

# Regra da Cadeia

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Cálculo de Funções de Várias Variáveis – I



<https://material-didatico.github.io/cfvv1>

# Conteúdo

Regra da Cadeia

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# Regra da Cadeia em uma variável

$$h(x) = f(g(x))$$

$$\frac{dh}{dx}(x) = \frac{df}{dg}(g(x)) \frac{dg}{dx}(x)$$

$$h'(x) = f'(g(x))g'(x)$$

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# Regra da Cadeia – Caso 1

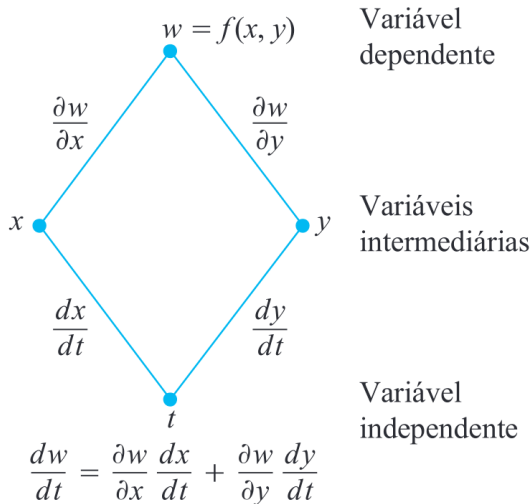
Uma variável independente e duas variáveis intermediárias

Se  $w = f(x, y)$  é diferenciável e se  $x = x(t)$  e  $y = y(t)$  são deriváveis

a composta  $w = f(x(t), y(t))$  será uma função derivável de  $t$  e

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt}(t)$$

# Regra da Cadeia – Caso 1



# Exemplo 1

Calcule a derivada de  $w = xy$  com relação a  $t$  ao longo do caminho

$$x = \cos(t) \quad y = \sin(t)$$

Qual o valor da derivada em  $t = \frac{\pi}{2}$

# Exemplo 1 – Solução

Pela regra da cadeia

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$



# Exemplo 1 – Solução

Calculando as derivadas separadamente

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xy) = y \frac{\partial x}{\partial x} = y$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xy) = x \frac{\partial y}{\partial y} = x$$

$$\frac{dx}{dt} = \frac{d}{dt} \cos(t) = -\text{sen}(t)$$

$$\frac{dy}{dt} = \frac{d}{dt} \text{sen}(t) = \cos(t)$$

## Exemplo 1 – Solução

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\&= \frac{\partial}{\partial x}(xy) \frac{d}{dt}(\cos(t)) + \frac{\partial}{\partial y}(xy) \frac{d}{dt}(\sin(t)) \\&= y(-\sin(t)) + x(\cos(t)) \\&= (\sin(t))(-\sin(t)) + (\cos(t))(\cos(t)) \\&= -\sin^2(t) + \cos^2(t) \\&= \cos(2t)\end{aligned}$$

## Exemplo 1 – Solução

$$\begin{aligned}\frac{dw}{dt} \left( \frac{\pi}{2} \right) &= \cos(2t) \Big|_{t=\frac{\pi}{2}} \\ &= \cos \left( 2 \frac{\pi}{2} \right) \\ &= \cos(\pi) \\ &= -1\end{aligned}$$

# Com Três Variáveis

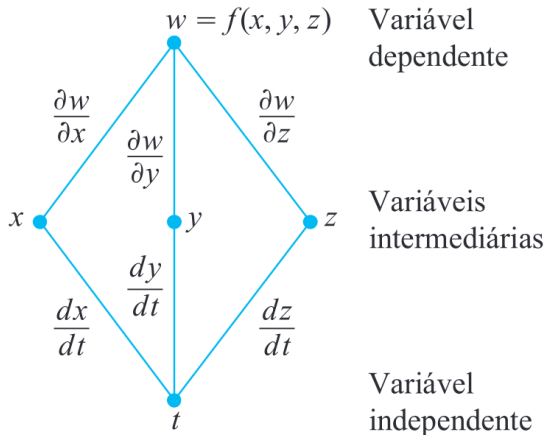
Se  $w = f(x, y, z)$  é diferenciável

e se  $x = x(t)$ ,  $y = y(t)$  e  $z = z(t)$  são deriváveis

a composta  $w = f(x(t), y(t), z(t))$  será uma função derivável de  $t$  e

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x}(x(t), y(t), z(t)) \frac{dx}{dt}(t) \\ &+ \frac{\partial f}{\partial y}(x(t), y(t), z(t)) \frac{dy}{dt}(t) \\ &+ \frac{\partial f}{\partial z}(x(t), y(t), z(t)) \frac{dz}{dt}(t)\end{aligned}$$

