

Introduction to Game Theory 5

Iterated Games

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Reading

- **Recommended**

- Shoham and Leyton-Brown: Chapter 5, sections 5.1

- **Optional**

- William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
- N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
- A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

Overview

Bargaining as example of sequential game

Repeated Games

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Bargaining as example of sequential game

Repeated Games

The Ultimatum Game

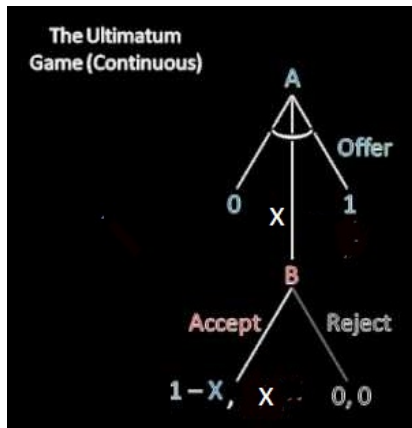
Ultimatum Game (UG): baseline (simplest non-trivial) model for **bargaining**: *take-it-or-leave-it!*

Assumptions

- **Surplus** can be divided continuously ($0 \leq x \leq 1$)
- **Two agents:**
 - A **proposes** split x versus $1 - x$, (**proposal power**)
 - B accepts or rejects;
- **No deal (conflict deal)** is considered **worst outcome**;
- Both agents aim to **maximize their utility**;

One round ultimatum game

- A makes single offer, B either accepts or rejects!

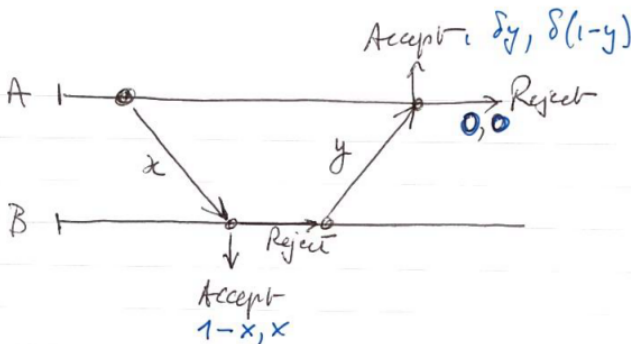


Two rounds ultimatum game with impatient players

Power of counter-offer:

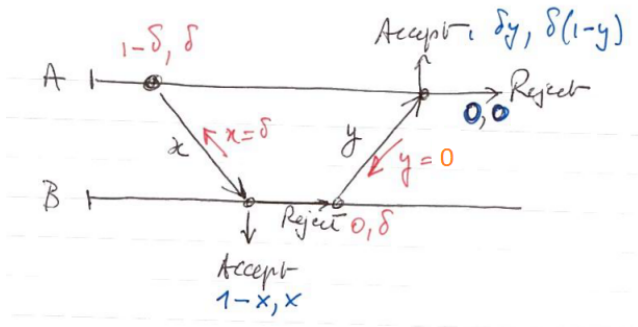
- A makes offer;
- B either accepts, or makes counter-offer.
- However, in each round the total is **reduced by factor $\delta < 1$**
(the icecream is melting!)

A makes offer, but B can make counter-offer!



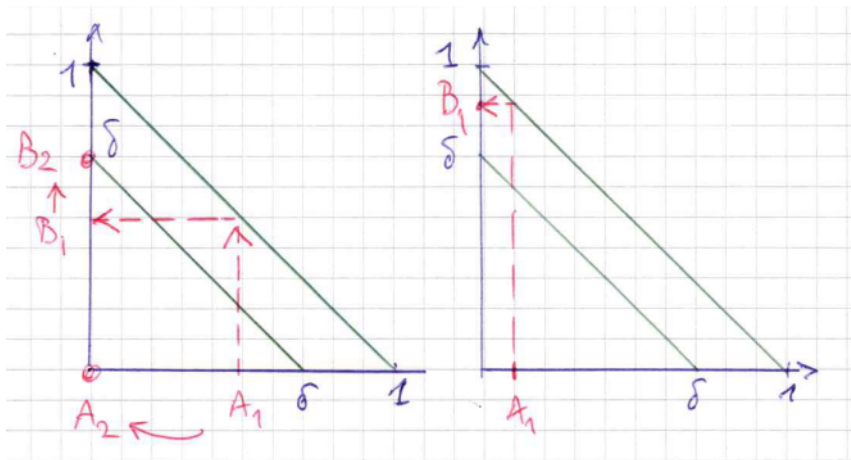
Two rounds ultimatum game: BI solution

- A makes offer, but B can make counter-offer!
- Use **backward induction** to find optimal solution.

