# Introduction to Game Theory 1

## Basic Concepts

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# Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 3, sections 3.1-3.3
- **Optional**
  - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
  - Joseph E. Harrington: Games, Strategies and Decision Making. Worth Publishers. *Lots of engaging examples and applications.*
  - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
  - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

# Game Theory Part 1: Outline

1. **Game Theory: Science of Strategic Thinking**
2. **Examples of interesting games**
3. Formalising games
4. Solution concept 1: Weak optimality
5. Solution concept 2: Strategies with (weak) guarantees
6. Solution concept 3: Nash equilibrium

# Overview

Game Theory: The Science of Strategic Thinking

Examples of interesting games

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# Games people play ...

**Games as entertainment or challenge**

- Board games: chess, backgammon, GO, etc.
- Card games
- Rock-paper-scissors, etc.
- ...

**Games as models**

- War games, simulations, etc.
- More generally: many **interactions in society or nature** share the same ingredients!
- *All models are wrong, but some are useful!* (G. Box)

# Game Theory: Science of Strategic Thinking

- Originally, tool in **economics**
  - 1944: von Neumann & Morgenstern, *Theory of Games and Economic Behavior,*
  - Nobel Prize in Economics was awared for GT-work in the years: 1994 (J. Harsanyi, J.Nash and R. Selten), 2005 (R. Aumann and T. Schelling), 2007 (L. Hurwicz, E. Maskin and R. Myerson), 2012 (A. Roth and L. Shapley), 2014 (J. Tirole), 2020 (P.R. Milgrom and R.B. Wilson)
- Game theory provides a level of abstraction appropriate to study a wide range of **socio-economic, political** and even **biological** phenomena.

# Game Theory: Different points of view

Philosopher John Elster (1982):

> *If one accepts that interaction is the essence of social life, then game theory provides solid micro foundations for the study of social structure and social change.*

Hargreaves-Heap and Varoufakis
"Game Theory. A critical Introduction" (1995)

> *Does game theory simply repeat what everyone already knows in a language that no one understands?*

# Game Theory: Science of Strategic Thinking

- GT is the **mathematical study** of interaction among independent, self-interested agents.
- **Self-interest**:
  - Each agent has its own interests and preferences (aims, goals);
  - Agents tend to have (partially) conflicting interests;
  - These interests are reflected in (numerical) utilities that are consistent with the preferences;

    if option A is preferred to B, then $u(A) > u(B)$

  - Agents act to maximise their utility
- **Coalitional/Cooperative** vs **non-coalitional/non-cooperative** GT

# Ingredients of interesting games

- Players: You against one or more opponent(s)
  - Opponent: other agents, other version of yourself, nature, lady luck, etc.
- Rules determine which actions can be taken, and what the corresponding pay-offs or utilities are;
  - actions and pay-offs: exogenous variables
- Maximize your pay-off: Everyone wants to win!
- **Competition and collaboration:** individuals or teams (non-cooperative and cooperative GT)

# Cooperative versus Non-Cooperative Games

- **Non-Cooperative**
  - Selfish individuals, only consider their own interest;
  - Do **not coordinate** their actions in groups
    - Emergence Coordination might happen as "accident" of selfish behaviour
  - Agreements need to be self-enforcing (no "contracts"!)

- **Cooperative:** Binding commitments (*"contracts"*) allow groups of players to coordinate their actions
  - Non-transferable utility: pay-off of each individual increases!
    *E.g. Stable marriage problem;*
  - Transferable utility: need to find a fair way to divide the additional value (e.g. money) generated by collaboration:
    *E.g. Shapley value*

# Simultaneous vs. Sequential Games

- **Simultaneous games:** players make their moves simultaneously, i.e. without knowing what the other players will do!
  - Rock-paper-scissors
  - Sealed bid auctions
- **Sequential games:** Sequence of successive moves by players who can see each other's moves:
  - Chess
  - Card games
  - Open cry auctions

# Encoding utilities of actions: Matrix form

**Simultaneous game: two players, finite number of actions**

| | Player 2 chooses *Left* | Player 2 chooses *Right* |
|---|---|---|
| Player 1 chooses *Up* | 4, 3 | −1, −1 |
| Player 1 chooses *Down* | 0, 0 | 3, 4 |

*Normal form or payoff matrix of a 2-player, 2-strategy game*

# Encoding utilities of actions: Extensive Form

**Sequential game: Decision tree**

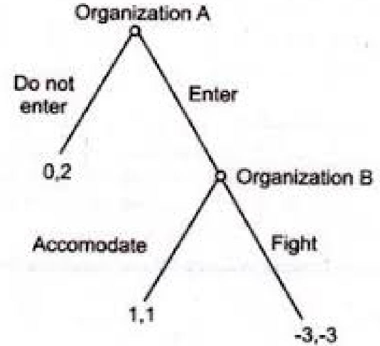


**Figure-2: Extensive Form Games**

# Encoding utilities of actions: General case

- Utility is a function of the joint action of players:
- Ultimatum game (one shot)
  - Player A can choose any fraction $0 \leq x \leq 1$ for himself, and offer the rest $(1 - x)$ to player B;
  - If player B accepts this offer, then that is the outcome. If he rejects it, then both get zero.

