Introduction to Game Theory 3

Solution Concepts: Regret Minimisation, Minimax and Maximin, and Nash Equilibrium

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Game Theory Part 1: Outline

1. Game Theory: Science of Strategic Thinking
2. Examples of interesting games
3. Formalising games
4. Solution concept 1: Weak optimality
5. **Solution concept 2: Strategies with (weak) guarantees**
    - Regret minimisation
    - Maximin
    - Minimax
6. **Solution concept 3: Nash equilibrium**

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Strategies with (weak) guarantees

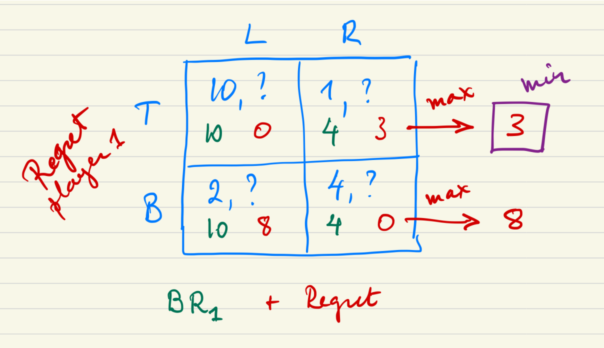
- Focus on outcomes that can be guaranteed by your own actions
- **Reasons:**
  - Opponent-agnostic: Opponent's utilities might not be known
  - Following a different/hidden agenda:
    e.g. threatening or punishing opponent;
  - ...others??
- **Strategies**
  1. Regret minimisation
  2. Safety (maximin) and Punishment (minimax) strategies

Regret minimisation

- What is my maximal possible regret if I take this action?
- Regret: diff. btw pay-offs best response and current action:

$$R_1(s_1, s_2) = u_1(BR_1(s_2)) - u_1(s_1, s_2)$$

- Take action to minimise max regret



Regret minimisation

- **Regret** (for agent i) is the difference between the actual and maximal pay-off for a given action profile (s_i, s_j)

$$R_i(s_i, s_j) = u_i(BR_i(s_j)) - u_i(s_i, s_j) = \max_{s_i'} u_i(s_i', s_j) - u_i(s_i, s_j)$$

- For each action s_i , there is maximum regret depending on s_j:

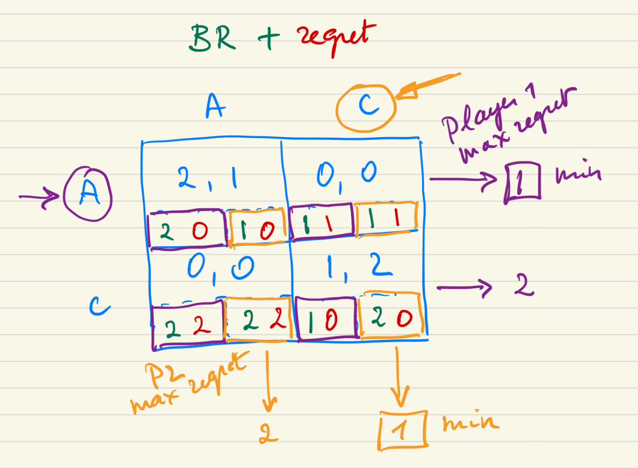
$$R_i^{max}(s_i) = \max_{s_j} R_i(s_i, s_j)$$

- **Regret minimisation** (minimax regret): agent i picks action s_i that mimimises max regret:

$$s_i^{rm} := \arg\min_{s_i} R_i^{max}(s_i) = \arg\min_{s_i} \max_{s_j} R_i(s_i, s_j).$$

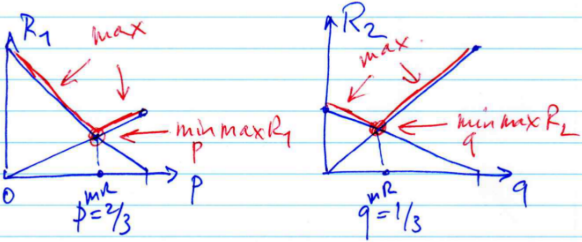
- Actual solution concepts: allows an agent to choose a strategy with specific guarantees/properties;

Example of regret minimisation for BoS



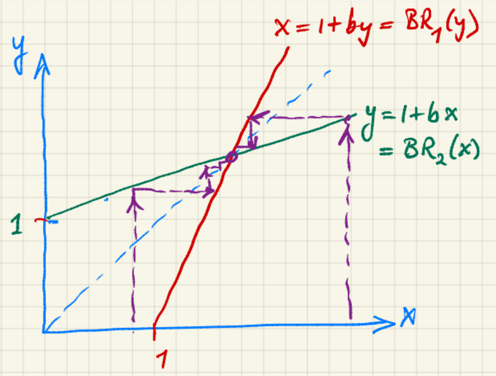
BoS: Regret minimisation for mixed strategies

	q	$1-q$	
p	$0, 0$	$1, 1$	$1-q$
$1-p$	$2, 2$	$0, 0$	$2q$
	$2(1-p)$	p	



Regret minimisation for partnership game

- Recall:



Regret minimisation for partnership game

Pay-off for player 1:

$$u(x,y) = 2(x+y+bxy) - x^2$$

$0 \leq b < 1 \qquad 0 \leq x,y \leq 4$

Recall: $\bar{x} = BR_1(y) = 1 + by$

Regret for player 1 if he plays x, opp. y

$$R(x,y) = u(\bar{x},y) - u(x,y)$$
$$= 2(\bar{x} - x + b(\bar{x}-x)y) - (\bar{x}^2 - x^2)$$
$$= (\bar{x}-x)\left[2(1+by) - (\bar{x}+x)\right]$$

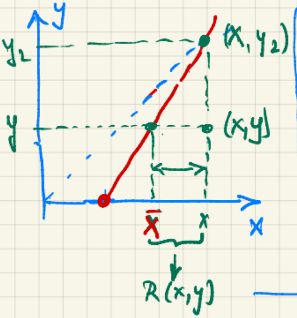
Regret minimisation for partnership game

$$R(x,y) = (\bar{x}-x)\left[2(1+by) - (\bar{x}+x)\right]$$

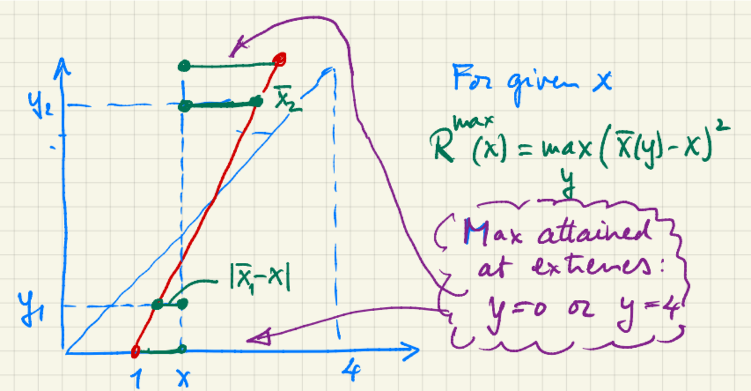
$$(\bar{x} = 1+by)$$

$$= (1+by-x)\left[2(1+by) - (1+by) - x\right]$$

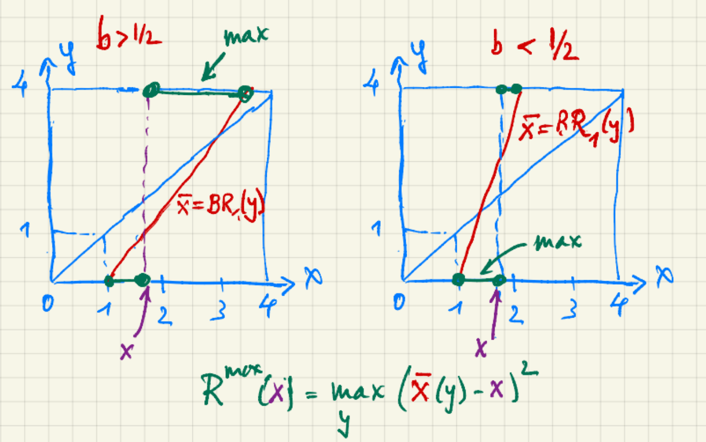
$$= (\underbrace{1+by}_{\bar{x}}-x)^2 = (\bar{x}-x)^2$$



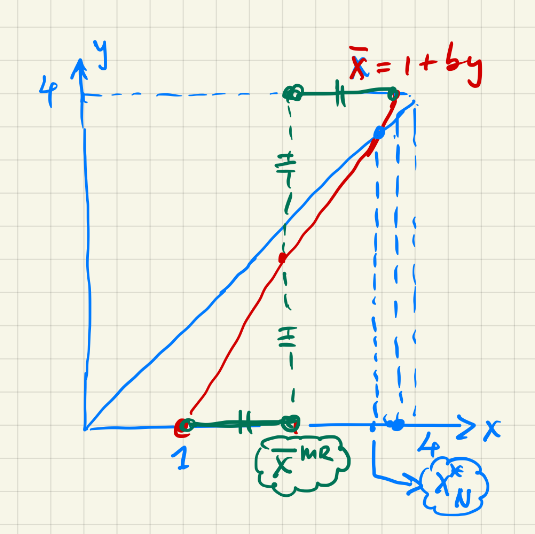
Regret minimisation for partnership game



