Introduction to Game Theory 6
Shapley Value for Coalitional Games

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Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 12, sections 12.1-12.2

Overview

Coalitional Game Theory

- Basic modelling unit is **group** rather than indvidual agent.
- Transferable vs. non-transferable utility
- **Coalitional game with transferable utility (N, v):**
  - N finite set of players:
  - $v : 2^N \longrightarrow \mathbb{R}$ pay-off function ($v(\emptyset) = 0$)
- Fundamental questions:
  - Which coalitions will form?
  - How should coalition divide its pay-off among its members?

Classes of coalitional games

- **Super-additive game ("synergy")**
  Game (N, v) is super-additive iff

$$\forall S, T \subset N : S \cap T = \emptyset \Longrightarrow v(S \cup T) \geq v(S) + v(T).$$

  In particular: $v(S \cup i) \geq v(S) + v(i)$ for any $S \subset N \setminus \{i\}$.

- As a consequence, for super-additive game, the grand coalition has the highest pay-off of all coalitional structures:

$$v(N) = v(S \cup S^c) \geq v(S) + v(S^c) \geq v(S).$$

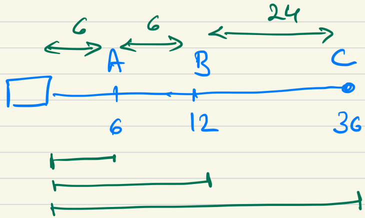
- Therefore focus on a fair redistribution of total pay-off among the members of the grand coalition.

Motivating example: Ways to allocate common benefits

- Three friends sharing a taxi cab
- Splitting the profit from an investment

Worked example: Fair division of taxi fare

- Alice, Bob and Charlize share a taxi to go home;
- The individual fares would be: A(6), B(12) and C(36);
- If they share the cab then they only have to pay the fare to the farthest destination (C = 36).
- What would be a fair way to share the fare?



A: $6/3 = 2$

B: $2 + 6/2 = 5$

C: $2 + 3 + 24 = 29$

Sanity check: $2+5+29 = 36$, and everyone is better off!

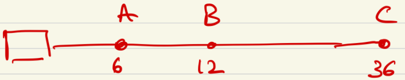
Worked example: Fair division of taxi fare

- Consider a sequential version of the problem in which A, B and C arrive in random order, and pay whatever is lacking (i.e. their marginal contribution);
- Permutation ACB indicates that coalition grows as follows:

$$A \to AC \to ACB$$

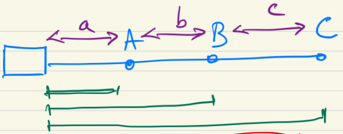
1. When A joins, he pays the fare to his destination: 6
2. When C joins, he pays the remainder to get to C: 36-6=24;
3. Finally, when B joins, everything is already paid for.

Worked example: Fair division of taxi fare



marginal contribution	A	B	C	
ABC	6	6	24	4
ACB	6	0	30	5
BAC	0	12	24	4
BCA	0	12	24	4
CAB	0	0	36	6
CBA	0	0	36	6
mean	$\frac{12}{6} = 2$	$\frac{30}{6} = 5$	$\frac{174}{6} = 29$	

Shapley: common sense vs. permutation solution



$\varphi_c = ?$

ABC	c
ACB	$b+c$
BAC	c
BCA	c
CAB	$a+b+c$
CBA	$a+b+c$
	$6c+3b+2a$

$$\varphi_c = \frac{6c+3b+2a}{3!} = \frac{c}{1} + \frac{b}{2} + \frac{a}{3}$$

sharers

Shapley value: Alternative definition

- Marginal contribution only depends on what precedes a contributor;
- Marginal contribution of player i to subset S:

$$\delta_i(S) = v(S \cup i) - v(S)$$

- Shapley value of player i: (denoting $\#N = n, \#S = s$)

$$\varphi_i(N, v) := \frac{1}{n} \sum_{S \subset N \setminus i} \binom{n-1}{s}^{-1} \delta_i(S)$$

- Amplification: see next slides!

Shapley Value: Amplification

- We focus on Shapley value $\varphi_i(N, v)$ for agent i;
- For any existing coalition S not including i, i.e.

$$S \subset N_i := N \setminus i$$

we consider the value increment due to i joining:

$$\delta_i(S) = v(S \cup i) - v(S)$$

- The size $s := \#S$ of the possible coalitions S that i joins, can range between $0 \leq s \leq n - 1$.

