

Fórmulas de De Moivre

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Cálculo de Funções de Várias Variáveis – I

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Potencias de Números Complexos

Raízes de Números Complexos

Exemplos

Lista Mínima

Justificativa

$$z^n = zz \cdots z$$

$$= \rho\rho \cdots \rho [\cos(\varphi + \varphi + \cdots + \varphi) + i \operatorname{sen}(\varphi + \varphi + \cdots + \varphi)]$$

$$= \rho^n [\cos(n\varphi) + i \operatorname{sen}(n\varphi)]$$

Primeira Fórmula de De Moivre

Se

$$z = \rho [\cos(\varphi) + i \operatorname{sen}(\varphi)]$$

e n um número natural

Primeira Fórmula de De Moivre

$$z^n = \rho^n [\cos(n\varphi) + i \operatorname{sen}(n\varphi)]$$

Exemplo 1

Calcule $z = \left[2 \left(\cos \left(\frac{\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi}{6} \right) \right) \right]^4$

Exemplo 1 – Solução

$$u = 2 \left(\cos \left(\frac{\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi}{6} \right) \right)$$

$$\rho = |u| = 2 \qquad \varphi = \arg(u) = \frac{\pi}{6}$$

$$z = u^4 = \rho^4 \left[\cos(4\varphi) + i \operatorname{sen}(4\varphi) \right]$$

$$= 2^4 \left[\cos \left(4 \frac{\pi}{6} \right) + i \operatorname{sen} \left(4 \frac{\pi}{6} \right) \right] = 16 \left[\cos \left(\frac{2\pi}{3} \right) + i \operatorname{sen} \left(\frac{2\pi}{3} \right) \right]$$

Exemplo 2

Calcule $z = \left(1 + \sqrt{3}i\right)^6$

Exemplo 2 – Solução

$$u = 1 + \sqrt{3}i \qquad a = \operatorname{Re}(u) = 1 \qquad b = \operatorname{Im}(u) = \sqrt{3}$$

$$\rho = |u| = \sqrt{a^2 + b^2} = \sqrt{1 + 3} = \sqrt{4} = 2$$

$$\operatorname{sen}(\varphi) = \frac{b}{\rho} = \frac{\sqrt{3}}{2}$$

$$\operatorname{cos}(\varphi) = \frac{a}{\rho} = \frac{1}{2}$$

$$\varphi = \arg(u) = 60^\circ = \frac{\pi}{3} \text{rad}$$

$$u = \rho (\cos(\varphi) + i \operatorname{sen}(\varphi)) = 2 \left(\cos\left(\frac{\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{3}\right) \right)$$

Exemplo 2 – Solução

$$u = 1 + \sqrt{3}i = 2 \left(\cos \left(\frac{\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi}{3} \right) \right)$$

$$\begin{aligned} z = u^6 &= \rho^6 [\cos(6\varphi) + i \operatorname{sen}(6\varphi)] \\ &= 2^6 \left[\cos \left(6\frac{\pi}{3} \right) + i \operatorname{sen} \left(6\frac{\pi}{3} \right) \right] \\ &= 64 [\cos(2\pi) + i \operatorname{sen}(2\pi)] \\ &= 64 [1 + i0] \\ &= 64 \end{aligned}$$

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Justificativa

Se $z = \rho [\cos(\varphi) + i \operatorname{sen}(\varphi)]$ e $n > 1$ é um número natural

A raiz n -ésima de z é um número

$$u = r [\cos(\alpha) + i \operatorname{sen}(\alpha)]$$

tal que

$$u^n = z$$

$$r^n [\cos(n\alpha) + i \operatorname{sen}(n\alpha)] = \rho [\cos(\varphi) + i \operatorname{sen}(\varphi)]$$

Justificativa

Encontrar r e α tais que

$$r^n = \rho$$

$$\cos(n\alpha) = \cos(\varphi)$$

$$\sin(n\alpha) = \sin(\varphi)$$

ou seja

$$r = \sqrt[n]{\rho}$$

$$n\alpha = \varphi + 2k\pi$$

$$\alpha = \frac{\varphi}{n} + 2\pi \frac{k}{n}$$

apenas

$$k = 0, 1, 2, \dots, n-1$$

fornecem soluções distintas

Segunda Formula de De Moire

Se $z = \rho [\cos(\varphi) + i \operatorname{sen}(\varphi)]$ e $n > 1$ é um número natural

