Fórmulas de De Moivre

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Cálculo de Funções de Várias Variáveis – I

https://material-didatico.github.io/cfvv1

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Conteúdo

Justificativa

$$z^n = zz\cdots z$$

$$= \rho\rho\cdots\rho\big[\cos(\varphi+\varphi+\cdots+\varphi) + i\,\mathrm{sen}(\varphi+\varphi+\cdots+\varphi)\big]$$

$$= \rho^n\big[\cos(n\varphi) + i\,\mathrm{sen}(n\varphi)\big]$$

Primeira Fórmula de De Moivre

Se

$$z = \rho\left[\,\cos(\varphi) + i\,\mathrm{sen}(\varphi)\right]$$

e n um número natural

Primeira Fórmula de De Moivre

$$z^n = \rho^n\left[\,\cos(n\varphi) + i\,\mathrm{sen}(n\varphi)\right]$$

Exemplo 1

Calcule $z = \left[2\left(\cos\left(\frac{\pi}{6}\right) + i\,\text{sen}\left(\frac{\pi}{6}\right)\right)\right]^4$

Exemplo 1 – Solução

$$u = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}\right)\right)$$

$$\rho = |u| = 2 \qquad\qquad \varphi = \arg(u) = \frac{\pi}{6}$$

$$z = u^4 = \rho^4\big[\cos(4\varphi) + i\,\mathrm{sen}(4\varphi)\big]$$

$$= 2^4\left[\cos\left(4\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(4\frac{\pi}{6}\right)\right] = 16\left[\cos\left(\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right)\right]$$

Exemplo 2

Calcule $z = \left(1 + \sqrt{3}i\right)^6$

Exemplo 2 – Solução

$$u = 1 + \sqrt{3}i \qquad a = \mathrm{Re}(u) = 1 \qquad b = \mathrm{Im}(u) = \sqrt{3}$$

$$\rho = |u| = \sqrt{a^2 + b^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\mathrm{sen}(\varphi) = \frac{b}{\rho} = \frac{\sqrt{3}}{2}$$

$$\cos(\varphi) = \frac{a}{\rho} = \frac{1}{2}$$

$$\varphi = \arg(u) = 60^\circ = \frac{\pi}{3}\mathrm{rad}$$

$$u = \rho\left(\cos(\varphi) + i\,\mathrm{sen}(\varphi)\right) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{3}\right)\right)$$

Exemplo 2 – Solução

$$u = 1 + \sqrt{3}i = 2\left(\cos\left(\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{3}\right)\right)$$

$$\begin{aligned}
z = u^6 &= \rho^6\left[\cos(6\varphi) + i\,\mathrm{sen}(6\varphi)\right] \\
&= 2^6\left[\cos\left(6\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(6\frac{\pi}{3}\right)\right] \\
&= 64\left[\cos\left(2\pi\right) + i\,\mathrm{sen}\left(2\pi\right)\right] \\
&= 64\left[1 + i0\right] \\
&= 64
\end{aligned}$$

Conteúdo

Justificativa

Se $z = \rho\big[\cos(\varphi) + i\,\mathrm{sen}(\varphi)\big]$ e $n > 1$ é um número natural

A raiz n-ésima de z é um número

$$u = r\big[\cos(\alpha) + i\,\mathrm{sen}(\alpha)\big]$$

tal que

$$u^n = z$$

$$r^n\big[\cos(n\alpha) + i\,\mathrm{sen}(n\alpha)\big] = \rho\big[\cos(\varphi) + i\,\mathrm{sen}(\varphi)\big]$$

Justificativa

Encontrar r e α tais que

$$r^n = \rho$$

$$\cos(n\alpha) = \cos(\varphi)$$

$$\operatorname{sen}(n\alpha) = \operatorname{sen}(\varphi)$$

ou seja

$$r = \sqrt[n]{\rho}$$

$$n\alpha = \varphi + 2k\pi$$

$$\alpha = \frac{\varphi}{n} + 2\pi\frac{k}{n}$$

apenas

$$k = 0, 1, 2, \ldots, n-1$$

fornecem soluções distintas

Segunda Formula de De Moire

Se $z = \rho\big[\cos(\varphi) + i\,\mathrm{sen}(\varphi)\big]$ e $n > 1$ é um número natural