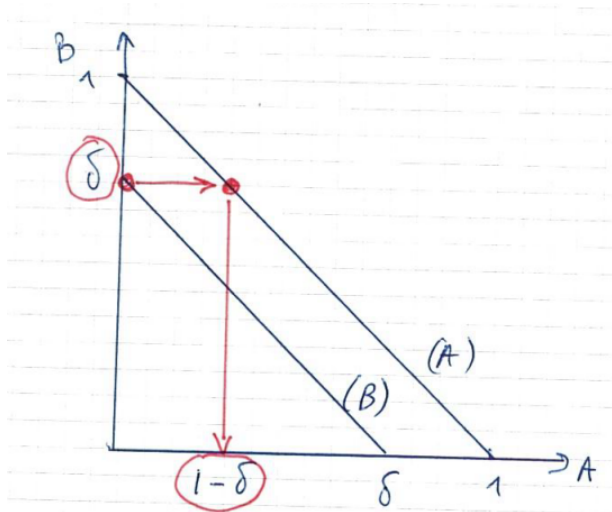


Conclusion: A offers split $(1 - \delta, \delta)$ which B accepts.

Alternative interpretation

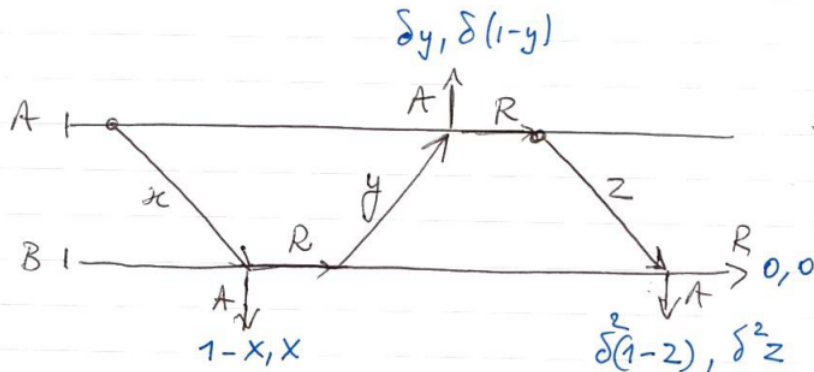


Two rounds ultimatum game: "Pareto Shuffle"

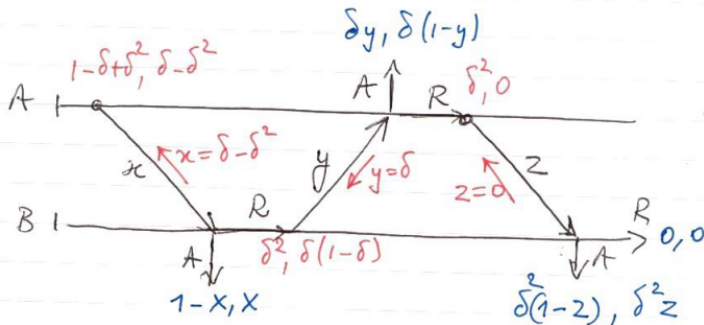


Three rounds ultimatum game

Room for two counter-offers!

