Math 115A HW 4

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1. (a) A function which is injective, but not surjective.

$$f(x) = (x, 1)$$

(b) A function which is surjective, but not injective.

$$f(x, y) = x + y$$

(c) A function which is bijective.

$$f(x) = x^3$$

2. Textbook 2.1 # 17

(a) If $\dim(V) < \dim(W)$, then T cannot be onto.
Proof.

$$\begin{gathered}
\text{Let } T \text{ be an onto linear transformation.} \\
\text{nullity}(T) + \text{rank}(T) = \dim(V) \text{ (Rank-Nullity Theorem)} \\
\text{Im}(T) = W \text{ (def. of onto)} \\
\therefore \text{rank}(T) = \dim(W) \\
\therefore \text{nullity}(T) + \dim(W) = \dim(V) \\
\text{nullity}(T) = \dim(V) - \dim(W) \\
\text{The nullity of } T \text{ cannot be negative, thus } \dim(V) \geq \dim(W) \text{ if } T \text{ is onto.} \\
\therefore \text{if } \dim(V) < \dim(W), \text{ then } T \text{ cannot be onto.} \quad \square
\end{gathered}$$

(b) If $\dim(V) > \dim(W)$, then T cannot be one-to-one.
Proof.

$$\begin{gathered}
\text{Let } T \text{ be a one-to-one linear transformation.} \\
\ker(T) = \{v \in V | T(v) = \vec{0}\}, \\
\therefore \text{ since } T(v_1) = T(v_2) = w \implies v_1 = v_2 \text{ (def. of one-to-one) and} \\
T(\vec{0}) = \vec{0} \text{ (def. of lin. transformation),} \\
v = \vec{0} \text{ if } T(v) = \vec{0} \\
\therefore \text{nullity}(T) = 0 \\
\text{nullity}(T) + \text{rank}(T) = \dim(V) \text{ (Rank-Nullity Theorem)} \\
0 + \text{rank}(T) = \dim(V) \\
\text{rank}(T) \leq \dim(W) \therefore \dim(V) \leq \dim(W) \text{ if } T \text{ is one-to-one.} \\
\therefore \text{if } \dim(V) > \dim(W), \text{ then } T \text{ cannot be one-to-one.} \quad \square
\end{gathered}$$

3. Textbook 2.1 # 21

(a) T (left shift) and U (right shift) are linear.
Proof.

$$\text{Let } a = (a_1, ..., a_n) \text{ and } b = (b_1, ..., b_n),\ \ a, b \in V$$
$$T(a) = T(a_1, a_2, ...) = (a_2, a_3, ...)$$
$$T(b) = T(b_1, b_2, ...) = (b_2, b_3, ...)$$
$$T(a) + T(b) = (a_2 + b_2, a_3 + b_3, ...)$$
$$T(a + b) = T(a_1 + b_1, a_2 + b_2, ...) = (a_2 + b_2, a_3 + b_3, ...)$$
$$\therefore T(a) + T(b) = T(a + b)$$

$$T(\lambda a) = T(\lambda a_1, \lambda a_2, ...) = (\lambda a_2, \lambda a_3, ...)$$
$$\lambda T(a) = \lambda T(a_1, a_2, ...) = \lambda(a_2, a_3, ...) = (\lambda a_2, \lambda a_3, ...)$$
$$\therefore T(\lambda a) = \lambda T(a)$$

Since $T(a) + T(b) = T(a + b)$ and $T(\lambda a) = \lambda T(a)$, T is linear.

$$U(a) = U(a_1, a_2, ...) = (0, a_1, a_2, ...)$$
$$U(b) = U(b_1, b_2, ...) = (0, b_1, b_2, ...)$$
$$U(a) + U(b) = (0, a_1 + b_1, a_2 + b_2, ...)$$
$$U(a + b) = U(a_1 + b_1, a_2 + b_2, ...) = (0, a_1 + b_1, a_2 + b_2, ...)$$
$$\therefore U(a) + U(b) = U(a + b)$$

$$U(\lambda a) = U(\lambda a_1, \lambda a_2, ...) = (0, \lambda a_1, \lambda a_2, ...)$$
$$\lambda U(a) = \lambda U(a_1, a_2, ...) = \lambda(0, a_1, a_2, ...) = (0, \lambda a_1, \lambda a_2, ...)$$
$$\therefore U(\lambda a) = \lambda U(a)$$

Since $U(a) + U(b) = U(a + b)$ and $U(\lambda a) = \lambda U(a)$, U is linear. □

(b) T is onto, but not one-to-one.
Proof.