As raízes n-ésimas de z são

$$u_k = \sqrt[n]{\rho}\left[\cos\left(\frac{\varphi}{n} + 2\pi\frac{k}{n}\right) + i\,\mathrm{sen}\left(\frac{\varphi}{n} + 2\pi\frac{k}{n}\right)\right]$$

com $k = 0, 1, 2, \ldots, n-1$

Conteúdo

Exemplo 3

Encontre as raízes cúbicas de 1

Exemplo 3 – Solução

$$z = 1 = 1 + 0i = 1\left[\cos(0) + i\,\mathrm{sen}(0)\right]$$

$$\rho = |1| = 1 \qquad\qquad \varphi = \arg(1) = 0$$

$$u_k = \sqrt[3]{\rho}\left[\cos\left(\frac{\varphi + 2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\varphi + 2k\pi}{3}\right)\right] \qquad\qquad k = 0, 1, 2$$

$$= \sqrt[3]{1}\left[\cos\left(\frac{0 + 2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{0 + 2k\pi}{3}\right)\right]$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2k\pi}{3}\right)$$

Exemplo 3 – Solução

$$
\begin{aligned}
u_0 &= \cos\left(\frac{2\times 0\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\times 0\pi}{3}\right)\\
&= \cos(0) + i\,\mathrm{sen}(0)\\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= \cos\left(\frac{2\times 1\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\times 1\pi}{3}\right)\\
&= \cos\left(\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right)\\
&= -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= \cos\left(\frac{2\times 2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\times 2\pi}{3}\right)\\
&= \cos\left(\frac{4\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{4\pi}{3}\right)\\
&= -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}
$$

Exemplo 4

Dado $z = \left(\sqrt{3} + i\right)^4$

calcule

a) a parte real de z,

b) a parte imaginária de z,

c) o módulo de z,

d) o argumento de z

Exemplo 4 – Avaliando u na forma polar

Convertendo $u = \sqrt{3} + i$ para a forma polar

$$\rho = |u| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$$

$$\operatorname{sen}(\varphi) = \frac{\operatorname{Im}(u)}{|u|} = \frac{1}{2} \qquad \cos(\varphi) = \frac{\operatorname{Re}(u)}{|u|} = \frac{\sqrt{3}}{2} \qquad \varphi = \arg(u) = 30^\circ = \frac{\pi}{6}$$

Assim

$$u = \rho\left[\cos\left(\varphi\right) + i \operatorname{sen}\left(\varphi\right)\right] = 2\left[\cos\left(\frac{\pi}{6}\right) + i \operatorname{sen}\left(\frac{\pi}{6}\right)\right]$$

Exemplo 4 – Avaliando z

$$
\begin{aligned}
z &= u^4 \\
&= \rho^4 \left[\cos\left(4\varphi\right) + i \operatorname{sen}\left(4\varphi\right)\right] \\
&= 2^4 \left[\cos\left(\frac{4\pi}{6}\right) + i \operatorname{sen}\left(\frac{4\pi}{6}\right)\right] \\
&= 16 \left[\cos\left(\frac{2\pi}{3}\right) + i \operatorname{sen}\left(\frac{2\pi}{3}\right)\right] \\
&= 16 \left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] \\
&= -8 + 8\sqrt{3}i
\end{aligned}
$$

Exemplo 4 – Avaliando z

Parte real de z

$$\operatorname{Re}(z) = -8$$

Parte imaginária de z

$$\operatorname{Im}(z) = 8\sqrt{3}$$

Módulo de z

$$|z| = 16$$

Argumento de z

$$\arg(z) = \frac{2\pi}{3}$$

Exemplo 5

Encontre as raízes cúbicas complexas do número $z = -27$

Exemplo 5 – Solução

Escrevemos $z = -27$ na forma polar

$$z = 27[1 + i0] = 27\left[\cos\left(\pi\right) + i\,\text{sen}\left(\pi\right)\right]$$

