Introduction to Game Theory 4:

Sequential Games

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Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 5, sections 5.1-5.3
- **Optional**
  - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
  - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
  - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

Overview

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Sequential games

- **Normal-form games:**
  - Simultaneous moves by players
  - Central solution concept: Nash equilibrium
- **Sequential games:**
  - Players move in succession, observe (at least partially) prior moves by opponents
  - **Perfect versus imperfect information:** what exactly is known about previous moves?
    - players have full knowledge of all the preceding moves (perfect information)
    - players might not know the complete game history till then (imperfect information);
  - **Model for many sequential interactions** in games, politics, economics, etc

Simultaneous vs. Sequential Games

- **Simultaneous games:** players make their moves simultaneously, i.e. without knowing what the other players will do!
  - Rock-paper-scissors
  - Sealed bid auctions
  - Cournot's duopoly model
- **Sequential games:** Sequence of successive moves by players who can see each other's moves (to some extent – see next slide):
  - Chess
  - Card games
  - Open cry auctions
  - Stackelberg's duopoly model
  - Negotiation (Rubinstein's model)

Extensive form representation of sequential game

- Visualisation of temporal relationships (game tree)
- **Extensive form** is finite game representation that **does not assume** that players act **simultaneously**;
- Can be converted in normal form representation (*possibly exponentially larger!*)
- **Game tree:** makes **temporal structure** explicit

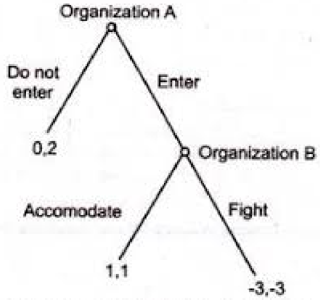


Figure-2: Extensive Form Games

Information in Game Theory

	PERFECT complete history known to all players	**IMPERFECT** unaware of actions taken by others
COMPLETE NO private info agents, actions, payoffs known	E.g. chess	Simultaneous games Information sets
INCOMPLETE private info e.g. private valuation	• Open cry auction • Different types of opponents	Sealed bid auction

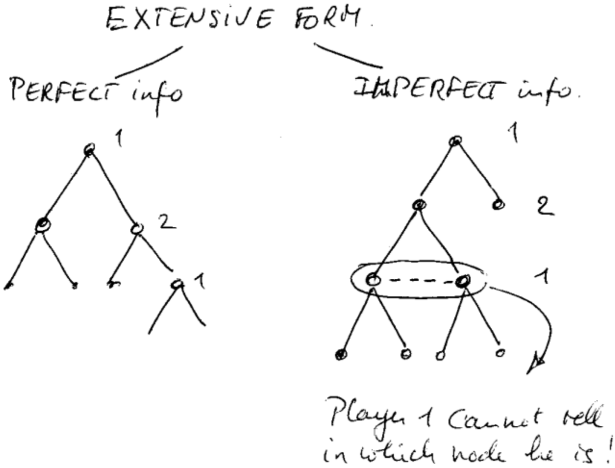
Aside: Types of knowledge

- **Mutual knowledge** is
  - known to all players,
  - but players do not know that others know
  - e.g. *the elephant in the room* , solutions to homeworks
- **Common knowledge** is
  - known to all players,
  - and all players know all others know . . .
  - and all players know all others know that all others know . . .
  - and so on . . .
  - e.g. In continental Europe one drives on the RHS of the road

Major Ideas in Non-Cooperative Game Theory

	SIMULTANEOUS (STATIC)	**SEQUENTIAL** (DYNAMIC)
Complete information NO private info	Nash equilibrium	Backwards induction, Subgame-perfect NE: *discard NE based on* *non-credible threats*
Incomplete information private info	Bayesian Nash eq.	Perfect Nash eq.

Sequential games: perfect vs. imperfect information



Perfect info: Chess

Imperfect info: Card games (e.g. poker),

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Sequential Games with Perfect Information

Solution strategy: Backward induction
Protoype: two-player, sequential-move game:

- Player 1 chooses action $a_{11} \in A_1$;
- Player 2 observes a_{11} and then chooses action $a_{21} \in A_2$;
- ...
- Hereafter, both players receive pay-off: $u_1(a_{1n}, a_{2n})$ and $u_2(a_{1n}, a_{2n})$ respectively;

Examples:

- Various board and card games (e.g. chess, go, etc)
- Stackelberg's sequential-move version of Cournot's duopoly;
- Rubinstein's bargaining model

Solving Perfect information Sequential Games using Backward Induction

Perfect info sequential games are easy: use **backward induction**!

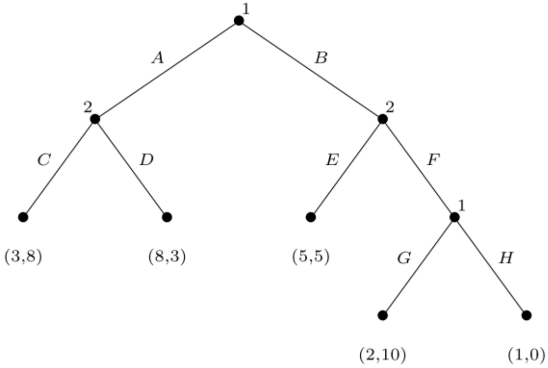
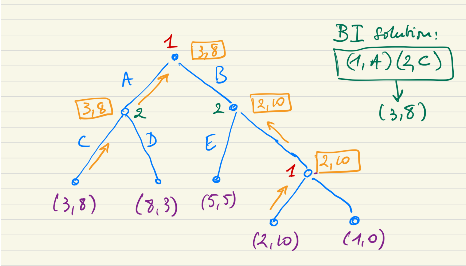


Figure 5.2: A perfect-information game in extensive form.