Regret minimisation for partnership game



Regret minimisation for partnership game



Maximin Value and Safety Strategy

- Safety strategy: What is the best outcome i can secure for myself, no matter what the opponent does?
- No need to know the corresponding opponent's pay-offs!
- We consider player i's point of view: (i.e. maximise over s_i)

$$\max_{s_i} \dots u_i(s_i, s_j)$$

Maximin strategy (safety strategy): Algorithm

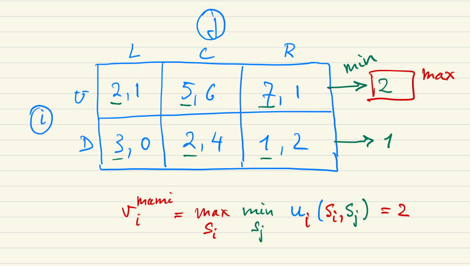
1. Ag_i computes for all his actions the worst possible outcome:

$$s_i \longrightarrow \min_{s_j} u_i(s_i, s_j)$$

2. Next, ag_i chooses action s_i to maximise his minimal pay-off:

$$v_i^{mami} := \max_{s_i} \min_{s_j} u_i(s_i, s_j)$$

Agent tries to maximise pay-off of worst possible outcome



Maximin Value and Strategy

Context: 2-player, general sum game

- Maximin **value** (or security level for WORST CASE) is the guaranteed minimal pay-off for agent i playing strategies in S_i:

$$v_i^{mami} := \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j)$$

- Maximin **strategy** (or safety strategy) for agent i maximizes his worst case pay-off (and therefore yields at least v^{mami} as utility):

$$s_i^{mami} = \arg \underbrace{\max_{s_i} \min_{s_j} u_i(s_i, s_j)}_{\text{security level}}$$

Maximin (safety) value and strategy

- **Why play maximin strategy?**
  - Pay-off guarantee! Highest pay-off agent i can guarantee for himself irrespective of the actions taken by other agent(s).
  - Worst case analysis: assume that opponent is malicious!

Example: both agents play maximin-strategy

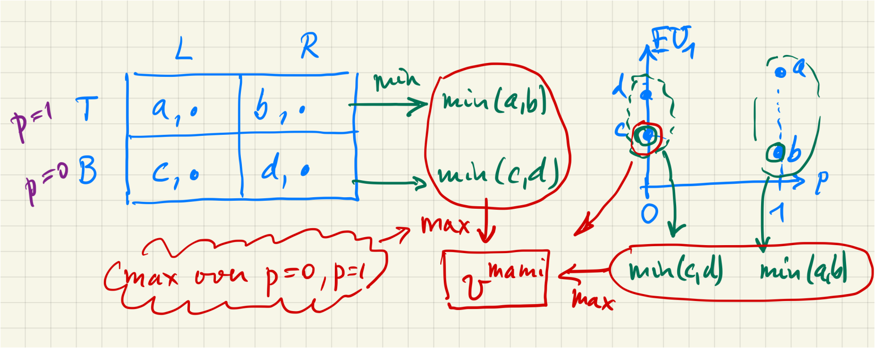
Both players play maximin.

1 \ 2	L	CL	CR	R	min	
U	3,1	4,6	8,1	5,2	$\boxed{3}$	max.
M	4,2	7,4	1,5	4,5	1	
D	6,2	1,3	7,0	0,4	0	
min	1	$\boxed{3}$	0	2		
		max				

$$\left.\begin{array}{ll} v_1^{maxi} = 3, & s_1^{maxi} = U \\ v_2^{maxi} = 3, & s_2^{maxi} = CL \end{array}\right\} \rightarrow \boxed{\begin{array}{l} u_1(U,CL) = 4 \\ u_2(U,CL) = 6 \end{array}}$$

From Pure to Mixed safety strategies

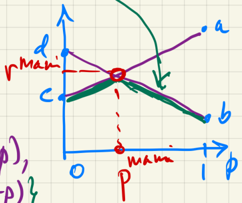
- safety value for pure safety strategy



From Pure to Mixed safety strategies

		L	R
$p=1$	T	$a, \bullet$	$b, \bullet$
$p=0$	B	$c, \bullet$	$d, \bullet$
	$pT+(1-p)B$	$ap+c(1-p)$	$bp+d(1-p)$

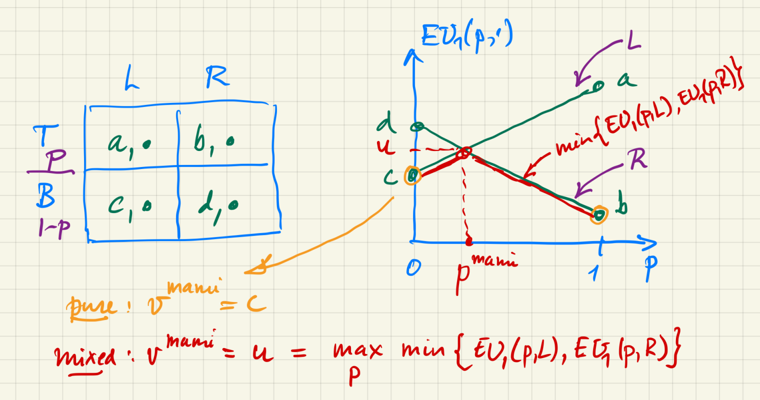
$$\xrightarrow{\min} \min\{ap+c(1-p),\ bp+d(1-p)\}$$



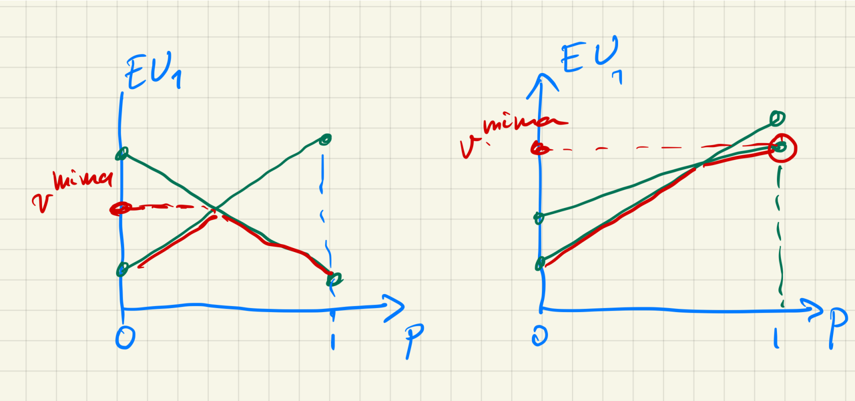
$$\boxed{v^{mami} = \max_p \min\{ap+c(1-p),\ bp+d(1-p)\}}$$

Mixed safety strategies

- If $EU(p, \cdot)$ graphs intersect, it might be better to look at mixed strategies



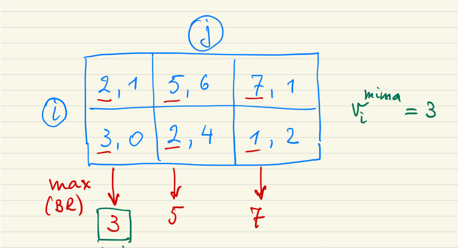
Sometimes pure safety strategy is best



Punishment (or minimax) Strategy (of player j) yields Minimax Value (for player i)

1. Player i computes best response utility for each of j's strategies s_j: $BR_i(s_j) = \max_{s_i} u_i(s_i, s_j)$;
2. Then player j (who wants to punish i) picks action to minimise player i best pay-off. This punishment strategy of **player j** results in the minimax value for **player i**:

$$v_i{}^{mima} := \min_{s_j} \max_{s_i} u_i(s_i, s_j) = \min_{s_j} u_i(BR_i(s_j))$$



Algo: Minimax Value and Punishment Strategy

- **Minimax value** for agent i:
  - Given the strategy s_j of his opponent, agent i will play its **best response**, resulting in a pay-off:

$$\max_{s_i} u_i(s_i, s_j)$$

  - The opponent is aware of this and wants to *"punish"* i by *minimizing* this pay-off, yielding the minimax value for player i:

$$v_i^{mima} := \min_{s_j} \max_{s_i} u_i(s_i, s_j)$$

- The corresponding minimising strategy s_j^{mima} is called the minimax strategy for player j.
- If j plays his minimax strategy s_j^{mima}, then i cannot do better than v_i^{mima} (even if i plays best response $BR_i(s_j^{mima})$.