Exemplo 5 – Raízes cúbicas

$$
\begin{aligned}
u_k &= \sqrt[3]{\rho}\left[\cos\left(\frac{\varphi + 2k\pi}{3}\right) + i\,\text{sen}\left(\frac{\varphi + 2k\pi}{3}\right)\right] \qquad k = 0, 1, 2 \\
&= \sqrt[3]{27}\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi + 2k\pi}{3}\right)\right] \\
&= 3\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi + 2k\pi}{3}\right)\right]
\end{aligned}
$$

Exemplo 5 – $k = 0$

$$
\begin{aligned}
u_0 &= 3\left[\cos\left(\frac{\pi + 2 \times 0\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi + 2 \times 0\pi}{3}\right)\right] \\
&= 3\left[\cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right)\right] \\
&= 3\left[\cos\left(60^\circ\right) + i\,\text{sen}\left(60^\circ\right)\right] \\
&= 3\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] \\
&= \frac{3}{2} + \frac{3\sqrt{3}}{2}i
\end{aligned}
$$

Exemplo 5 – $k = 1$

$$
\begin{aligned}
u_1 &= 3\left[\cos\left(\frac{\pi + 2 \times 1\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi + 2 \times 1\pi}{3}\right)\right] \\
&= 3\left[\cos\left(\pi\right) + i\,\text{sen}\left(\pi\right)\right] \\
&= 3\left[-1 + i0\right] \\
&= -3
\end{aligned}
$$

Exemplo 5 – $k = 2$

$$
\begin{aligned}
u_2 &= 3\left[\cos\left(\frac{\pi + 2 \times 2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi + 2 \times 2\pi}{3}\right)\right] \\
&= 3\left[\cos\left(\frac{5\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{3}\right)\right] \\
&= 3\left[\cos\left(300^\circ\right) + i\,\mathrm{sen}\left(300^\circ\right)\right] \\
&= 3\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] \\
&= \frac{3}{2} - \frac{3\sqrt{3}}{2}i
\end{aligned}
$$

Exemplo 6

Encontre as raízes cúbicas complexas do número $z = 1 + i$

Exemplo 6 – Solução

Escrevendo $z = 1 + i$ na forma polar

Módulo

$$\rho = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Argumento

$$\varphi = \arg(1 + i) = 45^\circ = \frac{\pi}{4}$$

Forma polar

$$z = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi}{4}\right)\right]$$

Exemplo 6 – Raízes cúbicas

$$
\begin{aligned}
u_k &= \sqrt[3]{\rho}\left[\cos\left(\frac{\varphi + 2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\varphi + 2k\pi}{3}\right)\right] \qquad\qquad k = 0, 1, 2\\
&= \sqrt[3]{\sqrt{2}}\left[\cos\left(\frac{1}{3}\left(\frac{\pi}{4} + 2k\pi\right)\right) + i\,\mathrm{sen}\left(\frac{1}{3}\left(\frac{\pi}{4} + 2k\pi\right)\right)\right]\\
&= \sqrt[6]{2}\left[\cos\left(\frac{\pi + 8k\pi}{12}\right) + i\,\mathrm{sen}\left(\frac{\pi + 8k\pi}{12}\right)\right]
\end{aligned}
$$

Exemplo 6 – Solução

$$
\begin{aligned}
u_0 &= \sqrt[6]{2}\left[\cos\left(\frac{\pi + 8 \times 0\pi}{12}\right) + i\,\mathrm{sen}\left(\frac{\pi + 8 \times 0\pi}{12}\right)\right] \\
&= \sqrt[6]{2}\left[\cos\left(\frac{\pi}{12}\right) + i\,\mathrm{sen}\left(\frac{\pi}{12}\right)\right]
\end{aligned}
$$