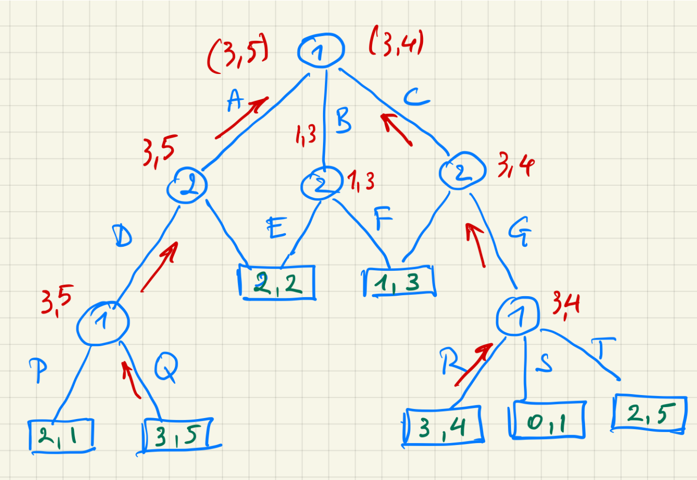
Solving Extensive Form Games using Backward Induction

Backward induction

- Basic assumption:
  - Players believe that all future play will be rational
  - and condition decisions on what they expect in future;
- Algorithm
  - Start at leaf-nodes: easy decision as only one player involved;
  - Propagate decisions and utilities to root;
  - The solution path (A, C) is called the equilibrium path

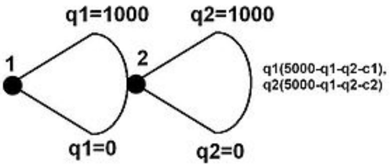


Backward induction with multiple equilibrium paths



Sequential game with continuous actions Stackelberg duopoly

- Sequential version of Cournout duopoly
- In a **Stackelberg game**,
  - one player (leader) moves first,
  - and all other players (followers) move after him.
  - Continuous action space but can also be solved using backward induction



Stackelberg Duopoly

- Two firms produce a bland product (e.g. bottled water, airline seats).
  - Bland product: customers don't care which firm they buy from.
- Firm 1 moves first and decides to produce a total quantity q_1.
- Firm 2 observes this move, then decides to produce q_2.
- The market price (per unit) decreases (linearly) as the total amount produced $(q_1 + q_2)$ increases:

$$P(q_1, q_2) = \alpha - \beta(q_1 + q_2) \qquad (\alpha, \beta > 0)$$

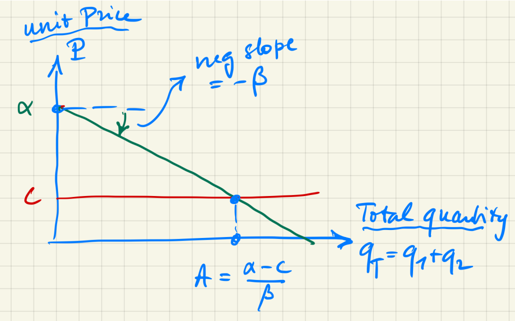
- Assume that both firms can produce the product at a fixed unit cost c. Hence the pay-off for each firm equals:

$$u_i(q_1, q_2) = (P(q_1, q_2) - c)q_i$$

Stackelberg Duopoly

$$
\begin{aligned}
u_i(q_1, q_2) &= (P(q_1, q_2) - c)q_i \\
&= (\alpha - \beta(q_1 + q_2) - c)q_i \\
&= \beta\left(\frac{\alpha - c}{\beta} - (q_1 + q_2)\right) q_i \\
&= \beta\,(A - (q_1 + q_2))\, q_i \qquad \text{where } A = (\alpha - c)/\beta.
\end{aligned}
$$

A is the total quantity at which price equals cost ($P = c$), hence a quantity in excess of A is not economically viable.



Solving Stackelberg competition using backward induction

- For an observed quantity q_1 we compute $q_2^* = BR_2(q_1)$:

$$\frac{\partial u_2}{\partial q_2} = \frac{\partial}{\partial q_2}\beta\left(A - (q_1 + q_2)\right) q_2 = \beta(A - (q_1 + 2q_2))$$

- Hence:

$$\frac{\partial u_2}{\partial q_2} = 0 \quad \Longrightarrow \quad q_2^* = \frac{1}{2}(A - q_1)$$

Solving Stackelberg competition using backward induction

- **Given the anticipated optimal response q_2^* of firm 2**, what is best action q_1^* for firm 1?
- Optimal utility:

$$
\begin{aligned}
u_1(q_1, q_2^*) &= \beta(A - q_1 - q_2^*))q_1 \\
&= \beta\,(A - q_1 - q_2^*))q_1 \\
&= \beta\left(A - q_1 - \frac{A - q_1}{2}\right))q_1 \\
&= \frac{\beta}{2}\,(A - q_1)\,q_1
\end{aligned}
$$

- Hence optimal quantities: $q_1^* = A/2$ and $q_2^* = A/4$.
- Notice: optimal total quantity $q_T^* = q_1^* + q_2^* = (3/4)A$

Solving Stackelberg competition using backward induction

Corresponding utilities The optimal total quantity is given by $q_T^* = q_1^* + q_2^* = (3/4)A$. This implies that $P(q_1^*, q_2^*) - c = (\alpha - c) - \beta q_T^* = (\beta/2)A$. From this we find

$$u_1^* = \frac{1}{8}A^2\beta \qquad \text{and} \qquad u_2^* = \frac{1}{16}A^2\beta$$

clearly showing the *first mover's advantage* for the leader (firm 1).

Comparison with Cournot's duopoly In the case of Cournot we found (previous homework):

$$q_1^* = (A - q_2)/2 \qquad \text{and} \qquad q_2^* = (A - q_1)/2.$$

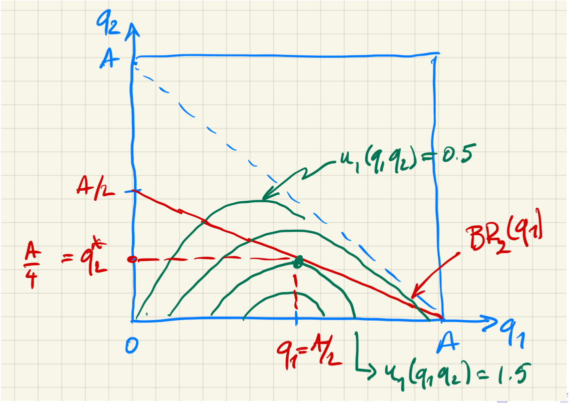
Substituting the 2nd one in the first, yields for the optimal quantities in the Cournot case:

$$q_1^* = q_2^* = \frac{1}{3}A \qquad \text{with corresponding optimal utilities} \qquad u_i^* = \frac{1}{9}A^2\beta.$$