As raízes n -ésimas de z são

$$u_k = \sqrt[n]{\rho} \left[\cos \left(\frac{\varphi}{n} + 2\pi \frac{k}{n} \right) + i \operatorname{sen} \left(\frac{\varphi}{n} + 2\pi \frac{k}{n} \right) \right]$$

com $k = 0, 1, 2, \dots, n-1$

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Exemplo 3

Encontre as raízes cúbicas de 1

Exemplo 3 – Solução

$$z = 1 = 1 + 0i = 1[\cos(0) + i\operatorname{sen}(0)]$$

$$\rho = |1| = 1 \qquad \varphi = \arg(1) = 0$$

$$\begin{aligned} u_k &= \sqrt[3]{\rho} \left[\cos \left(\frac{\varphi + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{\varphi + 2k\pi}{3} \right) \right] & k = 0, 1, 2 \\ &= \sqrt[3]{1} \left[\cos \left(\frac{0 + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{0 + 2k\pi}{3} \right) \right] \\ &= \cos \left(\frac{2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{2k\pi}{3} \right) \end{aligned}$$

Exemplo 3 – Solução

$$\begin{aligned}u_0 &= \cos\left(\frac{2 \times 0\pi}{3}\right) + i \operatorname{sen}\left(\frac{2 \times 0\pi}{3}\right) \\&= \cos(0) + i \operatorname{sen}(0) \\&= 1\end{aligned}$$

$$\begin{aligned}u_1 &= \cos\left(\frac{2 \times 1\pi}{3}\right) + i \operatorname{sen}\left(\frac{2 \times 1\pi}{3}\right) \\&= \cos\left(\frac{2\pi}{3}\right) + i \operatorname{sen}\left(\frac{2\pi}{3}\right) \\&= -\frac{1}{2} + \frac{\sqrt{3}}{2}i\end{aligned}$$

$$\begin{aligned}u_2 &= \cos\left(\frac{2 \times 2\pi}{3}\right) + i \operatorname{sen}\left(\frac{2 \times 2\pi}{3}\right) \\&= \cos\left(\frac{4\pi}{3}\right) + i \operatorname{sen}\left(\frac{4\pi}{3}\right) \\&= -\frac{1}{2} - \frac{\sqrt{3}}{2}i\end{aligned}$$

Exemplo 4

Dado $z = (\sqrt{3} + i)^4$

calcule

- a) a parte real de z ,
- b) a parte imaginária de z ,
- c) o módulo de z ,
- d) o argumento de z

Exemplo 4 – Avaliando u na forma polar

Convertendo $u = \sqrt{3} + i$ para a forma polar

$$\rho = |u| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$$

$$\operatorname{sen}(\varphi) = \frac{\operatorname{Im}(u)}{|u|} = \frac{1}{2} \quad \cos(\varphi) = \frac{\operatorname{Re}(u)}{|u|} = \frac{\sqrt{3}}{2} \quad \varphi = \arg(u) = 30^\circ = \frac{\pi}{6}$$

Assim

$$u = \rho [\cos(\varphi) + i \operatorname{sen}(\varphi)] = 2 \left[\cos\left(\frac{\pi}{6}\right) + i \operatorname{sen}\left(\frac{\pi}{6}\right) \right]$$

Exemplo 4 – Avaliando z

$$\begin{aligned} z &= u^4 \\ &= \rho^4 [\cos(4\varphi) + i \operatorname{sen}(4\varphi)] \\ &= 2^4 \left[\cos\left(\frac{4\pi}{6}\right) + i \operatorname{sen}\left(\frac{4\pi}{6}\right) \right] \\ &= 16 \left[\cos\left(\frac{2\pi}{3}\right) + i \operatorname{sen}\left(\frac{2\pi}{3}\right) \right] \\ &= 16 \left[-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right] \\ &= -8 + 8\sqrt{3}i \end{aligned}$$