Exemplo 6 – Solução

$$
\begin{aligned}
u_1 &= \sqrt[6]{2}\left[\cos\left(\frac{\pi + 8 \times 1\pi}{12}\right) + i\,\mathrm{sen}\left(\frac{\pi + 8 \times 1\pi}{12}\right)\right] \\
&= \sqrt[6]{2}\left[\cos\left(\frac{9\pi}{12}\right) + i\,\mathrm{sen}\left(\frac{9\pi}{12}\right)\right] \\
&= \sqrt[6]{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{4}\right)\right] \\
&= \sqrt[6]{2}\left[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] \\
&= -\frac{2^{1/6}}{2^{1/2}} + \frac{2^{1/6}}{2^{1/2}}i \\
&= -2^{-1/3} + 2^{-1/3}i
\end{aligned}
$$

Exemplo 6 – Solução

$$
\begin{aligned}
u_2 &= \sqrt[6]{2}\left[\cos\left(\frac{\pi + 8 \times 2\pi}{12}\right) + i\,\mathrm{sen}\left(\frac{\pi + 8 \times 2\pi}{12}\right)\right] \\
&= \sqrt[6]{2}\left[\cos\left(\frac{17}{12}\pi\right) + i\,\mathrm{sen}\left(\frac{17}{12}\pi\right)\right]
\end{aligned}
$$

Exemplo 7

Encontre as raízes cúbicas complexas do número $z = 27i$

Exemplo 7 – Forma polar

Escrevemos $z = 27i$ na forma polar

$$z = 27(0 + i) = 27\left[\cos\left(\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}\right)\right]$$

Exemplo 7 – Solução

$$
\begin{aligned}
u_k &= \sqrt[3]{\rho}\left[\cos\left(\frac{\varphi + 2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\varphi + 2k\pi}{3}\right)\right] \qquad k = 0, 1, 2 \\
&= \sqrt[3]{27}\left[\cos\left(\frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right)\right) + i\,\mathrm{sen}\left(\frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right)\right)\right] \\
&= 3\left[\cos\left(\frac{\pi + 4k\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi + 4k\pi}{6}\right)\right]
\end{aligned}
$$

Exemplo 7 – Solução

$$
\begin{aligned}
u_0 &= 3\left[\cos\left(\frac{\pi + 4 \times 0\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi + 4 \times 0\pi}{6}\right)\right] \\
&= 3\left[\cos\left(\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}\right)\right] \\
&= 3\left[\cos\left(30^\circ\right) + i\,\mathrm{sen}\left(30^\circ\right)\right] \\
&= 3\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] \\
&= \frac{3\sqrt{3}}{2} + \frac{3}{2}i
\end{aligned}
$$

Exemplo 7 – Solução

$$
\begin{aligned}
u_1 &= 3\left[\cos\left(\frac{\pi + 4 \times 1\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi + 4 \times 1\pi}{6}\right)\right] \\
&= 3\left[\cos\left(\frac{5\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{6}\right)\right] \\
&= 3\left[\cos\left(150^\circ\right) + i\,\mathrm{sen}\left(150^\circ\right)\right] \\
&= 3\left[-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] \\
&= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i
\end{aligned}
$$

Exemplo 7 – Solução

$$\begin{aligned}
u_2 &= 3\left[\cos\left(\frac{\pi + 4 \times 2\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi + 4 \times 2\pi}{6}\right)\right] \\
&= 3\left[\cos\left(\frac{9\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{9\pi}{6}\right)\right] \\
&= 3\left[\cos\left(\frac{3\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{3\pi}{2}\right)\right] \\
&= 3\left[\cos\left(270^\circ\right) + i\,\mathrm{sen}\left(270^\circ\right)\right] \\
&= 3\left[0 + i(-1)\right] \;\; = \;\; -3i
\end{aligned}$$

Conteúdo

Lista Mínima

Atenção: A prova é baseada no livro, não nas apresentações