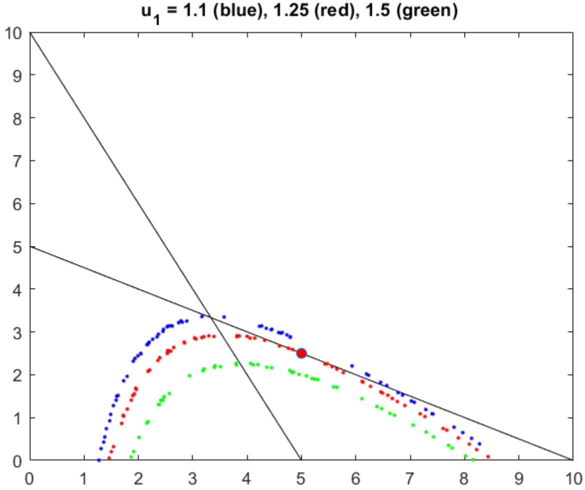
Leader versus follower

- Firm 2 plays BR $\Rightarrow$ solution on BR_2-curve: $q_2^* = (A - q_1)/2$
- Firm 1 optimises pay-off at $q_1 = A/2$, since:

$$u_1(q1, q_2^*) = \beta(A - q_1 - q_2^*)q_1 = (\beta/2)(A - q_1)q_1$$



Leader versus follower



Stackelberg (sequential) versus Cournot (simultaneous)

So compared to Cournot, in Stackelberg competition:

- the leader produces more and has higher profits ($u_1^* = (1/8)A^2\beta = 2u_2^*$)
- the follower produces less and has lower profits ($u_2^* = (1/16)A^2\beta$)

This is called a **first mover's advantage**:

- The leader (first mover) gets to optimise its utility (*anticipating BR from firm 2*);
- The follower just does the best it can (best response);
- Notice that the leader is **not** playing BR to the follower!

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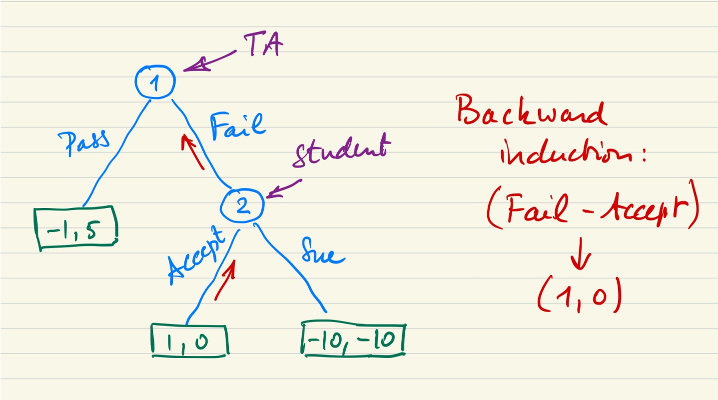
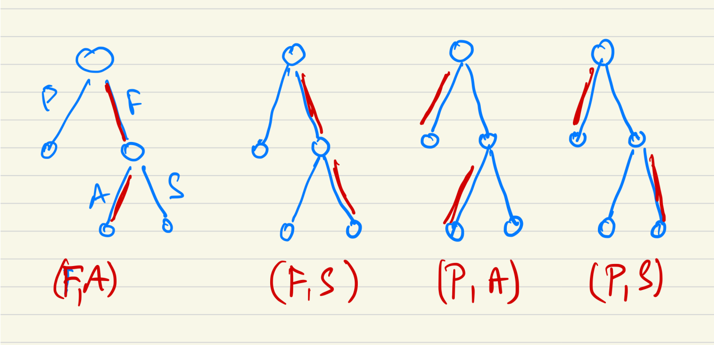
Bad homework: Nash Eq. in backward induction

Student is trying to bully TA into giving passing grade!



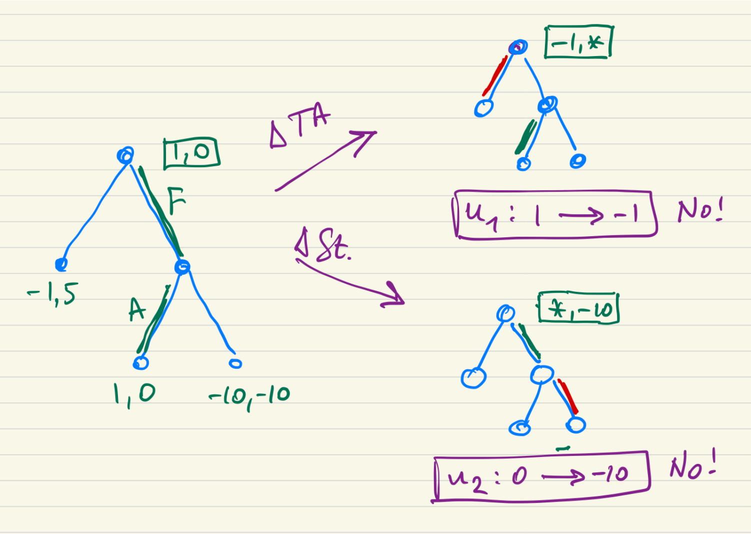
Does backward induction yield a Nash equilibrium?

- Other possible strategy profiles: see below
- **Recall:** a strategy specifies what to do in any node!

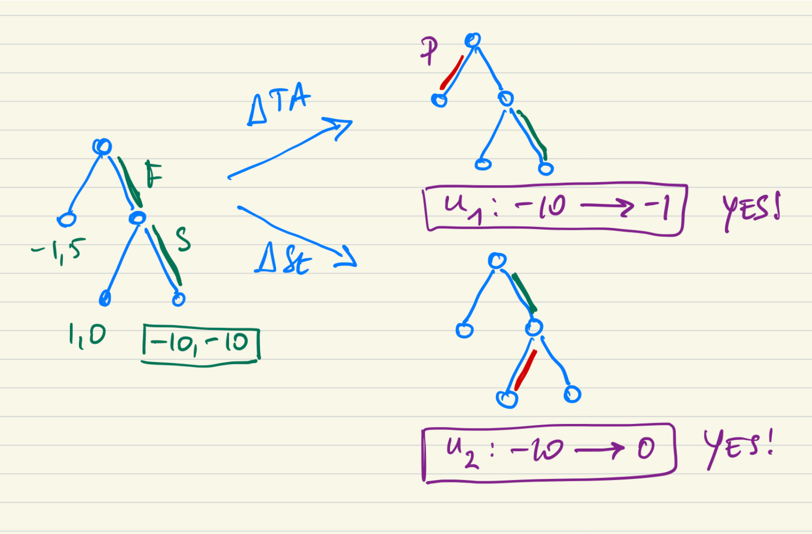


- Is backward induction solution (F, A) a Nash eq.?
- Are there other Nash eq.?