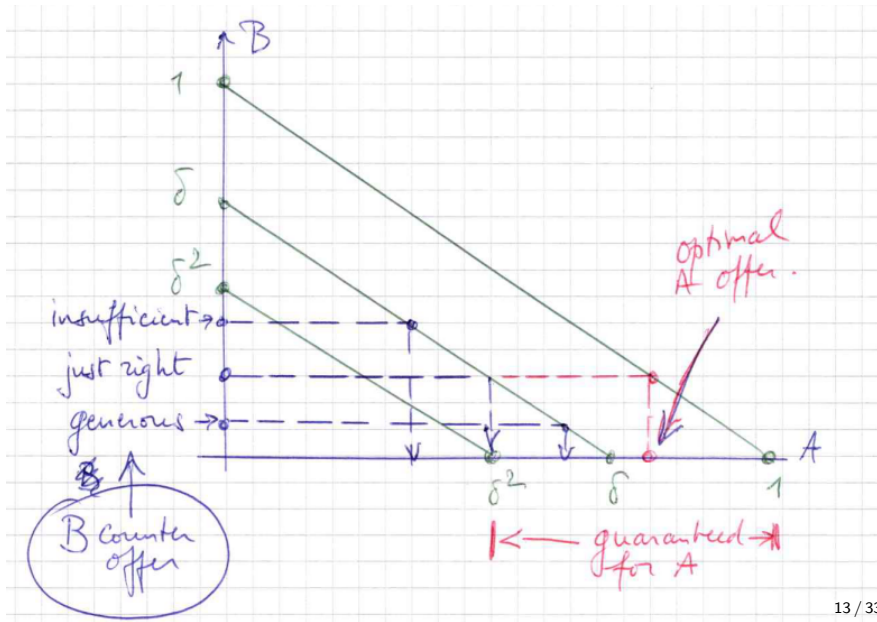


Three rounds ultimatum game: Backwards induction

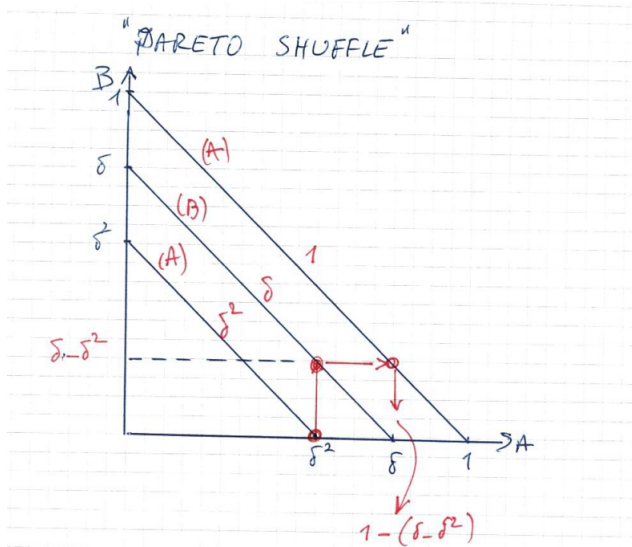


Conclusion: A proposes split $(1 - \delta + \delta^2, \delta - \delta^2)$ which B accepts.