Comparing minimax and maximin value for player i

- $\forall s_i, s_j$:

$$\underbrace{\min_{s_j} u_i(s_i, s_j)}_{\phi(s_i)} \leq u_i(s_i, s_j) \leq \underbrace{\max_{s_i} u_i(s_i, s_j)}_{\psi(s_j)}$$

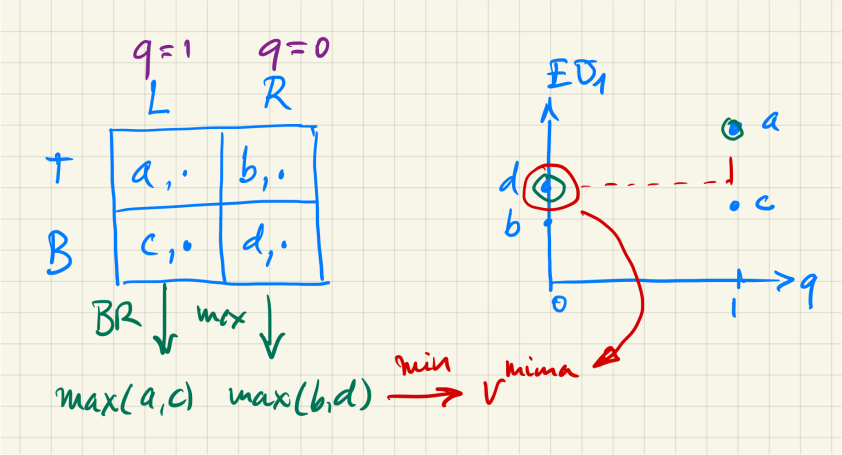
- Since the above inequality holds for all s_i and s_j, it follows:

$$\max_{s_i} \phi(s_i) \leq \min_{s_j} \psi(s_j)$$

- Hence:

$$v_i^{mami} = \max_{s_i} \min_{s_j} u_i(s_i, s_j) \quad \leq \quad \min_{s_j} \max_{s_i} u_i(s_i, s_j) = v_i^{mima}$$

Punishment strategy: Pure

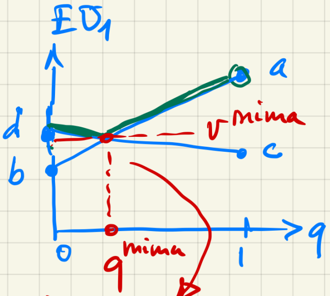


Punishment strategy: Mixed

	L	R	$qL+(1-q)R$
T	$a,\cdot$	$b,\cdot$	$aq+b(1-q)$
B	$c,\cdot$	$d,\cdot$	$cq+d(1-q)$

$$\downarrow$$

$$\max\{aq+b(1-q),\ cq+d(1-q)\} \xrightarrow{\min_q} v_M^{mima}$$

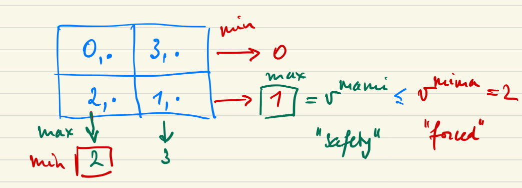


Minimax and Maximin Value for Player i

- In general:

$$v_i^{mami} = \max_{s_i} \min_{s_j} u_i(s_i, s_j) \quad \leq \quad \min_{s_j} \max_{s_i} u_i(s_i, s_j) = v_i^{mima}$$

- Your guaranteed pay-off is a is lower bound for the worst your opponent can force onto you!
- The worst your opponent can force onto you is an upper bound on your guaranteed pay-off.



Recap Safety and Punishment Strategy

- **Player i's maximin strategy is safety strategy**
  - player i concerned about his own safety
    - Malicious or adversarial opponent
    - Multi-agent setting
  - strategy yields highest guaranteed outcome for player i
  - Viable solution algorithm.

- **Player i's minimax strategy is punishment strategy**
  - i's strategy is directed against player j
  - player i tries to minimize best (i.e. maximum) pay-off for j
  - Useful as threat (e.g. in repeated games);
  - i's maximin **strategy** gives rise to j's maximin **value**

Which is better? Regret minimisation or Safety?

- Neither, not comparable!
- Comparison for $(0 \leq T \leq 2, -1 \leq S \leq 1)$ parametrisation for social dilemmas.

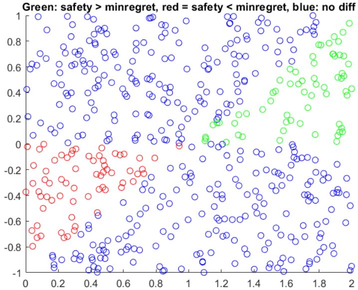


Figure: green: safety > min regret, red = min regret > safety, blue: no difference

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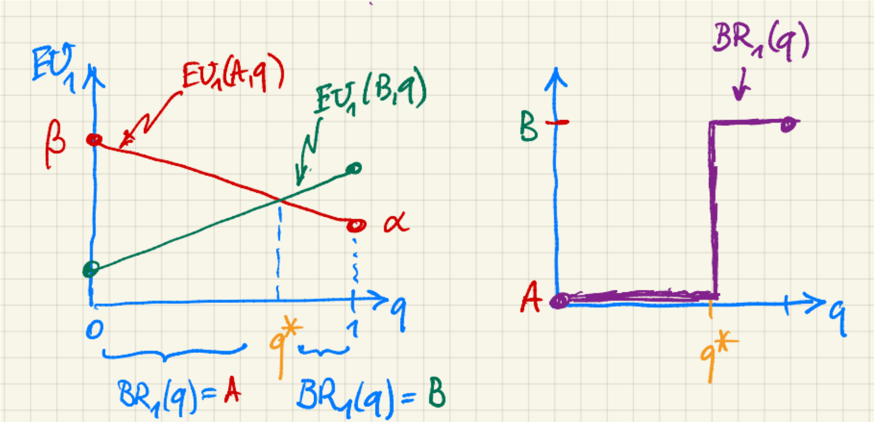
Nash equilibrium: Context

- Von Neumann's minimax theorem established game theory as a discipline;
- Nash equilibrium: Extension of von Neumann's minimax theorem
  1. From two person to n person game;
  2. from zero sum to general utilities
- For more info on the minimax theorem: See addendum;

Nash Equilibrium: Intro

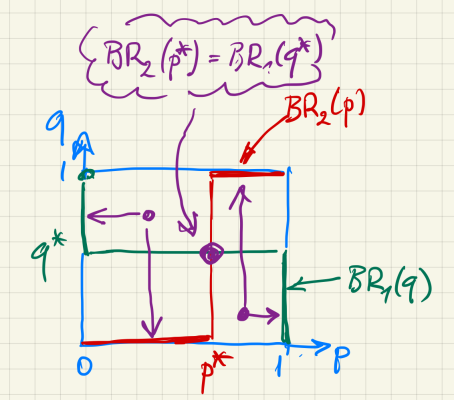
- Optimal choice no longer applies, as your outcome depends on opponent's actions.
  - Opposing "forces" at play;
- From optimum to equilibrium:
  - Opposing "forces" balance out;
- **Recall:** Best response (BR) dynamics . . .
  - sometimes converged to equilibrium position;
  - sometimes gave rise to cycles . . . Nash to the rescue!

Recap: Best response to mixed strategy



Nash equilibrium: intro

- Consider general mixed strategies in 2p-game, with two actions each;
- Imagine one player is allowed to change his mind!



Nash equilibrium (1)

- A **Nash equilibrium** (NE, 1950) is a solution concept based on *conditions* instead of an *algorithm*.
- **Mutual best response:** NE is joint strategy profile s^* such that **for each agent** i the strategy s_i^* is a **best response** to s_{-i}^*;
- **Formally:** A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a **strict NE** if:

$$\forall \text{agents}\, i,\ \forall s_i' \neq s_i^* : u(s_i^*, s_{-i}^*) > u(s_i', s_{-i}^*).$$

- **Strict** ($>$) versus **weak** ($\geq$) NE
- **No Regret/Self-enforcing:** a (strict) NE is a stable strategy profile for which no agent has an incentive to **unilaterally deviate**;

Nash equilibrium: Computation of NE (pure strategy)

Find mutual best responses:

- **Battle of the Sexes** (two pure strategy NEs)

	action	comedy
action	$\underline{2}, \underline{1}$	$0, 0$
comedy	$0, 0$	$\underline{1}, \underline{2}$

- **Prisoner's dilemma** (single NE, not Pareto-optimal!)

	hush	confess
hush	$-1, -1$	$-12, \underline{0}$
confess	$\underline{0}, -12$	$\underline{-8}, \underline{-8}$

Nash's Theorem

Existence of Nash Equilibrium (Nash, 1950)

A **finite strategic game** (i.e. finite number of players and actions) always has **at least one Nash equilibrium** (allowing mixed strategies).