$v = (v_1, v_2, ...), v_i \in F, v$ is any vector $\in V$.

Since v_i represents any field element, $(v_1, v_2, ...)$ represents the entirety of V.

$$T(v_1, v_2, ...) = (v_2, v_3, ...)\ \ \forall v \in V$$

$(v_2, v_3, ...)$ also represents the entirety of V since $\{v_2, v_3, ...\}$ can be any field element

$$\therefore \text{Im}(T) = V$$

$T : V \to V$ and $\text{Im}(T) = V \therefore T$ is onto.

$$\text{Let } a = (a_1, c_2, c_3, ...) \text{ and } b = (b_1, c_2, c_3, ...)$$
$$a \neq b$$
$$T(a) = (c_2, c_3, ...),\ \ T(b) = (c_2, c_3)$$

$\therefore T(a) = T(b)$A one-to-one transformation implies if $T(v_1) = T(v_2),\ \ v_1 = v_2$

$\therefore$ since $a \neq b$ and $T(a) = T(b),\ \ T$ is not one-to-one. □

(c) U is one-to-one, but not onto.

$$\text{Let } a = (a_1, a_2, ...),\ \ b = (b_1, b_2, ...) : a \neq b$$
$$U(a) = (0, a_1, a_2, ...),\ \ U(b) = (0, b_1, b_2, ...)$$
$$U(a) \neq U(b)\ \ \forall a, b \in V$$

$\therefore \nexists a, b : a \neq b,\ \ U(a) = U(b)$ and thus U is one-to-one.

$$v = (v_1, v_2, ...), v_i \in F, v \text{ is any vector } \in V.$$
$$U(v) = (0, v_1, v_2, ...)\ \forall v \in V$$
$$\text{Even if } \{v_1, v_2, ...\} \text{ represented all vectors in } V,\ U(v) \text{ does not.}$$
$$\text{Let } w = (w_1, w_2, ...),\ w \in V,\ w_1 \neq 0.$$
$$w \notin \mathrm{Im}(U) \text{ since all vectors in } U \text{ start with 0, } \therefore \mathrm{Im}(U) \neq V$$
$$\text{Since } \mathrm{Im}(U) \neq V,\ U \text{ is not onto.} \quad \square$$

4. Textbook 2.1 # 22

   (a) $T : \mathbb{R}^3 \to \mathbb{R}$ is linear. There exist scalars $a, b, c : T(x, y, z) = ax + by + cz\ \forall (x, y, z) \in \mathbb{R}^3$.
   Proof.

$$\{e_1, e_2, e_3\} \text{ is a basis for } \mathbb{R}^3$$
$$\therefore \forall (x, y, z) \in \mathbb{R}^3,\ (x, y, z) = xe_1 + ye_2 + ze_3 \text{ for some values } x, y, z \in \mathbb{R}$$

$$\text{Let } T(e_1) = a,\ T(e_2) = b,\ T(e_3) = c$$
$$T(xe_1) = xa,\ T(ye_2) = yb,\ T(ze_3) = zc \text{ (linearity)}$$
$$\therefore T(xe_1 + ye_2 + ze_3) = xa + yb + zc \text{ (linearity)}$$
$$\therefore T(x, y, z) = xa + yb + zc = ax + by + cz \quad \square$$

   (b) $T : F^n \to F$
   For a vector space F^n with basis $\{v_1, ..., v_n\}$ (in which every element can be represented by $x_1v_1 + ... + x_nv_n$), $T(x_1, ..., x_n) = a_1x_1 + ... + a_nx_n$ for some values $a_1, ..., a_n \in F$.