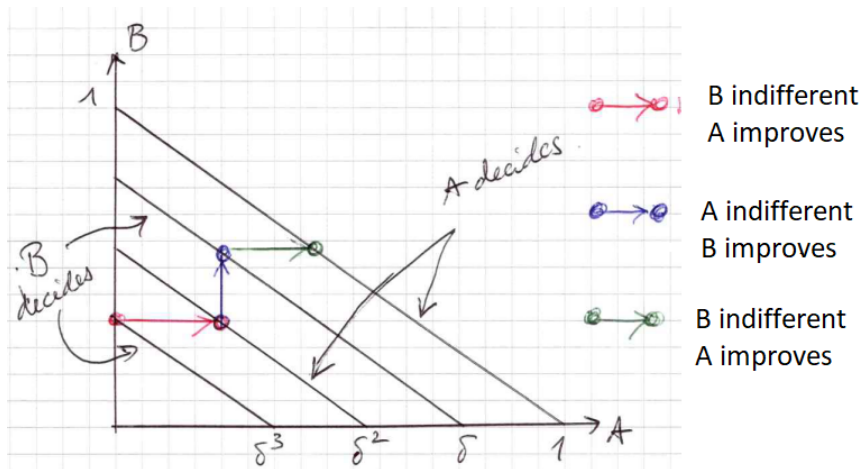
Pareto shuffle forward



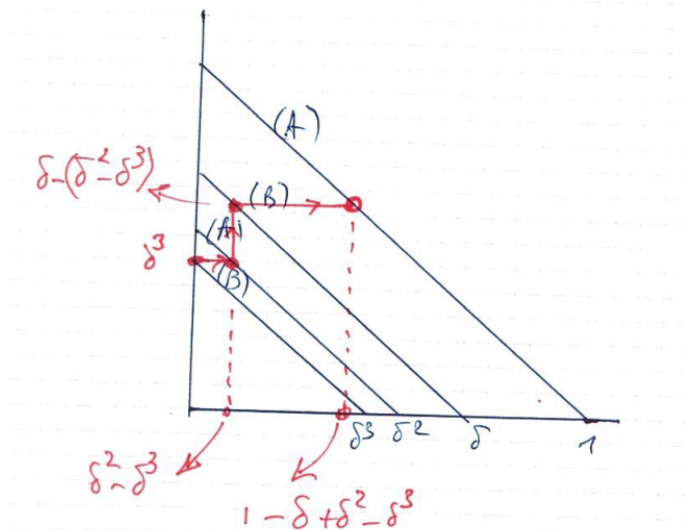
Three rounds ultimatum game: "Pareto Shuffle"



Four rounds ultimatum game: backward induction



Four rounds ultimatum game: "Pareto Shuffle"

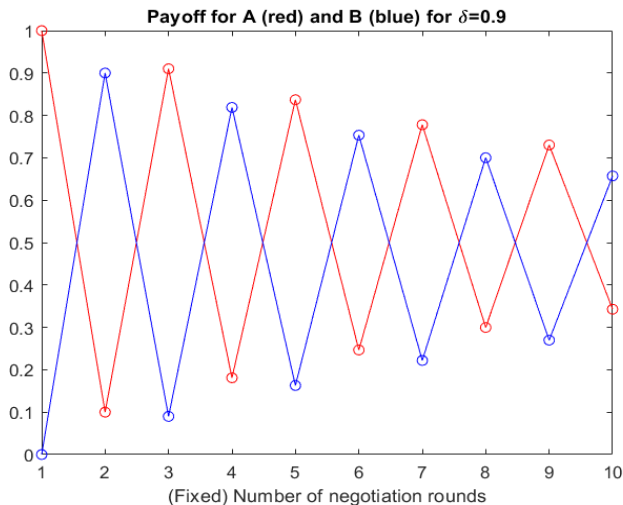


Alternating offers bargaining

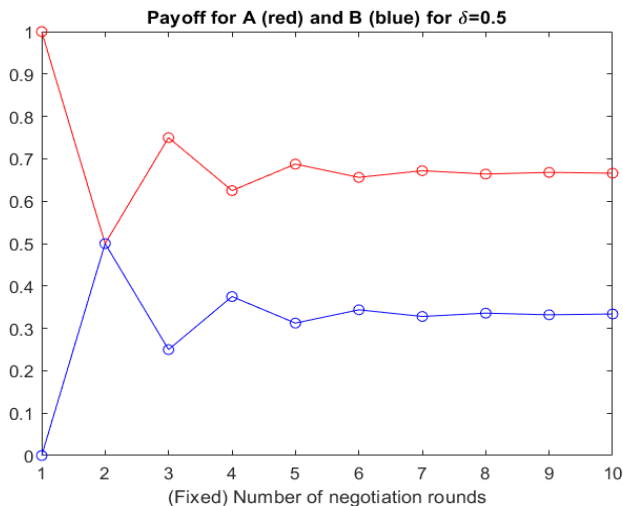
- Generalizing to negotiation with n rounds;
- When number of negotiation rounds is fixed at n , the equilibrium split is given in table below:

payoff for n rounds	$n = 1$	$n = 2$	$n = 3$	$n = 4$
for A	1	$1 - \delta$	$1 - \delta + \delta^2$	$1 - \delta + \delta^2 - \delta^3$
for B	0	δ	$\delta - \delta^2$	$\delta - \delta^2 + \delta^3$

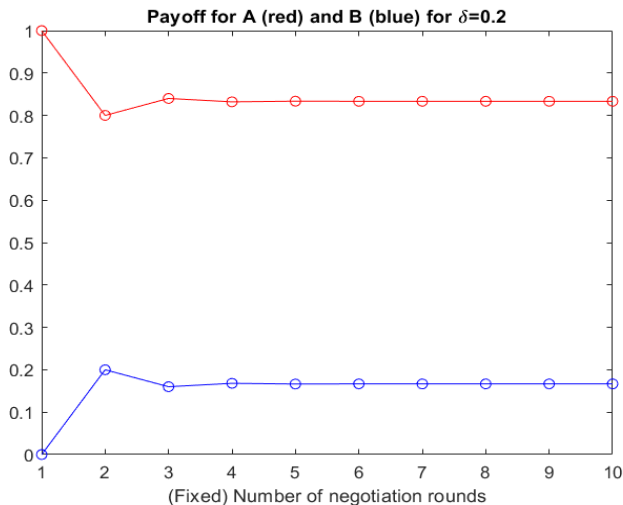
Alternating offers bargaining (mildly impatient, $\delta = 0.9$)