- A **pure** Nash equilibrium can be strict or weak;
- A **mixed** Nash equilibrium is necessarily weak;

Nash equilibrium: Nash's Theorem

- A **finite strategic game** is a game with a finite number of agents and a finite number of actions;
- A game may have **zero**, **one**, or **more pure-strategy NE**.
- If there's a **single** NE: natural solution concept, but might be (Pareto) sub-optimal!
- If there are **multiple NEs**: there might be no compelling reasons to pick a particular one; but . . .
  - Utility dominant NE, Schelling's focal points
- Since humans are not always rational, Nash equilibria might not agree with experiments or observations..

Note on **multiple** Nash Equilibria

- A unique best alternative is exceptional.
- In general, non-cooperative game theory is plagued by a multiplicity of equilibria.
- Hence, prescriptions of how to act without any coordination or cooperation are in general impossible.
- Non-cooperative game theory tells us what to exclude from choice.
- Since in non-coop GT, there are no binding contracts, a player's announcements and promises in a pre-play phase are credible only, if they are totally in line with his best interests.

Computation of NE

- **Pure NE** **for each agent** i the strategy s_i^* is a **best response** to s_{-i}^* (mutual best response);
  - Matrix games (discrete state/action) space;
  - Continuous action space
- **Mixed NE**: make opponent indifferent (matrix games only);

Nash equilibrium: Computation of **mixed** NE

Matching pennies (zero-sum game)

- **No pure strategy NE . . .**

	heads	tails
heads	$\underline{1}, -1$	$-1, \underline{1}$
tails	$-1, \underline{1}$	$\underline{1}, -1$

- . . . hence, at least one **mixed strategy NE!**
- **Intuitively** this is obvious; play each action with prob $= 1/2$:

$$s_1 = s_2 = \{(H, 1/2), (T, 1/2)\}$$

- **Expected utility (pay-off)**:

$$u_1(s_1, s_2) = \frac{1}{4}u_1(H, H) + \frac{1}{4}u_1(T, H) + \frac{1}{4}u_1(H, T) + \frac{1}{4}u_1(T, T) = 0.$$

Matching pennies: Computation of mixed NE

$$
\begin{array}{cc|c|c|}
 & & q & 1-q \\
 & & H & T \\
\hline
p & H & 1,-1 & -1,1 \\
\hline
1-p & T & -1,1 & 1,-1 \\
\hline
\end{array}
$$

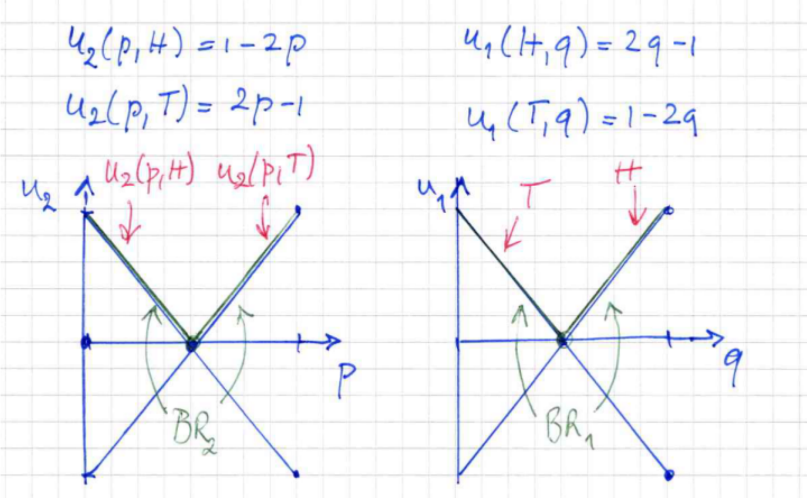
$$
\begin{aligned}
u_2(p,H) &= (-1)p + 1\cdot(1-p) \\
&= 1-2p.
\end{aligned}
$$

$$
\begin{aligned}
u_2(p,T) &= p+(-1)(1-p) \\
&= 2p-1
\end{aligned}
$$

$$u_1(H,q) = q-(1-q) = 2q-1$$

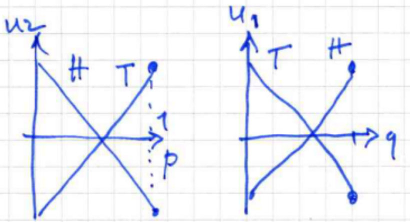
$$u_1(T,q) = -1.q+(1-q) = 1-2q$$

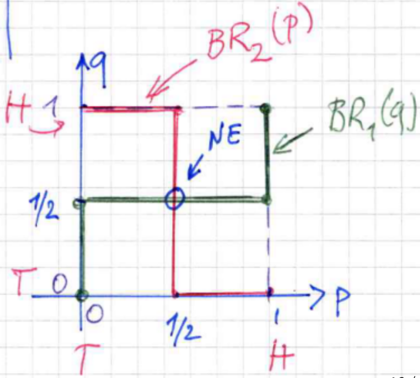
Matching pennies: Computation of best response



Matching pennies: Best response graph

		q	$1-q$
		H	T
p	H	1, −1	−1, 1
$1-p$	T	−1, 1	1, −1





Matching pennies: Mixed Nash Equilibrium

Nash equilibrium characteristics:

- At the intersection point $(p = 1/2, q = 1/2)$, players are simultaneously playing best response to each other;
- No player can do strictly better by unilaterally deviating:
  - If player 2 keeps playing $q = 1/2$ then player 1's utility $u_1(p, q = 1/2)$ can be computed as follows:

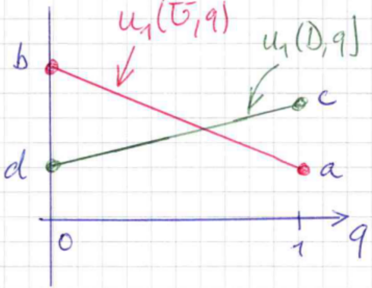
$$\begin{aligned} u_1(p, 1/2) &= (1 \cdot p + (-1) \cdot (1-p) + (-1) \cdot p + 1 \cdot (1-p)) \cdot \frac{1}{2} \\ &= 0 \qquad \textit{(independent of } p\textit{)} \end{aligned}$$

Player 1:no incentive to change his strategy (i.e. change p)

  - Same consideration for player 2.

Computation of mixed Nash equilibrium

		q	$1-q$
		L	R
p	U	a, A	b, B
$1-p$	D	c, C	d, D



(Expected) utility for player 1:

$$u_1(p,q) = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q)$$

Linear in p (for fixed q) and vice versa.

Computation of mixed Nash equilibrium

$$u_1(p,q) = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q)$$

Player 1: no incentive to change;

$$0 = \frac{\partial u_1}{\partial p} = \underbrace{\left(aq + b(1-q)\right)}_{= u_1(U,q)} - \underbrace{\left(cq + d(1-q)\right)}_{= u_1(D,q)}$$

$$u_1(U,q) - u_1(D,q) = 0$$

Player 2 needs to pick q such that player 1 is indifferent btw U and D.

Computing mixed Nash equilibrium

Player 1: T or B, player 2: L or R

- **Strategies:** Player 1: T or B, Player 2: L or R
- Player 1: fix $p = p^*$, such that player 2 is indifferent (between L and R):

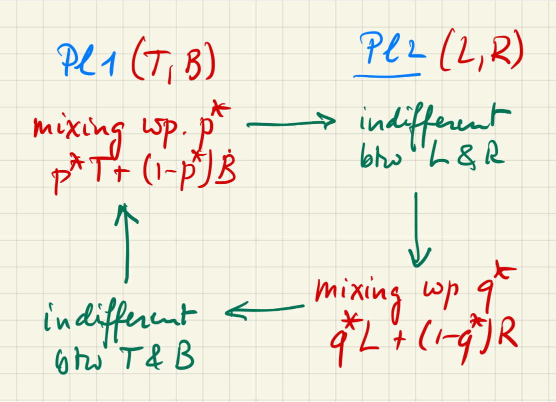
$$EU_2(p^*, L) = EU_2(p^*, R)$$

- Player 2: fix $q = q^*$, such that player 1 is indifferent (between T and B):

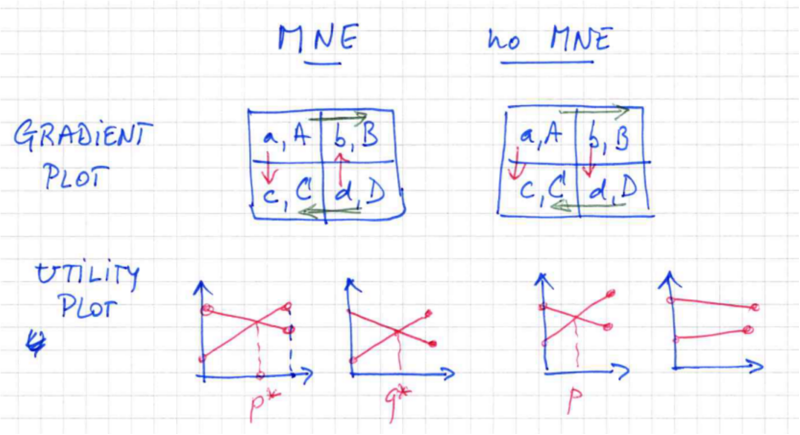
$$EU_1(T, q^*) = EU_1(B.q^*)$$

Mixed Nash equilibrium: “mutual stranglehold”

- P1 mixes (with prob p^*) to make P2 indifferent;
- Since P2 is indifferent, he’s OK to mix too!
- This allows P2 to mix with prob q^* to make P1 indifferent;
- As a consequence P1 is willing to mix with prob p^*.