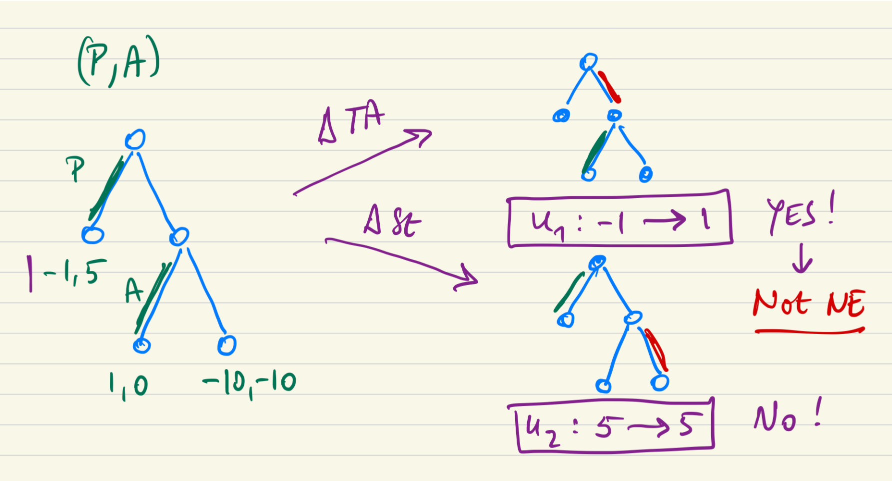
Backward Induction -solution (F,A) is Nash eq.



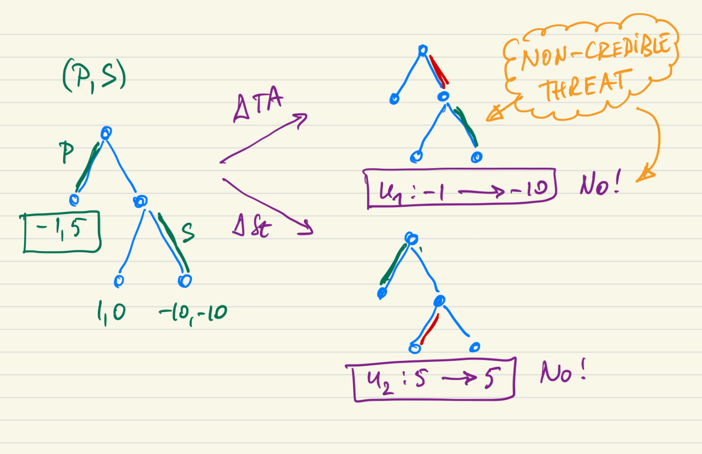
(Fail, Sue) is not NE



Pass-Accept is not NE

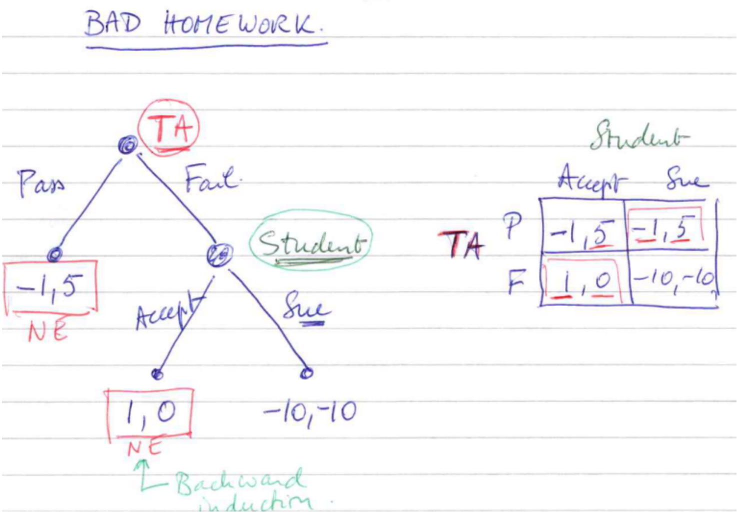


(Pass,Sue) is NE because of non-credible threat



- (Pass, Sue): no incentive for unilateral deviation: hence NE!
- . . . but due to non-credible threat (sue) by student.

Bad homework: from extensive to normal form



- NE in extensive form reappear in normal form.
- Checking NE in normal form is easier than in extensive form

Backward induction and Nash eq.

- Backward induction results in a Nash eq.
- There are additional Nash eq., however they are based on non-credible threats, i.e. choices in the game-tree (i.e. in subgames) that are not rational.
- Nash eq. that depend on non-credible threats should be eliminated.
- This gives rise to the concept of **subgame perfect (Nash) equilibrium**
- Convert sequential game to normal form: Often easier to check.

From Extensive for Normal Form (perfect information)

Aim: Transform sequential game in normal form to use standard methods to find NEs.

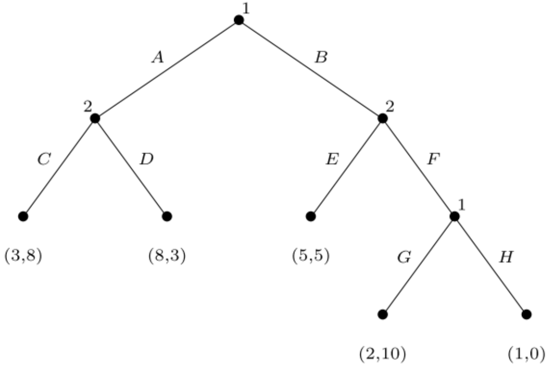
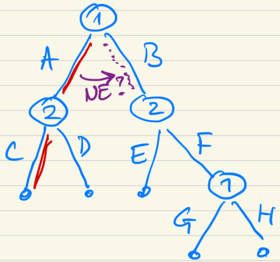


Figure 5.2: A perfect-information game in extensive form.

From Extensive for Normal Form (perfect information)



Naive approach:

P_1 $\left[\begin{array}{l} A \\ B \text{ if } P_2 \text{ chooses } E \\ BG, BH \text{ if } P_2 \text{ chooses } F \end{array}\right.$

P_2 : C $\longrightarrow$ game ends.

NE?: $A \longrightarrow B$ alternative.

Now we need to know what P_2 will do: E or F?

