Forma Polar Números Complexos

Luis Alberto D'Afonseca

Cálculo de Funções de Várias Variáveis – I

17 de agosto de 2025

Conteúdo

Plano Complexo

$$z = x + yi \qquad \in \mathbb{C}$$

$$(x, y) \qquad \in \mathbb{R}^2$$

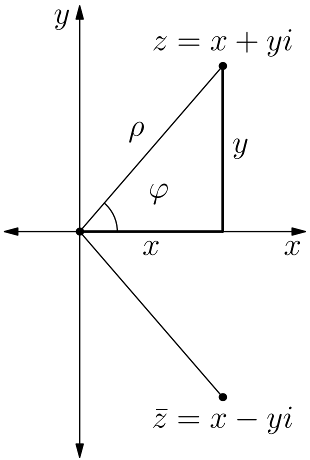
Parte real $\quad \mathrm{Re}(z) = x$

Parte imag $\quad \mathrm{Im}(z) = y$

Módulo $\quad \rho = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$

Argumento $\quad \varphi = \arg(z) = \operatorname{arctg}\left(\frac{y}{x}\right)$

Conjugado $\quad \bar{z} = x - yi$



Forma Polar ou Trigonométrica de um Número Complexo

Usando

$$x = \rho\cos(\varphi) \qquad \text{e} \qquad y = \rho\,\mathrm{sen}(\varphi)$$

Podemos escrever o número complexo $z = x + yi$ como

$$z = \rho\big(\cos(\varphi) + i\,\mathrm{sen}(\varphi)\big)$$

Abreviação

$$z = \rho\,[\cos\,(() + i\,\mathrm{sen}\,(()]\;\varphi)$$

Notação de Steinmetz (circuitos de corrente alternada)

$$z = \rho\angle\varphi$$

Exemplo 1

Escreva os números complexos na sua forma polar

1. $z_1 = \sqrt{3} + i$
2. $z_2 = -2 + 2\sqrt{3}i$

Exemplo 1 – Solução – 1

$$z_1 = \sqrt{3} + i \qquad a = \mathrm{Re}(z) = \sqrt{3} \qquad b = \mathrm{Im}(z) = 1$$

$$\rho = |z_1| = \sqrt{a^2 + b^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\mathrm{sen}(\varphi) = \frac{b}{\rho} = \frac{1}{2} \qquad \cos(\varphi) = \frac{a}{\rho} = \frac{\sqrt{3}}{2}$$

$$\varphi = \arg(z_1) = 30^\circ = \frac{\pi}{6}\mathrm{rad}$$

$$z_1 = \rho\left(\cos(\varphi) + i\,\mathrm{sen}(\varphi)\right) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}\right)\right)$$

Exemplo 1 – Solução – 2

$$z_2 = -2 + 2\sqrt{3}i \qquad a = \mathrm{Re}(z_2) = -2 \qquad b = \mathrm{Im}(z_2) = 2\sqrt{3}$$

$$\rho = |z_2| = \sqrt{a^2 + b^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\mathrm{sen}(\varphi) = \frac{\mathrm{Im}(z_2)}{|z_2|} = \frac{b}{\rho} = \frac{1}{2} \qquad \cos(\varphi) = \frac{\mathrm{Re}(z_2)}{|z_2|} = \frac{a}{\rho} = \frac{\sqrt{3}}{2}$$

$$\varphi = \arg(z_2) = 30^\circ = \frac{\pi}{6}\mathrm{rad}$$

$$z_1 = \rho\left(\cos(\varphi) + i\,\mathrm{sen}(\varphi)\right) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}\right)\right)$$

Conteúdo

Justificativa

$$
\begin{aligned}
z_1 z_2 &= \rho_1 \big[\cos(\varphi_1) + i\,\mathrm{sen}(\varphi_1)\big]\rho_2\big[\cos(\varphi_2) + i\,\mathrm{sen}(\varphi_2)\big] \\
&= \rho_1\rho_2\big[\cos(\varphi_1) + i\,\mathrm{sen}(\varphi_1)\big]\big[\cos(\varphi_2) + i\,\mathrm{sen}(\varphi_2)\big] \\
&= \rho_1\rho_2\big[\cos(\varphi_1)\cos(\varphi_2) + i\,\cos(\varphi_1)\,\mathrm{sen}(\varphi_2) \\
&\qquad + i\,\mathrm{sen}(\varphi_1)\cos(\varphi_2) + i^2\,\mathrm{sen}(\varphi_1)\,\mathrm{sen}(\varphi_2)\big] \\
&= \rho_1\rho_2\big[\cos(\varphi_1)\cos(\varphi_2) - \mathrm{sen}(\varphi_1)\,\mathrm{sen}(\varphi_2) \\
&\qquad + i\big(\cos(\varphi_1)\,\mathrm{sen}(\varphi_2) + \mathrm{sen}(\varphi_1)\cos(\varphi_2)\big)\big] \\
&= \rho_1\rho_2\big[\cos(\varphi_1 + \varphi_2) + i\,\mathrm{sen}(\varphi_1 + \varphi_2)\big]
\end{aligned}
$$