# Regra da Cadeia – Caso 1



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

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# Regra da Cadeia – Caso 2

Funções definidas em superfícies

Se  $w = f(x, y, z)$  é diferenciável

e se  $x = x(r, s)$ ,  $y = y(r, s)$  e  $z = z(r, s)$  são deriváveis

a composta  $w = f(x(r, s), y(r, s), z(r, s))$  será uma função das deriváveis  $r$  e  $s$

e terá duas derivadas parciais

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r}$$

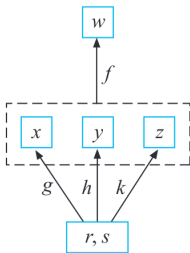
$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}$$

# Regra da Cadeia – Caso 2

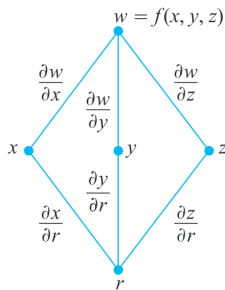
Variável dependente

Variáveis intermediárias

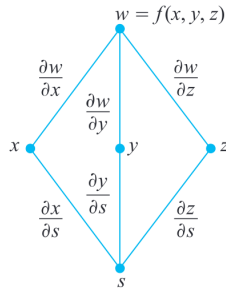
Variáveis independentes



$$w = f(g(r, s), h(r, s), k(r, s))$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$



## Exemplo 2

Expresse  $\frac{\partial w}{\partial r}$  e  $\frac{\partial w}{\partial s}$  em termos de  $r$  e  $s$

$$w = x + 2y + z^2 \quad x = \frac{r}{s} \quad y = r^2 + \ln(s) \quad z = 2r$$

## Exemplo 2 – Solução

Pela regra da cadeia

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

## Exemplo 2 – Solução

Calculando as derivadas separadamente

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} (x + 2y + z^2) = 1$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} (x + 2y + z^2) = 2$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} (x + 2y + z^2) = 2z$$

## Exemplo 2 – Solução

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r} \left( \frac{r}{s} \right) = \frac{1}{s}$$

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} \left( \frac{r}{s} \right) = \frac{\partial}{\partial s} (rs^{-1}) = -\frac{r}{s^2}$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r} (r^2 + \ln(s)) = 2r$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (r^2 + \ln(s)) = \frac{1}{s}$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r} (2r) = 2$$

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s} (2r) = 0$$

## Exemplo 2 – Solução

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\&= 1 \frac{\partial x}{\partial r} + 2 \frac{\partial y}{\partial r} + 2z \frac{\partial z}{\partial r} \\&= 1 \frac{1}{s} + 2(2r) + 2z2 \\&= \frac{1}{s} + 4r + 4z \\&= \frac{1}{s} + 4r + 4(2r) \\&= \frac{1}{s} + 12r\end{aligned}$$

## Exemplo 2 – Solução

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\&= 1 \frac{\partial x}{\partial s} + 2 \frac{\partial y}{\partial s} + 2z \frac{\partial z}{\partial s} \\&= 1 \left( \frac{-r}{s^2} \right) + 2 \frac{1}{s} + 2z \times 0 \\&= -\frac{r}{s^2} + \frac{2}{s}\end{aligned}$$

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## Exemplo 3

Expresse  $\frac{\partial w}{\partial r}$  e  $\frac{\partial w}{\partial s}$  em termos de  $r$  e  $s$

$$w = x^2 + y^2 \quad x = r^3 - s^2 \quad y = r^2 + s^3$$



## Exemplo 3 – Solução

Pela regra da cadeia

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

## Exemplo 3 – Solução

Calculando as derivadas

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2) = 2x$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} (x^2 + y^2) = 2y$$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r} (r^3 - s^2) = 3r^2$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r} (r^2 + s^3) = 2r$$

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} (r^3 - s^2) = -2s$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (r^2 + s^3) = 3s^2$$

## Exemplo 3 – Solução

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} \\&= 2x \frac{\partial x}{\partial r} + 2y \frac{\partial y}{\partial r} \\&= 2x3r^2 + 2y2r \\&= 6(r^3 - s^2)r^2 + 4(r^2 + s^3)r \\&= 6r^5 - 6s^2r^2 + 4r^3 + 4s^3r \\&= 6r^5 + 4r^3 - 6r^2s^2 + 4rs^3\end{aligned}$$

## Exemplo 3 – Solução

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \\ &= 2x \frac{\partial x}{\partial s} + 2y \frac{\partial y}{\partial s} \\ &= 2x(-2s) + 2y3s^2 \\ &= -4(r^3 - s^2)s + 6(r^2 + s^3)s^2 \\ &= -4r^3s + 4s^3 + 6r^2s^2 + 6s^5\end{aligned}$$