Exemplo 4 – Avaliando z

Parte real de z

$$\operatorname{Re}(z) = -8$$

Parte imaginária de z

$$\operatorname{Im}(z) = 8\sqrt{3}$$

Módulo de z

$$|z| = 16$$

Argumento de z

$$\arg(z) = \frac{2\pi}{3}$$

Exemplo 5

Encontre as raízes cúbicas complexas do número $z = -27$

Exemplo 5 – Solução

Escrevemos $z = -27$ na forma polar

$$z = 27[1 + i0] = 27 [\cos (\pi) + i \operatorname{sen} (\pi)]$$

Exemplo 5 – Raízes cúbicas

$$\begin{aligned}u_k &= \sqrt[3]{\rho} \left[\cos \left(\frac{\varphi + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{\varphi + 2k\pi}{3} \right) \right] & k = 0, 1, 2 \\&= \sqrt[3]{27} \left[\cos \left(\frac{\pi + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi + 2k\pi}{3} \right) \right] \\&= 3 \left[\cos \left(\frac{\pi + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi + 2k\pi}{3} \right) \right]\end{aligned}$$

Exemplo 5 – $k = 0$

$$\begin{aligned}u_0 &= 3 \left[\cos \left(\frac{\pi + 2 \times 0\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi + 2 \times 0\pi}{3} \right) \right] \\&= 3 \left[\cos \left(\frac{\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi}{3} \right) \right] \\&= 3 [\cos (60^\circ) + i \operatorname{sen} (60^\circ)] \\&= 3 \left[\frac{1}{2} + i \frac{\sqrt{3}}{2} \right] \\&= \frac{3}{2} + \frac{3\sqrt{3}}{2} i\end{aligned}$$

Exemplo 5 – $k = 1$

$$\begin{aligned}u_1 &= 3 \left[\cos \left(\frac{\pi + 2 \times 1\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi + 2 \times 1\pi}{3} \right) \right] \\&= 3 [\cos(\pi) + i \operatorname{sen}(\pi)] \\&= 3 [-1 + i0] \\&= -3\end{aligned}$$

Exemplo 5 – $k = 2$

$$\begin{aligned}u_2 &= 3 \left[\cos \left(\frac{\pi + 2 \times 2\pi}{3} \right) + i \operatorname{sen} \left(\frac{\pi + 2 \times 2\pi}{3} \right) \right] \\&= 3 \left[\cos \left(\frac{5\pi}{3} \right) + i \operatorname{sen} \left(\frac{5\pi}{3} \right) \right] \\&= 3 [\cos(300^\circ) + i \operatorname{sen}(300^\circ)] \\&= 3 \left[\frac{1}{2} - i \frac{\sqrt{3}}{2} \right] \\&= \frac{3}{2} - \frac{3\sqrt{3}}{2} i\end{aligned}$$

Exemplo 6

Encontre as raízes cúbicas complexas do número $z = 1 + i$

Exemplo 6 – Solução

Escrevendo $z = 1 + i$ na forma polar

Módulo

$$\rho = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Argumento

$$\varphi = \arg(1 + i) = 45^\circ = \frac{\pi}{4}$$

Forma polar

$$z = \sqrt{2} \left[\cos \left(\frac{\pi}{4} \right) + i \operatorname{sen} \left(\frac{\pi}{4} \right) \right]$$

Exemplo 6 – Raízes cúbicas

$$\begin{aligned}u_k &= \sqrt[3]{\rho} \left[\cos \left(\frac{\varphi + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{\varphi + 2k\pi}{3} \right) \right] & k = 0, 1, 2 \\&= \sqrt[3]{\sqrt{2}} \left[\cos \left(\frac{1}{3} \left(\frac{\pi}{4} + 2k\pi \right) \right) + i \operatorname{sen} \left(\frac{1}{3} \left(\frac{\pi}{4} + 2k\pi \right) \right) \right] \\&= \sqrt[6]{2} \left[\cos \left(\frac{\pi + 8k\pi}{12} \right) + i \operatorname{sen} \left(\frac{\pi + 8k\pi}{12} \right) \right]\end{aligned}$$

Exemplo 6 – Solução

$$\begin{aligned}u_0 &= \sqrt[6]{2} \left[\cos \left(\frac{\pi + 8 \times 0\pi}{12} \right) + i \operatorname{sen} \left(\frac{\pi + 8 \times 0\pi}{12} \right) \right] \\&= \sqrt[6]{2} \left[\cos \left(\frac{\pi}{12} \right) + i \operatorname{sen} \left(\frac{\pi}{12} \right) \right]\end{aligned}$$