Alternating offers bargaining (seriously impatient, $\delta = 0.5$)



Alternating offers bargaining (very impatient, $\delta = 0.2$)



Mathematical aside

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad (\text{for } |x| < 1)$$

$$\sum_{k=1}^{\infty} x^k = x + x^2 + x^3 + x^4 \dots = \frac{x}{1-x} \quad (\text{for } |x| < 1)$$

$$\begin{aligned} 1 + x + x^2 + \dots + x^n &= \frac{1}{1-x} - x^{n+1} \frac{1}{1-x} \\ &= \frac{1 - x^{n+1}}{1-x} \end{aligned}$$

General conclusions

- Limit behaviour for pay-offs:

$$A(n) = 1 - \delta + \delta^2 - \dots \pm \delta^{n-1} = \frac{1 - (-\delta)^n}{1 - (-\delta)}$$

$$\implies \lim_{n \rightarrow \infty} A(n) = \frac{1}{1 + \delta}$$

$$B(n) = 1 - A(n) \implies \lim_{n \rightarrow \infty} B(n) = \frac{\delta}{1 + \delta}$$

- First offer advantage:** $\lim A(n) \geq \lim B(n)$
- First offer advantage disappears for very patient negotiators:

$$\delta \longrightarrow 1 \implies A(n) \searrow 1/2, \quad B(n) \nearrow 1/2;$$

Rubinstein's Model: Infinite Horizon Bargaining

- 2 agents, infinite horizon (# rounds **not fixed in advance**);
- **Time is valuable:** discount factors for both agents (δ_1, δ_2);
- **Optimal split:**

$$u_1 = \frac{1 - \delta_2}{1 - \delta_1 \delta_2} \quad \text{and} \quad u_2 = \frac{\delta_2(1 - \delta_1)}{1 - \delta_1 \delta_2}$$

- **Tragedy of bargaining:**

The more time matters, the lower the share!

E.g. if $\delta_2 \approx 0$, then $u_2 \approx 0$, etc.

- Notice **first mover's advantage** for $\delta_1 = \delta_2 = \delta$:

$$u_1 = \frac{1 - \delta}{1 - \delta^2} = \frac{1}{1 + \delta} \quad \text{and} \quad u_2 = \frac{\delta(1 - \delta)}{1 - \delta^2} = \frac{\delta}{1 + \delta}$$

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Bargaining as example of sequential game

Repeated Games

Repeated Games

- Repeat the **same** one-shot (stage) game (special case of sequential game);
- At each stage, information about preceding games is known;
- **Finite number** n (known!) of repetitions;
 - Maximize total reward:

$$R_n = \sum_{t=1}^n r_t$$

- **Infinite (unlimited) number** of repetitions
 - Maximize **discounted** total reward:

$$R = \sum_{t=1}^{\infty} \delta^{t-1} r_t \quad \text{discount factor } 0 < \delta < 1$$

- equivalently: ending after **random number** of repetitions

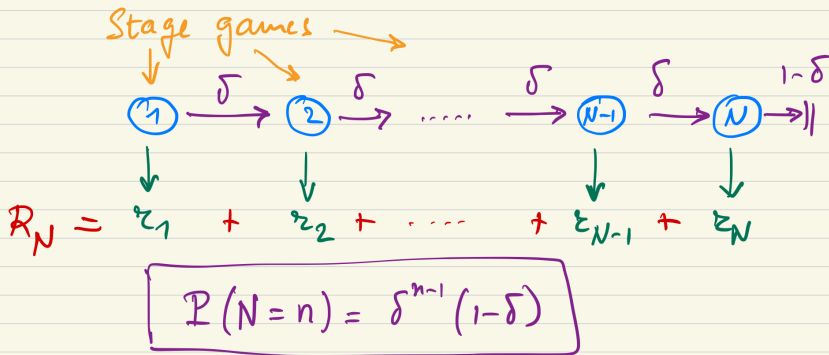
Mathematical aside

- Recall:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad (\text{for } |x| < 1)$$

$$\sum_{k=1}^{\infty} x^k = x + x^2 + x^3 + x^4 \dots = \frac{x}{1-x} \quad (\text{for } |x| < 1)$$

