Regra da Cadeia

Luis Alberto D'Afonseca

Cálculo de Funções de Várias Variáveis – I

Conteúdo

Regra da Cadeia em uma variável

$$h(x) = f\big(g(x)\big)$$

$$\frac{dh}{dx}(x) = \frac{df}{dg}\big(g(x)\big)\,\frac{dg}{dx}(x)$$

$$h'(x) = f'\big(g(x)\big)\,g'(x)$$

Conteúdo

Regra da Cadeia – Caso 1

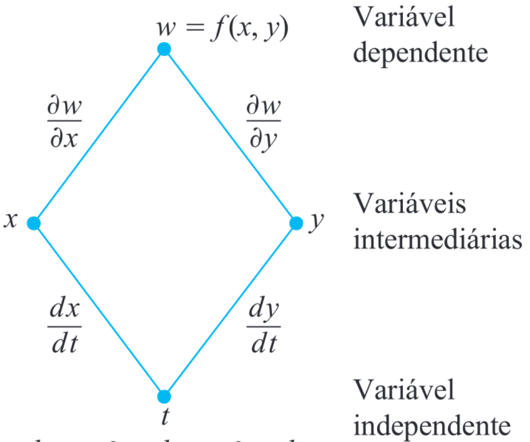
Uma variável independente e duas variáveis intermediárias

Se $w = f(x, y)$ é diferenciável e se $x = x(t)$ e $y = y(t)$ são deriváveis

a composta $w = f\big(x(t), y(t)\big)$ será uma função derivável de t e

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\big(x(t), y(t)\big)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(x(t), y(t)\big)\,\frac{dy}{dt}(t)$$

Regra da Cadeia – Caso 1



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Exemplo 1

Calcule a derivada de $w = xy$ com relação a t ao longo do caminho

$$x = \cos(t) \qquad y = \text{sen}(t)$$

Qual o valor da derivada em $t = \dfrac{\pi}{2}$

Exemplo 1 – Solução

Pela regra da cadeia

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Exemplo 1 – Solução

Calculando as derivadas separadamente

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy) = y\frac{\partial x}{\partial x} = y$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy) = x\frac{\partial y}{\partial y} = x$$

$$\frac{dx}{dt} = \frac{d}{dt}\cos(t) = -\operatorname{sen}(t)$$

$$\frac{dy}{dt} = \frac{d}{dt}\operatorname{sen}(t) = \cos(t)$$

Exemplo 1 – Solução

$$
\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial}{\partial x}(xy)\ \frac{d}{dt}\Big(\cos(t)\Big)\ +\ \frac{\partial}{\partial y}(xy)\ \frac{d}{dt}\Big(\operatorname{sen}(t)\Big) \\
&= y\Big(-\operatorname{sen}(t)\Big) + x\Big(\cos(t)\Big) \\
&= \Big(\operatorname{sen}(t)\Big)\Big(-\operatorname{sen}(t)\Big) + \Big(\cos(t)\Big)\Big(\cos(t)\Big) \\
&= -\operatorname{sen}^2(t) + \cos^2(t) \\
&= \cos(2t)
\end{aligned}
$$

Exemplo 1 – Solução

$$\begin{aligned}
\frac{dw}{dt}\left(\frac{\pi}{2}\right) &= \cos(2t)\Big|_{t=\frac{\pi}{2}} \\
&= \cos\left(2\frac{\pi}{2}\right) \\
&= \cos(\pi) \\
&= -1
\end{aligned}$$

Com Três Variáveis

Se $w = f(x, y, z)$ é diferenciável

e se $x = x(t)$, $y = y(t)$ e $z = z(t)$ são deriváveis

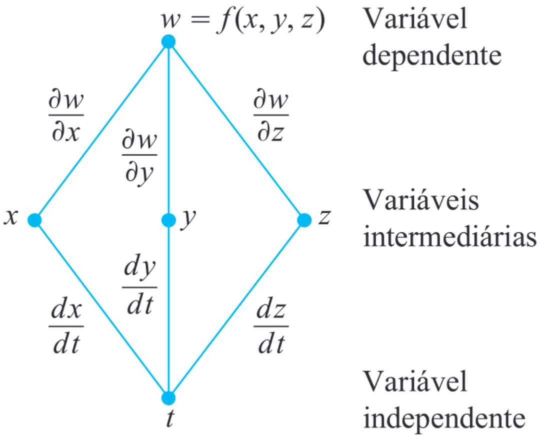
a composta $w = f\big(x(t), y(t), z(t)\big)$ será uma função derivável de t e