$$u_A(x) = \begin{cases} x \\ 0 \end{cases} \quad \text{and} \quad u_B(x) = \begin{cases} 1 - x & \text{if B accepts} \\ 0 & \text{if B rejects} \end{cases}$$

Important applicability issue: Rationality vs Behaviour (emotion)

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# Partnership game

- Two students work on project, divide profits evenly (50-50).
- Each student must decide how much effort (e.g. hours) he/she is contributing to the project. Hence, student $i$ chooses action $s_i$ = amount of effort (assume $0 \leq s_1, s_2 \leq 4$);
- Project generates reward: $4(s_1 + s_2 + bs_1 s_2)$
    - where $0 < b < 1$ measures the synergy
- The cost of the work to student $i$ equals $s_i^2$
- Pay-off for individual student (half total reward minus cost):

$$u_i(s_1, s_2) = 2(s_1 + s_2 + bs_1 s_2) - s_i^2$$

- How much effort should each student invest in the project?

# Economics: Cournot's Duopoly Model

- Two companies make an interchangeable product (e.g. bottled water).
- Without knowing the other company's strategy (i.e. simultaneously), both need to determine the quantity they will produce, say $q_1$ and $q_2$ respectively.
- The unit price $p$ of the product in the market depends on the total produced quantity $q_1 + q_2$; specifically

$$p(q_1, q_2) = \alpha - \beta(q_1 + q_2) \quad (\alpha, \beta > 0).$$

- Firm $i = 1, 2$ can produce the item at a unit-cost $c_i$.
- What quantity $q_i$ should each company produce in order to maximise its profit?

# Cournot vs. Stackleberg

- Simultaneous game: In a Cournot duopoly, firms make their moves at the same time, i.e. without knowing what the other will do.
- Sequential game: In Stackelberg duopoly, one firm becomes the leader and so make the first move. The other firm is the follower and can observe (and therefore react to) the decision of the leader.
- How does the simultaneous/sequential set-up impact the optimal strategies?

# Children dividing pie

*Two kids get a pie and need to divide it among themselves.*

- Zero- (or constant) sum game: my gain is your loss!
- **Mechanism design:** ("inverse GT")
  - How to set up **rules of game** so that **selfish individual behaviour** will lead to **social welfare**?
- Selfish agents, but rational (cutter knows that chooser will pick largest piece)
- Equilibrium Solution: minimax and maximin
  - Cutter: mitigating the worst result that chooser can enforce;
  - Chooser: maximizing his pay-off
  - Satisfactory equilibrium, notwithstanding selfish behaviour

# Prisoners' dilemma (RAND, 1950s)

Two suspects in a crime are put into separate cells. The police officer tells them: *Currently you're charged with trespassing which carries a jail sentence of one month. I know you were planning a robbery though, but cannot prove it – I need your testimony. If you confess, I will drop the charges against you, but your partner will be charged to the fullest extent of the law: 12 months in jail. I'm offering the same deal to him. If you both confess, your individual testimony is less valuable, and you will get 8 months each.*

| | quiet | confess |
|---|---|---|
| quiet | $-1, -1$ | $-12, 0$ |
| confess | $0, -12$ | $-8, -8$ |

What should the suspects do?

General model for all sorts of arms races

# Standard Parametrisation of Prisoner's Dilemma

- **Actions:** C = Cooperate (with each other), D = Defect
- **Pay-offs:**
  - If both cooperate, they both get reward $R = 1$
  - If both defect, they both get punishment $P = 0$
  - Full pay-off table:

| | C | D |
|---|---|---|
| C | 1, 1 | S, T |
| D | T, S | 0, 0 |

with $T > 1$ Temptation pay-off, $S < 0$ Sucker pay-off

- **Rational decision** Both agents play D (defect)

# Social Dilemmas

- PD is an example of a social dilemma
- **Social Dilemma**
  - Situations in which individual goals clash with the collective ones.
  - *More precisely:* Actions taken independently by individuals in pursuit of their own private objectives result in an inferior outcome compared to what could have been achieved if people had acted together (cooperation).
- **Central enigma:** why and under what circumsances do we get emergence of cooperation among selfish agents?