Some useful graphical representations



Nash equilibrium: Computation of mixed equilibrium

- **Battle of the Sexes** (two pure strategy NEs)
- Additional NE by **mixing pure strategies**?

$$s_1 = \{(A, p), (C, 1-p)\} \quad \text{and} \quad s_2 = \{(A, q), (C, 1-q)\}.$$

	Action (q)	Comedy ($1-q$)
Action (p)	2, 1	0, 0
Comedy ($1-p$)	0, 0	1, 2

- Determining the mixture parameters p and q
  - Ag 1 chooses p such that Ag 2 is **indifferent** btw actions A and C; if not Ag 2 would focus on the most lucrative option.

Condition for Ag 1: $u_2(s_1(p), A) = u_2(s_1(p), C)$

Nash equilibrium: Computation of mixed equilibrium

- **Battle of the Sexes** (mixed strategy NEs)

	Action (q)	Comedy ($1-q$)
Action (p)	2, 1	0, 0
Comedy ($1-p$)	0, 0	1, 2

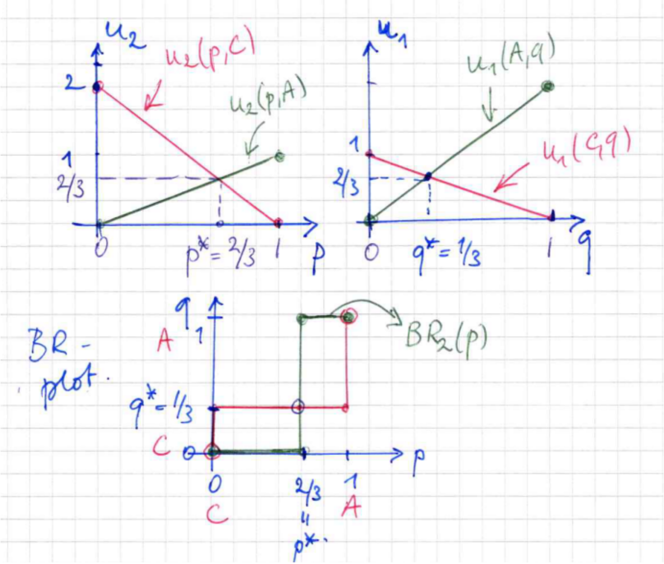
- Determining the mixture parameters p and q

Ag 1: $EU_2(s_1(p), A) = EU_2(s_1(p), C) \implies p = 2(1-p)$

Ag 2: $EU_1(A, s_2(q)) = EU_1(C, s_2(q)) \implies 2q = (1-q)$

Conclusion: $p = \frac{2}{3}, \ q = \frac{1}{3}$ $\qquad EU_1(s_1, s_2) = EU_2(s_1, s_2) = 2/3.$

Battle of the sexes: Mixed Nash Equilibrium



Nash equilibrium: Computation of mixed equilibrium

- **Prisoner's Dilemma** (does mixed strategy NE exist?)

	Quiet (q)	Confess ($1-q$)
Quiet (p)	$-1, -1$	$-12, 0$
Confess ($1-p$)	$0, -12$	$-8, -8$

- Determining the mixture parameters p and q

$$\text{Ag 1: } EU_2(s_1, Q) = EU_2(s_1, C) \Longrightarrow \quad -p - 12(1-p) = -8(1-p)$$

$$\text{Ag 2: } EU_1(Q, s_2) = EU_1(C, s_2) \Longrightarrow \quad -q - 12(1-q) = -8(1-q)$$

Conclusion: $p = q = \dfrac{4}{3}$ **impossible!**

Mixed NE for game with three actions

Rock-Paper-Scissors: no PURE NE!

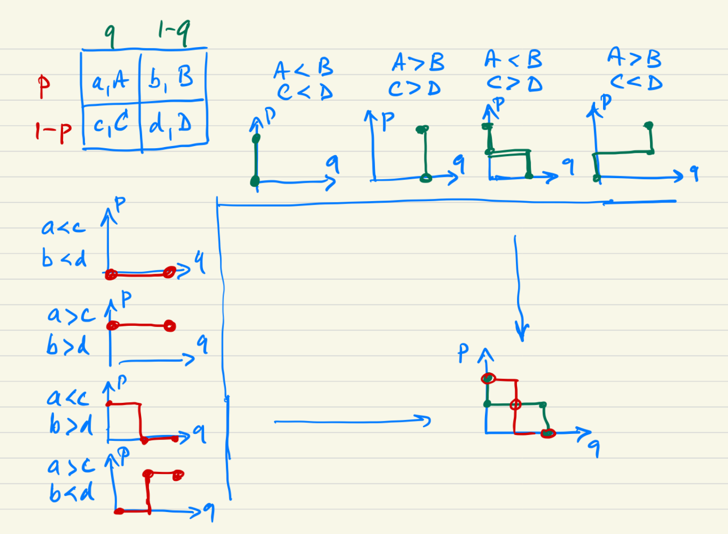
		v	w	$1-(v+w)$
		R	P	S
p	R	$0,0$	$-1,1$	$1,-1$
q	P	$1,-1$	$0,0$	$-1,1$
$1-(p+q)$	S	$-1,1$	$1,-1$	$0,0$

Determine p, q by insisting that player 2 is indifferent btw actions:

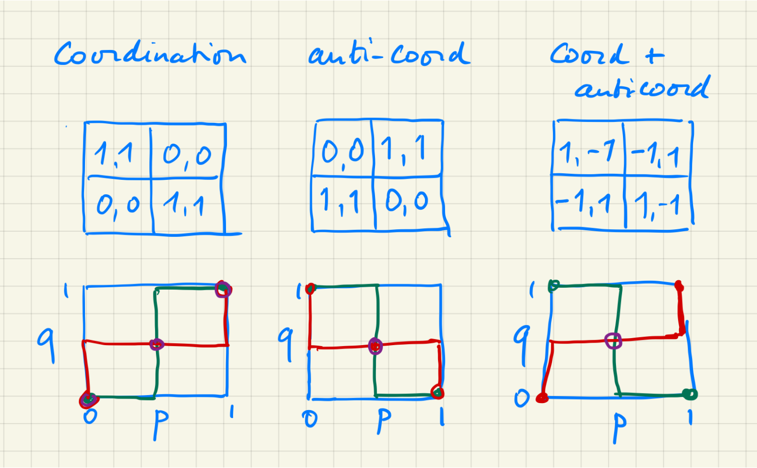
$$\begin{aligned} 0\cdot p+(-1)\cdot q+1\cdot(1-p-q) &= 1\cdot p+0\cdot q+(-1)\cdot(1-p-q) \\ 0\cdot p+(-1)\cdot q+1\cdot(1-p-q) &= -1\cdot p+1\cdot q+0\cdot(1-p-q) \end{aligned}$$

Result: $p=q=1/3$. Determine mixing parameters u, v similarly.

Nash equilibrium for 2 × 2 matrix game



Nash eq. for coordination and anti-coordination games



NE 2×3: Example of NE for 2×3 game

- Pay-off matrix and best response

		u	v	$1-u-v$
		L	C	R
p	T	3, 5	1, 1	2, 4
$1-p$	B	4, 1	0, 3	6, 2

- No pure Nash eq.
- There must be at least one mixed Nash eq.
  - Player 1: mixing: $(p, 1-p)$
  - Player 2: mixing: $(u, v, 1-(u+v))$