SOLUTION: Systematic approach:

$$P_1 \to \{A, B\} \times \{G, H\} = \{AG, AH, BG, BH\}, \quad P_2 \to \{C, D\} \times \{E, F\} = \{CE, CF, DE, DF\}.$$

From Extensive to Normal Form (perfect information)

- **A pure strategy** for player i in a (perfect information) sequential game is a complete plan of action specifying which action to take at each of its decision nodes ...
- ... irrespective of **whether or not that node can be reached** when playing the strategy!
- Mathematically: it's the **product space** of the possible actions in each decision node:
  - Node 1 has 2 decision nodes: hence $\{A, B\} \times \{G, H\} = \{(A, G), (A, H), (B, G), (B, H)\}$
  - Node 2 has 2 decision nodes: hence $\{C, D\} \times \{E, F\} = \{(C, E), (C, F), (D, E), (D, F)\}$

From extensive to normal form

- **Alternative perspective:**
  1. Extensive form: player “waits” till one of his nodes is reached, then decides what to do;
  2. Normal form: each player makes a complete contingent plan in advance.
- **Informally:**
  - It’s a **complete and contingent plan** instructing an assistant playing on your behalf, what to do in **each possible situation;**
  - Suppose that your assistant misunderstood and ended up in another node, then he still needs to know what to do.
  - Allows to explore whether unilateral deviation would be advantageous (Nash criterion)

From Extensive to Normal Form

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

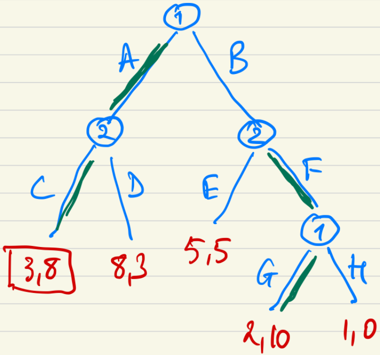
Figure 5.3: The game from Figure 5.2 in normal form.

Determining Nash eq. in Normal Form

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	$\boxed{3, 8}$	8, 3	8, 3
(A, H)	3, 8	$\boxed{3, 8}$	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	$\boxed{5, 5}$	1, 0	5, 5	1, 0

Figure 5.4: Equilibria of the game from Figure 5.2.

Checking game tree for normal form NE (AG, CF)



NE1 : (AG, CF)

P1 :

* A → B

u : (3,8) → (2,10) ↓

* G → H

u : (3,8) → (3,8) =

P2 * C → D

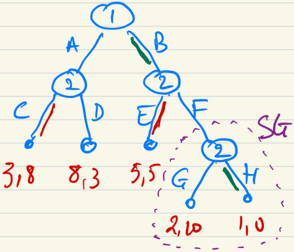
u : (3,8) → (8,3) ↓

* F → E

u : (3,8) → (3,8) =

Checking game tree for normal form NE (BH,CE)

. . . based on non-credible threat!



NE3: {BH, CE} OK

P1: B → A : (5,5) → (3,8) ↓

G → H : (5,5) → (5,5) =

↳ irrelevant

P2: C → D : irrelevant =

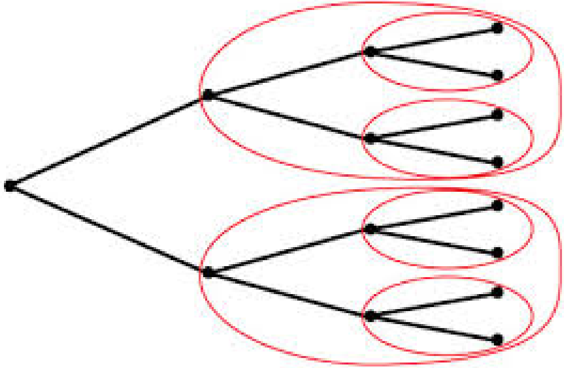
E → F : (5,5) → (1,0) ↓

HOWEVER: utility-decrease depends on P1 choosing H in final subgame

→ NON-CREDIBLE THREAT!

Introducing Subgames

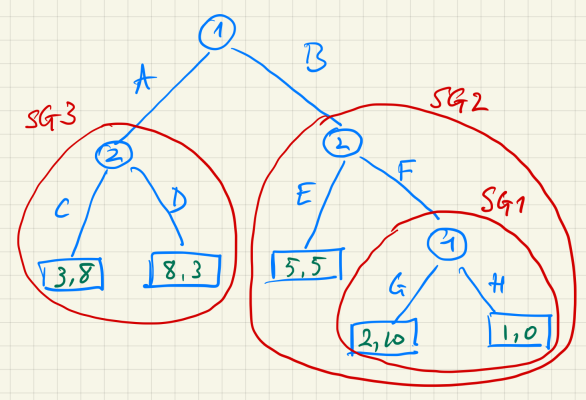
- Intuitively, subgames are subtrees of the game tree.



Introducing subgames for subgame perfection

- **Why subgames?** Allows us to translate the notion of non-credible threats to the normal form.
- **Continuation strategies** are a restrictions of a strategy for the original game to all of its subgames.
- **Subgame perfect Nash equilibrium (SPNE)** A strategy profile for an extensive-form game is a SPNE if it specifies a Nash equilibrium in each of its subgames.
- **SPNE** requires that what the players would do, conditional on being dropped in any of the subgames. should constitute a NE, even if a strategy profile dictates that certain subgames will not reached!
- This off-the-equilibrium-path behavior can be important, because it affects the incentives of players to follow the equilibrium.

Example of subgames and SPNE



Recall: Identified NE in normal form

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

Figure 5.3: The game from Figure 5.2 in normal form.

SPNE? Checking behaviour in subgames

Which of the three identified Nash eq. is a SPNE?

→ NE = (BH, CE) → $\boxed{\text{NOT SPNE}}$

SG1 $\xrightarrow{\text{induces}}$ H   NOK!

→ NE = (AH, CF) → $\boxed{\text{NOT SPNE}}$

SG1 → H   NOK!

→ NE = (AG, CF) → $\boxed{\text{SPNE}}$

SG1 → G   OK.