Multiplicação

$$
\begin{aligned}
z_1 z_2 \cdots z_n = {} & \rho_1 \big[\cos(\varphi_1) + i \operatorname{sen}(\varphi_1) \big] \\
& \rho_2 \big[\cos(\varphi_2) + i \operatorname{sen}(\varphi_2) \big] \\
& \vdots \\
& \rho_n \big[\cos(\varphi_n) + i \operatorname{sen}(\varphi_n) \big] \\
= {} & \rho_1 \rho_2 \cdots \rho_n \big[\cos(\varphi_1 + \varphi_2 + \cdots + \varphi_n) + i \operatorname{sen}(\varphi_1 + \varphi_2 + \cdots + \varphi_n) \big]
\end{aligned}
$$

Exemplo 2

Calcule o produto dos números complexos

$$z_1 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\operatorname{sen}\left(\frac{\pi}{6}\right)\right)$$

$$z_2 = 5\left(\cos\left(\frac{\pi}{3}\right) + i\,\operatorname{sen}\left(\frac{\pi}{3}\right)\right)$$

Exemplo 2 – Solução

$$z_1 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}\right)\right) \text{ e } z_2 = 5\left(\cos\left(\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{3}\right)\right)$$

$$\rho_1 = |z_1| = 2 \qquad \varphi_1 = \arg(z_1) = \frac{\pi}{6}$$

$$\rho_2 = |z_2| = 5 \qquad \varphi_2 = \arg(z_2) = \frac{\pi}{3}$$

$$\rho = |z_1 z_2| = \rho_1\rho_2 = 2 \times 5 = 10$$

$$\varphi = \arg(z_1 z_2) = \varphi_1 + \varphi_2 = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi + 2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

$$z = z_1 z_2 = \rho\left(\cos(\varphi) + i\,\mathrm{sen}(\varphi)\right) = 10\left(\cos\left(\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}\right)\right) = 10i$$

Exemplo 3

Calcule o produto dos números complexos

$$z_1 = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}\right)\right)$$

Exemplo 3 – Solução

$$z_1 = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right)\right) \text{ e } z_2 = \sqrt{2}\left(\cos\left(\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}\right)\right)$$

$$\rho_1 = 3 = |z_1| \qquad \varphi_1 = \arg(z_1) = \frac{2\pi}{3}$$

$$\rho_2|z_2| = \sqrt{2} \qquad \varphi_2 = \arg(z_2) = \frac{\pi}{2}$$

$$\rho = \rho_1\rho_2 = 3\sqrt{2}$$

$$\varphi = \varphi_1 + \varphi_2 = \frac{2\pi}{3} + \frac{\pi}{2} = \frac{4\pi + 3\pi}{6} = \frac{7\pi}{6}$$

$$z = z_1 z_2 = \rho\left(\cos(\varphi) + i\,\mathrm{sen}(\varphi)\right) = 3\sqrt{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{7\pi}{6}\right)\right)$$

Conteúdo

Divisão

A divisão de

$$z_1 = \rho_1 \left(\cos(\varphi_1) + i \operatorname{sen}(\varphi_1)\right)$$

por

$$z_2 = \rho_2 \left(\cos(\varphi_2) + i \operatorname{sen}(\varphi_2)\right)$$

é

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \left[\cos(\varphi_1 - \varphi_2) + i \operatorname{sen}(\varphi_1 - \varphi_2)\right]$$

Exemplo 4

Divida

$$z_1 = 6\left(\cos\left(\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{2\pi}{3}\right)\right)$$

por

$$z_2 = 3\left(\cos\left(\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}\right)\right)$$

Exemplo 4 – Solução

$$z_1 = 6\left(\cos\left(\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{2\pi}{3}\right)\right) \text{ e } z_2 = 3\left(\cos\left(\frac{\pi}{2}\right) + i\,\text{sen}\left(\frac{\pi}{2}\right)\right)$$

$$\rho_1 = 6 \qquad \varphi_1 = \frac{2\pi}{3} \qquad\qquad \rho_2 = 3 \qquad \varphi_2 = \frac{\pi}{2}$$

$$\rho = \frac{\rho_1}{\rho_2} = \frac{6}{3} = 2$$

$$\varphi = \varphi_1 - \varphi_2 = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{4\pi - 3\pi}{6} = \frac{\pi}{6}$$

$$z = \frac{z_1}{z_2} = \rho\left(\cos(\varphi) + i\,\text{sen}(\varphi)\right) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\,\text{sen}\left(\frac{\pi}{6}\right)\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 3 + i$$

Conteúdo

Lista Mínima

Atenção: A prova é baseada no livro, não nas apresentações