NE 2x3: Player 2 mixing to make Player 1 indifferent

Pl 2 mixing to make Pl. 1 indifferent:

$$EU_1(T,\pi_2) = 3u + v + 2(1-u-v) = u - v + 2$$

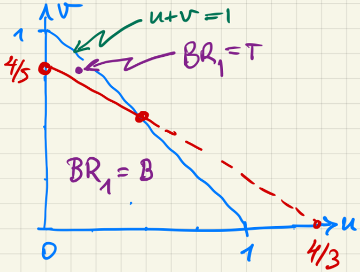
$$EU_1(B,\pi_2) = 4u + 0.v + 6(1-u-v) = -2u - 6v + 6$$

indifference

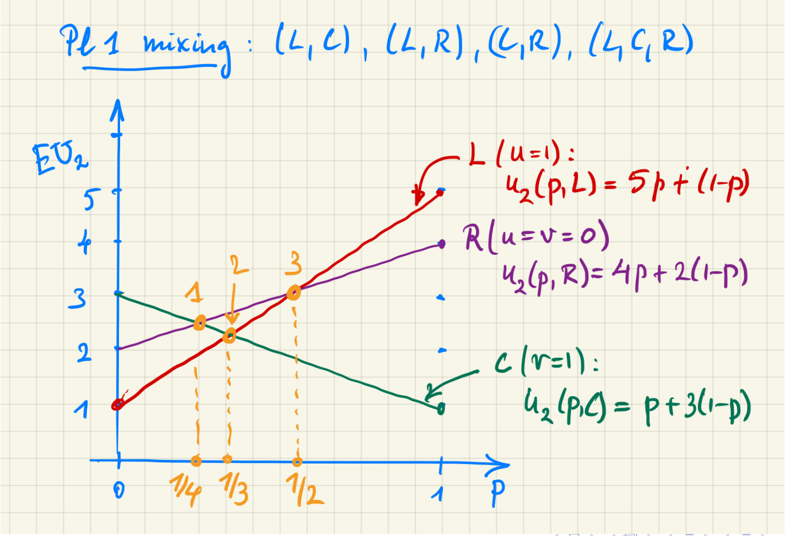
$\Downarrow$

$$u - v + 2 = -2u - 6v + 6$$

$$\Rightarrow \boxed{3u + 5v = 4}$$



NE 2x3: Player 1 mixing to make Player 2 indifferent



NE 2x3: Inspection of the Player 1 mixing options

- **M1: mixing R and C at** $p = 1/4$
  - Since L is not included in mix: $u = 0$
  - There is a unique point on indifference line for Player 1, corresponding to $u = 0$ and $v = 4/5$
  - Hence mixed NE at $p = 1/4$ and $u = 0, v = 4/5$
- **M2: mixing L and C at** $p = 1/3$
  - Not NE as deviating to R is better!
- **M3: Mixing L and R at** $p = 1/2$
  - Since C is not included: $v = 0$
  - No solution (point on indifference line outside feasible triangle!
- **Mixing all three strategies** dilutes pay-off and therefore is not optimal

NE 2x3: Expected utilities in Nash Eq.

3 5	1 1	2 4
4 1	0 3	6 2

$$
\begin{array}{cc}
 & \overbrace{\begin{array}{ccc} 0 & 4/5 & 1/5 \end{array}}^{\pi_2} \\
\pi_1 \left\{ \begin{array}{c} 1/4 \\ 3/4 \end{array} \right. & \begin{array}{|c|c|c|} \hline 0 & 4/20 & 1/20 \\ \hline 0 & 12/20 & 3/20 \\ \hline \end{array}
\end{array}
$$

$$u_1(\pi_1, \pi_2) = \frac{1}{20}\left[1 \cdot 4 + 0 \cdot 12 + 2 \cdot 1 + 6 \cdot 3\right]$$

$$= \frac{24}{20} = \frac{6}{5} = \underline{\underline{1.2}}$$

$$u_2(\pi_1, \pi_2) = \frac{1}{20}\left[1 \cdot 4 + 3 \cdot 12 + 4 \cdot 1 + 2 \cdot 3\right]$$

$$= \frac{50}{20} = \frac{5}{2} = \underline{\underline{2.5}}$$

NE 2x3: Compare safety strategy to Nash Eq.

$$
\begin{array}{|c|c|c|l}
\hline
3 \quad 5 & 1 \quad 1 & 2 \quad 4 & \xrightarrow{\text{min}} \boxed{1} \\
\hline
4 \quad 1 & 0 \quad 3 & 6 \quad 2 & \longrightarrow 0 \\
\hline
\end{array}
$$

$$\boxed{v_1^{\text{mami}} = 1}$$

min ↓: $1 \qquad 1 \qquad \boxed{2}$ max

$$\boxed{v_2^{\text{mami}} = 2}$$

Fictitions Play (FP)

- FP uses simulation of many game iterations to learn about equilibria;
- FP refers to a dynamic process where at each stage, players play a (pure) best response to the empirical distribution of their opponent's play
- **If FP converges (in distribution)**, the limit distribution coincides with mixed Nash strategies.
- Reminiscent of best response dynamics, but more general (includes mixed strategies)
- **Form of learning**, works even opponent's utilities are unknown;

Fictitious Play: NE 2 × 3 example

	L	C	R
T	3, 5	1, 1	2, 4
B	4, 1	0, 3	6, 2

Suppose after $n = 40$ iterations (simulations)

$$N_1 = \begin{array}{cc} T & B \\ \boxed{30} & \boxed{10} \end{array} \qquad \hat{\pi}_1 = \begin{array}{cc} \boxed{3/4} & \boxed{1/4} \end{array}$$

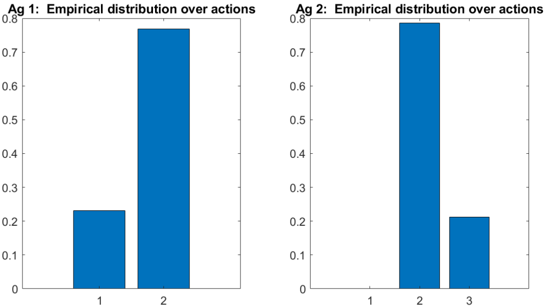
$$N_2 = \begin{array}{ccc} L & C & B \\ \boxed{15} & \boxed{5} & \boxed{20} \end{array} \qquad \hat{\pi}_2 = \begin{array}{ccc} \boxed{3/8} & \boxed{1/8} & \boxed{4/8} \end{array}$$

$$u_1(T, \hat{\pi}_2) = \frac{3}{8} \cdot 3 + \frac{1}{8} \cdot 1 + \frac{4}{8} \cdot 2 = \frac{9}{4}$$

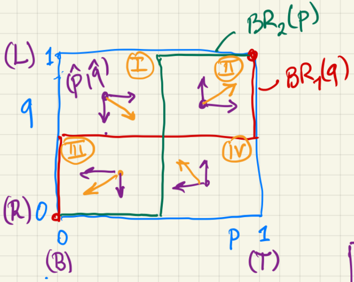
$$u_1(B, \hat{\pi}_2) = \frac{3}{8} \cdot 4 + \frac{1}{8} \cdot 0 + \frac{4}{8} \cdot 6 = \frac{18}{4} \longleftarrow (*)$$

Fictitious Play: NE 2×3 example

- Recall: MNE at P1 (1/4, 3/4) and P2 (0,4/5, 1/5)
- Results for FP (300 iterations)



FP 2 PNE and 1 MNE



After N iterations

Pl1 : $\to n : T$; $\to N-n : B$

Pl2 : $\to m : L$; $\to N-m : R$

$$\boxed{\hat{p} = \frac{n}{N}, \quad \hat{q} = \frac{m}{N}}$$

(I) : $BR_1(\hat{q}) = T \ (p=1) \Rightarrow n \to n+1 \Rightarrow \hat{p} \uparrow$

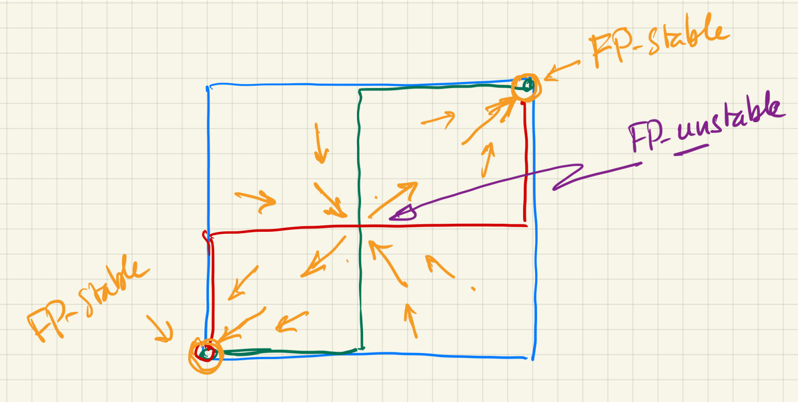
$BR_2(\hat{p}) = R \ (q=0) \Rightarrow m \to m \Rightarrow \hat{q} \downarrow$

(II) $BR_1(\hat{q}) = T \Rightarrow \hat{p} \uparrow \ ; \ BR_2(\hat{p}) = L \Rightarrow \hat{q} \uparrow$

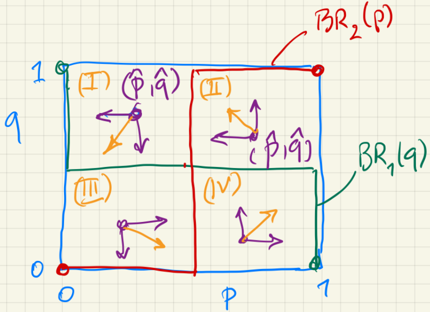
Mixed NE not found by FP.