SG2 → (G, F)   OK →

	E	F
G	5,5	2,10
H	5,5	1,0

SG3 → C   OK

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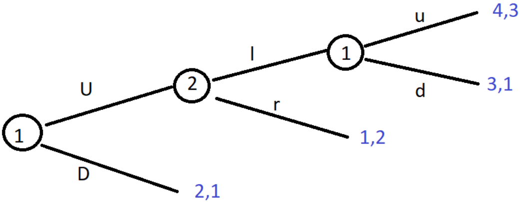
Example: Backward Induction vs. Nash Equilibria



	ℓ	r	
Uu	$\underline{4,3}$	$1,2$	backward induction!
Ud	$3,1$	$1,2$	
Du	$2,1$	$\underline{2,1}$	"r non-credible threat"
Dd	$2,1$	$\underline{2,1}$	"d non-credible threat"

Non-credible threats do not constitute NE in their subgame!

Subgame Perfect Nash Equilibrium

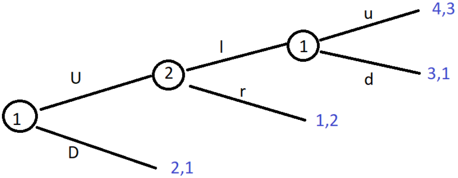
SGPE/SPNE : refinement of Nash Equilibrium:

Subgame-perfect equilibrium (SPGE, Selten 1965)

A Nash equilibrium s (of game G as a whole) is **subgame-perfect** iff for every subgame G' of G, the restriction of s to G' is also a Nash equilibrium.

- SGPE rules out Nash equilibria that rely on non-credible threats;
- Put differently: SGPE is the study of credible threats.

Example 2: Nash equilibria for subgames



	ℓ	r	
Uu	$\underline{4,3}$	$1,2$	backward induction!
Ud	$3,1$	$1,2$	
Du	$2,1$	$\underline{2,1}$	"r non-credible threat"
Dd	$2,1$	$\underline{2,1}$	"d non-credible threat"

$SG2$:

	ℓ	r
u	$\underline{4},\underline{3}$	$\underline{1},2$
d	$3,1$	$\underline{1},\underline{2}$

Notes

- d non-credible threat in SG1, implies r non-credible threat in SG2;
- Subgames (e.g. SG2) can have extra NE (comp. to full game)

Example 2, continued

Game has two non-trivial subgames:

- SG1 rooted at 2nd decision node of 1,
- SG2 rooted at decision node of 2

Normal form yields 3 NE's. Do they induce NE in all subgames?

- $NE1 = (Dd, r)$ with utility $(2, 1)$:
  induces action d in SG1 (not NE!)
- $NE2 = (Du, r)$ with utility $(2, 1)$:
  induces action (u, r) in SG2 (not NE!)
- $NE3 = (Uu, \ell)$ with utility $(4, 3)$:
  induces actions u in SG1, (u, ℓ) in SG2 (OK!)

Finding SGPE in perfect information games

Two approaches:

- **Backward induction (See next slide)**
  - works if NO infinite horizons!
- **Matrix form**
  1. Convert game-tree into matrix;
  2. Find all Nash equilibria;
  3. Eliminate the ones that depend on non-credible threats, i.e. do not induce a NE for each subgame;

Backwards induction

Algorithm to find subgame perfect equilibrium

- Consider each subgame of the game (in increasing order of inclusion)
- Find the NE for the subgame;
- Replace the subgame by a new terminal node that has the equilibrium pay-offs;

Zermelo's Thm (1913)

- With perfect information (one player in each iteration), a deterministic move is optimal. Hence there is a SGPE where each player uses a pure strategy.
- **For games with imperfect information**, a SGPE may require mixed strategies.

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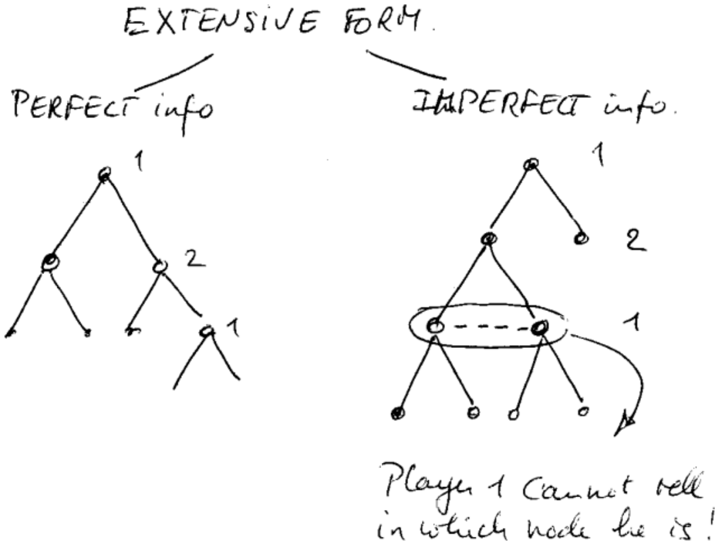
Imperfect information and information sets

- **Imperfect information: Intuition** Players need to act
  - with partial or no knowledge of actions taken by others,
  - with partial recall, i.e. limited memory of own past actions.
- An **imperfect- information game** is an extensive-form game in which each player's decision nodes are partitioned into **information sets;**
- Intuitively, if two decision nodes are in the same information set then the agent cannot distinguish between them.

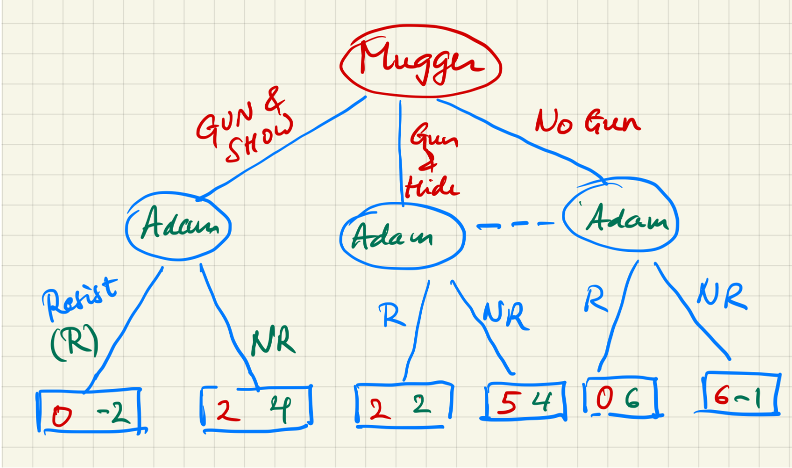
Sequential games: perfect vs. imperfect information



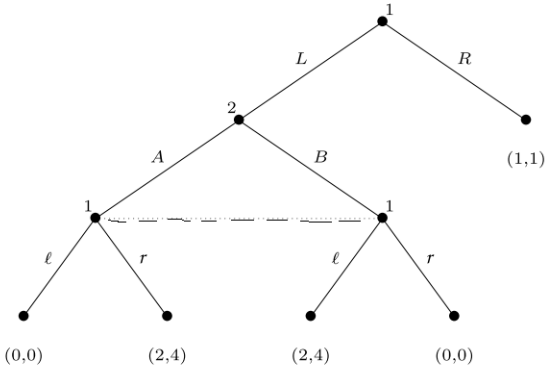
Mugger vs. Adam

- Mugger needs to decide whether to take a gun or not (aggrevated)
- In addition, if he takes a gun will he show it or keep it concealed?
- Adam is a fit young man, who might or might not resist.