Interpretation of discount factor



Repeated games: interpretation of discount factor

- After every stage-game there is a
 - probability $1 - \delta$ that game will be ended;
 - probability δ that game will proceed to next round
- Reward R now becomes random variable:

$$R_N = \sum_{t=1}^N r_t \quad \text{where} \quad P(N = n) = \delta^{n-1}(1 - \delta)$$

(Assuming at least one stage game is played, hence $N \geq 1$).

Then:

$$E(R_N) := E\left(\sum_{t=1}^N r_t\right) = r_1 + \delta r_2 + \delta^2 r_3 + \dots = \sum_{t=1}^{\infty} \delta^{t-1} r_t.$$

Interpretation of discount factor

$$\begin{aligned}
 E(R_N) &= E\left(\sum_{t=1}^N z_t\right) \\
 &= \sum_{n=1}^{\infty} E\left(\sum_{t=1}^N z_t \mid N=n\right) P(N=n) \\
 &\quad \downarrow \\
 &= \sum_{n=1}^{\infty} \left(\sum_{t=1}^n z_t\right) \underbrace{P(N=n)}_{\delta^{n-1}(1-\delta)} \\
 &= (1-\delta) \sum_{n=1}^{\infty} \left(\sum_{t=1}^n z_t\right) \delta^{n-1}
 \end{aligned}$$

Interpretation of discount factor

$$\begin{aligned}
 E(R_N) &= (1-\delta) \sum_{n=1}^{\infty} \left(\sum_{t=1}^n r_t \right) \delta^{n-1} \\
 &\quad \downarrow \\
 &= r_1 + \delta r_1 + \delta r_2 \\
 &\quad + \delta^2 r_1 + \delta^2 r_2 + \delta r_3 \\
 &\quad \downarrow \dots \\
 &\quad \left(\sum_{k=0}^{\infty} \delta^k \right) r_1 + \delta \left(\sum_{k=0}^{\infty} \delta^k \right) r_2 + \dots \\
 &= (1-\delta) \left[\frac{1}{1-\delta} r_1 + \delta \left(\frac{1}{1-\delta} \right) r_2 + \dots \right] \\
 &= r_1 + \delta r_2 + \delta^2 r_3 + \dots = \sum_{t=1}^{\infty} \delta^{t-1} r_t. \quad \square
 \end{aligned}$$

Example: Repeated Prisoner's dilemma

- Pay-off matrix for **stage game**:

	$C(oop)$	$D(effect)$
C	3, 3	1, 4
D	4, 1	2, 2

- For **finite number** of repetitions:
 - Single Nash eq.:** Play D-D for each repetition;
 - Can be proved using **backward induction**
- For **infinite number** of repetitions:
 - More interesting strategies, including **cooperation**
 - Examples: Tit-for-Tat, Grim Trigger

Grim Trigger Strategy for Repeated Prisoner's Dilemma

- **Grim Trigger and Tit-for-Tat strategy**

Start by cooperating . . .

- **GT**: continue cooperating, until someone defects; from then onwards, always defect!
- **TfT**: from then onwards, copy last move of opponent.
- If both parties play GT, is it rational to defect?

Consider player 1:

- Utility for continued **cooperation**:

$$u_1(C) = 3 + 3\delta + 3\delta^2 + \dots = \frac{3}{1 - \delta}$$

- Utility for **defection**:

$$u_1(D) = 4 + 2\delta + 2\delta^2 + \dots = 4 + 2\frac{\delta}{1 - \delta}$$

- Continued **cooperation is rational** if $u_1(C) > u_1(D)$.

Grim Trigger Strategy for Repeated Prisoner's Dilemma

- Continued **cooperation is rational** if $u_1(C) > u_1(D)$:

$$u_1(C) > u_1(D) \quad \Longleftrightarrow \quad \frac{3}{1-\delta} = 4 + 2\frac{\delta}{1-\delta}$$

$$\Longleftrightarrow \quad \delta > 1/2.$$

- Interpretation:** If the players are **sufficiently patient** (i.e. future rewards are sufficiently valuable) then it's **rational to cooperate**.