# Donation game: Model for Cooperation

Model for studying emergence of cooperation among selfish agents

- Donor (cost $c$) $\longrightarrow$ Recipient (benefit $b$)
- Assume $0 < c < b$ (otherwise not interesting)
- Pay-off matrix (C = cooperate, D = Defect)

| | Coop | Defect |
|---|---|---|
| Coop | $b-c, b-c$ | $-c, b$ |
| Defect | $b, -c$ | $0, 0$ |

- Version of Prisoner's dilemma: Defection is rational choice!

# Split or Steal: PD with cheap talk

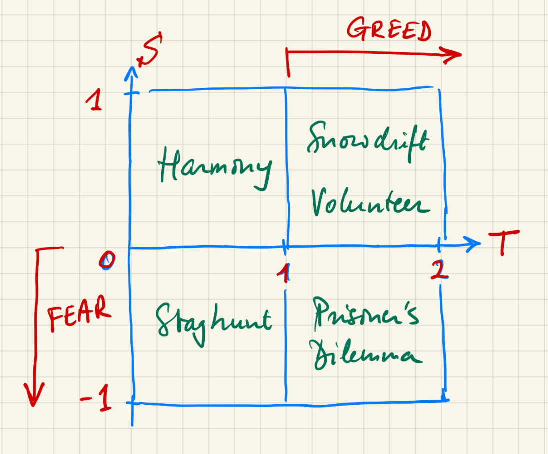
- Two contestants compete for a money prize;
- They each have two choices/decisions: split or steal
  - If they both split: each gets half of the money;
  - If they both steal: both go home empty-handed;
  - If one splits and one steals, the "stealer" gets everything;
- Cheap talk:
  - Classic prisoner's dilemma: the two prisoners can't communicate;
  - This version (split or steal): the two contestants are allowed to discuss with each other how they should pick.
  - But talk is cheap: no need to follow up on their promises!

# Split or Steal: Nick and Ibrahim

- Meet Ibrahim and Nick, who must decide how to split £13,600.
- During the discussion Nick tells Ibrahim:
  1. he's **definitely** going to steal, ...
  2. but he promises that if Ibrahim chooses split then he will give him half of the money after the show.
- What will happen ...? Link to BBC clip

# 2-parameter family of social dilemmas

| | C | D |
|---|---|---|
| C | 1, 1 | S, T |
| D | T, S | 0, 0 |



# Stag Hunt aka Common Interest Game

- Two hunters know that a stag follows a certain path.
- If two hunters cooperate to kill the stag there's plenty to eat.
- The hunters hide and wait for a long time, alas with no sign of the stag. However, a hare is spotted by all hunters.
- If a hunter shoots the hare, he will eat, but the stag will be alarmed and flee, and the other hunter will go hungry.
- If both hunters kill the hare, they share the little there is.
- **Dilemma:** Foregoing smaller reward for bigger one is risky!
- $(S = 0) < (P = 2) < (T = 4) < (R = 10)$ or $S < 0 < T < 1$

| | Stag | Hare |
|---|---|---|
| Stag | 10, 10 | 0, 4 |
| Hare | 4, 0 | 2, 2 |

| | C | D |
|---|---|---|
| C | 1, 1 | S, T |
| D | T, S | 0, 0 |

# Snowdrift game aka Volunteer's dilemma aka Chicken

- Two drivers are blocked by snow drift on the road,
- Both are reluctant to get out of the comfort of their car, to clear the road. They both hope the other driver will oblige.
- If both shovel, the discomfort for each is halved.
- **Dilemma:** Volunteering leads to a benefit for the whole community, but free-riding is tempting!
- $(P = 0) < (S = 3) < (R = 5) < (T = 10)$ or $0 < S < 1 < T$

| | Vol | FR |
|---|---|---|
| Volunteer | 5, 5 | 3, 10 |
| Free-ride | 10, 3 | 0, 0 |

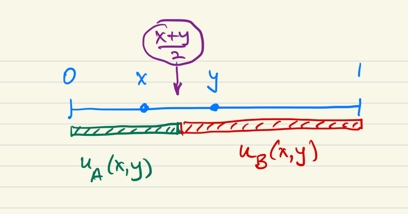
| | C | D |
|---|---|---|
| C | 1, 1 | S, T |
| D | T, S | 0, 0 |

# Ice cream time!

# Ice cream time! (Hotelling's game)

- **Two players** but **continuous (infinite) action space:**
  - each player can choose any position between 0 and 1.
  - Player A chooses $x$, player B chooses $y$ ($0 \leq x < y \leq 1$);
- **Utility**

$$
\begin{aligned}
u_A(x,y) &= x + \frac{y-x}{2} = \frac{x+y}{2} \\
u_B(x,y) &= 1 - y + \frac{y-x}{2} = 1 - \frac{x+y}{2}
\end{aligned}
$$




- Both simultaneous and sequential version (same outcome);