## Exemplo 4

Considere  $z = \sin(x^2 + y^2)$  onde  $x = st$  e  $y = 2s + t$

Encontre  $\frac{\partial z}{\partial s}$

## Exemplo 4 – Solução

Pela regra da cadeia

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

## Exemplo 4 – Solução

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \sin(x^2 + y^2) = \cos(x^2 + y^2) \frac{\partial}{\partial x} (x^2 + y^2) = \cos(x^2 + y^2) 2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \sin(x^2 + y^2) = \cos(x^2 + y^2) \frac{\partial}{\partial y} (x^2 + y^2) = \cos(x^2 + y^2) 2y$$

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} (st) = t$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (2s + t) = 2$$

## Exemplo 4 – Solução

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \\&= 2x \cos(x^2 + y^2) \frac{\partial x}{\partial s} + 2y \cos(x^2 + y^2) \frac{\partial y}{\partial s} \\&= 2x \cos(x^2 + y^2) t + 2y \cos(x^2 + y^2) 2 \\&= (2xt + 4y) \cos(x^2 + y^2) \\&= [2(st)t + 4(s + t)] \cos((st)^2 + (s + t)^2) \\&= (2st^2 + 4s + 4t) \cos(s^2t^2 + (s + t)^2)\end{aligned}$$



## Exemplo 5

Seja  $w = e^{xyz}$  onde  $x = r^2 + 4s$ ,  $y = 2r - 3s$ ,  $z = rs$

Encontre  $\frac{\partial w}{\partial r}$

## Exemplo 5 – Solução

Pela regra da cadeia

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

## Exemplo 5 – Solução

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} e^{xyz} = yze^{xyz}$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} e^{xyz} = xze^{xyz}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} e^{xyz} = xye^{xyz}$$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r} (r^2 + 4s) = 2r$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r} (2r - 3s) = 2$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r} (rs) = s$$

## Exemplo 5 – Solução

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\&= yze^{xyz} \frac{\partial x}{\partial r} + xze^{xyz} \frac{\partial y}{\partial r} + xye^{xyz} \frac{\partial z}{\partial r} \\&= e^{xyz} \left( yz \frac{\partial x}{\partial r} + xz \frac{\partial y}{\partial r} + xy \frac{\partial z}{\partial r} \right) \\&= e^{xyz} (yz2r + xz2 + xys) \\&= e^{xyz} [2(2r - 3s)(rs)r + 2(r^2 + 4s)(rs) + (r^2 + 4s)(2r - 3s)s] \\&= e^{xyz} [2r^2s(2r - 3s) + 2rs(r^2 + 4s) + s(r^2 + 4s)(2r - 3s)]\end{aligned}$$

## Exemplo 6

Dada  $u = \ln(x^2 + y^2 + z^2)$  com  $x = e^t$ ,  $y = e^{-t}$ ,  $z = t^2$

Encontre  $\frac{du}{dt}$

## Exemplo 6 – Solução

Pela regra da cadeia

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

## Exemplo 6 – Solução

$$\frac{\partial u}{\partial x} = \ln' (x^2 + y^2 + z^2) \frac{\partial}{\partial x} (x^2 + y^2 + z^2) = \frac{2x}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial y} = \ln' (x^2 + y^2 + z^2) \frac{\partial}{\partial y} (x^2 + y^2 + z^2) = \frac{2y}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial z} = \ln' (x^2 + y^2 + z^2) \frac{\partial}{\partial z} (x^2 + y^2 + z^2) = \frac{2z}{x^2 + y^2 + z^2}$$

## Exemplo 6 – Solução

$$\frac{dx}{dt} = \frac{de^t}{dt} = e^t$$

$$\frac{dy}{dt} = \frac{de^{-t}}{dt} = -e^{-t}$$

$$\frac{dz}{dt} = \frac{dt^2}{dt} = 2t$$



## Exemplo 6 – Solução

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} \\&= \frac{2x}{x^2 + y^2 + z^2} \frac{dx}{dt} + \frac{2y}{x^2 + y^2 + z^2} \frac{dy}{dt} + \frac{2z}{x^2 + y^2 + z^2} \frac{dz}{dt} \\&= \frac{2}{x^2 + y^2 + z^2} \left( x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt} \right) \\&= \frac{2}{x^2 + y^2 + z^2} (xe^t - ye^{-t} + 2zt) \\&= \frac{2e^{2t} - 2e^{-2t} + 4t^3}{x^2 + y^2 + z^2}\end{aligned}$$

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**Lista Mínima**

# Lista Mínima

Cálculo Vol. 2 do Thomas 12<sup>a</sup> ed. – Seção 14.4

1. Estudar o texto da seção
2. Resolver os exercícios: 3, 5, 6, 10, 12, 14, 17

Atenção: A prova é baseada no livro, não nas apresentações