FP No PNE, 1 MNE

PNE are stable but MNE is unstable under FP-dynamics



FP No PNE, 1 MNE



(I) $BR_1(\hat{q}) = (p=0) \Rightarrow (n \to n) \Rightarrow \hat{p}\downarrow$

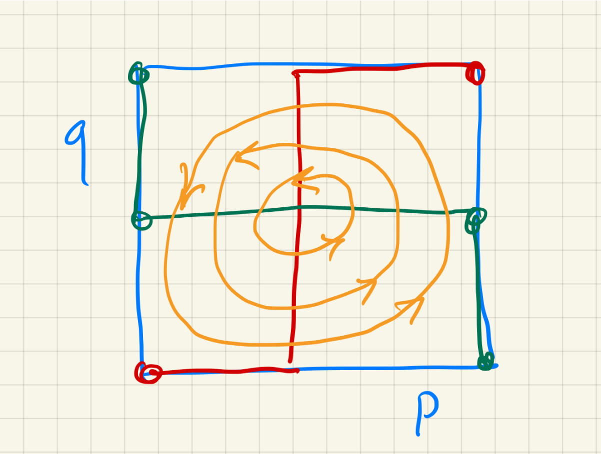
$BR_2(\hat{p}) = (q=0) \Rightarrow (m \to m) \Rightarrow \hat{q}\downarrow$

(II) $BR_1(\hat{q}) = (p=0) \Rightarrow (n \to n) \Rightarrow \hat{p}\downarrow$

$BR_2(\hat{p}) = (q=1) \Rightarrow m \to m+1 \Rightarrow \hat{q}\uparrow$

FP No PNE, 1 MNE

Cycle yields approximate empirical distribution



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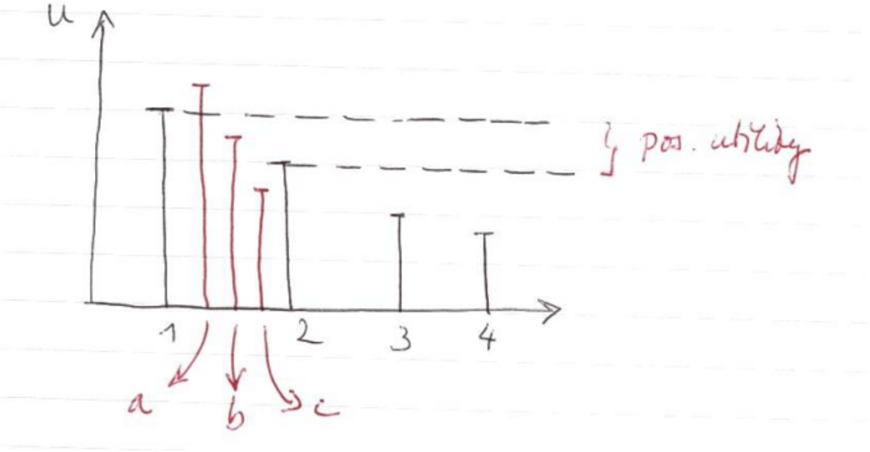
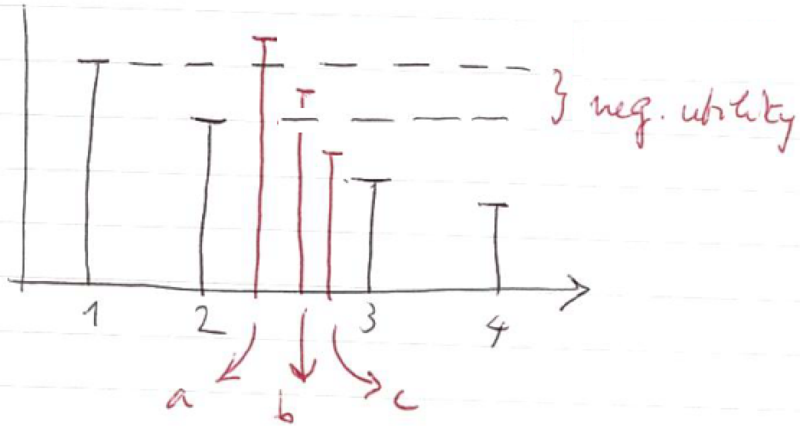
Vickrey auction: Second Price Auction

- **Vickrey Auction:**
  - n sealed bid auction for single item;
  - highest bid wins, but pays 2nd highest price;
- **Truth-telling** is (weakly) **dominant strategy;**
- NE: Neither winner nor loser(s) have incentive to deviate:
- **Winner:**
  - Higher: still winner, same price;
  - Lower: might lose, but if still winner, still paying 2nd price;
- **Loser:**
  - Higher: possibly winner, but at higher price;
  - Lower: still loser;
- Example of **mechanism design**.

Vickrey Auction: Alternatives for winner



Vickrey Auction: Alternatives for runner-up



Hawk or Dove

- Equilibria as a function of **exogenous pay-off variables**;
- **Exogenous variables** are imposed on the game (not by players);
- **Strategy: Hawk or dove:**
  - Two parties are in conflict over some good of value $2v > 0$;
  - Two doves share, each gets v;
  - Hawks fight, on average each gets half, but at a cost c (e.g. due to injury)
  - A dove is no match for a hawk and yields;
  - **Pay-off matrix:**

	Hawk	*Dove*
Hawk	$v-c,\ v-c$	$2v, 0$
Dove	$0, 2v$	$v,\ v$

- Different outcomes depending on cost of aggression c!

Hawk or Dove

- **Aggression is cheap ($c < v$)** H is dominant, hence: NE $=$ (H,H) with utility $(v - c, v - c)$;
- **Aggression is risky ($c > v$)**
  - Two PNE $=$ (H,D) and (D,H)
  - One MNE at $P(H) = p^* = v/c$.
  - If risk/cost increases $c \uparrow$, then $p^* \downarrow$.

Investment game

- Many agents (n) but only two strategies;
- **Strategies:** Each agent can either
  - I: invest 10 euro
  - N: not invest
- **Pay-offs:**
  - N: No investment: pay-off $= 0$;
  - I: Pay-off $= 15$:

$$\text{net pay-off} = \begin{cases} 5 & (= 15 - 10) & \text{if at least } 90\% \text{ of ag. invest} \\ -10 & (= 0 - 10) & \text{otherwise} \end{cases}$$

- **Coordination game!**

Strategic Effects

- **Symmetric penalty kick game:**

		goal keeper	
		left	right
kicker	left	$0, 0$	$1, -1$
	right	$1, -1$	$0, 0$

- Suppose kicker has weak left kick: **direct and indirect effect!**

		goal keeper	
		left	right
kicker	left	$0, 0$	$a, -a$
	right	$1, -1$	$0, 0$

$$0 < a < 1$$

Strategic Effects

- Kicker has weak left side, so is inclined to kick less with left;
- Goalkeeper knows this, so anticipates less kicks to the left, so will tend to jump less to the left ...
- Since kicker knows this, he might reconsider and kick more with left, since goalie has tendency to jump to the right ...,
- but goalie knows this too, so might reconsider ...
- and so on ...
- Is there a way out of infinite regress???

Strategic Effects

goalie

		(q) L	$(1-q)$ R
striker	(p) L	0, 0	$a, -a$
	$(1-p)$ R	1, −1	0, 0

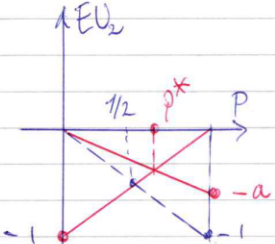
$0 < a < 1$.

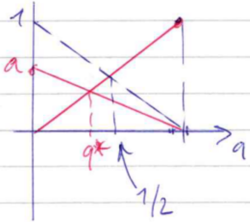
$$\left.\begin{aligned} EU_2(p, L) &= 0 \cdot p + (-1)(1-p) = p-1 \\ EU_2(p, R) &= -ap + 0(1-p) = -ap. \end{aligned}\right\} \text{goalie.}$$

$$\left.\begin{aligned} EU_1(L, q) &= 0 \cdot q + a(1-q) = a(1-q) \\ EU_1(R, q) &= q + 0 = q \end{aligned}\right\} \text{striker}$$

Strategic Effects

			q	$1-q$
			L	R
Striker	(p)	L	$0,0$	$a,-a$
	$(1-p)$	R	$1,-1$	$0,0$





$\boxed{a\downarrow \Rightarrow p^*\uparrow}$

$\boxed{a\downarrow \Rightarrow q^*\downarrow}$

Strategic Effects

- Mixed Nash equilibrium at:

$$p^* = \frac{1}{1+a} = 1 - q^* \quad \Rightarrow \quad p^* > 1/2, \quad q^* < 1/2$$

- Zero-sum game: Utilities for kicker and goalie:

$$EU_1(p^*, q^*) = \frac{a}{1+a} = -EU_2(p^*, q^*)$$

- Notice that $p^* \to 1$ as $a \to 0$, i.e. if left kick is powerless ($a \approx 0$), make sure to kick left ($p^* \approx 1$). Is NE the right tool to think about this??