   (c) $T : F^m \to F^n$

$$(x_1, ..., x_m) \in F^m$$
$$T(x_1, ..., x_m) = \begin{pmatrix} a_{11}x_1 + ... + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + ... + a_{nm}x_m \end{pmatrix}$$
$$\text{for some values } a_{ij} \in F$$

5. Textbook 2.1 # 23

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$\text{Let } (x, y, z) \text{ represent any vector in } \mathbb{R}^3$$
$$T(x, y, z) = ax + by + cz$$
$$\ker(T) = \{(x, y, z) : ax + by + cz = 0\}$$
$$ax + by + cz = 0$$

   This is the equation for a plane. Thus, the kernel of T is a plane in $\mathbb{R}^3$ that goes through the origin.

6. Textbook 2.1 # 24
   If $s \in K$, then $K = \{s\} + \ker(T)$.

Proof.

$$K = \{x \in V : T(x) = b\}$$
$$T(s) = b \text{ (def. of K)}$$
$$\text{Let } v = \text{ any vector in } \ker(T)$$
$$T(v) = \vec{0} \text{ (def. of kernel)}$$
$$\{s\} + \ker(T) = \{s + v; s \in K,\ v \in \ker(T)\}$$
$$\text{Let } w \in \{s + v; s \in K,\ v \in \ker(T)\}$$
$$w = s + v,\ T(w) = T(s + v) = T(s) + T(v) \text{ (linearity)}$$
$$= b + \vec{0} = b$$
$$\therefore w \in K \text{ and } K = \{s + v; s \in K,\ v \in \ker(T)\} \quad \square$$

7. Textbook 2.2 # 1
   (a) T (b) T (c) F
   (d) T (e) T (f) F

8. Textbook 2.2 # 2

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$
   3×2 matrix

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} c_{11}a_1 + c_{12}a_2 \\ c_{21}a_1 + c_{22}a_2 \\ c_{31}a_1 + c_{32}a_2 \end{pmatrix}$$

$$c_{11}a_1 + c_{12}a_2 = 2a_1 - a_2$$
$$c_{21}a_1 + c_{22}a_2 = 3a_1 + 4a_2$$
$$c_{31}a_1 + c_{32}a_2 = a_1$$

$$c_{11} = 2,\ c_{12} = -1$$
$$c_{21} = 3,\ c_{22} = 4$$
$$c_{31} = 1,\ c_{32} = 0$$

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(a_1, a_2, a_3) = (2a_1 + 3a_2, a_1 + a_3)$
   2×3 matrix

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c_{11}a_1 + c_{12}a_2 + c_{13}a_3 \\ c_{21}a_1 + c_{22}a_2 + c_{23}a_3 \end{pmatrix}$$

$$c_{11}a_1 + c_{12}a_2 + c_{13}a_3 = 2a_1 + 3a_2$$
$$c_{21}a_1 + c_{22}a_2 + c_{23}a_3 = a_1 + a_3$$

$$c_{11} = 2,\ c_{12} = 3,\ c_{13} = 0$$
$$c_{21} = 1,\ c_{22} = 0,\ c_{23} = 1$$

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(f) $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(a_1, a_2, ..., a_n) = (a_n, a_{n-1}, ..., a_1)$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

A matrix with a 1 in the diagonal starting from the upper-rightmost position toward the lower-leftmost.

(g) $T : \mathbb{R}^n \to \mathbb{R}$, $T(a_1, a_2, ..., a_n) = a_1 + a_n$

$$[T]_\beta^\gamma = (1, 0, \dots, 0, 1)$$

9. Textbook 2.2 # 4

$T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b) + (2d)x + bx^2$

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},\ \gamma = \{1, x, x^2\}$$

Since $\dim(\mathcal{P}_2(\mathbb{R})) = 3$ and $\dim(\mathrm{Mat}_{2\times 2}(\mathbb{R})) = 4$, $[T]_\beta^\gamma$ will be a 3×4 matrix.

Find the transformation of the ordered basis vectors.

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1 + 0)1 + (2 \cdot 0)x + (0)x^2 = (1)1 + (0)x + (0)x^2$$
$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0 + 1)1 + (2 \cdot 0)x + (1)x^2 = (1)1 + (0)x + (1)x^2$$
$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (0 + 0)1 + (2 \cdot 0)x + (0)x^2 = (0)1 + (0)x + (0)x^2$$
$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0 + 0)1 + (2 \cdot 1)x + (0)x^2 = (0)1 + (2)x + (0)x^2$$

Use the coefficients of the transformed ordered basis vectors as columns of $[T]_\beta^\gamma$.

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$