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\big(x(t), y(t), z(t)\big)\ \frac{dx}{dt}(t)$$

$$+ \frac{\partial f}{\partial y}\big(x(t), y(t), z(t)\big)\ \frac{dy}{dt}(t)$$

$$+ \frac{\partial f}{\partial z}\big(x(t), y(t), z(t)\big)\ \frac{dz}{dt}(t)$$

Regra da Cadeia – Caso 1



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Conteúdo

Regra da Cadeia – Caso 2

Funções definidas em superfícies

Se $w = f(x, y, z)$ é diferenciável

e se $x = x(r, s)$, $y = y(r, s)$ e $z = z(r, s)$ são deriváveis

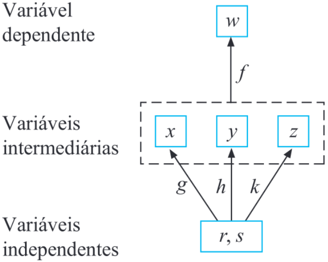
a composta $w = f\big(x(r, s), y(r, s), z(r, s)\big)$ será uma função das deriváveis r e s

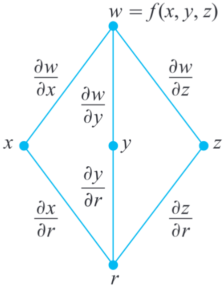
e terá duas derivadas parciais

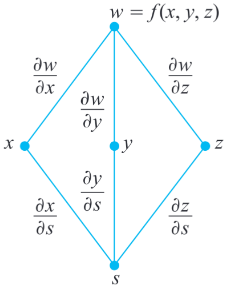
$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Regra da Cadeia – Caso 2







$$w = f(g(r, s), h(r, s), k(r, s))$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Exemplo 2

Expresse $\dfrac{\partial w}{\partial r}$ e $\dfrac{\partial w}{\partial s}$ em termos de r e s

$$w = x + 2y + z^2 \qquad x = \frac{r}{s} \qquad y = r^2 + \ln(s) \qquad z = 2r$$

Exemplo 2 – Solução

Pela regra da cadeia

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Exemplo 2 – Solução

Calculando as derivadas separadamente

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(x + 2y + z^2\right) = 1$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(x + 2y + z^2\right) = 2$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(x + 2y + z^2\right) = 2z$$

Exemplo 2 – Solução

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}\left(\frac{r}{s}\right) = \frac{1}{s} \qquad \frac{\partial x}{\partial s} = \frac{\partial}{\partial s}\left(\frac{r}{s}\right) = \frac{\partial}{\partial s}\left(rs^{-1}\right) = -\frac{r}{s^2}$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}\left(r^2 + \ln(s)\right) = 2r \qquad \frac{\partial y}{\partial s} = \frac{\partial}{\partial s}\left(r^2 + \ln(s)\right) = \frac{1}{s}$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(2r) = 2 \qquad \frac{\partial z}{\partial s} = \frac{\partial}{\partial s}(2r) = 0$$

Exemplo 2 – Solução

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= 1\frac{\partial x}{\partial r} + 2\frac{\partial y}{\partial r} + 2z\frac{\partial z}{\partial r} \\
&= 1\frac{1}{s} + 2(2r) + 2z2 \\
&= \frac{1}{s} + 4r + 4z \\
&= \frac{1}{s} + 4r + 4(2r) \\
&= \frac{1}{s} + 12r
\end{aligned}$$

Exemplo 2 – Solução

$$
\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= 1\frac{\partial x}{\partial s} + 2\frac{\partial y}{\partial s} + 2z\frac{\partial z}{\partial s} \\
&= 1\left(\frac{-r}{s^2}\right) + 2\frac{1}{s} + 2z \times 0 \\
&= -\frac{r}{s^2} + \frac{2}{s}
\end{aligned}
$$

Conteúdo

Exemplo 3

Expresse $\dfrac{\partial w}{\partial r}$ e $\dfrac{\partial w}{\partial s}$ em termos de r e s

$$w = x^2 + y^2 \qquad x = r^3 - s^2 \qquad y = r^2 + s^3$$

Exemplo 3 – Solução

Pela regra da cadeia

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Exemplo 3 – Solução

Calculando as derivadas

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y$$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}\left(r^3 - s^2\right) = 3r^2$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}\left(r^2 + s^3\right) = 2r$$

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}\left(r^3 - s^2\right) = -2s$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}\left(r^2 + s^3\right) = 3s^2$$

Exemplo 3 – Solução

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\
&= 2x\frac{\partial x}{\partial r} + 2y\frac{\partial y}{\partial r} \\
&= 2x3r^2 + 2y2r \\
&= 6(r^3 - s^2)r^2 + 4(r^2 + s^3)r \\
&= 6r^5 - 6s^2r^2 + 4r^3 + 4s^3r \\
&= 6r^5 + 4r^3 - 6r^2s^2 + 4rs^3
\end{aligned}$$

Exemplo 3 – Solução

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2x\frac{\partial x}{\partial s} + 2y\frac{\partial y}{\partial s}$$

$$= 2x\left(-2s\right) + 2y3s^2$$

$$= -4(r^3 - s^2)s + 6(r^2 + s^3)s^2$$

$$= -4r^3s + 4s^3 + 6r^2s^2 + 6s^5$$

Exemplo 4

Considere $z = \operatorname{sen}\left(x^2 + y^2\right)$ onde $x = st$ e $y = 2s + t$