Shapley Value: Amplification

- For fixed coalition size s there are

$$N_s := \binom{n-1}{s}$$

coalitions S of that size.

- Hence, the mean contribution $\overline{\Delta}_i$ of i to existing coalitions S of size s is given by:

$$\overline{\Delta}_i(s) := \frac{1}{N_s} \sum_{S:\#S=s} \delta_i(S) = \binom{n-1}{s}^{-1} \sum_{S:\#S=s} \delta_i(S).$$

Shapley Value: Amplification

- Finally, since $0 \le s \le n-1$ we compute the average over the n possible choices of s. This average is the Shapley value:

$$
\begin{aligned}
\varphi_i \quad &:= \quad \frac{1}{n}\sum_{s=0}^{n-1}\overline{\Delta}_i(s) = \frac{1}{n}\sum_{s=0}^{n-1}\binom{n-1}{s}^{-1}\sum_{S:\#S=s}\delta_i(S) \\
&= \quad \frac{1}{n}\sum_{s=0}^{n-1}\sum_{S:\#S=s}\binom{n-1}{s}^{-1}\delta_i(S) \\
&= \quad \frac{1}{n}\sum_{S\subset N\setminus i}\binom{n-1}{s}^{-1}\delta_i(S)
\end{aligned}
$$

- Double sum above is actually sum over all subsets $S \subset N \setminus i$.

Shapley Value: Amplification

Recall:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where} \quad n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1.$$

Hence:

$$\begin{aligned} \varphi_i &= \frac{1}{n} \sum_{S \subset N \setminus i} \binom{n-1}{s}^{-1} \delta_i(S) \\ &= \frac{1}{n} \sum_{S \subset N \setminus i} \frac{s!(n-1-s)!}{(n-1)!} \delta_i(S) \\ &= \frac{1}{n!} \sum_{S \subset N \setminus i} s!(n-1-s)!\, \delta_i(S) \end{aligned}$$

Claim: this corresponds to the permutation definition!

Shapley: equivalence of definitions

Permutation formula:

$$\varphi_i = \frac{1}{n!} \sum_{S \subset N \setminus i} s!\,(n-1-s)!\,\delta_i(S) \qquad (s = \#S)$$

$\varphi_3 = ?$

$$\pi_1 = \underbrace{2\ 4}_{S}\ \underline{\underline{3}}\ \underbrace{1\ 5}_{S^c} \longrightarrow S = \{2,4\},\quad S^c = \{1,5\}$$

$$\pi_2 = \underbrace{4}_{S}\ \underline{\underline{3}}\ \underbrace{2\ 1\ 5}\ \longrightarrow S = \{4\},\quad S^c = \{1,2,5\}$$

Shapley: equivalence of definitions

Recall: $\delta_i(S)$ depends on the **set** S, not on the permutation sequence of the elements in S!

- Hence, $\delta_3(S)$ has the same value in $\pi_1 = 24315$ and $\pi_2 = 42351$ as in both cases $S = \{2, 4\}$.
- Consider an arbitrary permutation of $1, 2, \dots, n$ and let's focus on i somewhere in the sequence, all numbers that appear to the left of i, constitute the set S. Likewise, all elements that appear to the right, are collected in the set S^c:

$$\pi = \underbrace{* * * \dots * **}_{S} \, i \, \underbrace{* * * \dots * **}_{S^c}$$

- Any permutation of the S-elements in π yields the same value $\delta_i(S)$ (see above), There are $s!$ such permutations.

Shapley: equivalence of definitions

- Likewise, $\delta_i(S)$ does **not depend** on the elements in S^c. Any permutation of these elements in S^c yields that same value $\delta_i(S)$. There are $(n-1-s)!$ such permutations.
- Hence, we can conclude that of the $n!$ permutations of $\{1, 2, \ldots, n\}$, a total of $s!(n-1-s)!$ give rise to the same value $\delta_i(S)$, which only depends on the **set** S.
- Averaging over all possible choices for $S \subset N_i \equiv N \setminus i$ yields:

$$\frac{1}{n!} \sum_{S \subset N_i} s!(n-1-s)!\, \delta_i(S) = \phi_i(i)$$

Shapley's Axioms: Some useful terminology

- Players i and j are **interchangeable** if their contributions to every coalition (subset) S is exactly the same:

