ since there is not a 0 in every position.}$$

Additionally, given an element $v = (v_1, ..., v_n) \in F^n$,

$$v = a_1 e_1 + ... + a_n e_n \text{ for values } a_i \in F$$

Thus, S generates and therefore spans F^n. □

(b) i. $\mathbb{Z}/2\mathbb{Z}$ has 2 elements

ii. $(\mathbb{Z}/2\mathbb{Z})^2$ has 4 elements.

iii. $(\mathbb{Z}/2\mathbb{Z})^3$ has 8 elements.

iv. $(\mathbb{Z}/2\mathbb{Z})^4$ has 16 elements.

v. $(\mathbb{Z}/2\mathbb{Z})^k,\ k \in \mathbb{N}$ has 2^k elements.

vi. $(\mathbb{Z}/3\mathbb{Z})^k, k \in \mathbb{N}$ has 3^k elements.

(c) Textbook 1.5 #11

The span denotes every linear combination of each vector, and as shown above there are $2^k, k \in \mathbb{N}$ vectors within a vector space with two possible field elements 0 and 1 and k basis vectors. These vectors can represent every linear combination of a set of n linearly independent vectors in S, therefore there are 2^n vectors in span(S).

8. (a) 3×3 upper triangular matrix with entries in $\mathbb{R}$

$$\begin{pmatrix} 3 & 2 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 9 \end{pmatrix}$$

3×3 not upper triangular matrix with entries in $\mathbb{R}$

$$\begin{pmatrix} 3 & 2 & 2 \\ 10 & 7 & 1 \\ 6 & 0 & 9 \end{pmatrix}$$

(b) The set of upper triangular matrices is a subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

Proof.

The 0 vector in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ is defined as

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is an upper triangular matrix, therefore the 0 vector is in the set of upper triangular matrices. Since matrix addition of two matrices A and B is defined as $A_{ij} + B_{ij}$, all values in an upper triangular matrix at or above the diagonal will be added with other values at or above the diagonal. This will result in a matrix with only zeroes below the diagonal (an upper triangular matrix) since any nonzero values in one matrix will have been added to a value in the same position in another matrix.

Scalar values (in $\mathbb{R}$) multiplied by zero will only yield zero, therefore multiplying an upper triangular matrix by a scalar will only change the nonzero values. The resulting matrix from any scalar multiplication is an upper triangular matrix.

Since the zero vector $\in$ the set of all upper triangular matrices, the set is closed under addition, and the set is closed under scalar multiplication, it is a subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$

This can be generalized to the case of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ since none of the proof above explicitly needs the size to be 2×2 to be true.

(c) The set

$$S := \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 4 & 6 \\ 0 & 0 \end{pmatrix} \right\}$$

is not a basis for 2×2 upper triangular matrices over $\mathbb{R}$.

Proof.
S is not linearly independent and is thus not a basis. Additionally, if $v \in V$, v is a linear combination of vectors in the basis for V. There is not a combination of elements $s_1, s_2, s_3 \in S$ such that $a_1 s_1 + a_2 s_2 + a_3 s_3 = v,\ a_1, a_2, a_3 \in \mathbb{R}\ \forall v \in$ the set of all triangular matrices, $\therefore S$ is not a basis for the set of all triangular matrices. $\square$

A basis for $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ can be

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

9. Textbook 1.6 #12
If $\{u, v, w\}$ is a basis for V, $\{u + v + w, v + w, w\}$ is also a basis for V.
Proof.
Since the basis is linearly independent,

$$a_1 u + a_2 v + a_3 w = 0 \iff a_1 = a_2 = a_3 = 0.$$

If $\{u + v + w, v + w, w\}$ is also a basis for V, the same should apply.

$$\begin{gathered} b_1(u + v + w) + b_2(v + w) + b_3 w = 0 \\ b_1 u + a_1 v + b_1 w + b_2 v + b_2 w + b_3 w = 0 \\ b_1 u + (b_1 + b_2)v + (b_1 + b_2 + b_3)w = 0 \\ \therefore \text{ since this is of the form } a_1 u + a_2 v + a_3 w = 0, a_1 = a_2 = a_3 = 0. \end{gathered}$$

$$\begin{aligned} b_1 \qquad\qquad &= 0 \\ b_1 + b_2 \qquad &= 0 \\ b_1 + b_2 + b_3 &= 0 \end{aligned}$$

$$\begin{aligned} b_2 \qquad &= 0 \\ b_2 + b_3 &= 0 \end{aligned}$$

$$b_3 = 0$$

Since $b_1(u+v+w)+b_2(v+w)+b_3 w = 0$ only when $b_1 = b_2 = b_3 = 0, \{u+v+w, v+w, w\}$ is a basis. $\square$

10. Textbook 1.6 #13

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0;\ \text{minus 2 times the 1st eqn.} \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0;\ \text{plus 2 times the 2nd eqn.} \\ x_2 - x_3 &= 0 \end{aligned}$$

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$

$$x_1 = x_2 = x_3$$

The solution is any vector (x_1, x_2, x_3) in which all are values are equal. Thus, the solution is $(\lambda, \lambda, \lambda)$ in which λ is any scalar in $\mathbb{R}$. The basis for this is $\{(1, 1, 1)\}$ since the span of $\{(1, 1, 1)\}$ generates $(\lambda, \lambda, \lambda)$ for any scalar λ.