Derivadas Parciais de Ordem Superior

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Cálculo de Funções de Várias Variáveis – I

17 de agosto de 2025

Conteúdo

Conceito

Derivada de uma derivada de uma função

Notações

Derivadas de segunda ordem

$$\frac{\partial^2 f}{\partial x^2} \qquad f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} \qquad f_{yy}$$

Derivadas de segunda ordem mistas

$$\frac{\partial^2 f}{\partial x \partial y} \qquad f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} \qquad f_{xy}$$

Atenção Para a Ordem das Derivadas Mistas

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \left(f_x\right)_y$$

Exemplo 1

Encontre as derivadas de segunda ordem da função

$$f(x, y) = x\cos(y) + ye^x$$

Note que f é contínua no plano

Precisamos calcular f_{xx}, f_{yy}, f_{xy} e f_{yx}

Exemplo 1 – f_x

$$
\begin{aligned}
f_x(x,y) &= \frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\left(x\cos(y) + ye^x\right) \\
&= \frac{\partial}{\partial x}\left(x\cos(y)\right) + \frac{\partial}{\partial x}\left(ye^x\right) \\
&= \cos(y)\frac{\partial x}{\partial x} + y\frac{\partial e^x}{\partial x} \\
&= \cos(y) + ye^x \qquad\qquad \text{(contínua no plano)}
\end{aligned}
$$

Exemplo 1 – f_y

$$
\begin{aligned}
f_y(x, y) &= \frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial y}\left(x\cos(y) + ye^x\right) \\
&= \frac{\partial}{\partial y}\left(x\cos(y)\right) + \frac{\partial}{\partial y}\left(ye^x\right) \\
&= x\frac{\partial}{\partial y}\cos(y) + e^x\frac{\partial y}{\partial y} \\
&= -x\,\mathrm{sen}(y) + e^x \\
&= e^x - x\,\mathrm{sen}(y) && \text{(contínua no plano)}
\end{aligned}
$$

Exemplo 1 – f_{xx}

$$
\begin{aligned}
f_{xx}(x, y) &= \frac{\partial^2 f}{\partial x^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\big(\cos(y) + ye^x\big) \\
&= \frac{\partial}{\partial x}\big(\cos(y)\big) + \frac{\partial}{\partial x}\big(ye^x\big) \\
&= 0 + y\frac{\partial e^x}{\partial x} \\
&= ye^x \qquad\qquad \text{(contínua no plano)}
\end{aligned}
$$

Exemplo 1 – f_{xy}

$$\begin{aligned}
f_{xy}(x,y) &= (f_x)_y\,(x,y) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\big(\cos(y) + ye^x\big) \\
&= \frac{\partial}{\partial y}\big(\cos(y)\big) + \frac{\partial}{\partial y}\big(ye^x\big) \\
&= -\operatorname{sen}(y) + e^x\frac{\partial y}{\partial y} \\
&= -\operatorname{sen}(y) + e^x \;=\; e^x - \operatorname{sen}(y) \qquad\qquad \text{(contínua no plano)}
\end{aligned}$$

Exemplo 1 – f_{yy}

$$\begin{aligned}
f_{yy}(x, y) &= \frac{\partial^2 f}{\partial y^2} \\
&= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\big(e^x - x\,\mathrm{sen}(y)\big) \\
&= \frac{\partial}{\partial y}\big(e^x\big) - \frac{\partial}{\partial y}\big(x\,\mathrm{sen}(y)\big) \\
&= 0 - x\frac{\partial}{\partial y}\,\mathrm{sen}(y) \\
&= -x\cos(y) && \text{(contínua no plano)}
\end{aligned}$$

Exemplo 1 – f_{yx}

$$
\begin{aligned}
f_{yx}(x, y) &= \left(f_y\right)_x (x, y) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(e^x - x\,\mathrm{sen}(y)\right) \\
&= \frac{\partial}{\partial x}e^x - \frac{\partial}{\partial x}\left(x\,\mathrm{sen}(y)\right) \\
&= e^x - \mathrm{sen}(y)\frac{\partial x}{\partial x} \\
&= e^x - \mathrm{sen}(y) && \text{(contínua no plano)}
\end{aligned}
$$

Exemplo 1 – Resposta

$$f_{xx}(x, y) = ye^x$$

$$f_{xy}(x, y) = e^x - \operatorname{sen}(y)$$

$$f_{yy}(x, y) = -x\cos(y)$$

$$f_{yx}(x, y) = e^x - \operatorname{sen}(y)$$

Exemplo 2

Encontre as derivadas de segunda ordem mistas da função $f(x, y) = \operatorname{arctg}\left(\dfrac{y}{x}\right)$

Dica $\dfrac{d}{dt}\operatorname{arctg}(t) = \operatorname{arctg}'(t) = \dfrac{1}{1+t^2}$

f não é contínua no plano

Precisamos calcular f_{xy} e f_{yx}

Exemplo 2 – f_x

$$
\begin{aligned}
f_x(x,y) &= \frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}\left(\operatorname{arctg}\left(\frac{y}{x}\right)\right) \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right) y\frac{\partial x^{-1}}{\partial x} \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right) y\left(-x^{-2}\right) \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right)\frac{-y}{x^2}
\end{aligned}
$$

Lembrando que $\operatorname{arctg}'(t) = \dfrac{1}{1+t^2}$

$$
\begin{aligned}
f_x(x,y) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{-y}{x^2} \\
&= \frac{1}{\frac{x^2+y^2}{x^2}}\,\frac{-y}{x^2} \\
&= \frac{x^2}{x^2+y^2}\,\frac{-y}{x^2} \\
&= \frac{-y}{x^2+y^2}
\end{aligned}
$$