Exemplo 6 – Solução

$$\begin{aligned}u_1 &= \sqrt[6]{2} \left[\cos \left(\frac{\pi + 8 \times 1\pi}{12} \right) + i \operatorname{sen} \left(\frac{\pi + 8 \times 1\pi}{12} \right) \right] \\&= \sqrt[6]{2} \left[\cos \left(\frac{9\pi}{12} \right) + i \operatorname{sen} \left(\frac{9\pi}{12} \right) \right] \\&= \sqrt[6]{2} \left[\cos \left(\frac{3\pi}{4} \right) + i \operatorname{sen} \left(\frac{3\pi}{4} \right) \right] \\&= \sqrt[6]{2} \left[-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right] \\&= -\frac{2^{1/6}}{2^{1/2}} + \frac{2^{1/6}}{2^{1/2}} i \\&= -2^{-1/3} + 2^{-1/3} i\end{aligned}$$

Exemplo 6 – Solução

$$\begin{aligned} u_2 &= \sqrt[6]{2} \left[\cos \left(\frac{\pi + 8 \times 2\pi}{12} \right) + i \operatorname{sen} \left(\frac{\pi + 8 \times 2\pi}{12} \right) \right] \\ &= \sqrt[6]{2} \left[\cos \left(\frac{17}{12} \pi \right) + i \operatorname{sen} \left(\frac{17}{12} \pi \right) \right] \end{aligned}$$

Exemplo 7

Encontre as raízes cúbicas complexas do número $z = 27i$

Exemplo 7 – Forma polar

Escrevemos $z = 27i$ na forma polar

$$z = 27(0 + i) = 27 \left[\cos \left(\frac{\pi}{2} \right) + i \operatorname{sen} \left(\frac{\pi}{2} \right) \right]$$

Exemplo 7 – Solução

$$\begin{aligned}u_k &= \sqrt[3]{\rho} \left[\cos \left(\frac{\varphi + 2k\pi}{3} \right) + i \operatorname{sen} \left(\frac{\varphi + 2k\pi}{3} \right) \right] & k = 0, 1, 2 \\&= \sqrt[3]{27} \left[\cos \left(\frac{1}{3} \left(\frac{\pi}{2} + 2k\pi \right) \right) + i \operatorname{sen} \left(\frac{1}{3} \left(\frac{\pi}{2} + 2k\pi \right) \right) \right] \\&= 3 \left[\cos \left(\frac{\pi + 4k\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi + 4k\pi}{6} \right) \right]\end{aligned}$$

Exemplo 7 – Solução

$$\begin{aligned}u_0 &= 3 \left[\cos \left(\frac{\pi + 4 \times 0\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi + 4 \times 0\pi}{6} \right) \right] \\&= 3 \left[\cos \left(\frac{\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi}{6} \right) \right] \\&= 3 [\cos (30^\circ) + i \operatorname{sen} (30^\circ)] \\&= 3 \left[\frac{\sqrt{3}}{2} + i \frac{1}{2} \right] \\&= \frac{3\sqrt{3}}{2} + \frac{3}{2}i\end{aligned}$$

Exemplo 7 – Solução

$$\begin{aligned}u_1 &= 3 \left[\cos \left(\frac{\pi + 4 \times 1\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi + 4 \times 1\pi}{6} \right) \right] \\&= 3 \left[\cos \left(\frac{5\pi}{6} \right) + i \operatorname{sen} \left(\frac{5\pi}{6} \right) \right] \\&= 3 [\cos (150^\circ) + i \operatorname{sen} (150^\circ)] \\&= 3 \left[-\frac{\sqrt{3}}{2} + i \frac{1}{2} \right] \\&= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i\end{aligned}$$

Exemplo 7 – Solução

$$\begin{aligned}u_2 &= 3 \left[\cos \left(\frac{\pi + 4 \times 2\pi}{6} \right) + i \operatorname{sen} \left(\frac{\pi + 4 \times 2\pi}{6} \right) \right] \\&= 3 \left[\cos \left(\frac{9\pi}{6} \right) + i \operatorname{sen} \left(\frac{9\pi}{6} \right) \right] \\&= 3 \left[\cos \left(\frac{3\pi}{2} \right) + i \operatorname{sen} \left(\frac{3\pi}{2} \right) \right] \\&= 3 [\cos (270^\circ) + i \operatorname{sen} (270^\circ)] \\&= 3 [0 + i(-1)] = -3i\end{aligned}$$

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Atenção: A prova é baseada no livro, não nas apresentações