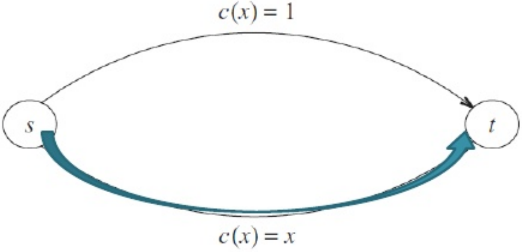
# Selfish routing in congestion games

**Congestion games:**

- **Context:** routing in network
- Single shot, $n$-player game
- Player chooses some resource (route) from **set of resources**;
- **Congestion:** Cost of resource depends on number of agents selecting this resource;

# Selfish routing in congestion games

- Pigou's example
  - Selfish players
  - Each player wants to minimize cost



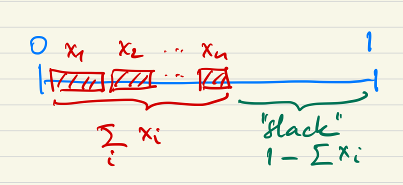

- Cost at equilibrium (1) exceeds optimal cost $(3/4 = 1/2 \cdot 1 + 1/2 \cdot 1/2)$
- **Price of anarchy:** $4/3$

# Tragedy of the Commons

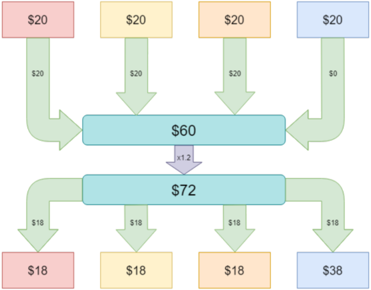
- $n$ players sharing some common resource (of total size 1)
  - E.g., village green, bandwidth in network, etc.
- Each player $i$ would like to have a big share ($0 \leq x_i \leq 1$)!
- However, individual utility also depends on what others do:

$$u_i(x_i, x_{-i}) = \begin{cases} x_i\,(1 - \sum_{j=1}^{n} x_j) & \text{if } \sum_j x_j < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Is there a optimal strategy?



# Public Goods Game (PGG)




- $n$ agents decide to contribute amount $0 \leq x_i \leq K$
- Free-riders ($x_i = 0$): no contribution, but share in the pay-off;
- Resulting pay-off for each agent after redistribution:

$$u_i = \frac{1}{n}\left(f \sum_{j=1}^{n} x_j\right) - x_i \qquad \text{where synergy factor } f > 1$$

# Public Goods Game

- Rationality dictates to not contribute (unless $f > n$);
  - Free-rider problem;
  - No gains for community!
- Observations from controlled experiments:
  - One-shot games: usually up to 50% of endowment;
  - Iterated games: contributions decrease but, instead of vanishing, there is always a residual contribution of 10–20%
  - Possibility of punishment of free-riders increases contributions; Comes at a cost (to the punishers)!

# Ultimatum game with impatient players

- Kids get a box of icecream to share among themselves;
- They can have all of it as long as they agree on the division;
- If they fail to agree, their parents take away all the ice cream (conflict deal);
- It's a hot day and the icecream is melting! The longer they argue, the less icecream there's left!

# Traveller's dilemma (Kaushik Basu, 1994)

- Airline severely damages identical antiques purchased by two different travelers.
- Management is willing to compensate them for the loss of the antiques, but since they have no idea about their value, they tell the two travelers to separately write down their estimate of the value as any number between \$2 and \$100 without conferring with one another.
- If both travelers write down the same number, they will be reimbursed that amount.
- If they write different numbers, management will pay both of them the lower figure, the person with the lower number will get a \$2 bonus for honesty, while the one who wrote the higher number will get a \$2 penalty.

# Traveller's dilemma

- Illustrates the paradox of rationality
- The rational strategy for both players is to choose the lowest possible payoff which results in both players receiving lower payoffs than they could achieve by following an irrational strategy.
- Experimental studies: people consistently chose higher payoffs and achieved better results than the rational strategy predicted by game theory.

# Penalty kicks

- **Kicker:** kick to left or right corner
- **Goalie:** jump to (defend) left or right corner
- Goalie wants to coordinate, kicker wants to anti-coordinate;
- Colorcode in table below: payoffs for kicker and goalie!

- Penalty-kick game
  - Soccer penalties have been studied extensively

| | Defend left | Defend right |
|---|---|---|
| Left | 0.58, -0.58 | 0.95, -0.95 |
| Right | 0.93, -0.93 | 0.70, -0.70 |

# Overview of Topics in Game Theory Course

- Non-cooperative games
  - Matrix games (2 players, finite action sets)
  - Sequential games, e.g. bargaining
- Cooperative (coalitional) games;
  - Shapley value;
- Mechanism design (inverse game theory)
  - Vickrey auction