Exemplo 2 – f_y

$$\begin{aligned}
f_y(x,y) &= \frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial y}\left(\operatorname{arctg}\left(\frac{y}{x}\right)\right) \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right)\frac{1}{x}\frac{\partial y}{\partial y} \\
&= \operatorname{arctg}'\left(\frac{y}{x}\right)\frac{1}{x}
\end{aligned}$$

Lembrando que $\operatorname{arctg}'(t) = \dfrac{1}{1+t^2}$

$$\begin{aligned}
f_y(x,y) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{1}{x} \\
&= \frac{1}{\frac{x^2+y^2}{x^2}}\,\frac{1}{x} \\
&= \frac{x^2}{x^2+y^2}\,\frac{1}{x} \\
&= \frac{x}{x^2+y^2}
\end{aligned}$$

Exemplo 2 – f_{xy}

$$\begin{aligned} f_{xy}(x,y) &= (f_x)_y\,(x,y) \\ &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \\ &= -\frac{(x^2+y^2)\frac{\partial y}{\partial y} - y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} \end{aligned}$$

$$\begin{aligned} f_{xy}(x,y) &= -\frac{x^2+y^2-y\left(\frac{\partial x^2}{\partial y}+\frac{\partial y^2}{\partial y}\right)}{(x^2+y^2)^2} \\ &= -\frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} \\ &= -\frac{x^2-y^2}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

Exemplo 2 – f_{yx}

$$\begin{aligned}
f_{yx}(x, y) &= \left(f_y\right)_x (x, y) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) \\
&= \frac{\left(x^2 + y^2\right)\dfrac{\partial x}{\partial x} - x\dfrac{\partial}{\partial x}\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^2}
\end{aligned}$$

$$\begin{aligned}
f_{yx}(x, y) &= \frac{x^2 + y^2 - x\left(\dfrac{\partial x^2}{\partial x} + \dfrac{\partial y^2}{\partial x}\right)}{\left(x^2 + y^2\right)^2} \\
&= \frac{x^2 + y^2 - x\left(2x\right)}{\left(x^2 + y^2\right)^2} \\
&= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}
\end{aligned}$$

Exemplo 2 – Resposta

$$f_{xy}(x, y) = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

$$f_{yx}(x, y) = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}$$

Conteúdo

Teorema das Derivadas Mistas

Se a função $f(x, y)$ e suas derivadas parciais f_x, f_y, f_{xy} e f_{yx}

forem definidas em uma vizinhança do ponto (a, b)

e todas forem contínuas, então

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Exemplo 3

Encontre $\dfrac{\partial^2 w}{\partial x \partial y}$ se $w = xy + \dfrac{e^y}{y^2+1}$

Assumindo que as derivadas são contínuas

Exemplo 3 – solução

Queremos calcular

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

Calculando a primeira derivada

$$\begin{aligned}\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y}\left(xy + \frac{e^y}{y^2+1}\right) \\ &= \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}\left(\frac{e^y}{y^2+1}\right)\end{aligned}$$

Precisamos calcular a derivada de uma divisão

Exemplo 3 – solução

Usando a hipótese de que as derivadas são contínuas

podemos usar o teorema

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$$

Temos então que calcular

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$

Exemplo 3 – solução

Calculando a derivada primeira

$$
\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1}\right) \\
&= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}\left(\frac{e^y}{y^2+1}\right) \\
&= y\frac{\partial x}{\partial x} + 0 \\
&= y \qquad\qquad \text{(contínua no plano)}
\end{aligned}
$$

Exemplo 3 – solução

Calculando a derivada segunda

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(y)$$

$$= 1 \qquad \text{(contínua no plano)}$$

Conteúdo

Notações

Derivadas terceiras

$$\frac{\partial^3 f}{\partial x^3} \qquad f_{xxx}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} \qquad f_{xxy}$$

Derivadas quartas

$$\frac{\partial^4 f}{\partial x^4} \qquad f_{xxx}$$

$$\frac{\partial^4 f}{\partial y^2 \partial x^2} \qquad f_{xxyy}$$

Ordem de Derivação

Desde que todas as derivadas envolvidas sejam contínuas

a ordem de derivação é irrelevante

Exemplo 4

Encontre a derivada $f_{yxyz}(x, y, z)$ da função $f(x, y, z) = 1 - 2xy^2z + x^2y$

Queremos calcular

$$f_{yxyz}(x, y, z) = \Big(\Big(\Big(f_y(x, y, z) \Big)_x \Big)_y \Big)_z = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)\right)$$

Exemplo 4 – solução

$$f(x, y, z) = 1 - 2xy^2z + x^2y$$

$$f_y(x, y, z) = \frac{\partial}{\partial y}\left(1 - 2xy^2z + x^2y\right) = 0 - 2x2yz + x^2 = -4xyz + x^2$$

$$f_{yx}(x, y, z) = \frac{\partial}{\partial x}\left(-4xyz + x^2\right) = -4yz + 2x$$

$$f_{yxy}(x, y, z) = \frac{\partial}{\partial y}\left(-4yz + 2x\right) = -4z + 0 = -4z$$

$$f_{yxyz}(x, y, z) = \frac{\partial}{\partial z}\left(-4z\right) = -4$$

Conteúdo

Lista Mínima

Cálculo Vol. 2 do Thomas 12ª ed. – Seção 14.3

1. Estudar o texto da seção
2. Resolver os exercícios: 43-45, 51-54, 57, 61

Atenção: A prova é baseada no livro, não nas apresentações