Mugger vs Adam



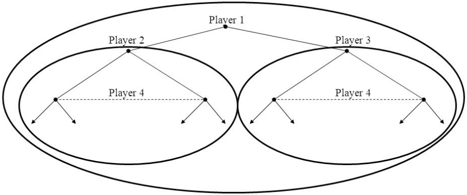
Subgames: Condition on information set



- at any information set, a player must have the same strategies regardless of how the player arrived there;
- Subgames cannot break up information sets!

Subgame of a Sequential Game with Imperfect Information

Example Subgames



How many subgames are there in this game?

1 + 1 + 1 = 3

Subgame definition:

- SG's initial node has singleton informationset;
- All successors are in SG;
- Any information-set is either completely in or out;

Pure strategies and induced normal form

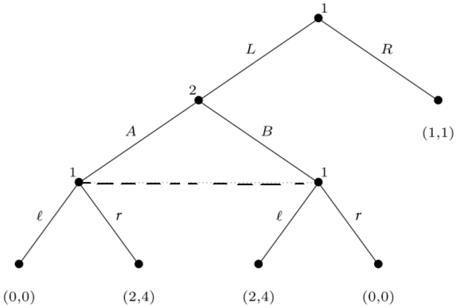


Figure 5.10: An imperfect-information game.

	A	B
$L\ell$	$0, 0$	$2, 4$
Lr	$2, 4$	$0, 0$
$R\ell$	$1, 1$	$1, 1$
Rr	$1, 1$	$1, 1$

Figure 5.14: The induced normal form of the game from Figure 5.10.

Pure actions are cartesian products over actions in information sets.

Subgame Perfect Nash Equilibrium for Imperfect Information Games

SGPE (aka SPE/SPNE) : refinement of Nash Equilibrium:

Subgame-perfect equilibrium (SPGE, Selten 1965)

A Nash equilibrium s (of game G as a whole) is **subgame-perfect** iff for every subgame G' of G, the restriction of s to G' is also a Nash equilibrium.

- SGPE rules out Nash equilibria that rely on non-credible threats;
- Put differently: SGPE is the study of credible threats.

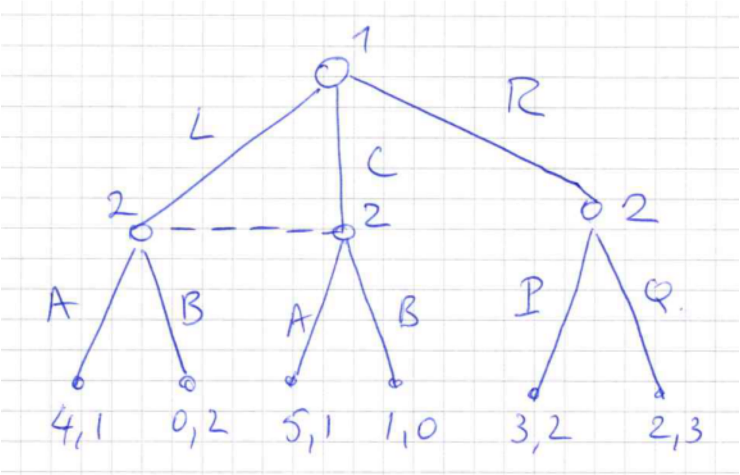
Generalized Backwards Induction for Imperfect Information Games

Systematically proceed as follows (if possible)

- Consider in turn each subgame of the game in increasing order of inclusion)

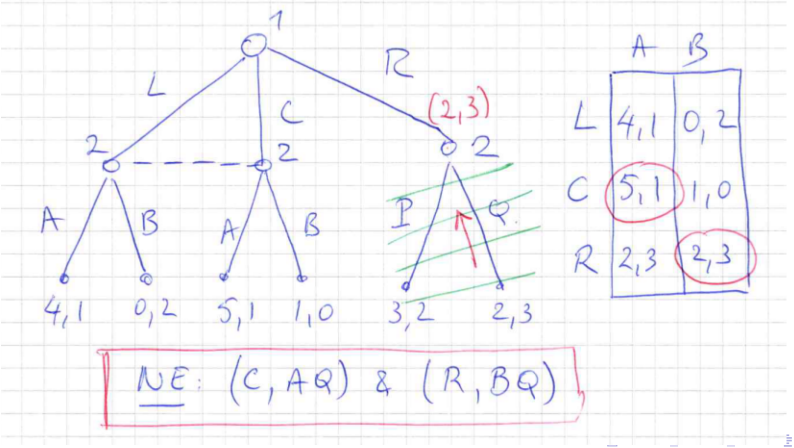
  Start at end of game, (no strategic interaction left)
  work backwards to beginning!
- Apply BI as far as you can;
- Replace the subgame by a new terminal node(s) that has the equilibrium pay-offs
  (we might need to consider different possibilities);
- If BI is not possible, use normal form solution techniques to find NE(s) for the remaining game (including mixed ones);

Generalized BI: Example 1

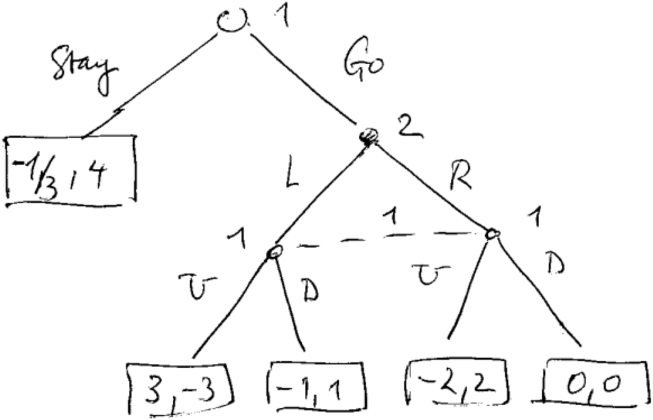


Generalized BI: Example 1, continued

- Not all "silly" NE (e.g. $(2, 3)$) can be eliminated (not enough subgames)
- Need for extra refinement (outside scope of this course).