$$\forall\, S \subset N \setminus \{i, j\} : \quad v(S \cup i) = v(S \cup j)$$

- A player i is a **dummy player** if the amount he contributes to any coalition is exactly the amount he's able to achieve alone:

$$\forall S \subset N \setminus \{i\} : v(S \cup i) = v(S) + v(i)$$

Shapley's Axioms

- **Symmetry:** If i and j are interchangeable then:

$$\psi_i(N, v) = \psi_j(N, v).$$

- **Dummy Player:** will only get what he can achieve on his own:

$$\psi_i(N, v) = v(i).$$

- **Additivity:** Consider two games $G_1 = (N, v)$, $G_2 = (N, w)$ and assume that we play G_1 with probability p and G_2 with prob $q = 1 - p$.
  Then

$$\psi_i(N, pv + qw) = p\psi_i(N, v) + q\psi_i(N, w)$$

Shapley's theorem

Shapley (1951)

Given a coalitional game (N, v), the Shapley values $\varphi_i, i = 1, \ldots, n$ specifies the unique distribution of the total value $v(N)$ that is both

- efficient, i.e. $\sum_i \varphi_i = v(N)$
- satisfies Shapley's axioms,
  i.e. *Symmetry, Dummy Player* and *Additivity*.

Shapley value: worked example

An AI expert (E) developed a powerful new algorithm. However, in order to implement his ideas, he needs to create a startup and hire a programmer (P) for 2 years. An angel investor (A) provides funding. The value that each coalition of these three stakeholders (E, P, A) can generate satisfies the following rules:

- Without both investor and expert, no value can be generated.
- If he has no assistance from a programmer, the expert's value equals 3, but if he can delegate the programming and focus on R&D, his value rises to 10.
- The value created by the programmer is 5. This is in addition to the rise in value of the expert.

The startup is sold to a large software company for serious money. How to split this money fairly among the three stakeholder?

Shapley value: Method 1

Shapley value computation: $\#N = n = 3$, $\#S = s$

$$\varphi_i = \frac{1}{n} \sum_{S \subset N \setminus i} \binom{n-1}{s}^{-1} \delta_i(S)$$

Expert (E)

$\underbrace{s=0}$ $\longrightarrow S = \phi \Rightarrow \delta_E(S) = v(E) - v(\phi) = 0.$

$\hookrightarrow \binom{n-1}{s} = \binom{2}{0} = 1$

$\underbrace{s=1}$ $\longrightarrow \delta_E(A) = v(AE) - v(A) = 3$

$\delta_E(P) = v(EP) - v(P) = 0$

$\hookrightarrow \binom{n-1}{s} = \binom{2}{1} = 2$

$\underline{s=2}$ $\longrightarrow \delta_E(AP) = v(APE) - v(AP) = 15$

$\hookrightarrow \binom{n-1}{s} = \binom{2}{2} = 1$

$$\varphi_E = \frac{1}{3}\left[\frac{1}{1}\cdot 0 + \frac{1}{2}3 + \frac{1}{1}.15\right] = \frac{11}{2}$$

Shapley value: Method 2

	E	P
AEP	$\underset{3}{v(AE)} - \underset{0}{v(A)} = 3$	$\underset{15}{v(AEP)} - \underset{3}{v(AE)} = 12$
APE	$\underset{15}{v(APE)} - \underset{0}{v(AP)} = 15$	$v(AP) - v(A) = 0$
EAP	$v(E) - v(\phi) = 0$	$v(AEP) - v(EA) = 12$
EPA	$v(E) - v(\phi) = 0$	$v(EP) - v(E) = 0$
PAE	$\underset{15}{v(AEP)} - \underset{0}{v(AP)} = 15$	$v(P) - v(\phi) = 0$
PEA	$v(EP) - v(P) = 0$	$v(P) - v(\phi) = 0$
	33	24

$$\boxed{\begin{array}{ll} \varphi_E = \frac{33}{6} = \frac{11}{2} & \varphi_P = \frac{24}{6} = 4 \\ \left(\varphi_A = \frac{11}{2}\right) & \text{Notice: } \frac{11}{2} + \frac{11}{2} + 4 = 15 \end{array}}$$

Figure: Notice distribution is efficient: $\varphi_E + \varphi_A + \varphi_P = 15 = v(N)$