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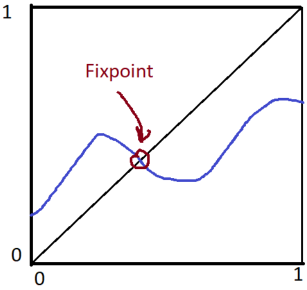
Aside: Sperner's Lemma (1928) and fix-point theorems

Defintion: Fix-point

Point $a \in A$ is a **fix-point** for function $f : A \longrightarrow A$, iff $f(a) = a$.



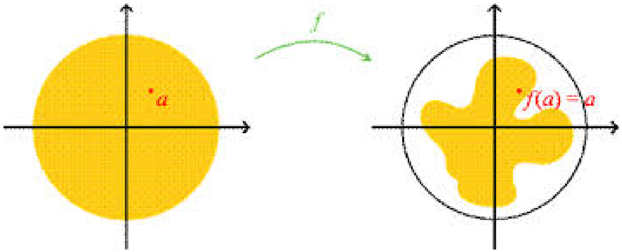
f: [0,1] ---> [0,1] continuous

Sometimes(!), fixpoints can be computed using **function iteration.**

Nash Theorem is based on Fixed-Point Theorem

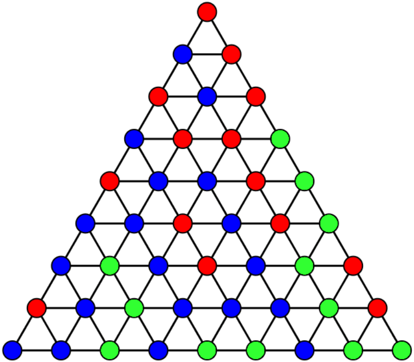
Brouwer's Fixed Point Thm

Let $K \subset \mathbb{R}^n$ be a compact and convex, and $f : K \longrightarrow K$ continuous. Then f has a fix-point in K, i.e. there exists a $x_0 \in K : f(x_0) = x_0$.



: **Non-constructive existence proof!**

Aside: Sperner's Lemma (1928) and fix-point theorems

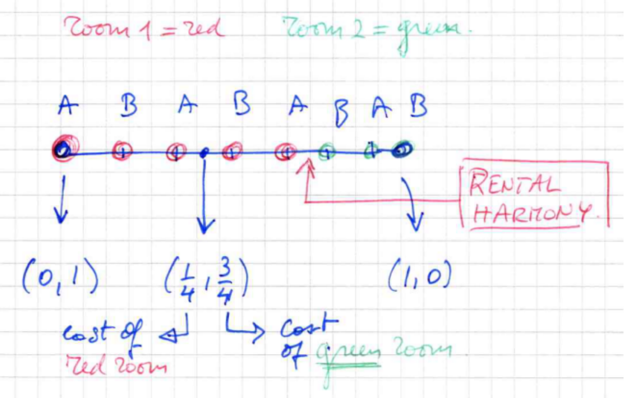


If in triangle:

- corner nodes have different colors
- nodes on outer edges have two possible colors (determined by corners)

Then there is a subtriangle with 3 differently colored corners.

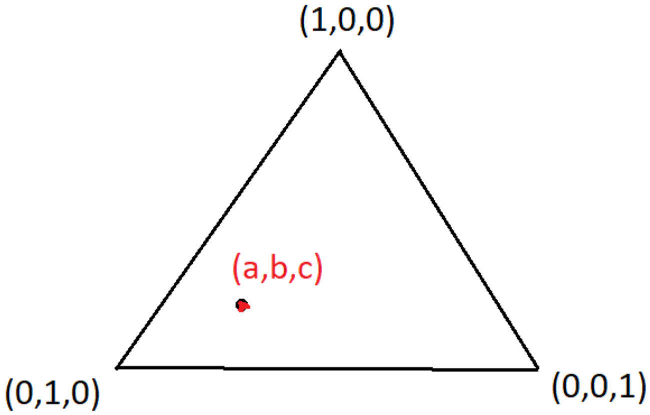
Sperner's lemma: Rental Harmony: Fair division of rent



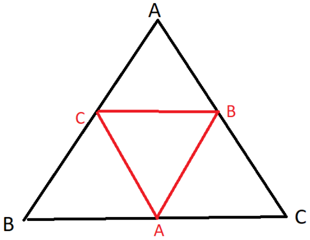
- Divide interval in segments by adding points
- Assign alternating decision makers (A and B)
- Decision maker decides which room he picks (for given rental division)

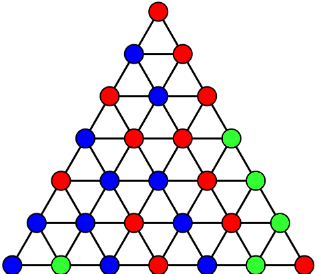
Sperner's lemma – Application 1: Rental Harmony

Flat sharing: three bedroom flat shared by tree friends but rooms are not of equal quality: how much should each contribute to rent?

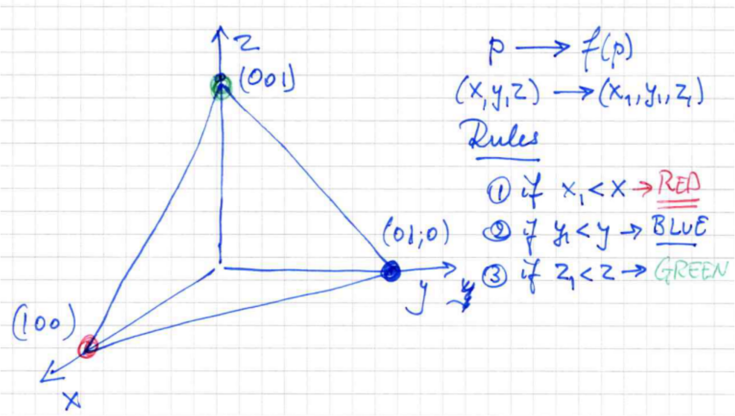


Sperner's lemma – Application 1: Rental Harmony





Sperner's lemma – Application 2: Proof of Brouwer's fixpoint thm



Youtube: Trefor Bazzett: A beautiful combinatorical proof of the Brouwer Fixed Point Theorem - Via Sperner's Lemma

Nash equilibrium: NE and IEWDS

Eliminating **weakly dominated** strategies might erase NEs!

	L	R
U	$2,3$	$4,3$
D	$3,3$	$1,1$

- Two NEs: (4,3) and (3,3)
- L weakly dominates R;
- Eliminating R would result in single solution (3,3);
- Notice that the Pareto-optimal NE would be eliminated.

Games with NO Nash Equilibrium

- **Dollar auction:** Sealed bid auction: highest bid gets item, 2nd highest pays this amount!
- More generally: One can construct games without NE by making sure that either
  - state space is **not compact**
  - utility function is **not continuous**
- **Ex. for 2-player games with continuous state spaces:**
  - Non-compact: $u_i(x, y) = xy \qquad \mathit{for} \quad 0 \leq x, y < 1$
  - Non-continuous: $0 \leq x, y \leq 1$ and

$$u_i(x, y) = \begin{cases} xy & \mathit{if} \quad x, y < 1 \\ 0 & \mathit{if} \quad x, y = 1 \end{cases}$$

Nash equilibrium: Computational aspects

- Finding a Nash equilibrium for 2-player zero-sum games can be done efficiently by formulating a linear program. Notice that in this case: NE = minimax = maximin.
- Finding a Nash equilibrium is not known to be NP-complete because it is not a decision problem
- PPAD (polynomial parity argument, directed version) is a class describing problems for which a solution always exists
- Daskalakis, Goldberg, and Papadimitriou showed that finding a sample Nash equilibrium of a general-sum finite game with two or more players is PPAD-complete (i.e. "difficult!").

Other solution concepts

- **Minimax equilibrium**: zero-sum special case of Nash
- **Trembling-hand perfect equilibrium**: each player's action is a best-response even if other players make small mistakes
- ϵ-**Nash equilibrium**: deviating benefits no agent more than ϵ
- **Correlated equilibrium**: agents can condition strategy on external signal: "If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium." -Myerson
- **Evolutionary stable state**: "A population is said to be in an evolutionarily stable state if its genetic composition is restored by selection after a disturbance, provided the disturbance is not too large." -Maynard Smith

Summary

- Game theory studies utility-based multiagent decision making.
- Solving a game means trying to predict its outcome.
- Rational agents never play strictly dominated actions.
- No agent has an incentive to **unilaterally deviate** from a **Nash equilibrium**.
- In finite games, there's always a NE (possibly in mixed strategies).
- Nash equilibria need not be Pareto optimal.

Some additional literature

- Presh Talwalkar: The Joy of Game Theory: An introduction to strategic thinking.
- Avinash K. Dixit, Barry J. Nalebuff: The Art of Strategy: A Game Theorist's Guide to Success in Business and Life
- William Poundstone: Prisoner's Dilemma: John von Neumann, Game Theory, and the Puzzle of the Bomb