Encontre $\dfrac{\partial z}{\partial s}$

Exemplo 4 – Solução

Pela regra da cadeia

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Exemplo 4 – Solução

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\operatorname{sen}\left(x^2+y^2\right) = \cos\left(x^2+y^2\right)\frac{\partial}{\partial x}\left(x^2+y^2\right) = \cos(x^2+y^2)2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\operatorname{sen}\left(x^2+y^2\right) = \cos\left(x^2+y^2\right)\frac{\partial}{\partial y}\left(x^2+y^2\right) = \cos(x^2+y^2)2y$$

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}(st) = t$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(2s+t) = 2$$

Exemplo 4 – Solução

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= 2x\cos\left(x^2+y^2\right)\frac{\partial x}{\partial s} + 2y\cos\left(x^2+y^2\right)\frac{\partial y}{\partial s} \\
&= 2x\cos\left(x^2+y^2\right)t + 2y\cos\left(x^2+y^2\right)2 \\
&= (2xt+4y)\cos\left(x^2+y^2\right) \\
&= [2(st)t + 4(s+t)]\cos\left((st)^2+(s+t)^2\right) \\
&= \left(2st^2+4s+4t\right)\cos\left(s^2t^2+(s+t)^2\right)
\end{aligned}
$$

Exemplo 5

Seja $w = e^{xyz}$ onde $x = r^2 + 4s$, $y = 2r - 3s$, $z = rs$

Encontre $\dfrac{\partial w}{\partial r}$

Exemplo 5 – Solução

Pela regra da cadeia

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Exemplo 5 – Solução

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x} e^{xyz} &= yze^{xyz} \\
\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y} e^{xyz} &= xze^{xyz} \\
\frac{\partial w}{\partial z} &= \frac{\partial}{\partial z} e^{xyz} &= xye^{xyz}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x}{\partial r} &= \frac{\partial}{\partial r} \left(r^2 + 4s\right) &= 2r \\
\frac{\partial y}{\partial r} &= \frac{\partial}{\partial r} (2r - 3s) &= 2 \\
\frac{\partial z}{\partial r} &= \frac{\partial}{\partial r} (rs) &= s
\end{aligned}
$$

Exemplo 5 – Solução

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= yze^{xyz}\frac{\partial x}{\partial r} + xze^{xyz}\frac{\partial y}{\partial r} + xye^{xyz}\frac{\partial z}{\partial r} \\
&= e^{xyz}\left(yz\frac{\partial x}{\partial r} + xz\frac{\partial y}{\partial r} + xy\frac{\partial z}{\partial r}\right) \\
&= e^{xyz}\left(yz2r + xz2 + xys\right) \\
&= e^{xyz}\left[2(2r-3s)(rs)r + 2(r^2+4s)(rs) + (r^2+4s)(2r-3s)s\right] \\
&= e^{xyz}\left[2r^2s(2r-3s) + 2rs(r^2+4s) + s(r^2+4s)(2r-3s)\right]
\end{aligned}$$

Exemplo 6

Dada $u = \ln\left(x^2 + y^2 + z^2\right)$ com $x = e^t,\ y = e^{-t},\ z = t^2$

Encontre $\dfrac{du}{dt}$

Exemplo 6 – Solução

Pela regra da cadeia

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Exemplo 6 – Solução

$$\frac{\partial u}{\partial x} = \ln'\left(x^2 + y^2 + z^2\right)\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) = \frac{2x}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial y} = \ln'\left(x^2 + y^2 + z^2\right)\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right) = \frac{2y}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial z} = \ln'\left(x^2 + y^2 + z^2\right)\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right) = \frac{2z}{x^2 + y^2 + z^2}$$

Exemplo 6 – Solução

$$\frac{dx}{dt} = \frac{de^t}{dt} = e^t$$

$$\frac{dy}{dt} = \frac{de^{-t}}{dt} = -e^{-t}$$

$$\frac{dz}{dt} = \frac{dt^2}{dt} = 2t$$

Exemplo 6 – Solução

$$
\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \\
&= \frac{2x}{x^2+y^2+z^2}\frac{dx}{dt} + \frac{2y}{x^2+y^2+z^2}\frac{dy}{dt} + \frac{2z}{x^2+y^2+z^2}\frac{dz}{dt} \\
&= \frac{2}{x^2+y^2+z^2}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) \\
&= \frac{2}{x^2+y^2+z^2}\left(xe^t - ye^{-t} + 2zt\right) \\
&= \frac{2e^{2t} - 2e^{-2t} + 4t^3}{x^2+y^2+z^2}
\end{aligned}
$$

Conteúdo

Lista Mínima

Cálculo Vol. 2 do Thomas 12ª ed. – Seção 14.4

1. Estudar o texto da seção
2. Resolver os exercícios: 3, 5, 6, 10, 12, 14, 17

Atenção: A prova é baseada no livro, não nas apresentações