Generalized BI, Example 2



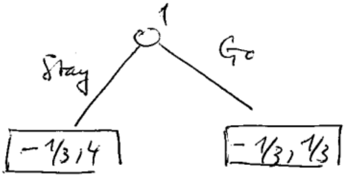
Generalized BI, Example 2 (cont'd)

$$
\begin{array}{ccc|c|c|}
 & & q & 1-q \\
 & & L & R \\
\hline
p & U & 3,-3 & -2,2 \\
\hline
1-p & D & -1,1 & 0,0 \\
\hline
\end{array}
$$

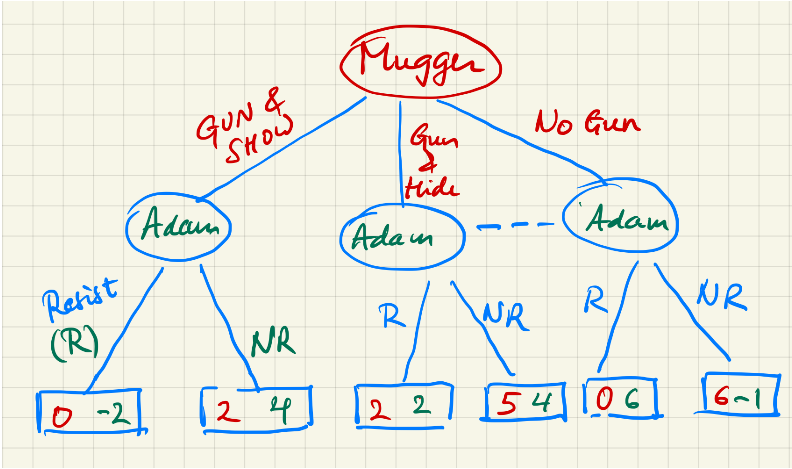
$p^* = 1/6 \qquad q^* = 1/3 .$

Generalized BI, Example 2 (cont'd)

$$
\begin{aligned}
EU_1 &= \frac{1}{6}\cdot\frac{1}{3}\cdot 3 + \frac{1}{6}\cdot\frac{2}{3}(-2) + \frac{5}{6}\cdot\frac{1}{3}(-1) + \frac{5}{6}\cdot\frac{2}{3}\,0 \\
&= \frac{1}{6} + \left(-\frac{2}{9}\right) + \left(\frac{-5}{18}\right) + 0 \\
&= \frac{3-4-5}{18} = -\frac{6}{18} = -1/3 .
\end{aligned}
$$



Mugger vs Adam



Mugger vs. Adam

$$
\begin{array}{ccc|c|c|}
 & & & (q) & (1-q) \\
 & & & \multicolumn{2}{c}{A} \\
 & & & R & NR \\
\hline
M & (p) & GH & \underline{2},2 & 5,\underline{4} \\
\hline
 & (1-p) & NG & 0,\underline{6} & \underline{6},-1 \\
\hline
\end{array}
$$

$$\boxed{\begin{aligned} EU_M(p^*,q^*) &= 6\left(1-\tfrac{1}{3}\right) = 4 \\ EU_A(p^*,q^*) &= 5\cdot\tfrac{5}{9} - 1 = \tfrac{26}{9} \end{aligned}}$$

$\underline{p^*?}$

$$2p + 6(1-p) = 4p - (1-p)$$

$$-4p + 6 = 5p - 1$$

$$\boxed{p^* = 7/9}$$

$\underline{q^* = ?}$

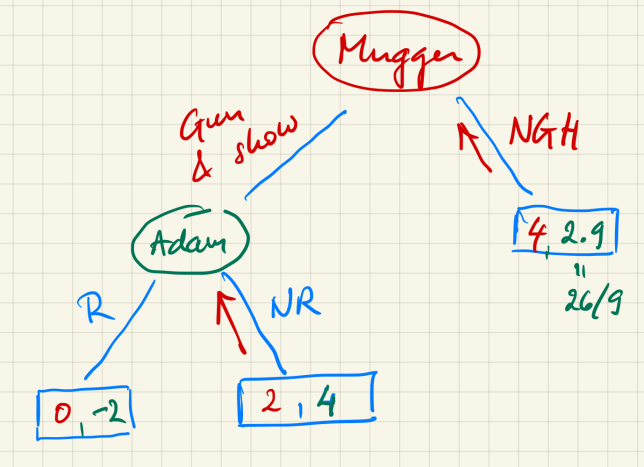
$$2q + 5(1-q) = 6(1-q)$$

$$2q - (1-q) = 0$$

$$\boxed{q^* = 1/3}$$

Mugger vs. Adam

- SPNE solution is NGH (No Gun or Hide)



Backward Induction vs. Subgame Perfection

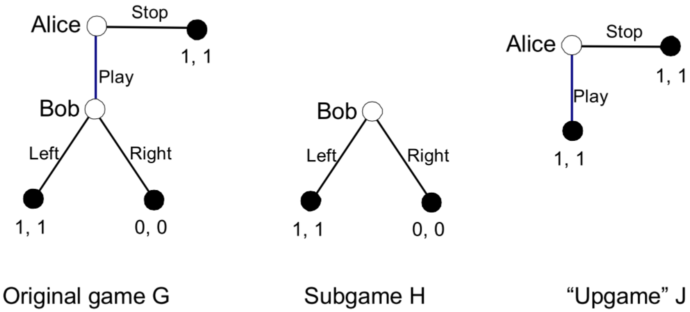


Figure 1. Backward induction versus subgame perfection.

The backward induction "upgame" J is NOT a subgame!

Ref: M. Kaminski: Generalized Backward induction, *Games* 2019, 10, 34

Backward Induction vs. Subgame Perfection

Finite sequential games with perfect information:

- All SPE can be found by backward pruning, ie.
  - systematic and incremental substitution of terminal subgames with Nash-eq pay-offs
- All BI solutions (backward pruning) are SPE

Backward Induction also works in some more general cases

e.g. some infinite games (e.g. Rubinstein)

More complex games require more restrictions on the Nash eq. solution to eliminate unreasonable solutions.

E.g. sequential rationality, perfect equilibrium